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A Directional-Entropy Framework for Polarization Disorder: Information-Theoretic Insights from von Mises–Fisher Statistics

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Abstract

The degree of polarization is usually obtained from the coherency matrix or, equivalently, from the mean Stokes vector of a partially polarized optical field. Here, we adopt a complementary geometric viewpoint by representing normalized local Stokes vectors as random directions on the Poincaré sphere. When these directions are described by an effective unimodal von Mises–Fisher distribution, the concentration parameter gives a direct one-to-one description of the degree of polarization through the mean resultant length. This formulation does not define a new independent polarization observable. Instead, it gives the degree of polarization a rotation-invariant information-theoretic meaning, expressed in terms of directional concentration and angular disorder. Within this framework, we derive closed-form expressions for the differential entropy of the von Mises–Fisher distribution and for the Kullback–Leibler divergence between two directional polarization states. The symmetrized divergence further incorporates both differences in concentration and relative orientation on the Poincaré sphere. We also discuss the assumptions, range of validity, and limitations of the single-vMF model, particularly in relation to Gaussian field statistics and more general directional models needed for anisotropic or multimodal polarization fluctuations. Overall, this formalism establishes a model-based theoretical framework for entropy and divergence descriptors of unimodal directional polarization and suggests natural extensions toward mixtures of vMF, Bingham, or Kent distributions.

Keywords: polarization; directional statistics; von Mises–Fisher distribution; entropy; Kullback–Leibler divergence; polarimetric imaging



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1. Introduction

In light scattering experiments, the polarization is closely linked to the microscopic organization and anisotropy of matter and provides information that goes beyond intensity-only measurements. When light is partially polarized, the electric-field components fluctuate randomly in amplitude and phase, and these fluctuations are conventionally described through the coherency matrix or, equivalently, through the ensemble-averaged Stokes vector [1–3]. The limitations of a purely second-order description, particularly for non-Gaussian polarization statistics, have also been emphasized [4]. The statistical properties of instantaneous and normalized Stokes parameters have been investigated in detail for partially polarized Gaussian fields [5–11]. Under the circular complex Gaussian assumption,

the coherency matrix provides a complete second-order description of the field, and the degree of polarization can be expressed through its invariant eigenvalue structure [2,3,12].

However, the classical coherency-matrix framework mainly characterizes the overall statistical mixing of the optical field and does not explicitly describe how polarization states are organized on the Poincaré sphere. Information-theoretic approaches have already provided important complementary views of partial polarization and coherence. Réfrégier et al. introduced entropy-based descriptions of partially polarized light and examined their relevance for statistical optical processing [13]. This approach was later extended to non-Gaussian optical fields through a Kullback-relative-entropy-based definition of the degree of polarization [14]. For Gaussian fields, the degree of polarization was related to the Kullback divergence between the observed field distribution and a fully depolarized reference state [15,16], and to the corresponding differential Shannon entropy [17]. Invariant and mutual-information-based degrees of coherence were also proposed to separate basis-dependent correlations from intrinsic polarization properties [18,19]. These studies established a rigorous link between polarization, coherence, and information theory, but they were formulated primarily in the electric-field probability space rather than directly on the Poincaré sphere. Invariant polarimetric contrast measures have also been extended to three-dimensional Gaussian optical fields [20].

This distinction requires some care. In standard coherence theory, the Stokes parameters are ensemble-averaged second-order quantities and define a single mean polarization state. A distribution of Stokes directions becomes meaningful only when the population from which local or instantaneous Stokes vectors are sampled has been clearly specified, such as a temporal sub-ensemble, repeated realizations, or a spatial neighborhood in a polarimetric image. The directional model introduced below should therefore be understood as a complement to the coherency-matrix formalism. It provides an effective spherical description of the dispersion of normalized Stokes directions under explicitly stated assumptions. Entropy-based representations have also been developed in optical scattering and Mueller polarimetry, notably through the Cloude–Pottier formalism [21].

Directional statistics provide a natural framework for random variables constrained to circles and spheres [22–24], with the von Mises–Fisher distribution serving as a standard rotationally symmetric model on $\mathbb{S}^2$ [23,24]. When applied to polarization [25], the normalized Stokes vector can be treated as a random point on the Poincaré sphere, fluctuating around a mean orientation owing to the temporal or spatial variability of the medium. The vMF model then provides a compact description of these fluctuations through a mean direction and concentration parameter, which can be interpreted as a directional analog of coherence and is in one-to-one correspondence with the DoP.

This study aims to bring together the directional-statistics description of polarization with an entropy-based analysis of polarization disorder. To this end, we derive analytical expressions for the differential entropy of polarization states described by a von Mises–Fisher distribution, as well as for the Kullback–Leibler divergence between two such states. The goal is not to define a new polarization observable that would be independent of the degree of polarization nor to suggest that a single-vMF model can account for all higher-order statistics of the optical field. Rather, we use this model to give a geometrically consistent interpretation of polarization order, directional entropy, and inter-state divergence on the Poincaré sphere. This formulation also clarifies where the single-vMF approximation is useful and where it becomes limited. It thereby motivates more general directional models for anisotropic, axial, or multimodal polarization fluctuations.

2. Background: Polarization, Entropy, and Coherence

Partially polarized light is most conveniently described by the coherency matrix of the electric field [1–3]. For a quasi-monochromatic wave with complex vector field $\mathbf{E} = [E_x, E_y]^T$, the coherency matrix is defined as

$$\mathbf{W} = \begin{pmatrix} \langle |E_x|^2 \rangle & \langle E_x E_y^* \rangle \\ \langle E_x^* E_y \rangle & \langle |E_y|^2 \rangle \end{pmatrix} \quad (1)$$

where $\langle \cdot \rangle$ denotes ensemble or temporal averaging, and * indicates complex conjugation. For Gaussian fluctuations, $\mathbf{W}$ fully specifies the polarization state. The associated degree of polarization (DoP), ρ , is obtained from the invariants of $\mathbf{W}$ as

$$\rho = \sqrt{1 - 4 \frac{\det(\mathbf{W})}{[\text{tr}(\mathbf{W})]^2}} \quad \text{with } 0 \leq \rho \leq 1 \quad (2)$$

where $\rho = 1$ corresponds to a fully polarized (rank-1) state, and $\rho = 0$ indicates complete depolarization [2,3,12]. This scalar quantity is widely used as a compact indicator of the polarization order.

2.1. Polarization and Entropy

Réfrégier and Goudail [15,16] proposed an information-theoretic interpretation of the degree of polarization by considering the Kullback relative entropy between the probability density functions of optical fields.

Let $P_{obs}(\mathbf{E})$ denote the Gaussian PDF associated with an observed field of coherency matrix $\mathbf{W}_{obs}$, and let $P_u(\mathbf{E})$ denote the PDF of a totally unpolarized reference field with the same total intensity. The Kullback relative entropy is defined as [15]

$$H_K(P_{obs}||P_u) = \int P_{obs}(\mathbf{E}) \log \frac{P_{obs}(\mathbf{E})}{P_u(\mathbf{E})} d\mathbf{E} \quad (3)$$

which is non-negative and vanishes only when the two distributions coincide. In this framework, the DoP can be recast as

$$\rho^2 = 1 - \exp[-H_K(P_{obs}||P_u)] \quad (4)$$

This relation provides a direct link between the statistical distance to a completely depolarized state and the measured degree of polarization [15,16].

2.2. Shannon Entropy of Partially Polarized Light

In a complementary approach, Réfrégier and Morio [17] introduced the Shannon entropy of partially polarized Gaussian light as a global disorder measure. For a normalized field PDF $P(\mathbf{E})$, the differential Shannon entropy is

$$H_S(P) = - \int P(\mathbf{E}) \log P(\mathbf{E}) d\mathbf{E}. \quad (5)$$

When the field is Gaussian with coherency matrix $\mathbf{W}$, this leads to:

$$H_S = \log [(\pi e)^2 \det(\mathbf{W})] \quad (6)$$

or equivalently, in terms of the total intensity and DoP:

$$H_S = 2\log(eI) + \log(1 - \rho^2) + \log\left(\frac{\pi^2}{4}\right) \quad (7)$$

using $I = \text{tr}(\mathbf{W})$ and $\det(\mathbf{W}) = \frac{I^2}{4}(1 - \rho^2)$.

The additive constant $\log\left(\frac{\pi^2}{4}\right)$ depends on the normalization convention used for the complex Gaussian field. This constant affects the absolute value of the differential entropy but not its dependence on the total intensity or on the degree of polarization.

2.3. Gaussian Field Statistics and Higher-Order Polarization Information

This distinction between field-based and directional descriptions must therefore be made carefully. The Stokes parameters are bilinear functions of the electric-field components and, after ensemble averaging, constitute second-order field quantities. In contrast, the probability distribution and covariance of instantaneous Stokes parameters involve fourth- and higher-order field moments. Such induced statistics have been derived explicitly for Gaussian optical fields by Barakat, Brosseau, Eliyahu, and co-workers [5–11].

For a proper circular complex Gaussian field, these higher-order moments are fixed by second-order correlations through Gaussian moment factorization. Consequently, the statistics of the instantaneous Stokes variables are fully determined by the coherency matrix, and their directional dispersion does not provide independent information beyond this second-order description [4–11]. In this regime, the directional representation should therefore be understood as a geometric reparameterization of information already contained in the coherency matrix.

By contrast, independent higher-order polarization information can arise for non-Gaussian fields, in which higher-order cumulants are not constrained by the coherency matrix [4,14]. The single-vMF model used here, however, remains a two-parameter directional description defined by a mean direction and a concentration parameter. Once the relation between the degree of polarization and the concentration parameter is imposed, these quantities are determined by the mean normalized Stokes vector and do not, by themselves, encode arbitrary non-Gaussian higher-order statistics.

This interpretation is consistent with Charnotskii's analysis [4], which shows that polarization information extending beyond the Stokes vector and the coherency matrix requires statistical descriptors that cannot be reduced to the Gaussian second-order field model.

3. The von Mises–Fisher (vMF) Model on the Poincaré Sphere

3.1. Model Assumptions, Definition, and Physical Interpretation

The von Mises–Fisher formulation is introduced here as an effective directional model for polarization fluctuations on the Poincaré sphere. Its use first requires specifying the statistical population to which the model applies. In standard coherence theory, the Stokes vector obtained from the coherency matrix is an ensemble-averaged second-order quantity

$$\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \langle |E_x|^2 \rangle + \langle |E_y|^2 \rangle \\ \langle |E_x|^2 \rangle - \langle |E_y|^2 \rangle \\ 2 \text{Re}(\langle E_x^* E_y \rangle) \\ -2 \text{Im}(\langle E_x^* E_y \rangle) \end{pmatrix} \quad (8)$$

and therefore represents a mean polarization descriptor rather than a distribution of states. In the present framework, the random variable is instead the normalized local Stokes direction

$$\mathbf{s} = (S_1, S_2, S_3)^T / \sqrt{S_1^2 + S_2^2 + S_3^2} \quad (9)$$

sampled from a clearly defined sub-ensemble, such as a short temporal window, repeated realizations of a fluctuating field, or a spatial neighborhood in a polarimetric image. Under this interpretation, the classical degree of polarization can be identified with the mean resultant length of the directional distribution only as a modeling assumption, not as a general consequence of the coherency matrix alone. This assumption is appropriate when local realizations may be regarded as instantaneously, or locally, fully polarized states whose directions fluctuate on the Poincaré sphere and when the intensity is either constant or statistically independent of these directional fluctuations. If intensity and direction are statistically independent, the measured degree of polarization may be written as

$$\rho = \|\langle \mathbf{s} \rangle\| \quad (10)$$

If the directional distribution is further assumed to be rotationally symmetric about a single dominant mean direction, the degree of polarization can be mapped onto the concentration of a von Mises–Fisher distribution. This mapping should therefore not be read as an identity that follows generally from the coherency matrix. Rather, it defines an effective directional representation in which the measured degree of polarization determines a concentration parameter under the single-vMF assumption (Figure 1).

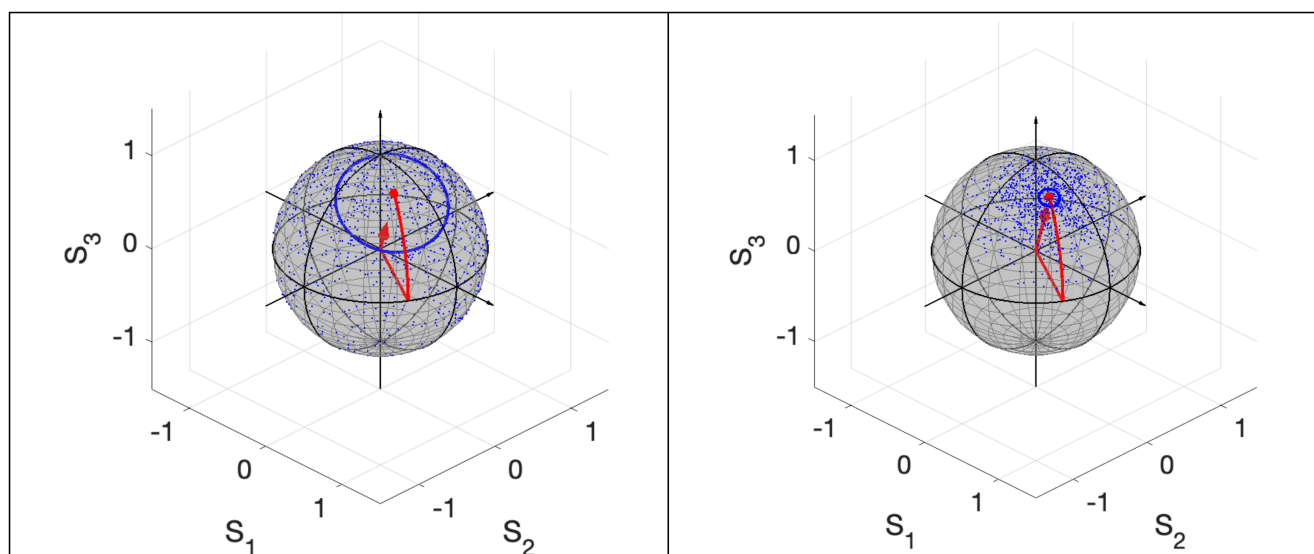


Figure 1. Samples from vMF distributions on the Poincaré sphere with mean direction specified by $\theta = 20^\circ$ and $\varphi = -45^\circ$. The red arrow represents the mean resultant vector; **(left)** $\kappa = 1.79$ corresponding to DoP = 0.5; **(right)** $\kappa = 10.0$ corresponding to DoP = 0.900. The circle on the sphere locates the points $\mathbf{x}$ where: $\kappa \mu^T \mathbf{x} = 1$, with superscript T symbolizing vector transpose.

Following the formulation introduced by Salazar-Ariza and Torres [25], the state of polarization is then represented by the normalized Stokes vector

$$\mathbf{s} = (s_1, s_2, s_3)^T; \quad \|\mathbf{s}\| = 1 \quad (11)$$

which defines a point on the unit Poincaré sphere.

For partially polarized light, this direction is not fixed: it fluctuates around one or several preferred orientations because of temporal or spatial variability of the field. A

probability density on the unit sphere is therefore required to describe the associated directional statistics.

Among spherical distributions, the von Mises–Fisher law provides the simplest rotationally symmetric model around a mean direction while remaining invariant under rotations of the Poincaré sphere associated with lossless SU(2) transformations (retarders or polarization rotators) acting on the Jones vector [23,24]. Exact distributions of the Stokes parameters and normalized Stokes variables induced by Gaussian fields have been derived in Refs. [5–11]. In the present work, the vMF model is adopted as an effective directional model when the data exhibit a single dominant mean polarization direction without preferred azimuthal anisotropy around it. Under these assumptions, the von Mises–Fisher distribution is the maximum-entropy distribution on $\mathbb{S}^2$ for a prescribed mean resultant vector.

It is well-suited to unimodal, rotationally symmetric directional fluctuations, but it cannot describe axial, ring-like, elliptical, or multimodal distributions. Its density is written as

$$f_{\text{vMF}}(\mathbf{s}; \boldsymbol{\mu}, \kappa) = c_3(\kappa) \exp(\kappa \boldsymbol{\mu}^T \mathbf{s}) \quad (12)$$

where $\boldsymbol{\mu}$ is a unit vector ($\|\boldsymbol{\mu}\| = 1$) specifying the mean polarization direction and $\kappa \geq 0$ is the concentration parameter [23,24]. The normalization constant for the sphere $\mathbb{S}^2$ is

$$c_3(\kappa) = \frac{\kappa}{4\pi \sinh \kappa} \quad (13)$$

so that the density is normalized over the sphere.

Physically, the concentration parameter quantifies how tightly the instantaneous polarization directions cluster around the mean direction. In the limit of vanishing concentration, the distribution tends to the uniform law on the sphere, corresponding to complete depolarization. In the opposite limit, the density becomes increasingly localized around the mean direction and approaches a fully polarized configuration. The vMF concentration parameter, therefore, provides a scalar measure of directional order that is consistent with the spherical geometry of polarization space.

3.2. Relation Between the Concentration Parameter and the Degree of Polarization

A central property of the vMF distribution is that the expectation of $\mathbf{s}$ has the closed-form expression

$$\langle \mathbf{s} \rangle = A(\kappa) \boldsymbol{\mu} \quad (14)$$

where

$$A(\kappa) = \coth \kappa - \frac{1}{\kappa} = \frac{I_{3/2}(\kappa)}{I_{1/2}(\kappa)} \quad (15)$$

and I_ν denotes the modified Bessel function of the first kind and order ν .

The quantity $A(\kappa)$ is the mean resultant length of the directional distribution and satisfies $0 \leq A(\kappa) < 1$.

Under the assumptions stated above, the degree of polarization can be identified with the norm of the mean normalized Stokes vector:

$$\rho = \|\langle \mathbf{s} \rangle\| = A(\kappa) \quad (16)$$

In directional statistics, $A(\kappa)$ is the mean resultant length of the vMF distribution. Within this model, the degree of polarization is mapped onto a directional concentration parameter [24,25].

This relation establishes a one-to-one mapping between κ and ρ : isotropic distributions with $\kappa \rightarrow 0$ correspond to $\rho \rightarrow 0$, whereas strongly concentrated states with $\kappa \rightarrow \infty$ yield

$\rho \rightarrow 1$. For small κ , a leading-order expansion $A(\kappa) \approx \kappa/3$ describes weakly polarized fields, while for large κ , $A(\kappa) \approx 1 - 1/\kappa$ characterizes nearly pure polarization.

In practice, this analytical relation allows κ to be inferred directly from a measured DoP by numerical inversion $\kappa = A^{-1}(\rho)$ using standard approximations or iterative procedures [26].

3.3. Interpretation in Terms of Coherency Matrix

Beyond the first moment, the vMF model also provides a compact expression for the second-order moment of the Stokes direction. From standard results in directional statistics [24], this moment is given by

$$\langle \mathbf{s}\mathbf{s}^T \rangle = \frac{A(\kappa)}{\kappa} \mathbf{I}_3 + \left(1 - \frac{3A(\kappa)}{\kappa}\right) \boldsymbol{\mu}\boldsymbol{\mu}^T \quad (17)$$

where $\mathbf{I}_3$ denotes the 3×3 identity matrix.

The directional second-order moment $\langle \mathbf{s}\mathbf{s}^T \rangle$ should not be confused with the optical coherency matrix $\mathbf{W}$. The latter describes second-order correlations between the complex electric-field components, whereas $\langle \mathbf{s}\mathbf{s}^T \rangle$ characterizes the angular dispersion of normalized Stokes directions on the Poincaré sphere. These quantities therefore describe different statistical objects: one is a field-correlation descriptor, while the other is a directional descriptor constructed from normalized polarization states.

Within the vMF model, $\langle \mathbf{s}\mathbf{s}^T \rangle$ describes the spread of local polarization directions around $\boldsymbol{\mu}$. It does not replace $\mathbf{W}$ and cannot, by itself, reconstruct the full coherency matrix. The two descriptions are linked only through the mean Stokes vector and the degree of polarization, and this link holds under the effective assumptions stated above.

Because the vMF model is defined directly on $\mathbb{S}^2$, all realizations of $\mathbf{s}$ remain physically admissible, $\|\mathbf{s}\| = 1$. This contrasts with some Cartesian or Gaussian approximations, which may generate unphysical states with $\|\mathbf{s}\| > 1$. This geometric consistency is essential when comparing polarization maps, performing spatial statistics on the sphere, or interpreting directional disorder in biomedical and atmospheric scattering applications. The resulting nonlinear relationship between the degree of polarization and the effective concentration parameter is illustrated in Figure 2.

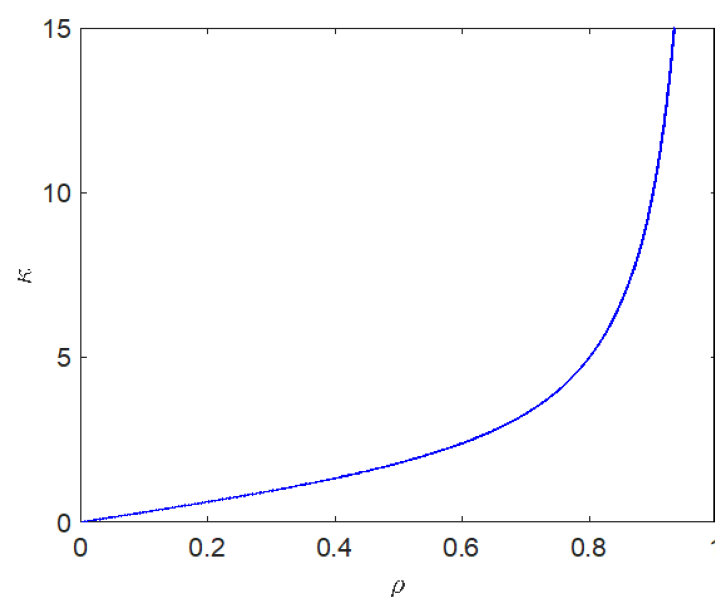


Figure 2. Concentration parameter κ as a function of the DoP under the single-vMF assumption $\rho = A(\kappa)$. The low-DoP regime is approximately linear.

4. Entropic Analysis in the von Mises–Fisher Framework

The von Mises–Fisher model offers a direct way to connect the geometry of the Poincaré sphere with the information-theoretic measures of polarization disorder. In this section, we derive closed-form expressions for the differential entropy of the vMF-distributed polarization states and for the Kullback–Leibler (KL) divergence between two such states. We also discuss how these quantities describe the randomness of, and similarity between, polarization states.

4.1. Differential Entropy of the vMF Distribution

Consider a vMF density $f_{\text{vMF}}(\mathbf{s}; \boldsymbol{\mu}, \kappa)$ on $\mathbb{S}^2$ as defined in Section 3. The differential Shannon entropy of this distribution is as follows:

$$H_{\text{vMF}}(\kappa) = - \int_{\mathbb{S}^2} f_{\text{vMF}}(\mathbf{s}) \log f_{\text{vMF}}(\mathbf{s}) d\mathbf{s} \quad (18)$$

Using the explicit form of the density and following the standard definition of differential entropy for directional distributions [24], this integral admits the closed-form expression

$$H_{\text{vMF}}(\kappa) = -\log\left(\frac{\kappa}{4\pi \sinh \kappa}\right) - \kappa A(\kappa) \quad (19)$$

where $A(\kappa) = \coth \kappa - 1/\kappa$ is the mean resultant length introduced previously. This formula shows that the entropy depends only on the concentration parameter κ , and therefore only on the degree of polarization $\rho = A(\kappa)$.

As expected, $H_{\text{vMF}}(\kappa)$ decreases monotonically with κ : as the distribution becomes more concentrated around its mean direction, directional disorder is reduced and information becomes more localized on the sphere [24]. The asymptotic behaviors highlight two limiting regimes:

$$H_{\text{vMF}}(\kappa) \approx \begin{cases} \log(4\pi) - \frac{1}{6}\kappa^2, & \kappa \ll 1, \\ 1 + \log(2\pi) - \log(\kappa), & \kappa \gg 1, \end{cases} \quad (20)$$

showing that for weakly polarized states ($\kappa \rightarrow 0$), $H_{\text{vMF}} \rightarrow \log(4\pi)$, which corresponds to an isotropic uniform distribution on the sphere, while for strongly polarized states ($\kappa \rightarrow \infty$) the entropy decreases logarithmically as the distribution collapses toward a single direction.

As a differential entropy defined with respect to the surface-area measure on $\mathbb{S}^2$, H_{vMF} may become negative for highly concentrated distributions; this does not imply an unphysical negative disorder [24].

By combining Equations (19) and (16), one obtains the entropic dependence on the degree of polarization

$$H_{\text{vMF}}(\rho) = -\log\left[\frac{A^{-1}(\rho)}{4\pi \sinh A^{-1}(\rho)}\right] - A^{-1}(\rho)\rho \quad (21)$$

This provides an explicit model-dependent mapping between the measurable degree of polarization and the directional entropy under the single-vMF assumption. It should be distinguished from the Gaussian field entropy of Equations (6) and (7): the latter quantifies the statistical mixing of the electric field through $\det \mathbf{W}$, whereas H_{vMF} quantifies the angular spread of normalized Stokes directions on the Poincaré sphere.

4.2. Kullback–Leibler Divergence Between Polarization States

We now consider two polarization states modeled by vMF distributions $f_1(\mathbf{s}) = f_{\text{vMF}}(\mathbf{s}; \boldsymbol{\mu}_1, \kappa_1)$ and $f_2(\mathbf{s}) = f_{\text{vMF}}(\mathbf{s}; \boldsymbol{\mu}_2, \kappa_2)$. Their Kullback–Leibler (KL) divergence is defined as

$$D_{\text{KL}}(f_1 \| f_2) = \int_{\mathbb{S}^2} f_1(\mathbf{s}) \log \frac{f_1(\mathbf{s})}{f_2(\mathbf{s})} d\mathbf{s} \quad (22)$$

Evaluating the log-ratio of the two vMF densities gives

$$D_{\text{KL}}(f_1 \| f_2) = \log \frac{c_3(\kappa_1)}{c_3(\kappa_2)} + \kappa_1 A(\kappa_1) - \kappa_2 A(\kappa_1) (\boldsymbol{\mu}_1^T \boldsymbol{\mu}_2) \quad (23)$$

The first two terms reflect the differences in concentration (i.e., directional entropy) between the two states, whereas the last term introduces their geometric alignment on the Poincaré sphere [24]. Thus, D_{KL} jointly accounts for changes in both the amount of directional disorder and the relative orientation of the polarization states. More generally, using statistical divergences to describe the geometry of parametric distributions is fully consistent with the principles of information geometry [27,28].

Because the KL divergence is not symmetric, practical image-comparison applications may benefit from symmetrized forms such as the symmetric KL divergence

$$D_{\text{sym}}(f_1, f_2) = \frac{1}{2} [D_{\text{KL}}(f_1 \| f_2) + D_{\text{KL}}(f_2 \| f_1)] \quad (24)$$

which yields

$$D_{\text{sym}}(f_1, f_2) = \frac{1}{2} (\kappa_1 \boldsymbol{\mu}_1 - \kappa_2 \boldsymbol{\mu}_2)^T (A(\kappa_1) \boldsymbol{\mu}_1 - A(\kappa_2) \boldsymbol{\mu}_2) \quad (25)$$

For visualization purposes, it can be useful to introduce a compensated divergence

$$D_c(f_1, f_2) = \frac{D_{\text{sym}}(f_1, f_2)}{\kappa_1 + \kappa_2} \quad (26)$$

provided that $\kappa_1 + \kappa_2 > 0$. When $\kappa_1 = \kappa_2 = 0$, both distributions are uniform and we set $D_c(f_{\text{uniform}}, f_{\text{uniform}}) = 0$.

This normalization should be regarded as a practical contrast index rather than as a fundamental information-theoretic distance. It compensates the growth of D_{sym} with concentration and emphasizes angular differences between mean directions. For finite values of κ_1 and κ_2 , D_c remains below its asymptotic upper bound. In the special case of antipodal mean directions, D_c tends to 1 only in the limit $\kappa_1, \kappa_2 \rightarrow \infty$.

In Figure 3, the mean polarization direction is parametrized by the azimuth ψ that controls the orientation of the polarization ellipse and χ its ellipticity, according to

$$\boldsymbol{\mu}(\psi, \chi) = \begin{pmatrix} \cos(2\chi) \cos(2\psi) \\ \cos(2\chi) \sin(2\psi) \\ \sin(2\chi) \end{pmatrix} \quad (27)$$

The symmetrized divergence D_{sym} provides a practical descriptor for comparing polarization states before and after interaction with anisotropic, turbid, or structured media. By combining sensitivity to directional separation and polarization concentration, it can capture changes induced by birefringence, optical activity, transmission, or reflection processes, while also supporting the analysis of spatially heterogeneous polarimetric responses. In this sense, the proposed directional divergences may complement biomedical Mueller-polarimetric descriptors that are sensitive to structural tissue alterations, inflammation, and dysplasia [29], and may also be useful for polarization control, correction, diagnostic imaging, and polarization-encoded image clustering.

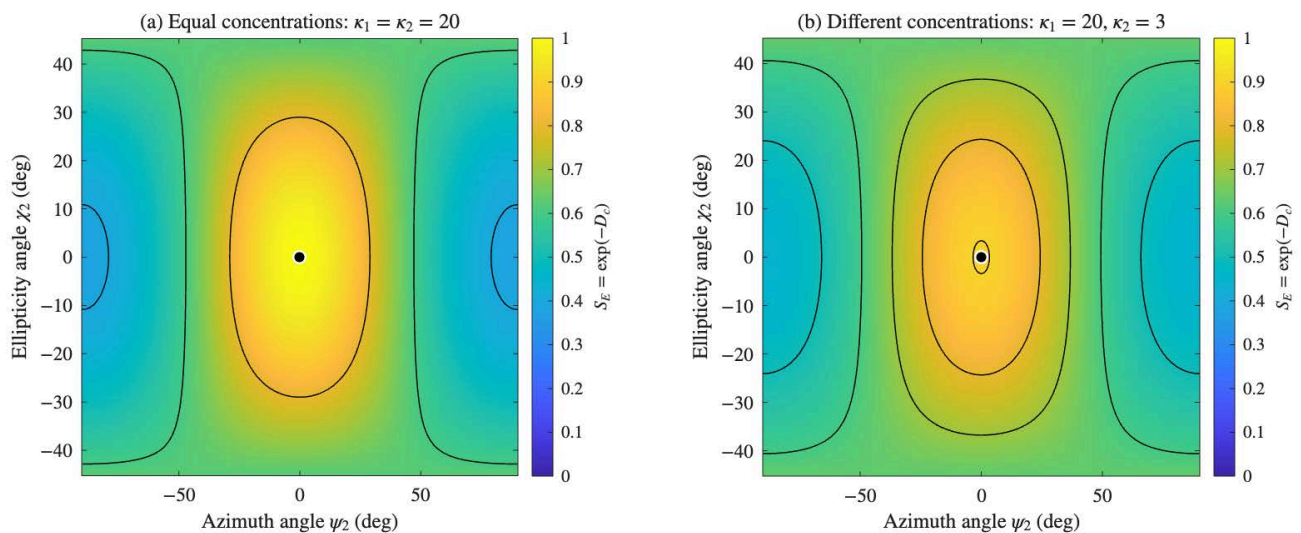


Figure 3. Similarity $S_E = \exp(-D_c)$ between a reference vMF polarization distribution centered on horizontal linear polarization, $\mu_1 = (1, 0, 0)^T$, and a second distribution whose mean direction spans the Poincaré sphere through the azimuth ψ_2 and the ellipticity angle χ_2 . (a) Equal concentration parameters, $\kappa_1 = \kappa_2 = 20$, such that the similarity variation results exclusively from the relative orientation of the mean polarization states. (b) Different concentration parameters, $\kappa_1 = 20$ and $\kappa_2 = 3$, illustrating the combined influence of directional separation and polarization concentration. The black dot identifies the reference polarization state, and the contour lines indicate constant-similarity levels.

This symmetric or compensated divergence can be converted into a similarity index bounded between 0 and 1, for instance,

$$S_E = \exp[-D_c(f_1, f_2)] \quad (28)$$

which takes the value 1 when the two states coincide and decreases as their mean directions become different. Since the compensated divergence is bounded above by unity, S_E remains in the interval $[e^{-1}, 1]$.

4.3. Physical Interpretation and Comparison with Classical Entropy

Within the classical Gaussian framework, entropy is computed from the eigenvalues of the coherency matrix and measures the degree of mixing of the optical field, irrespective of the distribution of polarization orientations on the Poincaré sphere. In contrast, the vMF entropy H_{vMF} characterizes the angular spread of the Stokes vector itself, i.e., how polarization directions populate the sphere.

Gaussian field entropy and the directional entropy considered here quantify distinct statistical objects and should therefore be viewed as complementary descriptors [17]. Entropy-based measures have also been introduced in Mueller polarimetry and optical scattering, notably through the Cloude–Pottier formalism [21]; however, these scattering entropies must be clearly distinguished from the differential entropy of a directional probability density.

Within the single-vMF model, directional entropy quantifies the angular dispersion of polarization states on the Poincaré sphere. Because this entropy depends only on the concentration parameter κ , which is in one-to-one correspondence with the degree of polarization (DoP), it is fully determined by the DoP under this assumption. The value of the model, therefore, lies not in distinguishing distributions with the same DoP but in providing a clear geometric interpretation of polarization disorder when the data are organized around a single dominant direction. More complex angular structures, including

axial, annular, or multimodal distributions, would require more general directional models such as the Bingham or Kent distributions, or mixtures of vMF laws. In this sense, the vMF framework enriches entropy-based polarimetric analysis by incorporating the spherical geometry of polarization, thereby yielding physically meaningful and rotation-invariant descriptors for imaging and scattering applications.

5. Numerical and Experimental Validation

The entropic quantities introduced in the von Mises–Fisher framework, namely the directional entropy $H_{\text{vMF}}(\kappa)$, the symmetric divergence D_{sym} , the compensated divergence D_c , and the associated similarity index S_E , can be related to standard polarimetric observables and used to build practical imaging metrics. In typical experiments, partially polarized fields are characterized by measuring the Stokes parameters or the coherency matrix, from which the degree of polarization $\rho(\mathbf{r})$ can be estimated at each spatial position $\mathbf{r}$. Within this model, ρ and κ are in one-to-one correspondence through $\rho = A(\kappa)$. Therefore, when the fluctuations of the normalized Stokes vector are reasonably described by a unimodal vMF distribution, measurements of the DoP can be used to estimate an effective concentration parameter and the corresponding H_{vMF} .

5.1. Reconstruction of Local vMF Parameters

Starting from a measured Stokes image, the DoP is calculated as

$$\rho(\mathbf{r}) = \frac{\sqrt{S_1^2(\mathbf{r}) + S_2^2(\mathbf{r}) + S_3^2(\mathbf{r})}}{S_0(\mathbf{r})} \quad (29)$$

The local concentration parameter is then retrieved by numerical inversion

$$\kappa(\mathbf{r}) = A^{-1}[\rho(\mathbf{r})] \quad (30)$$

using standard iterative schemes, such as Newton or Halley iterations, or established approximations for the inverse mean resultant relation [26]. Once $\kappa(\mathbf{r})$ is obtained, H_{vMF} is given by

$$H_{\text{vMF}}(\mathbf{r}) = -\log \left[\frac{\kappa(\mathbf{r})}{4\pi \sinh \kappa(\mathbf{r})} \right] - \kappa(\mathbf{r}) A[\kappa(\mathbf{r})] \quad (31)$$

The local mean polarization direction is calculated from the measured Stokes parameters as

$$\boldsymbol{\mu}(\mathbf{r}) = \frac{(S_1(\mathbf{r}), S_2(\mathbf{r}), S_3(\mathbf{r}))^T}{\sqrt{S_1^2(\mathbf{r}) + S_2^2(\mathbf{r}) + S_3^2(\mathbf{r})}} \quad (32)$$

The local mean direction is required for all divergence-based quantities, since D_{KL} , D_{sym} , D_c , and S_E depend on the scalar product $\boldsymbol{\mu}_1^T \boldsymbol{\mu}_2$. Consequently, entropy maps can be reconstructed from the DoP alone under the single-vMF assumption, whereas divergence and similarity maps require both the local degree of polarization and the local normalized Stokes direction.

In spatial imaging applications, the statistical population associated with each position may be defined using a sliding neighborhood. However, neighboring Stokes vectors may be correlated, so the number of pixels in a sliding window should not automatically be interpreted as the number of statistically independent realizations.

Within a window $W(\mathbf{r})$ of N_W pixels, the sample mean resultant vector is calculated as

$$\bar{\mathbf{R}}(\mathbf{r}) = \frac{1}{N_W} \sum_{i=1}^{N_W} \mathbf{s}_i \quad (33)$$

from which the finite-sample estimators are obtained as

$$\hat{\boldsymbol{\mu}}(\mathbf{r}) = \frac{\bar{\mathbf{R}}(\mathbf{r})}{\|\bar{\mathbf{R}}(\mathbf{r})\|}, \quad A(\hat{\kappa}(\mathbf{r})) = \|\bar{\mathbf{R}}(\mathbf{r})\| \quad (34)$$

The concentration inferred directly from the pixelwise DoP,

$$\kappa_\rho(\mathbf{r}) = A^{-1}(\rho(\mathbf{r})) \quad (35)$$

and the local directional concentration,

$$\hat{\kappa}_{dir}(\mathbf{r}) = A^{-1}(\|\bar{\mathbf{R}}(\mathbf{r})\|) \quad (36)$$

describe different statistical properties and need not coincide in general. The experimental clustering and the local vMF maps presented below were computed from $\hat{\kappa}_{dir}(\mathbf{r})$ and $\hat{\boldsymbol{\mu}}(\mathbf{r})$ estimated within the sliding windows. Consequently, both the concentration map reported in Figure 4f and the directional-entropy map shown in Figure 4c were derived from the local directional concentration $\hat{\kappa}_{dir}$, rather than from the pixelwise concentration κ_ρ . These maps therefore characterize the spatial dispersion of normalized Stokes directions within each analysis window, rather than the pixelwise depolarization inferred from the local degree of polarization.

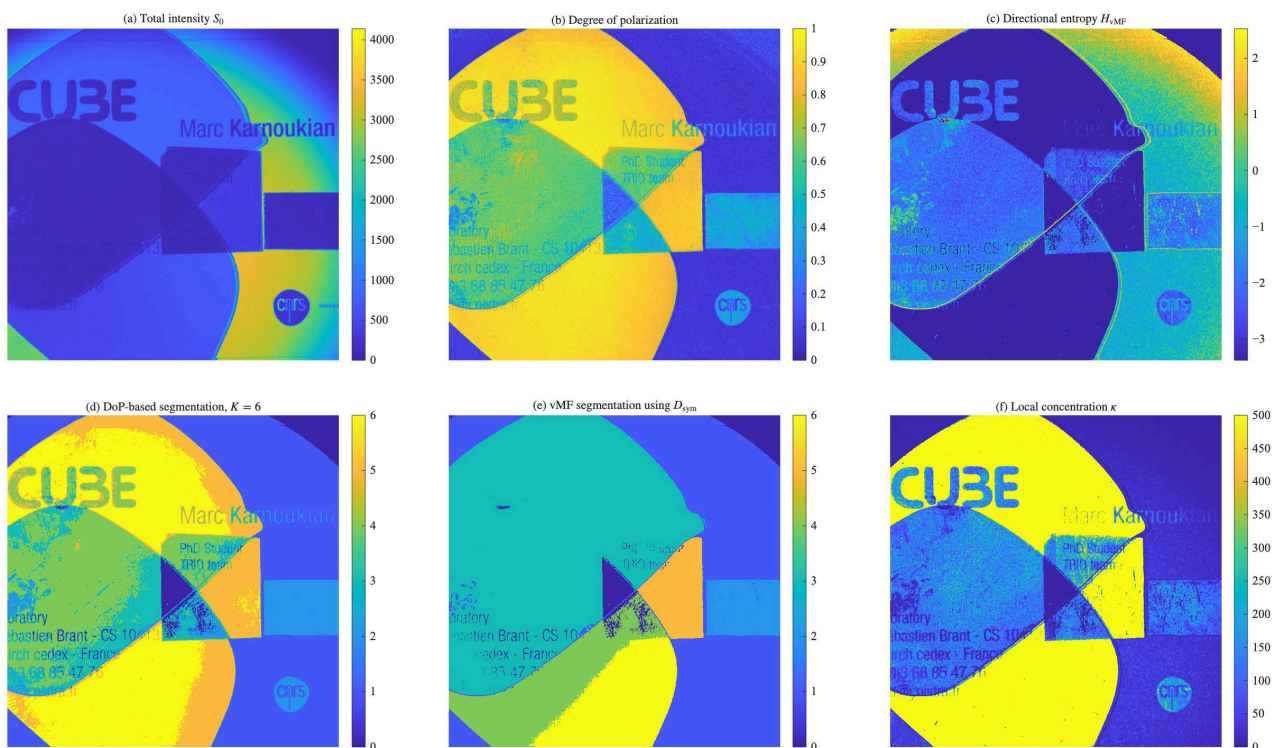


Figure 4. Experimental polarimetric image segmentation using the proposed vMF framework. (a) Total-intensity image S_0 . (b) Degree-of-polarization map. (c) Directional-entropy map H_{vMF} . (d) Reference segmentation based only on the degree of polarization. (e) Segmentation obtained using the divergence-based vMF clustering procedure. (f) Local concentration parameter $\hat{\kappa}_{dir}$. The divergence-based approach distinguishes polarization regions having comparable degrees of polarization but different mean directions on the Poincaré sphere.

The mean direction was considered undefined when the local resultant length was below a prescribed threshold.

This formulation makes explicit the statistical population represented by the local vMF distribution and provides a direct implementation pathway for polarimetric imaging.

Multidimensional statistics, spatial autocorrelation functions, and power spectral densities of Stokes parameters in polarization-speckle fields have been investigated in Refs. [30,31].

5.2. Numerical Consistency Verification

To assess the internal consistency of the proposed reconstruction procedure, synthetic ensembles of normalized Stokes vectors were generated from von Mises–Fisher distributions with known parameters on the Poincaré sphere. Four configurations were considered, combining different concentration parameters κ and distinct mean polarization directions μ . These configurations were chosen to represent several levels of polarization order together with clearly separated polarization states.

For each prescribed vMF distribution, $N = 150$ normalized Stokes vectors were independently generated at each pixel of a 60×60 -pixel region according to

$$f_{\text{vMF}}(s; \mu, \kappa) = c_3(\kappa) \exp(\kappa \mu^T s) \quad (37)$$

Each parameter configuration was therefore evaluated from $N_R = 3600$ independent pixelwise realizations. For every realization, the sample mean resultant vector was used to estimate the mean direction, the resultant length, and the concentration parameter $\hat{\kappa}$. The values reported in Table 1 correspond to averages over the 3600 realizations associated with each prescribed distribution. The quantities following the $\pm$ sign denote one standard deviation of the estimated concentration parameter across these realizations. The directional entropy $\hat{H}_{\text{vMF}}$ was evaluated separately for each realization from its estimated concentration parameter $\hat{\kappa}$ and was subsequently averaged over the corresponding region.

Table 1. Summary of the prescribed and reconstructed parameters.

True κ	Estimated $\hat{\kappa}$	True ρ	Estimated $\hat{\rho}$	True H_{vMF}	Estimated $\hat{H}_{\text{vMF}}$	Angular Error in $\hat{\mu}$
0.50	0.543 ± 0.143	0.164	0.178	2.49	2.48	20.54°
2.00	2.033 ± 0.196	0.537	0.543	2.05	2.04	5.63°
5.00	5.077 ± 0.423	0.800	0.803	1.23	1.21	2.93°
12.00	12.174 ± 1.003	0.917	0.918	0.35	0.34	1.79°

Numerical consistency verification of the single-vMF reconstruction procedure. For each prescribed distribution, the estimates were obtained from 3600 independent realizations, each containing $N = 150$ normalized Stokes vectors. Reported values are regional averages, and uncertainties on $\hat{\kappa}$ denote one standard deviation across the 3600 realizations.

For each realization, the reconstructed directional entropy was calculated as

$$\hat{H}_{\text{vMF}} = -\log c_3(\hat{\kappa}) - \hat{\kappa} A(\hat{\kappa}). \quad (38)$$

The numerical results support the internal consistency of the reconstruction procedure under the single-vMF assumption. For each synthetic distribution, the prescribed concentration parameter and mean polarization direction were estimated from finite ensembles of normalized Stokes vectors through the sample mean resultant. As expected, the reconstructed directional entropy decreases as the distribution becomes more concentrated, in agreement with its analytical dependence on the concentration parameter.

The larger angular error observed for weak concentration is also consistent with the underlying statistics: when the distribution is broad, the mean direction is less sharply

defined and is therefore more sensitive to finite-sample fluctuations. As the concentration increases, the samples cluster more tightly around the prescribed orientation, leading to a more stable estimate of the mean direction.

Taken together, these results verify the numerical implementation of the reconstruction pipeline

$$\mathbf{s}_i \rightarrow \left(\hat{\boldsymbol{\mu}}, \hat{\kappa} \right) \rightarrow H_{vMF}, D_{sym}, D_c, S_E \quad (39)$$

when the data satisfy the assumptions of the single-vMF model.

5.3. Experimental Polarimetric Image Segmentation

To illustrate the practical value of the proposed framework, a proof-of-concept experiment was carried out on a measured Stokes image containing orthogonal linear polarizers, imperfect left- and right-circular polarizers, and a globally depolarizing support. This configuration was chosen to test whether the directional information introduced by the vMF framework can distinguish polarization states with comparable degrees of polarization but distinct locations on the Poincaré sphere.

The experimental Stokes image consisted of 2048×2048 pixels and was acquired at a wavelength of 650 nm. Local von Mises–Fisher distributions were estimated in 15×15 -pixel sliding windows, yielding maps of the mean polarization direction, the concentration parameter, and the directional entropy. Pixels with $S_0 < 0.03 \max(S_0)$, or with a local mean resultant length lower than 0.03, were excluded. The remaining pixels were partitioned into $K = 6$ clusters using D_{sym} as the assignment dissimilarity. The first prototype was randomly selected among the valid local distributions, and the remaining prototypes were initialized sequentially using a farthest-point strategy based on D_{sym} .

The choice $K = 6$ was made to slightly over-segment the scene, thereby allowing the robustness of the proposed similarity measure to be assessed. The near-unity similarity between prototypes 3 and 4 observed in Figure 5 confirms that these clusters correspond to nearly identical polarization states and would naturally merge if a smaller number of clusters were used.

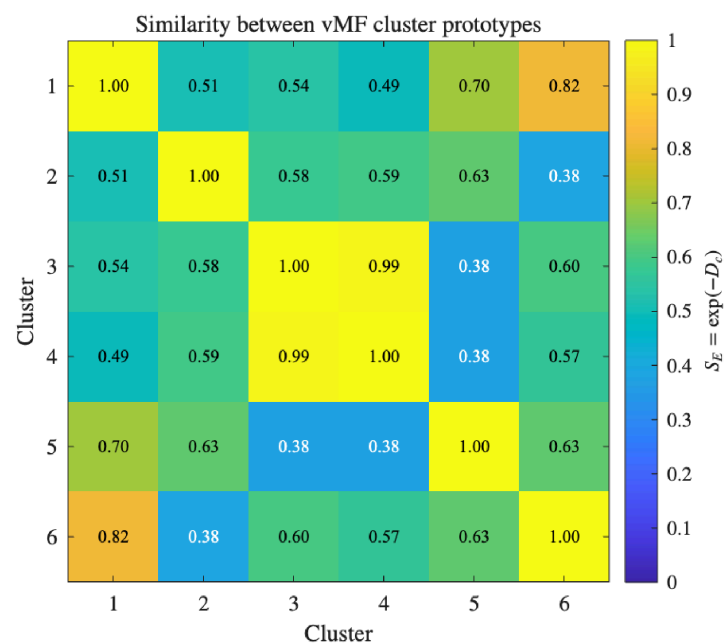


Figure 5. Similarity matrix between the reconstructed vMF cluster prototypes computed using S_E . The similarity jointly accounts for differences in concentration and angular separation between the mean polarization directions.

For each cluster, the prototype mean resultant vector was updated by averaging the vectors $A(\kappa_i)\mu_i$ associated with the local distributions assigned to that cluster. The updated mean direction was obtained by normalizing this average vector, whereas the concentration parameter was recovered by numerical inversion of its norm through $A(\kappa)$.

The procedure was stopped when the fraction of reassigned pixels fell below 0.4% of the valid pixels, or after 30 iterations. The segmentation reported in Figure 4e was obtained using D_{sym} as the assignment dissimilarity. The compensated divergence D_c and the associated similarity $S_E = \exp(-D_c)$ were subsequently used to compare the reconstructed cluster prototypes. The alternative dissimilarity

$$d_E = 1 - S_E = 1 - \exp(-D_c) \quad (40)$$

may also be used in the same iterative clustering framework but was not employed for the segmentation shown in Figure 4e.

Figure 4 summarizes the full processing pipeline and the corresponding segmentation results. The total-intensity image S_0 in Figure 4a identifies the polarization elements placed on the depolarizing support. The degree-of-polarization map in Figure 4b clearly separates the highly polarized regions associated with the linear and circular polarizers from the support, which exhibits substantially lower polarization values.

The directional-entropy map in Figure 4c is consistent with the expected behavior of the single-vMF model. Highly ordered polarization states lead to low entropy values, whereas the depolarizing support lies closer to the high-entropy regime associated with a broad distribution of directions on the Poincaré sphere. The overlapping region of circular polarizers occupies an intermediate regime, reflecting a moderate level of directional concentration.

The reference segmentation based only on the degree of polarization is shown in Figure 4d. Because the DoP is a scalar descriptor, regions with comparable polarization amplitudes may be grouped together even when their mean directions differ on the Poincaré sphere. By contrast, the proposed vMF-based segmentation in Figure 4e uses both the concentration parameter and the mean polarization direction. It can therefore distinguish orthogonal linear polarizers and opposite circular polarization states, even when their degrees of polarization are comparable. The concentration map in Figure 4f further highlights the spatial variation in directional order and the transition between the polarized elements and the depolarizing support.

Figure 5 presents the similarity matrix computed between the reconstructed cluster prototypes using $S_E = \exp(-D_c)$.

Identical clusters exhibit unit similarity, whereas orthogonal linear polarization states and opposite circular polarization states produce lower similarity values because of their large angular separation on the Poincaré sphere. The similarity additionally varies when the distributions have different concentration parameters. The proposed similarity index therefore jointly accounts for differences in directional concentration and mean polarization orientation.

Figure 6 displays the mean polarization directions associated with the reconstructed vMF clusters on the Poincaré sphere. The linear polarizers are located near opposite positions on the equatorial plane, whereas the left- and right-circular polarizers are located close to opposite poles. The color scale represents the estimated concentration parameter κ , thereby providing a simultaneous visualization of polarization orientation and directional order. The depolarizing support is characterized by a low concentration, for which the estimated mean direction has limited physical significance.

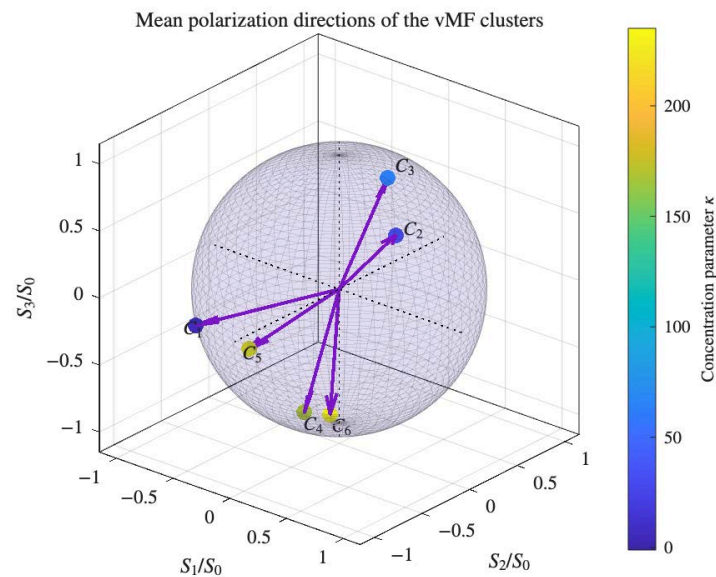


Figure 6. Mean polarization directions of the reconstructed vMF cluster prototypes represented on the Poincaré sphere. The color scale indicates the estimated concentration parameter κ , providing a joint representation of polarization orientation and directional concentration.

The experimental results show that the divergence-based representation can distinguish polarization regions having similar degrees of polarization but different mean directions on the Poincaré sphere. The scope, limitations, and possible extensions of this result are discussed in the following section.

6. Discussion and Outlook

The numerical verification confirms the internal consistency of the reconstruction procedure under the single-vMF assumption, while the experimental proof of concept shows the practical usefulness of directional divergences for polarimetric image analysis. The framework proposed here should therefore be understood as an extension of classical polarization descriptors toward a geometric and information-theoretic interpretation of disorder on the Poincaré sphere. Within this model, the concentration parameter encodes directional order, whereas the associated entropy and divergences express this order through rotation-invariant information measures.

This formulation is not intended to introduce a new polarization observable or an alternative theory of polarization. Under the single-vMF assumption, the directional entropy is entirely determined by the concentration parameter and, consequently, by the degree of polarization. Its value, therefore, lies not in adding an independent scalar descriptor but in providing a geometrically consistent interpretation of the degree of polarization in terms of angular concentration and directional disorder.

The additional discrimination observed in the experimental segmentation arises from the directional divergences rather than from entropy alone. These divergences combine differences in concentration with the relative orientation of the mean polarization directions. Thus, regions with similar degrees of polarization, and therefore similar directional entropies, can still be separated when their mean directions occupy different positions on the Poincaré sphere. In this way, the proposed descriptors extend conventional scalar measures by introducing a geometrically meaningful comparison between polarization distributions.

This distinction also clarifies the relation between field-based and directional descriptions. Entropies derived from the coherency matrix quantify the statistical mixing of the optical field through the eigenvalues of the coherency matrix, whereas the present formulation describes the angular organization of normalized local Stokes vectors within a specified temporal, spatial, or ensemble population. These descriptions concern different statistical objects and should not be treated as equivalent. In particular, the directional second-order moment is not, in general, identical to the 3×3 polarization submatrix of a Mueller matrix; any rigorous connection between them would require an explicit statistical model for the corresponding input and output polarization states.

The experimental results demonstrate the usefulness of the approach in a controlled configuration containing linear and circular polarization elements. They do not, however, imply that arbitrary scattering or biological media are intrinsically described by a single von Mises–Fisher distribution. The single-vMF model assumes a unimodal, rotationally symmetric population around one dominant mean direction and therefore cannot represent axial, annular, anisotropic, or multimodal directional structures that may occur in complex scattering media or heterogeneous tissues.

For this reason, the effective concentration parameter and entropy must be interpreted with caution when the local distribution is broad or nearly isotropic. In such cases, the mean direction is weakly defined and more sensitive to finite-sample fluctuations. In spatial imaging, the estimates also depend on the size of the analysis window and on correlations between neighboring pixels, which should be considered when interpreting local vMF parameters.

More general directional models offer a natural extension of this work. Mixtures of vMF distributions could describe multiple preferred polarization directions, while Bingham or Kent distributions [23,24] could account for axial symmetry and anisotropic angular spread. Model-selection criteria and goodness-of-fit tests would help determine whether a local population is adequately represented by a single-vMF law before computing the associated entropy and divergences.

From an imaging perspective, the proposed similarities and divergences can be incorporated into segmentation, classification, image retrieval, or machine-learning workflows. Their main advantage is that they compare local polarization distributions through both directional concentration and relative orientation, while remaining invariant under a common rotation of the polarization reference frame. Future work should evaluate these descriptors on scattering phantoms, biological tissues, and Mueller-polarimetric data and compare them with established measures such as the degree of polarization and polarization indices [2,3], as well as Cloude–Pottier entropy [21].

More broadly, this formulation provides a basis for developing an information geometry of polarization distributions on the Poincaré sphere. Extensions based on Fisher–Rao metrics, geodesic distances, statistical hypothesis testing, or optimal transport could offer richer tools for analyzing spatial, spectral, and temporal variations in polarization-sensitive measurements.

7. Conclusions

This work presents an effective directional-statistics framework for describing ensembles of normalized local Stokes vectors on the Poincaré sphere. Under the explicitly stated single-vMF assumptions, the degree of polarization can be mapped onto a concentration parameter and an associated directional entropy. This entropy should not be regarded as an independent polarization observable. Its role is instead to provide a geometric and information-theoretic interpretation of the degree of polarization in terms of angular concentration and directional disorder.

The main practical contribution lies in the proposed divergences, which combine differences in concentration with the relative orientation of the mean polarization directions. Numerical tests and an experimental proof of concept show that these quantities can support the comparison and segmentation of polarization distributions. The proposed framework, therefore, complements, rather than replaces, the coherency-matrix formalism. Its extension to anisotropic or multimodal directional models will be necessary for more complex polarization fluctuations.

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