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Analytical and Semi-Analytical Formulas for the Self and Mutual Inductances of Concentric Coplanar Ordinary and Bitter Disk Coils

Slobodan Babic

Independent Researcher, 53 Berlioz, 101, Montreal, QC H3E 1N2, Canada; e-slobobob@yahoo.com

Abstract: In this paper, analytical and semi-analytical formulas are presented for the self- and mutual inductance of thin ordinary disk coils and thin Bitter disk coils. The coils lie concentrically in a plane. The ordinary coils are coils with constant current density. The current density of a current carrying Bitter disc is not uniform across its cross-sectional area, but it is a function of the ratio of the inner diameter of the disk to an arbitrary radius within the disk. In this paper, we show the possibility to calculate the mutual and self-inductance of thin disk coils from the real coils of the cross-sections using some valuable conditions. The formulas for the mutual inductance and the self-inductance were obtained in the semi-analytic form as the combination of the elliptic integral of the second kind and a simple integral for the ordinary disk coils. The mutual inductance and self-inductance were obtained in the analytical form as the elliptic integral of the second kind for the Bitter disk coils. The formula for the self-inductance of the ordinary full disk was obtained in the close form. All formulas are given in remarkably simple form and give perfectly accurate results with a significantly small computational time. All cases of either regular or singular (disks in contact or overlapping) are covered. Many presented examples show the excellent numerical agreement with previously published methods.

Keywords: ordinary disk coils; Bitter disk coils; mutual inductance; self-inductance

1. Introduction

Several monographs and papers are devoted to calculating the self and the mutual inductance for ordinary circular coils (massive coils of the rectangular cross-section, thin wall solenoids, disk coils) with the azimuthal current density [1–13]. The conventional coils used in many applications, such as all ranges of transformers, generators, motors, current reactors, magnetic resonance applications, antennas, coil guns, medical electronic devices, superconducting magnets, tokamaks, electronic and printed circuit board design, plasma science, etc., are very well-known. Additionally, there are circular coils with a not uniform current density (massive coils of the rectangular cross-section, disk coils) which are interesting from an engineering aspect. These coils are the well-known Bitter coils [14–24], which supply extremely high magnetic fields of up to 45 T. Bitter magnets are constructed of circular conducting metal plates and insulating spacers stacked in a helical configuration rather than coils of wire. The current flows in an azimuthal path through the plates and is not uniform. The current density is the inverse function of its radius, which changes between the fixed radii of the real coil. In this paper, we provide quite a simple method to calculate the mutual and self-inductance for these two types of coils under some conditions when these real coils could be treated as thin disk coils. For this statement, the previous assumptions must be satisfied.

In Ref. [12], J.T. Conway proposes the analytical solutions for the self- and mutual inductances of ordinary concentric coplanar disk coils. He gives the excellent solution as generalized hypergeometric functions which are closely related to elliptic integrals. The method used is a Legendre polynomial expansion of the inductance integral, which

renders all integrations straightforward. In this paper, we provide remarkably simple solutions for calculating the self- and mutual inductance of the ordinary concentric coplanar and the Bitter disk coils. In the case of the ordinary concentric coplanar disks, the solutions are obtained in semi-analytical and analytical form. The solutions are obtained over the elliptic integral of the second kind and one simple integral the kernel of which is integrable over all intervals of integration. In the case of the concentric coplanar Bitter disk coils, the solutions are obtained in the analytical form over the elliptic integral of the second kind. All cases given in Ref. [12] are verified and confirmed by the presented approach. Additionally, the methods given in Refs. [8–11] are used to confirm all calculations for the ordinary concentric coplanar disks. The self- and the mutual inductance of the concentric coplanar disk coils are verified by the methods given in Refs. [17–19]. With obtained amazingly easy analytical and semi-analytical expressions, we solved many examples which show excellent agreement with already given results for the real coils of the rectangular cross-section under some valuable conditions.

2. Basic Expressions

2.1. Ordinary Concentric Coplanar Disks

The ordinary coils of the rectangular cross-section are coils with constant azimuthal current densities. The mutual inductance between ordinary concentric circular coils of the rectangular cross-section is given by [8]:

$$M = \mu_0 \int_{R_1}^{R_2} \int_{R_3}^{R_4} \int_{z_1}^{z_2} \int_{z_3}^{z_4} \int_0^{\pi/2} \frac{J_I J_{II} \cos(\theta) r_1 dr_1 r_2 dr_2 z_I dz_I z_{II} dz_{II} d\theta}{r}, \tag{1}$$

where

$$r = \sqrt{r_1^2 - 2r_1 r_2 \cos(\theta) + r_2^2 + (z_{II} - z_I)^2},$$

$$J_I = \frac{N_1}{(R_2 - R_1)(z_2 - z_1)} = \text{const.},$$

$$J_{II} = \frac{N_1}{(R_4 - R_3)(z_4 - z_3)} = \text{const.},$$

and

- N_1 and N_2 are equally spaced turns of coils,
- R_1 and R_2 are radii (m) of the first coil with the axial height $(z_2 - z_1)$ (in m),
- R_3 and R_4 are radii (m) of the second coil with the axial height $(z_4 - z_3)$ (in m),
- J_I is the azimuthal current density in the first coil (in A/m^2),
- J_{II} is the azimuthal current density in the second coil (in A/m^2),
- $\mu_0 = 4\pi \cdot 10^{-7}$ H/m,
- θ is the angle in cylindrical coordinates.

Here and further on, the cylindrical coordinates are used.

The difference between the real coil circular plate (which can be considered as the coil of the rectangular cross-section) and the thin circular disk is in the dependence on the relation of thickness b to diameter D. There is a rule of thumb like the empirical formulae:

(A) if $D/b \leq 10$, then it is a plate;
(B) if $D/b > 10$, then it is a disk.

If $(z_2 - z_1) = b_1$, $(z_4 - z_3) = b_2$, and $D_1 = 2R_1$, $D_2 = 2R_2$, $D_3 = 2R_3$ and $D_4 = 2R_4$ satisfy the condition (B), then Equation (1) becomes the mutual inductance between two ordinary coplanar disks [10,11]. In Figure 1, the constant azimuthal current is shown for one thin disk coil. Then:

$$M = \frac{\mu_0 N_1 N_2}{(R_2 - R_1)(R_4 - R_3)} \int_0^\pi \int_{R_1}^{R_2} \int_{R_3}^{R_4} \frac{r_1 r_2 \cos(\theta) dr_1 dr_2 d\theta}{\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta) + r_2^2}}, \tag{2}$$

where

- N_1 and N_2 are equally spaced turns of disks,
- R_1 and R_3 are inner radii of disks (in m),
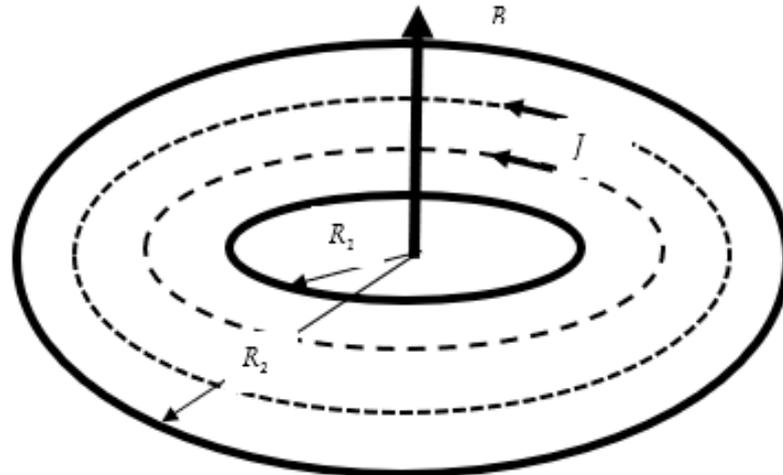- R_2 and R_4 are outer radii of disks (in m).

Figure 1. Thin disk coil carrying the constant azimuthal current density, *J*.

The self-inductance of a disk coil with equally spaced turns N and R_1 and R_2 as inner and outer radii, respectively, is given by [10]:

$$L = \frac{\mu_0 N^2}{(R_2 - R_1)^2} \int_0^\pi \int_{R_1}^{R_2} \int_{R_1}^{R_2} \frac{r_1 r_2 \cos(\theta) dr_1 dr_2 d\theta}{\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta) + r_2^2}}, \tag{3}$$

This formula can be obtained directly from Equation (2) with $R_1 = R_3$, $R_2 = R_4$, $N_1 = N_2 = N$.

2.2. Concentric Coplanar Bitter Disks

The Bitter coils of the rectangular cross-section are coils, where the azimuthal current densities in the coils conductor are inversely proportional to their radii. The mutual inductance between two concentric Bitter circular coils of the rectangular cross-section is given by [17–19]:

$$M = \mu_0 \int_{R_1}^{R_2} \int_{R_3}^{R_4} \int_{z_1}^{z_2} \int_{z_3}^{z_4} \int_0^{\pi/2} \frac{J_I J_{II} \cos(\theta) r_1 dr_1 r_2 dr_2 z_I dz_I z_{II} dz_{II} d\theta}{r}, \tag{4}$$

where

$$r = \sqrt{r_1^2 - 2r_1 r_2 \cos(\theta) + r_2^2 + (z_{II} - z_I)^2}$$

and J_I, J_{II} are the current densities, which are not constant [13].

The total current through one Bitter coil [13] is:

$$I = \int_{R_1}^{R_2} h(r) \, j(r) dr, \tag{5}$$

where $j(r)$ is the current density pro-unit area, and $h(r)$ is the thickness of the coil.

The value of the total current could be arbitrary. Obviously, the higher this value is, the greater is the value of the magnetic field intensity inside the Bitter plate. However, the upper value of the total current is strongly limited by different factors. The most important factors are the conductive heating of the coil and the stresses in the coil due to the Lorenz pressure of the magnetic field. The analytical approach can only be applied to the coil having cylindrical symmetry, which means that there is no change in geometry when

rotating about one axis and when magnetoresistance phenomenon, eddy currents, plastic deformations, and thermal stresses are neglected [13,14].

The current density distribution (Figure 2) that guarantees the constant electric potential over all contours with different radii is [13]:

$$j(r) = \frac{C}{r},$$ (6)

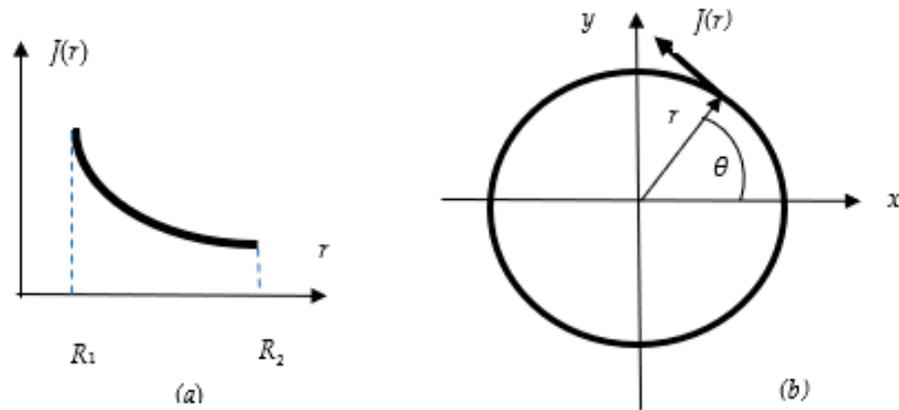where C is the constant which depends on the thickness function $h(r)$.

Figure 2. (**a**) The non-constant current density in the Bitter coil. (**b**) Dependence of the azimuthal current density of the angle θ.

If the thicknesses of the first and second Bitter coil of the rectangular cross-section are constants from [17–19], we obtain the current densities in them:

$$J_I = \frac{N_1}{(z_2 - z_1)\ln\frac{R_2}{R_1}} \cdot \frac{1}{r_1} = \frac{C_1}{r_1},$$ (7)

$$J_{II} = \frac{N_2}{(z_4 - z_3)\ln\frac{R_4}{R_3}} \cdot \frac{1}{r_2} = \frac{C_2}{r_2},$$ (8)

where

$$C_1 = \frac{N_1}{(z_2 - z_1)\ln\frac{R_2}{R_1}} = \text{const.}, \quad C_2 = \frac{N_2}{(z_4 - z_3)\ln\frac{R_4}{R_3}} = \text{const.}$$

and

- N_1 and N_2 are equally spaced turns of coils,
- R_1 and R_2 are radii (in m) of the first coil with the axial height $(z_2 - z_1)$ (in m),
- R_3 and R_4 are radii (in m) of the second coil with the axial height $(z_4 - z_3)$ (in m),
- J_I is the non-constant current density in the first coil (in A/m^2),
- J_{II} is the non-constant current density in the second coil (in A/m^2),

If $(z_2 - z_1) = b_1$, $(z_4 - z_3) = b_2$, and $D_1 = 2R_1$, $D_2 = 2R_2$, $D_3 = 2R_3$ and $D_4 = 2R_4$ satisfy the condition (B), Equation (4) becomes the mutual inductance between two coplanar Bitter disks [22]. In Figure 3, the non-uniform azimuthal current is shown for one thin disk coil. Then,

$$M_B = \frac{\mu_0 N_1 N_2}{\ln\frac{R_2}{R_1}\ln\frac{R_4}{R_3}} \int_0^\pi \int_{R_1}^{R_2} \int_{R_3}^{R_4} \frac{\cos(\theta)dr_1 dr_2 d\theta}{\sqrt{r_1^2 - 2r_1 r_2\cos(\theta) + r_2^2}},$$ (9)

and the self-inductance of the Bitter disk coil is given by [19]:

$$L_B = \frac{\mu_0 N^2}{\left[\ln \frac{R_2}{R_1}\right]^2} \int_0^\pi \int_{R_1}^{R_2} \int_{R_1}^{R_2} \frac{\cos(\theta)dr_1 dr_2 d\theta}{\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta) + r_2^2}}. \tag{10}$$
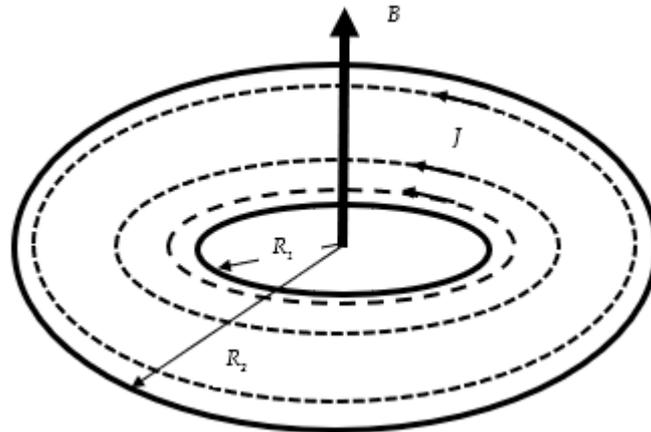
Figure 3. Thin disk Bitter coil carrying the azimuthal current density *J* and which has a variation according to the radius *r*, $R_1 \leq r \leq R_2$ which is different.

This formula can be obtained directly from Equation (9), putting $R_1 = R_3$, $R_2 = R_4$, $N_1 = N_2$.

3. Analytical Calculations

3.1. Ordinary Concentric Coplanar Disks

Before the first integration, let us make the following substitution $\theta = \pi - 2\beta$, so that Equation (2) reads:

$$M = \frac{-2\mu_0 N_1 N_2}{(R_2 - R_1)(R_4 - R_3)} \int_0^{\pi/2} \int_{R_1}^{R_2} \int_{R_3}^{R_4} \frac{r_1 r_2 \cos(2\beta)dr_1 dr_2 d\beta}{\sqrt{r_1^2 + 2r_1 r_2 \cos(2\beta) + r_2^2}}. \tag{11}$$

The integration over r_1 in Equation (11) [24,25] gives:

$$I_{r_1} = \int_{R_1}^{R_2} \frac{r_1 dr_1}{\sqrt{r_1^2 + 2r_1 r_2 \cos(2\beta) + r_2^2}} = \left[\sqrt{r_1^2 + 2r_1 r_2 \cos(2\beta) + r_2^2} - r_2 \cos(2\beta)\mathrm{arsinh}\frac{r_1 + r_2 \cos(2\beta)}{r_2 \sin(2\beta)}\right]\Bigg|_{R_1}^{R_2}.$$

The integration over r_2 in Equation (11) gives [24,25]:

$$I_{r_2} = \int_{R_3}^{R_4} r_2 I_{r_1} dr_2 = \frac{1}{3} \sum_{n=1}^4 (-1)^{n-1}\left[(l_n^2 + \rho_n^2)\sqrt{l_n^2 + 2l_n \rho_n \cos(2\beta) + \rho_n^2} - l_n^3 \cos(2\beta)\mathrm{arsinh}\frac{\rho_n + l_n \cos(2\beta)}{l_n \sin(2\beta)}\right.$$
$$\left. - \rho_n^3 \cos(2\beta)\mathrm{arsinh}\frac{l_n + \rho_n \cos(2\beta)}{\rho_n \sin(2\beta)}\right],$$

where $\rho_1 = \rho_2 = R_1$, $\rho_3 = \rho_4 = R_2$, $l_1 = l_4 = R_3$, $l_2 = l_3 = R_4$.
The left integration in Equation (11) gives [24,25]:

$$I_3 = \int_0^{\pi/2} \cos(2\beta)I_{r_2} d\theta = \frac{1}{6} \sum_{n=1}^4 (-1)^{n-1}[l_n \rho_n (l_n + \rho_n)E(k_n)$$

$$-\int_0^{\frac{\pi}{2}} l_n^3 \text{arsinh} \frac{\rho_n + l_n \cos(2\beta)}{l_n \sin(2\beta)} d\beta - \int_0^{\pi/2} \rho_n^3 \text{arsinh} \frac{l_n + \rho_n \cos(2\beta)}{\rho_n \sin(2\beta)} d\beta]$$

Finally, the mutual inductance between two coplanar ordinary disks is:

$$M = \frac{\mu_0 N_1 N_2}{3(R_2 - R_1)(R_4 - R_3)} \sum_{n=1}^{4} (-1)^n [l_n \rho_n (l_n + \rho_n) E(k_n) - I_n], \tag{12}$$

where

$$I_n = \int_0^{\pi/2} \left[l_n^3 \text{arsinh} \frac{\rho_n + l_n \cos(2\beta)}{l_n \sin(2\beta)} + \rho_n^3 \text{arsinh} \frac{l_n + \rho_n \cos(2\beta)}{\rho_n \sin(2\beta)} \right] d\beta,$$

$$k_n^2 = \frac{4\rho_n l_n}{(\rho_n + l_n)^2},$$

$$\rho_1 = \rho_2 = R_1, \quad \rho_3 = \rho_4 = R_2, \quad l_1 = l_4 = R_3, \quad l_2 = l_3 = R_4$$

and $E(k)$ is the elliptic integral of the second kind [25,26].

The mutual inductance is obtained in a remarkably simple form over the elliptic integral of the second kind and one simple integral whose kernel function is the continuation and integrability over the domain of the integration, so that Equation (6) is applicable in the regular or the singular cases (disks are in the contact or overlap).

For calculating the self-inductance of the ordinary disk coil, one can use Equation (2) or directly put $R_1 = R_3$, $R_2 = R_4$ and $N_1 = N_2 = N$ into Equation (8), which leads to:

$$L = \frac{2\mu_0 N^2}{3(R_2 - R_1)^2} \left\{ \left(R_1^3 + R_2^3 \right) (2G - 1) + R_2 R_1 (R_2 + R_1) E(k) - I_0 \right\}, \tag{13}$$

where

$$I_0 = \int_0^{\pi/2} \left[R_1^3 \text{arsinh} \frac{R_2 + R_1 \cos(2\beta)}{R_1 \sin(2\beta)} + R_2^3 \text{arsinh} \frac{R_1 + R_2 \cos(2\beta)}{R_2 \sin(2\beta)} \right] d\beta,$$

$$k^2 = \frac{4R_1 R_2}{(R_1 + R_2)^2},$$

and $G = 0.9159655941772190150546035149323841\ldots$ is the Catalan's constant.

We put $R_2 = \alpha R_1$ ($\alpha > 1$) and Equation (13) reads:

$$L = \frac{2\mu_0 N^2 R_1}{3(\alpha - 1)^2} \left\{ \left(\alpha^3 + 1 \right) (2G - 1) + \alpha(\alpha + 1) E(k) - I_0(\alpha) \right\}, \tag{14}$$

where

$$I_0 = \int_0^{\pi/2} \left[R_1^3 \text{arsinh} \frac{R_2 + R_1 \cos(2\beta)}{R_1 \sin(2\beta)} + R_2^3 \text{arsinh} \frac{R_1 + R_2 \cos(2\beta)}{R_2 \sin(2\beta)} \right] d\beta,$$

$$k^2 = \frac{4\alpha}{(\alpha + 1)^2}.$$

From Equation (13) or Equation (14) it is possible to obtain the self-inductance of the full disk ($R_1 = 0$ and $R_2 = R$):

$$L = \frac{2\mu_0 N^2 R}{3} (2G - 1). \tag{15}$$

3.2. Concentric Coplanar Bitter Disks

Before the integration, let us make the substitution, $\theta = \pi - 2\beta$, so that Equation (9) reads:

$$M_B = \frac{-2\mu_0 N_1 N_2}{\ln \frac{R_2}{R_1} \ln \frac{R_4}{R_3}} \int_0^\pi \int_{R_1}^{R_2} \int_{R_3}^{R_4} \frac{\cos(2\beta) dr_1 dr_2 d\beta}{\sqrt{r_1^2 + 2r_1 r_2 \cos(2\beta) + r_2^2}}. \tag{16}$$

The integration over r_1 in Equation (16) gives [24,25]:

$$I_{r_1} = \int_{R_1}^{R_2} \frac{dr_1}{\sqrt{r_1^2 + 2r_1 r_2 \cos(2\beta) + r_2^2}} = \left[\text{arsinh} \frac{r_1 + r_2 \cos(2\beta)}{r_2 \sin(2\beta)} \right] \Bigg|_{R_1}^{R_2}.$$

The integration over r_2 in Equation (16) gives [24,25]:

$$I_{r_2} = \int_{R_3}^{R_4} I_{r_1} dr_2 = \sum_{n=1}^{n=4} (-1)^{n-1} \left[\rho_n \text{arsinh} \frac{l_n + \rho_n \cos(2\beta)}{\rho_n \sin(2\beta)} + l_n \text{arsinh} \frac{\rho_n + l_n \cos(2\beta)}{l_n \sin(2\beta)} \right],$$

where $\rho_1 = \rho_2 = R_1, \rho_3 = \rho_4 = R_2,$ $l_1 = l_4 = R_3$, and $l_2 = l_3 = R_4$.

The integration over β in Equation (16) gives [24,25]:

$$I_3 = \int_{R_3}^{R_4} \cos(2\beta) I_{r_2} d\theta = \sum_{n=1}^{4} (-1)^{n-1} (l_n + \rho_n) E(k_n).$$

Finally, the mutual inductance between two coplanar Bitter disks is:

$$M_B = \frac{2\mu_0 N_1 N_2}{\ln \frac{R_2}{R_1} \ln \frac{R_4}{R_3}} \sum_{n=1}^{4} (-1)^n [l_n + \rho_n] E(k_n), \tag{17}$$

where

$$k_n^2 = \frac{4\rho_n l_n}{(\rho_n + l_n)^2}$$

with $\rho_1 = \rho_2 = R_1,$ $\rho_3 = \rho_4 = R_2,$ $l_1 = l_4 = R_3,$ $l_2 = l_3 = R_4$.

The mutual inductance is obtained in an amazingly simple form over the elliptic integral of the second kind. It is applicable in the regular or the singular cases (disks are in contact or overlap).

For calculating the self-inductance of the Bitter disk coil, one can use Equation (10) or directly put $R_1 = R_3$, $R_2 = R_4$ and $N_1 = N_2 = N$ into Equation (17), which leads to:

$$L_B = \frac{4\mu_0 N^2 (R_1 + R_2)}{\left[\ln \frac{R_2}{R_1} \right]^2} [E(k) - 1], \tag{18}$$

where

$$k^2 = \frac{4R_1 R_2}{(R_1 + R_2)^2}.$$

Thus, either the mutual inductance or the self-inductance for the Bitter disks are obtained in the close form over the elliptic integral of the second kind.

4. Numerical Validation

In this work, all calculations were made in Mathematica programming [27].

Example 1

Calculate the self-inductance of the ordinary real coils with the real thickness $d = 0.02$ m, and the radii, $R_1 = 1$ m, $R_2 = 3$ m, and $N_1 = 1000$.

From [9,11,23], the self-induction of the real coil of the rectangular cross section is:

$$M = 4.094233684833548 \text{ H}.$$

Obviously, $b \ll D_i$, $i = 1, 2$, and the condition (B) is satisfied, so that the real coil can be practically considered as the concentric coplanar thin disk coil, for which applying Equation (13), the self-inductance is:

$$M = 4.120247770949786 \text{ H}.$$

The discrepancy between these two calculations is 0.6314%. The results are in particularly good agreement. We conformed the validity of our approach.

Example 2

Calculate the mutual inductance between two ordinary real coils with the real thickness $d = 0.001$ (m), with the radii $R_1 = 1$ m, $R_2 = 2$ m, $R_3 = 3$ m, $R_4 = 4$ m, and $N_1 = N_2 = 1000$.

From [8,10], the mutual induction of two ordinary real coils of the rectangular cross-section is:

$$M = 1.459023283104694534983992 \text{ H}.$$

Obviously, for $b_i \ll D_i$, $i = 1, 2, 3$, and 4, the two real coils can be considered as the two concentric coplanar thin disk coils for which, applying Equation (12),

$$M = 1.459023375744836092948731 \text{ H}.$$

These results are in excellent agreement.

Example 3

Calculate the self-inductance of the real ordinary coil [9,11] and the self-inductance of the thin disk coil (14). The coil dimensions and the number of turns are as follows:

$$R_1 = 0.1 \text{ m}, \ R_2 = 0.4 \text{ m}, \ N = 100 \ .$$

The self-inductances as the thickness b of the real coil is changing are given in Table 1 compared with the earlier calculations [9,11].

Table 1. The self-inductance when b is extremely decrasing.

b	L_{coli} (mH) [9,11]	$L_{\text{thin-disk}}$ (mH), Equations (13) or (14) (This Work)
10^{-3}	4.642676770970589	4.653592975459863
10^{-4}	4.652496987024428	4.653592975459863
10^{-5}	4.653483320871182	4.653592975459863
10^{-6}	4.653582009322975	4.653592975459863
10^{-7}	4.653591878838192	4.653592975459863
10^{-8}	4.653592865797608	4.653592975459863
10^{-9}	4.6535929644936402	4.653592975459863
10^{-10}	4.653592974363245	4.653592975459863
10^{-11}	4.653592975350210	4.653592975459863
10^{-12}	4.653592975449353	4.653592975459863

Equations (13) or (14) give the self-inductance of the ordinary concentric coplanar thin disk coil (Table 1),

$$L_{\text{thin-disk}} = 4.653592975459863 \text{ mH}.$$

All results are in exceptionally good agreement with the result obtained in the semi-analytical form (14). Thus, the ordinary real coils with the thicknesses which are remarkably

smaller than their radii can be considered as the thin concentric coplanar disk coils. Again, our assumption is proven.

Example 4

In Ref. [12], Conway gives the calculation of normalized self-inductance in terms of the shape factor $R_2 = \alpha R_1$ as $L_N = L/N^2/R_1$ (Table 2).

Table 2. Self-inductance given as $L_N = L/N^2/R_1$ in µH/m.

α	[4]	[12], Equation (58)	Equation (14), This Work
1.5	3.9375	3.937556957309482	3.937556957309482
3	4.1202	4.120247770949786	4.120247770949786
4	4.6535	4.653592975459867	4.653592975459867
7	6.5440	6.544182694169381	6.544182694169381
9	7.8795	7.879438694940537	7.879438694940537
19	14.740	14.73964824153050	14.73964824153050
39	28.628	28.62677689946369	28.62677689946369

Table 2 shows the comparison between the inductance given by Kalantarov [4] and that calculated in [12], with Equation (58) as well as with Equation (14) from this work. The agreement is excellent in all cases.

Example 5

Table 3 shows calculations of the self-inductance of the ordinary disk for various shape factors and for values remarkably close to the logarithmic singularity as $\alpha \to 1$. These calculations are compared to two sets of the results obtained by the Spielrein method, see Ref. [5] and Equation (58) in Ref. [12].

Table 3. Comparison of calculations for the self-inductance (in µH/m) for values of the shape factor, both close to and far from the singularity $\alpha \to 1$.

α	[5]	[12], Equation (58)	Equation (14), This Work
50	36.2822050627	36.28220506268449	36.28220506268449
10	8.5558078657	8.555807865723495	8.555807865723494
3	4.1202478984	4.120247770949786	4.120247770949786
1.5	3.9375565536	3.937556957309482	3.937556957309482
1.1	5.1875898298	5.187589829874826	5.187589829874826
1.01	7.8169836166	7.816983616632973	7.816983616632973
1.001	10.6712873756	10.67128737563754	10.67128737563754
1.00001	16.4524421475	16.45244214746880	16.45244214746880
1.000001	19.3458776688	19.34587766869611	19.34587766869611
1.0000001	22.2393823064	22.23938230721058	22.23938230721058

Again, there is excellent agreement between the results of Equation (14) of this work and those of Equation (58) of Ref. [12], as well as the close agreement is with the results of Ref. [5]. Also, Equation (14) gives much more accurate results (for $\alpha - 1 \to 10^{-5}, 10^{-6}, \ldots$) than Equation (3) of Ref. [11].

Example 6

In this example, we show the performance of the Equation (8) when the singularity is approached ($\alpha - 1 \to 10^{-16}$). We compare the results of Equation (14) with those obtained by the asymptotic Equations (12) and (58) of Ref. [12], see Table 4.

Table 4. Comparison of the results of Equation (14) with those from [12] (Equaition (58) and the asymptotic formula (20) there) as the singularity at the unit shape factor is approached. The results are given in μH/m.

$\alpha - 1$	[12], Equation (58)	[12], Equation (20)	Equation (14), This Work
10^{-1}	4.998068846886786	5.187589829874826	5.187589829874826
10^{-2}	7.784316262102376	7.816983616632973	7.816983616632973
10^{-3}	10.66658208170639	10.67128737563754	10.67128737563754
10^{-6}	19.34586862407991	19.34587766869611	19.34587766869611
10^{-8}	25.13289490994222	25.13289502932351	25.13289502932351
10^{-10}	30.91992242743389	30.91992242891705	30.91992242891705
10^{-12}	36.70694995724185	36.70694995725958	36.70694995725958
10^{-15}	45.38749125213915	45.38749125213918	45.38749125213918
10^{-16}	48.28100501710534	48.28100501710534	48.28100501710534

All the results are in a remarkably good agreement. This shows that Equation (14) has a large range of applications.

Example 7

Let us provide the graphical presentation of the kernel function of the integral I_0 which appears in Equations (13) and (14), for which $R_1 = 1$ m , $R_2 = 2$ m, or $\alpha = 2$.

The kernel function is given by:

$$T_0 = \frac{dI_0}{d\theta} = \text{arsinh}\frac{\alpha + \cos(2\beta)}{\sin(2\beta)} + \alpha^3 \text{arsinh}\frac{1 + \alpha\cos(2\beta)}{\alpha\sin(2\beta)} = \text{arsinh}\frac{2 + \cos(2\beta)}{\sin(2\beta)} + 8\,\text{arsinh}\frac{1 + 2\cos(2\beta)}{2\sin(2\beta)},$$

and is displayed in Figure 4 in the interval $\beta \in [0,\ \pi/2]$. From Figure 4 one can conclude about two discontinuities of the type II because T_0 does not exist at $\beta = 0$, $\beta = \pi/2$.
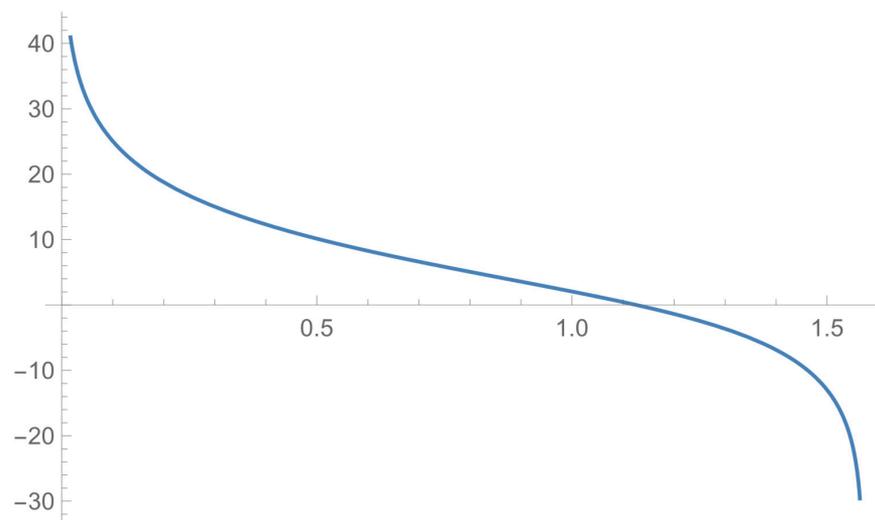

Figure 4. T_0 in the interval $\beta \in [0,\ \pi/2]$.

Using the numerical integration in MATLAB programming, we obtain:

$$I_0 = \int_0^{\frac{\pi}{2}} T_0 d\beta = 9.640116560153466.$$

Using the numerical integration in Mathematica programming, we obtain:

$$I_0 = \int_0^{\frac{\pi}{2}} T_0 d\beta = 9.64011655154581.$$

Both numerical integrations are in very good agreement.

Example 8

Here, we calculate the self-inductance of the full disk coil, for which $R = 1$ m and $N = 100$.

From Equation (15), the self-inductance of the full disk is:

$$L_{\text{full}-\text{disk}} = 6.969570425670744 \text{ mH}.$$

To verify this result, we use Equation (14), and for $R_1 = 10^{-17}$m , $R_2 = 1$ m, $N = 100$, that gives:

$$L_{\text{full}-\text{disk}} = 6.969570425670744 \text{ mH}.$$

We obtain the identical result.

Example 9

Table 5 gives the calculations of the mutual inductance in millihenrys for two concentric coplanar thin disk coils, where the radii are given the fixed values $R_2 = 1$ m, $R_3 = 0.4$ m, and $R_4 = 0.6$ m, whereas R_1 varies between 0.1 and 0.9 m. The number of turns in disks is 100 [12]. This covers all the generic overlap cases.

Table 5. Mutual inductance for all overlap cases.

R_1 (m)	M (mH) [12]	M (mH) Equation (12), This Work
0.1	8.3452226401271722326	8.3452226401271722326
0.2	9.2662149362835422651	9.2662149362835422651
0.3	10.178800706150685814	10.178800706150685814
0.4	10.751261394420867987	10.751261394420867987
0.5	10.014392107083652030	10.014392107083652030
0.6	8.2517781770123374536	8.2517781770123374536
0.7	7.1724760612076564190	7.1724760612076564190
0.8	6.5028431844372044556	6.5028431844372044556
0.9	6.0061106298597996959	6.0061106298597996959

Table 5 shows that all results are identical, obtained by Ref. [12] and Equation (12) from this work. The findings confirm that the relatively simple Equation (12) covers all overlap cases.

Example 10

We calculate the mutual inductance between two disk coils for which $R_1 = 1$ m, $R_2 = 2$ m, $R_3 = 1$ m, $R_4 = 2$ m, and $N_1 = N_2 = N = 1000$.

Obviously, the disks completely overlap, and it is a case of self-inductance. Let us apply Equation (12) for the mutual inductance of the disk coils. It gives:

$$M = L = 3.794801607308613 \text{ H}.$$

Applying Equation (13) for the self-inductance of the disk coil, we obtain:

$$L = 3.794801607308613 \text{ H}.$$

We obtain the identical result.

Thus, we show that simple Equation (13) is general for any case, either for the regular or for the singular. This is also confirmed in the previous examples.

Example 11

Here, we calculate the self-inductance of the real Bitter coil with the real thickness $b = 0.02$ m, the radii $R_1 = 2$ m, $R_2 = 3$ m, and $N = 1000$.

From [17,23], the self-inductance is:

$$M_{\text{B-real}} = 7.656835021915879 \text{ H}.$$

Obviously, as $b \ll D_i$, $i = 1, 2$, then condition (B) is satisfied, so that the real Bitter coil can be practically considered as the thin Bitter disk coil, for which applying Equation (18), the self-inductance is:

$$M_{\text{B-disk}} = 7.720464217225742 \text{ H}.$$

The discrepancy between these two calculations is 0.8242%. The results are in a very good agreement. This confirms the validity of our approach.

Example 12

Here, we calculate the mutual inductance of the two real Bitter coils with the real thickness $d = 0.001$ m, the radii $R_1 = 1$ m, $R_2 = 2$ m, $R_3 = 3$ m, $R_4 = 4$ m, and $N_1 = N_2 = 1000$. For these two real Bitter coils [17], the mutual inductance is:

$$M_{\text{B-real-coils}} = 1.3560448126675355151160466 \text{ H}.$$

By Ren's method [18,19], the mutual inductance is:

$$M_{\text{B-Ren}} = 1.35604559122892 \text{ H}.$$

By finite-element method (FEM), the mutual inductance is:

$$M_{\text{B-Ren(FEM)}} = 1.356045548 \text{ H}.$$

Thus, the results obtained by three different methods are in excellent agreement.

Obviously, two real Bitter coils can be practically considered as the two thin Bitter disk coils because the condition (B) is satisfied.

Applying Equation (17), the mutual inductance is:

$$M_{\text{B(thin-disks)}} = 1.3560447857140560667943 42 \text{ H}.$$

We see that the obtained result is in an excellent agreement with the earlier ones.

Example 13

The mutual inductance for two Bitter disk coils is given in Table 6, where the radii are given as $R_2 = 10$ m, $R_3 = 4$ m, and $R_4 = 6$ m, and R_1 varying between 1 and 9 m. The number of turns for the disk is 1000. This covers all the overlap cases.

Table 6. Mutual inductance for all overlap cases.

R_1 (m)	M_B (H), Equation (17), This Work	M_B (H) [18,19]
1	6.176953480899784	6.176890611593179
2	8.442109820575922	8.441929573292805
3	10.32822091283136	10.32810057022684
4	11.43930901799091	11.44063208654759
5	10.40642626084792	10.40633203780527
6	8.263044997910772	8.263044761562908
7	7.051548638612127	7.051548623257631
8	6.338568884058207	6.338568878810852
9	5.83274907953413	5.832749077642109

From Table 6, we can observe very good agreement of all results. There is negligible discrepancy between the presented results because the filament method [18,19] is approximative and depends on the number of divisions of the disk coils. Note that the more disk divisions are more computing time is needed.

Example 14

In this example, the self-inductance of the Bitter coil are calculated for the thickness $b = 0.001$, with the radii $R_1 = 1$ m, $R_2 = 2$ m, and $N = 1000$. Calculating the self inductance of this coil.

From [23,24], the self-inductance of the real Bitter disk coil is:

$$L_\mathrm{B} = 3.5680173362402784867397894 2784 \text{ H.}$$

By Ren's method [18,19], the self-inductance is:

$$L_\mathrm{B-Ren} = 3.57200018 \text{ H.}$$

By FEM, the self-inductance is:

$$L_\mathrm{B-Ren(FEM)} = 1.356838 \text{ H.}$$

Equation (17) gives the mutual inductance between two Bitter disk coils. Let us take $R_1 = 1$ m, $R_2 = 2$ m, $R_3 = 1$ m, $R_4 = 2$ m, and $N_1 = N_2 = 1000$. Applying Equation (17), we obtain:

$$M_\mathrm{B} = L_\mathrm{B} = 3.569912886724817 \text{ H.}$$

Applying Equation (18) for the self-inductance of the Bitter coil, we have:

$$L_\mathrm{B} = 3.569912886724817 \text{ H.}$$

We showed that all results for the self-inductance, obtained by the different methods, are in particularly good agreement.

Example 15

In Table 7, the self-inductance of the real Bitter coil [23,24] are given with the coil dimensions $R_1 = 0.3$ m, $R_2 = 0.4$ m, and the number of turns $N = 100$, for widely changing thickness of the real coils.

Table 7. The self-inductance for the real Bitter coil and the thin Bitter disk coil with strongly varying thickness.

b	$L_\mathrm{B-coli}$ (mH) [23,24]	$L_\mathrm{B-thin-disk}$ (mH), Equation (18), This Work
10^{-3}	12.31719134402740	12.36243889748211
10^{-4}	12.35787124164836	12.36243889748211
10^{-5}	12.36198154993393	12.36243889748211
10^{-6}	12.36243432318282	12.36243889748211
10^{-7}	12.36243844005104	12.36243889748211
10^{-8}	12.36243885173889	12.36243889748211
10^{-9}	12.36243889290775	12.36243889748211
10^{-10}	12.36243889701627	12.36243889748211
10^{-11}	12.36243889743836	12.36243889748211
10^{-12}	12.36243889749853	12.36243889748211

The Equation (18) for the thin Bitter disk coil gives the self-inductance:

$$L_\mathrm{thin-disk} = 12.36243889748211 \text{ mH.}$$

All results [23,24] are in a very good agreement with the result obtained in the analytical form (18). Thus, the real Bitter coils with thicknesses which are remarkably smaller than their radii can be considered as the concentric coplanar thin Bitter coils.

From the previous examples, we proved that it is possible to treat the real circular coils of the rectangular cross-section, whose thicknesses are remarkably smaller than

their radii (conditions (A) and (B)), as the thin disk coils (pancakes). In these cases, we use the amazingly easier analytical and semi-analytical expressions. It validates our presented method.

5. Conclusions

In this paper, we presented remarkably simple formulas for calculating the self- and mutual inductance of ordinary concentric coplanar disk coils with azimuthal current density and concentric coplanar Bitter disk coils with non-uniform current density. All presented formulas were obtained under some valuable conditions for the real coils of the rectangular cross-section, either for ordinary real coils or the real Bitter coils. For the concentric coplanar ordinary disks, the formulas were obtained in the semi-analytical form as the combination of the elliptic integral of the second kind and a simple integral. For the concentric coplanar thin Bitter disks, the formulas were obtained in the analytical form as the elliptical integral of the second kind. All cases of either the regular or the singular (disks overlap) were covered with obtained formulas. Many examples presented are shown to approve the correctness of the presented method. The presented formulas are simple to use for potential readers which are not familiar with complicated special functions, such as Bessel functions, generalized hypergeometric functions, or Mellin transform.

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