

Review

Accelerated Expansion of the Universe as a Quantum Gravity Phenomenon

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Abstract

It has been known for about hundred years that the universe is expanding, but an impressive discovery came at the end of 20th century, when it was found that the universe is expanding with acceleration. Since then, many models have been developed in cosmology to explain this phenomenon. One of the possible elucidations is that the theory of gravity must be modified at the classical level. However, many such models have already been excluded by gravitational wave experiments. Therefore, one must put a more critical question: could accelerated expansion of the universe be a phenomenon of quantum gravity? Here, we review the basic models of how one explains the origin of dark energy or the cosmological constant in metastring theory, discrete approaches to quantum gravity, group field theory, non-commutative geometry, causal dynamical triangulation, asymptotic safety, and models based on holography and entropic gravity. At the end of a newly formed approach to the quantization of gravity, we mention the ring paradigm, which may, after application to cosmology, naturally model the late-time accelerated expansion epoch in the universe. What is most remarkable, though, is that the final formulation of this theory may ultimately solve the old problem of the cosmological constant.

Keywords: dark energy; cosmological constant; accelerated expansion of the universe; string theory; loop quantum gravity; group field theory; causal set approach; non-commutative geometry; causal dynamical triangulation; asymptotic safety; entropic gravity

1. Introduction

Tracking down the origin of modern physics, one should definitely mention the breakthrough year 1687, when the *Principles* of Isaac Newton has been published [1] (English translation in Ref. [2]). The three-volume monograph mathematical reformulations followed. Finally, quantum theory was born in 1900 [3], as was the special theory of relativity in 1905 [4]. The new theory of gravity, the general relativity (GR), was formulated in 1915 [5]. However, without the long work of mathematicians such as, for example, Jean d’Alembert, Joseph-Louis Lagrange, William Rowan Hamilton, or Bernhard Riemann, these discoveries at the beginning of 20th century would hardly have been possible. One may therefore assume that there is still the period of discovery of a new mathematical apparatus, which will be needed for the formulation of the quantum gravity (QG) theory, unifying quantum field theory (QFT) [6] and GR [5]. This is also the reason why the extensively



Received: 12 January 2026

Revised: 19 February 2026

Accepted: 31 March 2026

Published: 1 July 2026

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studied string theory [7–10], as a candidate for a unified theory of all interactions, has been so successful so far, as it is connected with highly attractive mathematical physics [11–18].

Yet, mathematics is not the only point one focuses on in modern theoretical physics. It is often said that today we live in the golden age of cosmology, the science about the origin and development of the whole universe [19]. One can straightforwardly test the modern physical theories on cosmological models, which provide us with valuable data by new technologies, and this to be exactly the case for modern gravity theories. Further, it is already known quite long time ago (since Franc Zwicky [20] pointed out the problem in the 1930s) that some type of matter is missing in the universe, which is called dark matter (DM). Yet one to not be unexpected that the complete composition of the massive objects around is unknown (DM creates approximately 27% [21]), as soon as we are studying what we are actually made of. Much more unexpected is that there also exists dark energy (DE), which creates approximately 68% of the energetic composition of the universe [21], and fascinatingly, this may allude to the point that not yet everything is known about the theory of gravity.

The Big Bang model in cosmology is based on three discoveries. One of them is Hubble- revelation of the expansion of the universe, which was measured by Edwin Hubble in 1929 using the 100-inch Hooker telescope on Mount Wilson in California (USA) [22]. Hubble discovered that the farther a galaxy is, the faster it is receding into space away. If $H = \dot{a}/a$, with a the scale factor and the dot on top denoting the time derivative, is set for Hubble constant, which has a value of approximately 72 km/(s·Mpc) (there was some discussion concerning the true value of the constant [23]), $\vec{v}$ as the velocity of the receding of the galaxy, and $\vec{r}$ as the distance from the observer the formula reads

$$\vec{v} = H\vec{r}. \quad (1)$$

This finding only tells us that the universe had to have some beginning, and the expansion is a remnant of the initial explosion, but this was not the end of the story.

Albert Einstein initially added the cosmological constant Λ to the Einstein equations of GR as aimed to obtain the static universe [24]. As it was found later that the model must be dynamical, the reason for keeping the constant in the equations has disappeared [22]. The observations of supernovas (of type IA) in 1998 gave the most direct evidence for mimicking the positive Λ [25,26]. Let us briefly consider how to measure the sign of $\ddot{a}$ [27]. Actually, one can deduce the value of the acceleration of the scale factor $\ddot{a}$ from the plot of luminosity versus redshift, and one only needs to analyze the behavior of electromagnetic waves in curved spacetime to understand the effect. The key equation for the “luminosity distance”, or normalized “dimness”, d_L of a known source that is not especially far away is

$$Hd_L = z + \frac{1}{2} \left(1 + \frac{a\ddot{a}}{\dot{a}^2} \right) z^2 + \mathcal{O}(z^3), \quad (2)$$

where z is the redshift of the light from the supernova. One can straightforwardly deduce both $\dot{a}$ and $\ddot{a}$ if it is known how d_L varies with z . Moreover, one can see as soon as other quantities are equal, that a larger $\ddot{a}$ reduces an image that is dimmer at equal redshift. Thus, the graph of d_L as a function of z shifts upward, and this has been obtained.

A sign of this effect can also be deduced from the equivalence principle, where one uses only what is known of the Doppler shift in flat spacetime. According to the equivalence principle, spacetime is flat in our universe neighborhood. So, what is commonly described as the expansion of space, can be reinterpreted locally as the flow of galaxies away from us. When such a coordinate system is considered, two supernovas of equal brightness are at equal spatial distance from observer. Therefore, their light was emitted at the same time t_0

in the past. Now, let us consider these two stars into two different spacetimes with the same expansion rate H but different accelerations $\ddot{a}$. In the spacetime with larger acceleration, the expansion rate at t_0 was smaller; therefore, the recession rate of the supernova was smaller, hence its redshift z was also less. One can equivalently say the supernova appears dimmer at the same redshift.

The observations [25,26] tell that the expansion in the last five billion years is accelerating. The key question is: what is pumping energy into the universe? Acceleration needs some driving force. Further, how long will the universe be accelerating? Is the mechanism behind this late-time cosmic acceleration somehow connected to the mechanism behind cosmological inflation at the beginning of the evolution of the universe? Let us now briefly consider some points concerning physics of that direction.

As we already mentioned, GR is considered the classical theory of gravity [5]. So far, it is a paradigm, and it is highly consistent with experiments. To this end, the results of GR have to be considered with higher credibility, especially its applications to cosmology. Meantime, it is already known that the Friedmann models, the first attempts at cosmological models based on Einstein equations, were excluded by observations. Those models were gradually replaced with the Λ CDM model [28], which is currently the most recognized model in cosmology. However, even this model has considerable shortcomings.

We do not comment that one is completely unsure what DM is [29]. Quite the trouble is actually the value of the cosmological constant, and there are considerable complications connected with it. When one just writes down the Einstein equations with a cosmological constant, one gets

$$G_{\mu\nu} + \Lambda_b g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (3)$$

where $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ represents the energy–momentum tensor, $G_{\mu\nu}$ is the Einstein tensor, with the Greek letters of indexes taking the values 0 (for time component), 1, 2 and 3 (for space components), G is the Newtonian constant of gravitation, and c denotes the speed of light in vacuum. The constant Λ_b is responsible for the accelerated expansion ($\ddot{a} > 0$), but it is quite a mystery why it has so small value [30–33]. According to QFT, the value should be approximately 10^{120} orders of magnitude larger. Sometimes this complication is called the greatest failure in modern theoretical physics. Another puzzle is the following: matter dilutes during the universe's expansion, but the vacuum energy density remains constant. The constant Λ_b currently provides a dominant contribution to the overall energy content in the universe, so why should its value be of almost the same order of magnitude as the density of the non-relativistic matter? The latter is called the coincidence problem.

This is exactly the reason for conducting research on modified theories of gravity which model the late-time accelerated expansion in an alternative way [34]. The basic division is on models, which just add relevant terms with some form of scalar field to the right-hand side (RHS) of Einstein Equation (3). These models are motivated by the investigations into particle physics. Another possibility is to modify the Einstein–Hilbert action: either left-hand side (LHS) of Einstein equations (an example is $f(R)$ theories, which even allow building a model of late-time cosmic acceleration together with the cosmological inflation at the beginning of the evolution of the universe), or to use models with extra dimensions, where gravitons can leak out into the Einstein equations (an example is DGP models). To note is that, though, most of the modified gravity scenarios have already been excluded by the LIGO–Virgo–KAGRA collaboration [35]. Therefore, searching for other models is necessary.

Surely, studying another model means resorting to a new shell of physics, namely QG. Yet, is there any reason to rely on the observation fact that the accelerated expansion of the universe is indeed a phenomenon of QG? First of all, one is completely on safe ground from that point of view: if this were not the case, much more advanced models would be built, which are always helpful in classical physics. Simply, a highly sophisticated theory can bring further knowledge, although it has nothing to do with solving the given problem. But there exist quite reasons to suppose that accelerated expansion of the Universe may not be explained by classical physics.

There was recently developed a new approach to QG [36–39], which actually solves the problem of DE and brings us first hints on how to resolve the old problem of the cosmological constant. If these results are confirmed, it would signify that some phantom field is present in the universe (and what would be far more remarkable, it may probably be the phantom field necessary for building the cyclic model of Paul Steinhardt and Neil Turok [38,40]). Further, the no-go theorems in QFT must somehow be obeyed because the model is based on Lorentz breaking in the gravity sector. This would have quite the consequences for the current commonly known approaches to QG: the appearance of cosmological constants is not random, and models of DE may be found in applications of these theories in a completely generic way. In this review, we consider the current state of knowledge in cosmology concerning models of the late-time accelerated expansion applied to QG theories, and look for a common thread in often highly differed approaches, some of which are even contradictory, such as the string theory and the causal set approach. If we were to find the presence of this phenomenon in most of the paradigms, it would be an indication of indeed a solid result. Then one may be almost ensured that some of the outcomes are based on the true models of Nature. The complication is that today there are many differently developed models of QG. However, there are not too many models of accelerated expansion in cosmology that are based on them.

The paper is organized as follows. First, a short introduction is given to string theory (ST) with AdS/CFT correspondence, loop quantum gravity LQG, group field theory (GFT), the causal set approach (CSA), asymptotic safety (AS), causal dynamical triangulations (CDT), non-commutative geometry (NG), and entropic gravity (EG). Then, follows the consideration of the models of accelerated expansion in the applications of these paradigms. Finally, we discuss a new approach to QG and its consequences for the models in cosmology.

2. Approaches to Quantum Gravity

2.1. String Theory with AdS/CFT Correspondence

We begin by making some remarks about ST [7–10]. The first paper in this field was written by Gabriel Veneziano in 1968 to explain the physics of strong interactions [41]. The elementary particles are described as a vibration of strings having the spacetime coordinates $X^\mu, \mu = 0, 1, \dots, d$ here. The superstring theories are defined in 10 or 11 spacetime dimensions (so $d = 10$ or $d = 11$, which is obtained as a result of the application of the Lorentz invariance). $X^\mu = X^\mu(\tau, \sigma)$ is a function of two parameters τ and σ . The first step in the formulation of the theory leads to a prescription of the action (the Nambu–Goto action of the relativistic string):

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \, d\sigma d\tau, \quad (4)$$

where T_0 denotes the tension of the string and the notations

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \quad \text{and} \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma} \quad (5)$$

are used for derivatives. The equations of motion follow the action (4), and then the theory is quantized. It is considered a new feature in phenomenology that the strings must be attached somewhere (when the boundary conditions are established); therefore, the introduction of branes is inevitable. ST comes as the only QG paradigm so far with a straightforward explanation of what a graviton is, and therefore, it may be a candidate for the unified theory of all interactions. The mathematical foundations of ST stimulated research in comparably large areas of algebraic geometry ([11–14]) as well.

As we already stressed, the origin of ST was connected with particle physics, and cosmology did not play a big enough role in the first stages of its development [42]. These conditions quickly changed with the observation of accelerated expansion: It looked as if the cosmological constant $\Lambda = \Lambda_b$ (the notation Λ for Λ_b is used onward in the paper)—or something that caused a similar behavior—vanished on the scales of particle physics, but there is a mechanism allowing a tiny non-zero value; therefore, the anthropic principle started to be used. (It in some sense says that the value of the cosmological constant, which one assumes to randomly vary from one region to another, has a fine-tuned value, which is a property of a local region ensuring suitable conditions for life.) Then, the investigations into the multiverse in ST have started. Other parts of cosmology have been developed as well, and the cosmological inflation has been considered to be a first look at the multiverse. However, ST has been heavily criticized. That said, how could such a theory be falsifiable when it contains approximately 10^{500} different vacua? One may argue that the landscape exists and its properties are somewhat known, but what if this is not confirmed?

One may fall into circular reasoning: observing the de Sitter space (because of accelerated expansion), one supposes that ST is a highly suitable candidate for QG, and concludes that it has to contain the de Sitter space, but one knows about quite a few complications as to how to incorporate it into this theory. An unexpected result was published in Ref. [43] suggesting that one can successfully solve the problem of DE and the observed de-Sitter spacetime in a generic, non-commutative, generalized geometric phase space formulation of ST. The application of this minor modification of ST, to cosmology, called metastring theory, is the topic of the present review.

The AdS/CFT correspondence is a duality relating QFT and gravity. More precisely, the correspondence relates the quantum physics of strongly correlated many-body systems to the classical dynamics of gravity in one higher dimension. This duality is also frequently referred to as the holographic duality or the gauge/gravity correspondence. In its original formulation [44–46], the correspondence related a four-dimensional conformal field theory to the geometry of an AdS space in five dimensions. As the collective phenomena have been studied in condensed matter physics, it is quite common that when a system is strongly coupled, it reorganizes itself in such a way that new weakly coupled degrees of freedom emerge dynamically. Then, the system can be better described in terms of fields representing the emergent excitations and the AdS/CFT correspondence is a new example of this paradigm. The highly unexpected feature is that the emergent fields live in a space with one extra dimension and that the dual theory is a gravity theory. Actually, the extra dimension is related to the energy scale of the QFT. The holographic description is a geometrisation of the quantum dynamics of the systems with quite a large number of degrees of freedom, which makes it apparent that there are profound connections between quantum mechanics and gravity.

The gauge/gravity duality has been discovered in the context of ST, where it is now quite natural to realize gauge field theories on hypersurfaces embedded in a higher-dimensional space in a theory that contains gravity. However, the study of the correspondence has been extended to include quite various areas of research, such as the investigations of the strong coupling dynamics of quantum chromodynamics and the

electroweak theories [47–49], the physics of black holes and QG [50–52], relativistic hydrodynamics [53–55], or different applications in condensed matter physics (holographic superconductors [56], quantum phase transitions [57], cold atoms [58], etc.). This review focuses on the applications to cosmology.

2.2. Loop Quantum Gravity

The second most extensively studied approach to QG so far is LQG which represents an approach based on the quantization of GR [59,60]. The first announcement on this approach was given by Carlo Rovelli and Lee Smolin at the conference in Goa, India in 1987 [61] (see extended version in Ref. [62]) based on the works of Abhay Ashtekar (see, e.g., [63]). We start the computation with the Einstein–Hilbert action

$$S = \frac{c^4}{16\pi G} \int \sqrt{-g} R d^4x, \quad (6)$$

where g is the determinant of the metric tensor and R is the Ricci scalar in the 4-dimensional x -space. It is necessary to address quantum mechanics, therefore the $(3 + 1)$ -splitting is used. We choose some 3-dimensional surface Σ and define on it the 6-component induced Riemannian metric q_{ab} . The remaining four degrees of freedom create the lapse function N and the shift vector N^a . The action reads (after performing the Legendre transformation)

$$S[q_{ab}, \pi^{ab}, N, N_a] = \frac{c^4}{16\pi G} \int \int_{\Sigma} \left\{ \pi^{ab} \dot{q}_{ab} + 2N_b \nabla_a^{(3)} (q^{-\frac{1}{2}} \pi^{ab}) + N \left[q^{\frac{1}{2}} (R^{(3)} - q^{-1} \pi_{cd} \pi^{cd} + \frac{1}{2} q^{-1} \pi^2) \right] \right\} d^3x dt, \quad (7)$$

where π^{ab} is the momentum conjugate to q_{ab} and $\pi = \pi^{ab} q_{ab}$ ($\nabla_a^{(3)}$ denotes a covariant derivative corresponding to the metric q_{ab}), t denotes the time and $R^{(3)}$ is the 3-dimensional Ricci scalar. The form (7) of the action is more suitable when one considers to incorporate the commutation relations of quantum mechanics and use the Ashtekar–Barbero variables.

One also addresses attractive mathematics while dealing with loop quantum gravity [64], but the most constructive achievements come from cosmology [65]. It is believed to provide an explanation of what was happening at the beginning of the evolution of the universe because the loop quantum cosmology modifies the Friedmann equation

$$H^2 = \frac{\dot{p}^2}{4p^2} = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}} \right), \quad (8)$$

where ρ denotes the current universe density, p is its momentum, the critical density is $\rho_{\text{crit}} = \frac{3}{8\pi G \beta^2 \mu^2 p}$, with $\mu = 3\sqrt{3}/2$ and β labelling the Barbero–Immirzi parameter. One can see that the expression (8) for H^2 is not always positive-definite. In particular, H^2 equals zero when $\rho = \rho_{\text{crit}}$. If one runs the evolution of the universe backwards, a point with $H^2 = 0$ is reached. This points that the bounce replaces the Big Bang. Also, a solution to the black hole singularity problem was suggested in LQG [66]; therefore, one may point to it as to a paradigm with non-trivial phenomenological results.

2.3. Group Field Theory

With the study of Gerhard 't Hooft [67], it was recognized that the QFT of an $N \times N$ matrix field can be formally expanded into Feynman amplitudes indexed by ribbon graphs or combinatorial maps [68,69]. Those two equivalent combinatorial structures formalize the notion of a graph embedded into a canonically associated surface, and the genus of that surface indexes a topological large- N expansion in which $\frac{1}{N^2}$ takes the role of a formal relatively small parameter. 't Hooft's original point of using this expansion to probe the

non-perturbative sector of QFT found numerous applications in enumerative combinatorics and, more specifically, in two-dimensional QG. Indeed, formal matrix integrals can be used to efficiently manipulate the generating functions of combinatorial maps, and asymptotic enumeration results for such objects can be inferred by studying the critical properties of the underlying matrix model. One can see from a physics point of view that such a critical regime can be interpreted as defining the continuum limit of a two-dimensional random geometry which is equivalent to Liouville QG.

Tensor models were discovered in the early 1990s as an attempt to establish an analogous correspondence between random tensors and Euclidean QG in dimension $D \geq 3$. These early studies pointed to a prolific correspondence between the Feynman expansion of such a tensor model and discrete geometries in dimension D . However, no generalization of the large- N expansion of matrix models could be identified at the time. This impeded further progress on tensor models and their relationship to random geometry.

On the other hand, dynamical triangulations and spin foam models rely on a path integral formulation of QG when one defines the path integral over metrics:

$$\Psi(g|_{\partial\mathcal{M}}) = \sum_{\substack{\text{topologies} \\ \partial\mathcal{M} \text{ fixed}}} \int_{g|_{\partial\mathcal{M}} \text{ fixed}} [Dg] \exp \frac{iS[g]}{\hbar}, \quad (9)$$

with $\hbar$ the reduced Planck constant. Equation (9) involves an integration over metrics' modulo diffeomorphisms on a space-time manifold $\mathcal{M}$ which is reduced to a fixed metric on the boundary. As Equation (9) is a functional of the boundary metric, it can be considered as the wave function of the quantum gravitational field. Let us note that fluctuations of the topology of space-time are also allowed, even if these can be omitted in a first approximation.

Next, GFT arises from the merging of tensor models and spin foam models. The first model was proposed by Dmitri Boulatov for three-dimensional QG (in 1992) [70], which is actually the topological BF theory. Then, the generalization to BF theory in four dimensions was made by Hiroshi Ooguri [71]. GFT is actually a QFT whose Feynman graph expansion reproduces spin foam amplitudes:

$$\int [D\Phi] \exp S[\Phi] = \sum_{\substack{\text{triangulations } T \\ \Leftrightarrow \text{Feynman graphs}}} \frac{1}{C_T} A_T. \quad (10)$$

The GFT Feynman graphs are in one-to-one correspondence with 2-complexes and is dual to triangulations. C_T denotes the symmetry factor of the graph corresponding to T and A_T denotes the spin foam amplitude of the triangulation. The tensors in the models generating dynamical triangulations are replaced by functions over D copies of a group $G = SU(2), SO(3), SO(4), SL(2, \mathbb{C}), M_{i_1, \dots, i_D} \rightarrow \Phi(g_1, \dots, g_D)$. The action can be written symbolically as $S[\Phi] = \Phi^2 + \Phi^{D+1}$. The field Φ represents a $(D-1)$ -simplex and the interaction between a D -simplex and $D+1$ $(D-1)$ -simplexes. The application of Wick's theorem implements the glueing of D -simplexes along the boundary $(D-1)$ -simplexes. Finally, the Feynman graph amplitudes, expressed as an integral over group elements, reproduce the spin foam amplitudes.

2.4. Causal Set Approach

Present physics view indicate that QG may be a discrete theory [72]. However, there are issues with infinities of GR and QFT caused by the lack of a short-distance cut-off in degrees of freedom. Although the renormalization procedure is refining these problems in QFT, the tasks return with naive attempts to quantize gravity. Additionally, technical

issues arise in the definition of a path-integral on a continuous history space that have never been fully solved. Finally, the history space of Lorentzian manifolds presents specific issues of its own kind. A discrete history space provides a well-defined path integral that enables avoiding these issues.

The CSA is based on the consideration of the discretization of spacetime into points. The essential studies that started this program were those by Stephen Hawking (in 1976) [73] and David Malament in 1977 [74]).

The irreflexive CSA is founded on the following six axioms [75].

1. Binary Axiom. One may model the classical spacetime as a set S , whose elements represent spacetime events, together with a binary relation $\prec$ on S ; the elements of S represent causal relations between pairs of spacetime events.
2. Measure Axiom. The volume of a spacetime region corresponding to a subset C of S is equal to the cardinality of C in fundamental units, up to Poisson-type fluctuations.
3. Countability. The set S is countable.
4. Transitivity. For three elements x , y , and z in S , if $x \prec y \prec z$, then $x \prec z$.
5. Interval Finiteness. For every pair of elements x and z in S , there exists finite cardinality for the open interval $\langle\langle x, z \rangle\rangle \equiv \{y \in S | x \prec y \prec z\}$.
6. Irreflexivity. Elements of S are not self-related with respect to $\prec$; this means $x \not\prec x$.

The CSA [76] has highly intriguing connections with graph theory and graphons [77], but the most remarkable results are again in phenomenology because it is also changing the view on the Big Bang, as one can see in Figure 1 in Ref. [78], and an explanation of the origin of the cosmological constant in the late universe is obtained as well [79,80]. Generally speaking, the kinematics are well formulated, but no much is known about the dynamics, although the analog of the Einstein–Hilbert action (Benincasa–Dowker action) was already found [81].

2.5. Non-Commutative Geometry

The study by Alain Connes (year 1994) has highly attractive applications to the Standard Model of particle physics (SM), and it later also found its use in QG. To begin with, let us make some remarks concerning the mathematical consideration. The correspondence between geometric spaces and commutative algebras is a familiar and basic point of algebraic geometry. The purpose of Ref. [82] is to extend this correspondence to the non-commutative case within the framework of real analysis. The theory—non-commutative geometry—is based on two essential points. First of all, it is the existence of many natural spaces for which the classical set-theoretic tools of analysis, such as measure theory, topology, calculus, and metric, lose their relevance but which correspond quite naturally to a non-commutative algebra. Further, it relies on the extension of the classical tools, such as measure theory, topology, differential calculus, and Riemannian geometry, to the non-commutative case. This extension involves an algebraic reformulation of the tools listed.

However, passing from the commutative to the non-commutative case is never straightforward. On the one hand, completely novel phenomena arise in the non-commutative case, where a canonical time evolution for a non-commutative measure space exists. On the other hand, the constraint of developing the theory in the non-commutative framework leads to a new point of view and new tools even in the commutative case, such as the theory of cyclic cosmology and quantized differential calculus which, unlike the theory of distributions, is especially adapted to products and gives meaning to expressions such as $\int f(Z) |dZ|^p$, where Z is not differentiable (and p is not necessarily an integer).

Commenting on physical applications, Connes' contribution [82] does not shed any new light on the theoretical problems of the SM because it is limited to the classical level. However, it specifies quite precisely what modification of the continuum space it leads

to. Actually, its replacement by a product with a finite space entails the Lagrangian of quantum electrodynamics becoming the Lagrangian of the SM. As shown in Ref. [82], the geometry of the finite set is specified by its Dirac operator, and this is an operator in a finite-dimensional Hilbert space encoding both the masses of the elementary particles and the Kobayashi–Maskawa mixing parameters. Once the structure of this finite space F is given, one merely applies the general action to the space (which is a continuum) $\times F$ to get the SM action. In many ways, the theory should be regarded as an interpretation of a geometric nature of all the complicatedness of the most accurate phenomenological model of high-energy physics; it does confirm that high-energy physics unveils the fine structure of space-time. Finally, it gives a status to the Higgs boson as just another gauge field corresponding to a finite difference rather than a differential.

Connecting mathematics and physics, one needs to prescribe the dictionary. Standardly, in the basic data of NG, $(\mathcal{A}, H, D)$, the Hilbert space H and the operator D have straightforward interpretations:

$$H: \text{the Hilbert space of Euclidean fermions}, \quad (11)$$

$$D: \text{the inverse of the Euclidean propagator of fermions}. \quad (12)$$

The algebra $\mathcal{A}$ is related to the gauge group of local gauge transformations. The relation is as follows:

$$\ast\text{-algebra } \mathcal{A} \rightarrow \text{unitary group } \mathcal{U}(\mathcal{A}). \quad (13)$$

Let us comment on the functor $\mathcal{U}$ from $\ast$ -algebras to groups. When the functor is applied to the algebra of $n \times n$ matrices over $\mathcal{A}$, $M_n(\mathcal{A})$ and one also retains the corresponding inclusion of $U(n) = \mathcal{U}(M_n(\mathbb{C})) \subset G = \mathcal{U}(M_n(\mathcal{A}))$, one then recovers uniquely the algebra $\mathcal{A}$ from the pair of groups $U(n) \subset G$, which is provided by $n > 2$.

This result shows that the replacement of an algebra by its associated groups loses just little information, at least in the non-abelian case. When one prescribes the algebra $\mathcal{A}$, from which the group $G = \mathcal{U}(\mathcal{A})$ comes, it determines quite a narrow class of representations of G . Indeed, we have a natural mapping

$$\text{Rep}(\mathcal{A}) \xrightarrow{\text{Res}} \text{Rep}(G), \quad (14)$$

which associates with every unitary representation π of $\mathcal{A}$ on a Hilbert space H_π its restriction to the unitary group $\mathcal{U}(\mathcal{A})$. This mapping is compatible with direct sums, but while group representations $\rho_1, \rho_2 \in \text{Rep}(G)$ can be tensored, which gives a representation $\rho = \rho_1 \otimes \rho_2$ of G on $H_1 \otimes H_2$, there is no corresponding operation for representations of an involutive algebra.

Actually, the above mapping $\text{Rep}(\mathcal{A}) \xrightarrow{\text{Res}} \text{Rep}(G)$, is, in general, far from surjective. When one wants to illustrate it, we take $\mathcal{A} = M_n(\mathbb{C})$, so that $G = U(n)$, and then the range of Res consists of the multiples of the fundamental representation φ of $U(n)$ in $\mathbb{C}^n$. To stress the result, to assume that a representation ρ of G comes from a representation of $\mathcal{A}$ by restriction is a highly limiting condition.

Now, let us return to physics consideration. The group G does have a understandable significance as the group of local gauge transformations, and one shall take it as a characteristic of the group representation in the one-particle space of Euclidean fermions that this representation of $\mathcal{U}(\mathcal{A})$ is derived by restriction from a representation of $\mathcal{A}$. One can also notice that it is only through the group $G = \mathcal{U}(\mathcal{A})$ and the corresponding algebra $\mathcal{A}$ that the Euclidean space-time enters the picture, because in QFT the space-time points only play

the role of indices or labels except in their occurrence in the construction of the group of local gauge transformations. Therefore, the last piece of dictionary is

$$*-algebra \mathcal{A} \rightarrow \mathcal{U}(\mathcal{A}): \text{Group of local gauge transformations on Euclidean space-time.} \quad (15)$$

This whole construction works for the Glashow–Weinberg–Salam model for leptons, as is shown in Ref. [82], and further, the incorporation of quarks and strong interactions requires the setup of Poincaré duality and bimodules.

In this review, we just mention one model of Tejinder Singh [83], which is inspired by non-commutative geometry. Singh introduces a DE particle, called a mitron, which shows a nonlocal behavior. Further, non-commutative geometry also found applications in connection with M-theory. It was conjectured in Ref. [84] that M-theory can be defined as a matrix of quantum mechanics obtained from 10-dimensional supersymmetric Yang–Mills theory by means of reduction to $(0 + 1)$ -dimensional theory, where the size of the matrix tends to infinity. Another matrix model that can be obtained by reduction of 10-dimensional supersymmetric Yang–Mills theory to a point, was suggested in Ref. [85]. The two models, known as the BFSS Matrix model and the IKKT matrix model, are closely related. Actually, the goal of these studies is to formulate the IKKT and BFSS matrix models in order to make more precise the relation between these models and to study their toroidal compactifications.

2.6. Causal Dynamical Triangulations

An intriguing and rather unconventional approach to QG is that of CDT [86,87] (the first papers in this subject was published by Jan Ambjørn, Jerzy Jurkiewicz, and Renata Loll in 2000 and 2001). This is a nonperturbative approach to QG that aims to define a well-behaved gravitational path integral by making a discrete spacetime into elementary building blocks while preserving a causal structure. The central point is that the spacetime geometry can be approximated by glueing together simplicial manifolds, in particular four-dimensional simplices, in a manner that maintains a consistent global time foliation. Unlike the earlier theory of Euclidean dynamical triangulations which lost information about causal order by using Euclidean signature metrics, CDT imposes a Lorentzian causal structure from the outset and only later performs a Wick rotation to the Euclidean sector for computational purposes. The starting point is the formal gravitational path integral over all geometries [88,89]:

$$Z = \int \mathcal{D}[g] e^{iS_{\text{EH}}[g]}, \quad (16)$$

where S_{EH} is the Einstein–Hilbert action. In CDT, the integral (16) is replaced by a discrete sum over triangulated manifolds constructed from simplices, leading to a well-defined and finite combinatorial formulation.

The discrete analogue of the Einstein–Hilbert action used in CDTs is given by the Regge action, which encodes curvature through deficit angles at the edges (or hinges) of the triangulated spacetime. The Regge action can be expressed as [90,91]

$$S_{\text{Regge}} = \frac{1}{8\pi G} \sum_h A_h \delta_h - \Lambda \sum_{\sigma} V_{\sigma}, \quad (17)$$

where the sum over hinges (h) runs over all hinges (two-dimensional subsimplices), A_h is the area associated with a hinge, δ_h is the deficit angle measuring the curvature concentrated at a hinge and the second term accounts for the cosmological constant Λ with V_{σ} being the volume of each four-simplex σ . In causal dynamical triangulation, spacetime is foliated into layers of spatial hypersurfaces of fixed topology (often S^3) labeled by discrete proper time steps. This leads to ensuring that each simplex connects neighboring time slices in

a causal fashion, and the triangulations are thus restricted to respect global hyperbolicity, meaning that the causal order between spatial slices is maintained.

The path integral in causal dynamical triangulation then becomes a discrete sum over all causal triangulations weighted by the Regge action [92]

$$Z = \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} \cdot e^{iS_{\text{Regge}}[\mathcal{T}]}, \quad (18)$$

where the sum runs over all triangulations $\mathcal{T}$ compatible with the chosen foliation, and $C_{\mathcal{T}}$ is a symmetry factor that accounts for the automorphisms of each triangulation. To perform numerical simulations, a Wick rotation is applied to transform the Lorentzian action to a Euclidean form, $iS_{\text{Regge}} \rightarrow -S_{\text{Regge}}^E$ which allows Monte Carlo techniques to be used. The Euclidean version of the partition function is then given as

$$Z_E = \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} e^{-S_{\text{Regge}}^E[\mathcal{T}]}, \quad (19)$$

which serves as the foundation for exploring the emergent large-scale properties of quantum spacetime through computational simulations.

An essential outcome of CDT is that when averaged over ensembles of causal triangulations, spacetime dynamically exhibits features resembling a smooth 4-dimensional universe at large scale, despite being composed of discrete building blocks. The expectation value of the spatial 3-volume $V_3(t)$ as a function of proper time (t) often shows a behavior well approximated by a semiclassical de Sitter universe, corresponding to an effective minisuperspace action of the form

$$S_{\text{eff}}[V_3] = \frac{1}{24\pi G} \int dt \left[\frac{(\dot{V}_3(t))^2}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t) \right] \quad (20)$$

(with k_2 the spatial curvature coefficient and λ the cosmological constant), where the first term keeps the kinetic behavior of the spatial volume and the latter terms correspond to curvature and cosmological constant contributions, respectively. This result is quite remarkable, as it shows that a smooth, 4-dimensional spacetime geometry can emerge dynamically from a sum over purely discrete and microscopic causal structures without assuming a background metric. Consequently, CDTs provide a promising route toward a background-independent and UV-finite theory of QG in which spacetime geometry and dimensionality themselves arise as emergent phenomena from an underlying causal and combinatorial foundation.

2.7. Asymptotic Safety

The AS program emerged with the groundbreaking paper by Martin Reuter in 1998 [93], where functional renormalization tools were taken over by QG. First, it is to define the high-energy behavior of QFT [94–96]. The theory space is built on a set of fields, their symmetries, and a class of action functionals depending on fields η and couplings g_i . The key relations are $\tilde{g}_i = k^{d_i} g_i$, where k is a momentum cut-off and d_i is the mass dimension of g_i . One takes the real numbers g_i as coordinates in the theory space. The couplings g_i are defined in an ideal case in terms of physical observables such as cross-sections and decay rates. Further, one does not include redundant couplings (that could be eliminated through field re-definitions). It is assumed that the renormalization group flow has been prescribed

on the theory space, and it describes the dependence of the action on some energy scale k . Then, it is supposed that the action has the following form:

$$\Gamma_k(\eta, g_i) = \sum_i g_i(k) O_i(\eta), \quad (21)$$

where O_i typically denotes local operators that are constructed with the fields η and that are compatible with the symmetries of the theory. Then, one identifies the theories with renormalization group trajectories.

AS is a bottom-up approach to QG, because the discussion starts within the theory space of an effective field theory. Then, one continues and notes that if the universe corresponds to a trajectory of a special type, the effective description can be forced to arbitrarily high energy. Therefore, AS theory can be presented as the continuation of an effective theory at higher energy scales. As a result, it has the high advantage that if it occurs in nature, it becomes naturally in agreement with the knowledge about the low-energy world. This may be considered quite a contrast to ST and LQG, which are top-down approaches. It would be highly complicated for ST and LQG to make any connection with the known low-energy phenomenology. To conclude, AS has already provided us with attractive results concerning black hole singularities [97], and it has applications to the development of models of cosmological inflation [98]. The natural disadvantage is the absence of the introduction of new mathematics.

2.8. Entropic Gravity

To explain the basic point of a new approach to QG called entropic gravity—the beginnings of which are dated back to the year 2010 and are associated with the study by Erik Verlinde [99]—the starting assumption is directly motivated by Bekenstein's original thought experiment, from which Jacob Bekenstein obtained his entropy formula. Bekenstein considered a particle with mass m attached to a fictitious “string” that is lowered towards a black hole [99]. Just before the horizon, the particle is dropped in. Due to the infinite redshift, the mass increment of the black hole can be made arbitrarily small when one considers classical physics. Considering a thermal gas of particles, this feature leads to problems with the second law of thermodynamics. Bekenstein solved this by arguing that when a particle is one Compton wavelength from the horizon, it can be considered a part of the black hole. Therefore, it increases the mass and horizon area by a relatively small amount, which Bekenstein identified with one bit of information. This led to the discovery of the area law for the black hole entropy.

Let us mimic this reasoning not near a black hole horizon but in a flat, non-relativistic space. Therefore we consider a comparably small piece of a holographic screen and then a particle of mass m that approaches it from the side, at which space-time has already emerged. Eventually, the particle merges with the microscopic degrees of freedom on this screen, but before the particle does so, it already influences the amount of information that is stored on the screen.

Let us postulate that the change in entropy associated with the information on the boundary equals

$$\Delta S = 2\pi k_B \quad \text{when} \quad \Delta x = \frac{\hbar}{mc}, \quad (22)$$

where k_B denotes the Boltzmann constant. We now rewrite the formula as follows:

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x \quad (23)$$

Then, how does the force arise? The basic approach is to use the analogy of osmosis across a semi-permeable membrane. When a particle has an entropically justified reason to be on one side of the membrane and the membrane carries a temperature T , then the particle experiences an effective force defined by

$$F\Delta x = T\Delta S. \quad (24)$$

F is called the entropic force. So, in order to have a non-zero force, one needs to have a non-zero temperature. From Newton's law, it is known that a force leads to a non-zero acceleration. Meantime, Bill Unruh showed that an observer in an accelerated frame experiences a temperature

$$T = \frac{1}{2\pi} \frac{\hbar a}{ck_B}. \quad (25)$$

Let us consider the temperature (25) as the temperature associated with the bits on the screen. Then, one obtains from Equations (23)–(25) Newton's force law

$$F = ma. \quad (26)$$

It is supposed that the boundary is not infinitely extended but forms a closed surface and we assume it represents a sphere. For the following, the Unruh law (25) may not be considered, since it is not needed. (The Unruh law (25) only served as a further motivation for Equation (23).) The key statement is that one must have a temperature in order to have a force. Since we consider to understand the origin of the force, it is necessary to understand where the temperature comes from.

One can think about the boundary as a storage device for information. Assuming that the holographic principle holds, the maximal storage space, or total number of bits, is proportional to the area A . Specifically, in a theory of emergent space, this is how area may be defined such that each fundamental bit occupies, by definition, one unit cell. Now, let us denote the number of used bits by N . It is natural to assume that this number is proportional to the area. So, one writes

$$N = \frac{Ac^3}{G\hbar}, \quad (27)$$

where a new constant G is introduced. Eventually, this constant is finally to be identified with Newton's constant, but since nothing is assumed yet about the existence of a gravitational force, one can just regard Equation (27) as the definition of G .

Further, suppose there is a total energy E present in the system. Let us now make quite the simple assumption that the energy is divided evenly over the bits N . The temperature then is determined by the equipartition rule

$$E = \frac{1}{2}Nk_B T \quad (28)$$

as the average energy per one bit, then one also needs the equation

$$E = Mc^2. \quad (29)$$

Here, M represents the mass that would emerge in the part of space which is enclosed by the screen. Although the mass can not be directly visible in the emerged space, its presence is noticed through its energy.

Now, we eliminate E from Equations (28) and (29). Let us plug in for N in the resulting equation and express from this equation T . Then, everything is inserted into the Equation (24) and rearranged with the help of the formula

$$A = 4\pi R^2, \quad (30)$$

where R denotes the radius of the screen. The result is the Newton's force law

$$F_G = G \frac{mM}{R^2}. \quad (31)$$

Certainly, the reason the gravitational force (31) is obtained from the first principles is straightforward: it was used in the derivation of black hole thermodynamics and produces results concerning holography. Here, we somehow reversed the process, but the revelation that one can build a QG paradigm in this manner is quite unexpected, especially that one can, after application to cosmology, model the accelerated expansion of the universe.

3. Models of the Late-Time Accelerated Expansion in Current Quantum Gravity Approaches

3.1. Origin of the Cosmological Constant in the Metastring Theory

We claim that one can successfully explain the problem of dark energy and the observed de Sitter space-time in a generic, non-commutative generalized geometric phase-space formulation of ST, so-called metastring theory. Actually, the curvature and size of the canonically conjugate dual space are the cosmological and gravitational constants in the observed spacetime. Then, the three scales associated with

- (i) the cosmological constant,
- (ii) the Planck units, and
- (iii) the effective particle physics

are related by a seesaw-like formula via T-duality. This is an essential feature of the ST that reciprocally relates short and long distances and precisely the generic prediction of certain toy-models [43].

The generalized geometric formulation of ST we consider has recently been discussed in Ref. [43] and derives from the underlying chiral world sheet Hamilton's action for the strings

$$S_{\text{string}} = \frac{1}{4\pi} \int_{\Sigma} [\partial_{\tau} \mathbb{X}^A (\eta_{AB} + w_{AB}) \mathbb{P}^B - \mathbb{P}^A H_{AB} \mathbb{P}^B] d^2\sigma, \quad (32)$$

where $\mathbb{X}^A$ combine the sum (x^a) and the difference ($\tilde{x}_a$) of the left- and right-movers on the string and $\mathbb{P}^A = \partial_{\sigma} \mathbb{X}^A$ are closely related to the chiral generalized momenta. $\partial_{\zeta} = \partial/\partial\zeta$. The mutually compatible dynamical fields w_{AB} , η_{AB} and H_{AB} are the antisymmetric symplectic structure, the symmetric polarization metric η_{AB} , and the doubled symmetric metric H_{AB} . Precisely, this new framework for ST based on a quantum space-time captures the essential quantum non-locality of any quantum theory. Further, w_{AB} governs the Hilbert structure of a quantum theory, which is generally ignored in the standard spacetime interpretation of ST, whereas the Kalb–Ramond ($B_{\mu\nu}$) field is associated with the symplectic structure w_{AB} , rather than the doubled metric.

The quantization renders the doubled “phase-space” operators $\hat{\mathbb{X}}^A = (\hat{x}^a/\lambda, \hat{\tilde{x}}_b/\lambda)$ inherently non-commutative, inducing, in particular,

$$[\hat{\mathbb{X}}^a, \hat{\mathbb{X}}^b] = i w^{ab} : \quad [\hat{x}^a, \hat{x}^b] = 0, \quad [\hat{x}^a, \hat{\tilde{x}}_b] = 2\pi i \lambda^2 \delta^a_b, \quad [\hat{\tilde{x}}_a, \hat{\tilde{x}}_b] = 0, \quad (33)$$

where δ_b^a is the Kronecker delta and $\epsilon = 1/\lambda$ denotes the fundamental length scale, such as the Planck scale, so that ϵ gives the corresponding fundamental energy scale. The latter one finds by examining the simplest example of the canonical free string compactified on a circle in an intrinsically T-duality covariant formulation of the Polyakov string. The full spacetime covariance is actually maintained in this description, and the string tension is naturally the ratio of the fundamental length and energy scales, $\alpha' = \lambda/\epsilon$. This fundamental non-locality is actually independently confirmed by examining the algebra of vertex operators in the 2-dimensional CFT of a free string compactified on a circle [100,101].

The generalized geometric formulation of ST provides for an effective description of DE that is consistent with a de Sitter spacetime due to its chirally doubled realization of the target space and the non-commutative structure in Equation (33). Let us note that the natural stringy effective action on the doubled spacetime in terms of the coordinates $(x^a, \tilde{x}^a)$ takes the form

$$S_{\text{eff}}^{\text{nc}} = \int \text{Tr} \sqrt{g(x, \tilde{x}) [R(x, \tilde{x}) + \dots]}, \quad \text{with} \quad [x, \tilde{x}] = i\lambda^2, \quad (34)$$

where the dots denote higher-order curvature R terms induced by ST. Returning to Equation (33), the effective action $S_{\text{eff}}^{\text{nc}}$ (34) expands into many other terms, which upon $\tilde{x}$ -integration and from the x -space point gives rise to interactions that may lead to various forms of DE. Therefore, these effects may provide some results in the recent conflicting measurements of the Hubble constant.

Further, to the lowest order, the expansion of $S_{\text{eff}}^{\text{nc}}$ takes the form

$$S_d = - \int \sqrt{-g(x)} \sqrt{-\tilde{g}(\tilde{x})} [R(x) + \tilde{R}(\tilde{x})], \quad (35)$$

a result which is obtained by effectively setting $w_{AB} \rightarrow 0$ in Equation (33) by assuming that $[\hat{x}, \tilde{x}] = 0$. In this limit, the $\tilde{x}$ -integration in the first term of Equation (33) defines the gravitational constant G and in the second term produces a positive cosmological constant $\Lambda > 0$. It also follows that the weakness of gravity is determined by the size of the canonically conjugate dual space, while the smallness of the cosmological constant should be given by its curvature. However, these results from the commutative limit are not stable under loop corrections, which has been recently addressed in the paper of Nemanja Kaloper and Antonio Padilla (called the sequester mechanism).

The intrinsic non-commutativity of the zero modes x and $\tilde{x}$ in Equation (33) corrects these results in several ways. One may specifically ask whether the non-zero λ in Equation (33) stabilizes the cosmological constant. The fully non-commutative analysis is complicated, but an encouraging indication emerges as follows. One simplifies the conformal metrics, $g_{\mu\nu} = \phi^2 \delta_{\mu\nu}$, and then the actions in Equations (34) and (35) produce a non-commutative $\Lambda\phi^4$ theory. Unlike the theory's commutative limit, the intriguing results of Harald Grosse and Raimar Wulkenhaar [102,103] demonstrate that the $\Lambda\phi^4$ -theory is not perturbatively solvable, explicitly showing the finite renormalization of Λ in terms of the bare coupling. At least in this highly simplified, conformal degree limit, non-commutativity thus can afford a quite small, radiatively and perhaps even non-perturbatively stable cosmological constant for the non-commutative form of the “doubled” effective action.

Finding de Sitter spacetime within ST has been an ongoing quest over the past two decades. Among the vast number of constructions, Per Berglund, Tristan Hübsch and Djordje Minić developed a discretuum of toy models [104,105] (and references therein), that turned out to naturally capture several of the features of the above non-commutatively generalized phase-space reformulation of quantum gravity [106]. One of the essential features of the toy model developed is S -duality, which is built in the $SL(2; \mathbb{Z})$ monodromy

properties of the axion-dilaton models we consider. In generalizations where various moduli fields replace the axion dilaton system, this directly implies T-duality, which is in turn covariantly realized in the phase space approach. Actually, the T-duality maps $\hat{x}$ into $\tilde{\hat{x}}$, and vice versa, and thus a covariant representation calls for a phase space formulation that involves both $\hat{x}$ and $\tilde{\hat{x}}$. The seesaw formula is a backstroke of such a covariant phase space formulation, and it can be explicitly derived in the context of their toy models. The overall effect is closely related to the old observation of Edward Witten [107] that supercharges need not be globally defined in the presence of conical defects, and the mass splitting between superpartners is controlled by the strength of the conical defect. The details are presented for the corresponding four-dimensional generalization and relation to the cosmological constant in Refs. [108,109].

Models in this class produce a seesaw formula for the cosmological constant

$$M_\Lambda \sim M^2/M_P, \quad (36)$$

relating the mass scales of the vacuum energy/cosmological constant (M_Λ), particle physics (M), and the Planck scale (M_P). Especially, the see-saw formula (36) follows from very particular geometric properties which relate both the volume and the curvature of the space to the string length scale [104]. Identifying M_Λ and M_P as the IR and UV cut-offs, respectively, the renormalization group flow identifies a self-dual fixed point [110–112]. When one considers the phase-space formulation [100,101] as a T-duality covariant description of the ST, this naturally relates $M_P \rightarrow M^2/M_P$ under T-duality. Then, the prediction of the models, in Refs. [105,106], of $M_\Lambda \sim M^2/M_P$ satisfies these conditions, with $M_P \sim \epsilon$ as the fundamental energy scale corresponding to the fundamental length λ . This produces the commonly known formula for the observed DE scale, provided M is a TeV scale. Further, this illustrates one of the significant points, namely the novel realization of DE and de Sitter spacetime, in both the phase-space formulation of ST and the toy models in this given class [104,106].

3.2. Emergence of Cosmological Constant in Discrete Approaches to Quantum Gravity

In GR, local energy-momentum conservation $\nabla^b \langle T_{ab} \rangle$ is always a consequence of the field equations, both at classical and semi-classical levels [113,114]. This should be straightforward from the semi-classical version of Einstein's equation

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4} \langle T_{ab} \rangle, \quad (37)$$

where $\langle T_{ab} \rangle$ is the expectation value of the energy momentum operator in the corresponding quantum state of the matter fields—and the feature that the Bianchi identities make the geometric side divergence free.

The previous restriction can be circumvented as one considers a quite simple modification of GR (already studied by Einstein) called unimodular gravity:

$$R_{ab} - \frac{1}{4}Rg_{ab} = \frac{8\pi G}{c^4} (T_{ab} - \frac{1}{4}Tg_{ab}) \quad (38)$$

Equation (38) can be derived from the Einstein–Hilbert action by restricting to variations preserving the volume form.

This breaks the diffeomorphism symmetry down to volume-preserving diffeomorphism, whose infinitesimal version is given by divergence-free vector fields ζ^a ,

$$\nabla_a \zeta^a = 0. \quad (39)$$

This definite restriction on general covariance allows for violations of energy-momentum conservation of a certain form. To see this, let us consider an action for matter S_m , which is invariant under volume-preserving diffeomorphisms, we introduce the stress-energy tensor $T_{ab} \equiv -2|g|^{-\frac{1}{2}}\delta S_m/\delta g^{ab}$, and its energy-momentum violation current $J_a \equiv \nabla^b T_{ab}$. Then the variation of the action under an infinitesimal diffeomorphism ξ^a is

$$\delta S_m = - \int T_{ab} \nabla^a \xi^b \sqrt{-g} d^4x = \int J_a \xi^a \sqrt{-g} d^4x, \quad (40)$$

where the matter field equations are assumed to hold. When one inserts the general solution of (39), $\xi = \epsilon^{abcd} \nabla_b w_{cd}$ —for an arbitrary two-form w —the requirement that the action is invariant under volume-preserving diffeomorphisms implies that $dJ = 0$. Therefore, the violations of energy-momentum conservation are allowed in unimodular gravity as long as they are of this integrable type.

This condition reduces to

$$J_a = \nabla_a Q \quad (41)$$

for some scalar field Q in the case of simply connected spacetimes. Thus, if the matter action is only invariant under volume preserving diffeomorphisms, then $J \neq 0$ introduces deviations from GR.

Let us now consider the semiclassical version of Equation (38), where the energy-momentum tensor and its trace are now replaced by the corresponding expectation values in a quantum state of the matter fields. If Bianchi identities are used, one deduces that

$$\frac{1}{4} \nabla_a R = \frac{8\pi G}{c^4} \left(\nabla^b \langle T_{ab} \rangle - \frac{1}{4} \nabla_a \langle T \rangle \right), \quad (42)$$

which, after integration, can be used to rewrite Equation (38) as

$$R_{ab} - \frac{1}{2} R g_{ab} + \left(\Lambda_{-\infty} + \frac{8\pi G}{c^4} Q \right) g_{ab} = \frac{8\pi G}{c^4} \langle T_{ab} \rangle, \quad (43)$$

where $\Lambda_{-\infty}$ is a constant of integration, and Q is defined by Equation (41). As we expect, when the stress-energy tensor is conserved, i.e., $Q = 0$, Equation (43) reduces to Einstein's equation (37), with a cosmological constant equal to $\Lambda_{-\infty}$.

We emphasize that both semiclassical GR and its unimodular version are considered here as an effective and emergent description of more fundamental degrees of freedom. The violation of energy-momentum conservation in the scenario considered would have to admit a description in terms of the more fundamental QG degrees of freedom. In brief, the approach is as follows. Due to the interaction of matter with the space-time foam (discrete structure) at the Planck scale, one obtains the non-conservation of energy-momentum. Specifically, the effect on propagation of massive fields must be realized in a deviation from the geodesic motion of free particles. The force must be distinctly proportional to R . Further, the force should depend on the mass m , the 4-velocity u^μ of the particle and time-like unit vector ξ^μ specifying the local frame defined by the matter that curves the spacetime. For example, in cosmology, $\xi = \partial_t$ is naturally associated with the time-arrow of the comoving cosmic fluid. In addition, the force should be proportional to the particle's mass, which gives us a characteristic length scale, the Compton wave-length. The dimensional analysis provides us with an essentially unique expression, which is compatible with the above requirements:

$$u^\mu \nabla_\mu u^\nu = \alpha \frac{m}{M_P^2} \text{sign}(s \cdot \xi) R s^\nu, \quad (44)$$

where $\alpha > 0$ is a dimensionless coupling and s is a spin-vector.

The factor $\text{sign}(s \cdot \xi)$ ensures that the force is genuinely friction-like. This is straightforward when one considers the change of the mechanical energy $E \equiv -mu^\nu \xi_\nu$ along the particle's world-line, specifically,

$$\begin{aligned} \dot{E} &\equiv -mu^\mu \nabla_\mu (u^\nu \xi_\nu) \\ &= -\alpha \frac{m^2}{M_{\text{P}}^2} |(s \cdot \xi)| R - mu^\mu u^\nu \nabla_{(\mu} \xi_{\nu)}. \end{aligned} \quad (45)$$

The last term in Equation (45) encodes the standard change of E associated with the non-Killing character of ξ^μ . The first term on the RHS comprises the friction that damps out any motion with respect to ξ^μ . We see that the energy is lost into the fundamental granularity until $u^\mu = \xi^\mu$, and the particle is at rest with the cosmological fluid.

The simplest dynamics for the spin which is consistent with (44), the conservation of $s \cdot s$ and $s \cdot p = 0$ is

$$u^\mu \nabla_\mu s^\nu = \alpha \frac{m}{M_{\text{P}}^2} \text{sign}(s \cdot \xi) R (s \cdot s) u^\nu. \quad (46)$$

It is only essential to note that this is a minimalistic solution. Certainly, other terms can be added, but they do not affect the main point of this section.

Let us return back to cosmology, and we will consider a homogeneous, isotropic, and spatially flat FLRW universe, $ds^2 = -c^2 dt^2 + a^2 d\vec{x}^2$. Then, the modified Friedmann equation reads

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \rho(t) + \frac{\Lambda^{\text{eff}}(t)c^2}{3} \quad (47)$$

where the effective cosmological “constant”

$$\Lambda^{\text{eff}}(t) \equiv \Lambda_{-\infty} + \frac{8\pi G}{c^4} \int^t J \quad (48)$$

detects the possible violations of energy-momentum conservation in the past history of the universe. Actually, as is shown in Ref. [115], quite small violations of energy momentum conservation—that may be inaccessible to current tests of local physics—may nevertheless have significant cosmological effects at late times in the form of a nontrivial contribution to the present value of the cosmological constant. It can be determined that the cosmological constant, which is derived due to the calculations in Ref. [115], as

$$\Lambda^{\text{eff}}(t) \approx 4\alpha \Lambda_{\text{obs}}, \quad (49)$$

where Λ_{obs} denotes the observed Λ value. The finding (49) is a noteworthy result.

3.3. Group Field Theory Condensate Cosmology

GFT is an approach to QG which is relative to LQG. It provides, after applying the formalism to cosmology, a unique model solving the Big Bang singularity and simultaneously the problem of the late-time acceleration of the universe. One obtains from these models the existence of some phantom field, which may even lead to the solution of the Hubble tension.

In the framework of the GFT of cosmology, the evolution of the universe is captured by the transformation of a spatial slice of spacetime, where this transformation is to be found with respect to the relational time ϕ . To incorporate the dependence of observables on the clock variable, one just focuses on the states, which are sharply peaked around

a specific relational time ϕ_0 . Further, it is also necessary for these states to support the appropriate observables, such that the continuum type of spacetime emerges under suitable limits [116,117]. These requirements naturally lead to the introduction of the coherent peak states, which provide us with an effective framework for developing the dynamics at the quantum geometric level:

$$|\sigma_\epsilon; \phi_0, \pi_0\rangle = N(\sigma) \exp\left(\int \sigma_\epsilon(g_v, \phi; \phi_0, \pi_0) \hat{\phi}^\dagger(g_v, \phi) (dg)^4 d\phi\right) |0\rangle, \quad (50)$$

with $N(\sigma)$ as a constant responsible for achieving the correct normalization and $|0\rangle$ as the vacuum state. g_v here denotes the group element associated with the vertex v and the obelisk $\dagger$ denotes the Hermite conjugation. The condensate wavefunction $\sigma_\epsilon(g_v, \phi; \phi_0, \pi_0)$ is peaked at $\phi = \phi_0$ and can be written as

$$\sigma_\epsilon(g_v, \phi; \phi_0, \pi_0) = \eta_\epsilon(\phi - \phi_0, \pi_0) \tilde{\sigma}(g_v, \phi), \quad (51)$$

where $\eta_\epsilon(\phi - \phi_0, \pi_0)$ represents a peaking function, which is typically chosen as a Gaussian and which is centered around ϕ_0 with a characteristic width controlled by ϵ . The parameter π_0 further governs the fluctuations of the operator corresponding to the conjugate momentum of the scalar field ϕ . The reduced condensate function $\tilde{\sigma}(g_v, \phi)$, which serves as the primary dynamical variable in the hydrodynamic approximation, does not alter the peaking behavior of the function $\sigma_\epsilon(g_v, \phi; \phi_0, \pi_0)$. Next, it remains true that the condensate state defined in Equation (50) is still an eigenstate of the GFT field operator such that

$$\hat{\phi}(g_v, \phi) |\sigma_\epsilon(g_v; \phi_0, \pi_0)\rangle = \sigma_\epsilon(g_v, \phi; \phi_0, \pi_0) |\sigma_\epsilon(g_v; \phi_0, \pi_0)\rangle. \quad (52)$$

Next, motivated by geometric considerations, one can see that the condensation wave condition should be invariant under both right and left diagonal group actions, so one needs to impose the following condition on $\tilde{\sigma}(g_v, \phi)$:

$$\tilde{\sigma}(hg_vk, \phi) = \tilde{\sigma}(g_v, \phi), \quad \forall h, k \in SU(2). \quad (53)$$

The task now is to derive the cosmological dynamics of homogeneous and isotropic universes from the hydrodynamics of the GFT condensate, with the evolution of the universe's volume serving us as the primary observable. Therefore, an additional constraint is introduced on the condensate wavefunction: the isotropy.

From the perspective of discrete geometry, as it is encoded in the GFT Fock space, this implies that the wave function $\sigma(g_I, \phi)$ must only have support over equilateral tetrahedra [116]. This condition enforces the requirement that the spin labels be equal, and the intertwiners should be chosen such that the volume takes the largest value allowed by this choice of spins. If one takes into account both left and right invariance, the condensate wavefunction then assumes the form

$$\tilde{\sigma}(g_I, \phi) = \sum_j \tilde{\sigma}_j(\phi) \tilde{\mathcal{I}}_{\mathbf{m}}^{j, \iota_+} \tilde{\mathcal{I}}_{\mathbf{n}}^{j, \iota_+} d(j)^2 \prod_{l=1}^4 D_{m_l n_l}^j(g_l). \quad (54)$$

Here, j is a shorthand notation for the collection of spins $\mathbf{j} = (j_1, j_2, j_3, j_4) = (j, j, j, j)$, and similarly for $\mathbf{m}, \mathbf{n}$, $\tilde{\mathcal{I}}_{\mathbf{m}}^{j, \iota_+}$ represent the intertwiner labeled by ι , while $d(j) = 2j + 1$ is the dimension of the spin j representation. The functions $D_{m_l n_l}^j(g_l)$ are the Wigner representation functions. After these considerations, the time evolution of the isotropic

state is solely encrypted in the function $\sigma_j(\phi)$ for each mode j . Finally, recalling the definition of the annihilation operator in Equation (50), one has

$$\hat{c}_x(\phi)|\sigma_\epsilon; \phi_0, \pi_0\rangle = \eta_\epsilon(\phi - \phi_0, \pi_0)\tilde{\sigma}_j(\phi)\tilde{\mathcal{I}}_m^{j,\mu+}. \quad (55)$$

Since the peaking function $\eta_\epsilon(\phi - \phi_0, \pi_0)$ generally is chosen as Gaussian, the time evolution of the condensate can be traced once the reduced condensate function $\tilde{\sigma}(g_v, \phi)$ is known. At the mean-field level, the evolution of $\tilde{\sigma}(g_v, \phi)$ is encoded in an effective action

$$S(\tilde{\sigma}, \tilde{\sigma}) = \int \langle \sigma_\epsilon; \phi_0, \pi_0 | S(\hat{\phi}^\dagger, \hat{\phi}) | \sigma_\epsilon; \phi_0, \pi_0 \rangle = \int \left\{ \sum_j \left[\tilde{\sigma}_j(\phi_0) \tilde{\sigma}_j''(\phi_0) - 2i\tilde{\pi}_0 \tilde{\sigma}_j(\phi_0) \tilde{\sigma}_j'(\phi_0) - \right. \right. \\ \left. \left. - \tilde{\xi}_j^2 \tilde{\sigma}_j(\phi_0) \tilde{\sigma}_j(\phi_0) \right] + V(\tilde{\sigma}, \tilde{\sigma}) \right\} d\phi_0, \quad (56)$$

where $\tilde{\pi}_0 = \frac{\pi_0}{\epsilon\pi_0^2 - 1}$, $\tilde{\sigma}$ and $\tilde{\sigma}$ denote the classical or background field and $\tilde{\xi}_j$ is an effective parameter that encodes the details of the kinetic term in the fundamental GFT action under the isotropic restriction. The derivatives denoted by the prime represent derivatives with respect to ϕ_0 . Finally, $V(\tilde{\sigma}, \tilde{\sigma})$ is the interaction kernel, which is determined by the underlying GFT model [116].

Let us now address a phenomenological approach, where interactions are modelled by a term in a simplified form, which makes the computations more tractable:

$$V(\tilde{\sigma}, \tilde{\sigma}) = \sum_j \left(\frac{2\lambda_j}{n_j} |\tilde{\sigma}_j(\phi_0)|^{n_j} + \frac{2\mu_j}{n_j'} |\tilde{\sigma}_j(\phi_0)|^{n_j'} \right), \quad (57)$$

where the interaction couplings λ_j and μ_j correspond to each mode j , and they should satisfy the conditions $|\mu_j| \ll |\lambda_j| \ll |m_j^2|$. Additionally, it is assumed that $n_j' > n_j > 2$. This specific model shares similarities with certain microscopic GFTs, such as those related to the EPRL model [118].

The dynamics are encoded in the action (3.3), and for later convenience, one can transform the equation of motion into a more familiar hydrodynamic form. (In what follows, the subscript 0 is omitted for ϕ_0 for simplicity as soon as no confusion is introduced.) Varying the action (3.3), with respect to $\tilde{\sigma}_j$ one finds

$$\tilde{\sigma}_j'' - 2i\tilde{\pi}_0 \tilde{\sigma}_j - \tilde{\xi}_j^2 \tilde{\sigma}_j + 2\lambda_j |\tilde{\sigma}_j|^{n_j-2} \tilde{\sigma}_j + 2\mu_j |\tilde{\sigma}_j|^{n_j'-2} \tilde{\sigma}_j = 0. \quad (58)$$

When the decomposition $\tilde{\sigma}_j(\phi) = \rho_j(\phi)e^{i\theta_j(\phi)}$, with where $\rho_j \in \mathbb{R}^+$ representing the condensate density and θ_j representing its condensate phase, is used—it transforms the complex dynamical equations into coupled hydrodynamic relations. The global $U(1)$ symmetry $\theta_j \rightarrow \theta_j + \alpha$ in the effective action generates a conserved Noether current, and the corresponding conserved charge reads $Q_j = (\theta_j' - \tilde{\pi}_0)\rho_j^2$. Using the definition of Q_j one can express the equation of motion as follows:

$$\rho_j'' - \frac{Q_j^2}{\rho_j^3} - m_j^2 \rho_j + \lambda_j \rho_j^{n_j-1} + \mu_j \rho_j^{n_j'-1} = 0. \quad (59)$$

In Equation (59), a new constant $m_j = \tilde{\xi}_j - \tilde{\pi}_j$ is introduced. By integrating the equation once, one gets another conserved quantity:

$$E_j = \frac{1}{2}(\rho_j')^2 - \frac{1}{2}m_j^2 \rho_j^2 + \frac{Q_j^2}{2\rho_j^2} + \frac{\lambda_j}{n_j} \rho_j^{n_j} + \frac{\mu_j}{n_j'} \rho_j^{n_j'}. \quad (60)$$

The evolution of the universe can be seen from the volume. In particular, for the condensate state $|\sigma\rangle$, the expectation value of the volume operator $\hat{V}$ can be calculated as follows:

$$\begin{aligned} V(\phi_0) &= \langle \sigma_\epsilon; \phi_0, \pi_0 | \hat{V} | \sigma_\epsilon; \phi_0, \pi_0 \rangle = \\ &= \langle \sigma_\epsilon; \phi_0, \pi_0 | \int \sum_{\mathbf{x}, \mathbf{x}'} V(\mathbf{l}, \mathbf{l}') \delta_{\mathbf{x}-\{\mathbf{l}\}, \mathbf{x}'-\{\mathbf{l}'\}} \hat{c}_{\mathbf{x}}^\dagger(\phi) \hat{c}_{\mathbf{x}'}(\phi) | \sigma_\epsilon; \phi_0, \pi_0 \rangle \approx \sum_j V_j \rho_j(\phi_0)^2. \end{aligned} \quad (61)$$

The volume contribution from each quantum in the spin j representation is given by $V_j \propto L_P^3 j^{3/2}$, where L_P denotes the Planck length. We also employ the intertwiner normalization condition $\sum_{\mathbf{m}} \mathcal{I}_{\mathbf{m}}^{j, \mathbf{l}+} \bar{\mathcal{I}}_{\mathbf{m}}^{j, \mathbf{l}+} = \delta_{\mathbf{l}, \mathbf{l}'}$. The approximation used in this analysis consists of retaining only the dominant contribution to the saddle-point approximation of the peaking function, which arises from the choice of state (to avoid a confusion, the subscript 0 is reintroduced for the given relational time ϕ_0 at this stage).

Having derived the equation of motion (59) of condensate density ρ_j , one can write the equations for the volume dynamics in the form of modified FLRW equations:

$$\left(\frac{V'}{3V} \right)^2 = \frac{2 \sum_j V_j \sqrt{2E_j \rho_j^2 - Q_j^2 + m_j^2 \rho_j^4 - \frac{2}{n_j} \lambda_j \rho_j^{n_j+2} - \frac{2}{n'_j} \mu_j \rho_j^{n'_j+2}}}{3 \sum_k V_k \rho_k^2} \quad (62)$$

and

$$\frac{V''}{V} = \frac{2 \sum_j V_j \left[2E_j + 2m_j^2 \rho_j^2 - \left(1 + \frac{2}{n_j}\right) \lambda_j \rho_j^{n_j} - \left(1 + \frac{2}{n'_j}\right) \mu_j \rho_j^{n'_j} \right]}{\sum_k V_k \rho_k^2} \quad (63)$$

First, let us discuss the classical regime. During the pre-interaction-dominated phase of volumetric expansion, the system transitions through an intermediate regime, where the dynamics approximate the FLRW cosmology with a free massless scalar field. This happens under the hierarchy of scales $\rho_j^2 \gg E_j/m_j^2$, $\rho_j^3 \gg Q_j^2/m_j^2$ and $|\mu_j| \rho_j^{n'_j-2} \ll |\lambda_j| \rho_j^{n_j-2} \ll m_j^2$, under which the Friedmann Equations (62) and (63) read, respectively,

$$\left(\frac{V'}{3V} \right)^2 = \left(\frac{2 \sum_j V_j m_j \rho_j^2}{3 \sum_k V_k \rho_k^2} \right) \quad (64)$$

and

$$\frac{V''}{V} = \frac{\sum_j V_j (4m_j^2 \rho_j^2)}{\sum_k V_k \rho_k^2}. \quad (65)$$

For a dominant spin mode $\tilde{j}$ with approximately constant $m_{\tilde{j}}$, let us identify

$$m_{\tilde{j}}^2 = 3\pi G, \quad (66)$$

where G represents the effective dimensionless Newton's constant. Then, this yields the characteristic FLRW Equations (64) and (65) in the relational time:

$$\left(\frac{V'}{V} \right)^2 = \frac{V''}{V} = 12\pi G. \quad (67)$$

The mode dominance dynamics ensure a rapid convergence to the lowest spin mode j_0 , making $m_{j_0}^2 = 3\pi G$ sufficient for the classical recovery. This establishes GFT condensate

hydrodynamics as a viable pathway to classical cosmology at macroscopic scales, provided that the interaction terms remain subdominant. Also, single-mode dominance further induces a direct proportionality between the Hubble constant and volume evolution rate:

$$H = \frac{Q_1}{3} \frac{\mathcal{V}'}{\mathcal{V}}. \quad (68)$$

The critical relationship (68) then enables the extraction of the cosmological constant from microscopic GFT parameters.

We now mention the emergent cosmological dynamics using the framework, which are expressed via the relational clock evolution. For a homogeneous, isotropic metric with scale factor $a(t)$, the Hubble constant is $H = \dot{a}/a$. The effective equation of state becomes $w = -1 - 2\dot{H}/H^2$. Using the relational clock, it takes the form $w = 3 - 2VV''/(V')^2$, where $V = a^3$ is the volume and now primes denote derivatives with respect to the relational time ϕ .

Let us study w from single-mode GFT condensates. We investigate the effective equation of state behavior under single-mode dominance. While multi-mode scenarios introduce richer transitional dynamics and modified convergence to asymptotic w values, the single-mode framework already reveals key insights into how the GFT interactions shape emergent cosmology.

From Ref. [116], one gets from the single j mode that

$$w = \frac{-3Q^2 + 4E\rho^2 + m^2\rho^4 + (1 - \frac{4}{n})\lambda\rho^{n+2} + (1 - \frac{4}{n'})\mu\rho^{n'+2}}{-Q^2 + 2E\rho^2 + m^2\rho^4 - \frac{2}{n}\lambda\rho^{n+2} - \frac{2}{n'}\mu\rho^{n'+2}}, \quad (69)$$

where mode indices are omitted for simplicity. The direct proportionality $V \propto \rho^2$ enables applying the $w(V)$ analysis without solving the equations of motion.

Firstly, let us consider the bounce. In the interaction-free limit ($\lambda = \mu = 0$) at relatively small condensate modulus ρ , the equation of state reduces to

$$w = \frac{-3Q^2 + 4E\rho^2 + m^2\rho^4}{-Q^2 + 2E\rho^2 + m^2\rho^4} \quad (70)$$

The bounce condition $-Q^2 + 2E\rho^2 + m\rho^4 = 0$ determines the minimum volume

$$\rho_b = \frac{1}{m} \sqrt{\sqrt{E^2 + m^2Q^2} - E}. \quad (71)$$

At ρ_b , the numerator becomes negative, driving $w \rightarrow -\infty$ —characteristic of post-bounce acceleration. Crucially, this inflationary phase terminates quite rapidly, with acceleration ending at volumes comparable to V_b [116]. One can see that when there is a bounce, i.e., when $\rho_b = 0$, one must have $Q = 0$. More generally, there is a bounce if at least one Q_j is non-vanishing in all condensate modes.

Further, we study the phantom divide crossing. The single-mode dynamics permit limited phantom behavior.

1. $\mu < 0$ regime: asymptotic w approaches $2 - n'/2$ with

$$w \rightarrow -1(n' = 6) \text{ or } w' < -1(n' > 6). \quad (72)$$

2. Late-time singularities: $n' > 6$ leads to Big Rip.
3. Multi-mode resolution: two-mode systems enable phantom crossing ($w < -1$), while avoiding singularities through phantom de Sitter analogs.

The critical distinction lies in mode competition—multiple modes regulate energy density divergence despite phantom behavior, as we show in the further consideration here. In the presence of interactions, it becomes much more complicated to find an exact solution to the equation of motion (60), when actually two modes contribute. However, an approximate solution in the relatively large volume region can be obtained. When ρ is large enough, the E_j , Q_j and m_j terms can be ignored in the equation of motion (60); then one obtains approximately

$$\rho'_j(\phi) = \sqrt{\frac{-\lambda_j}{3}} \rho_j(\phi)^3, \quad (73)$$

which can be solved to give

$$\rho_j(\phi) = \frac{3^{\frac{1}{4}}}{\sqrt{2\sqrt{-\lambda_j}(\phi_{j\infty} - \phi)}}, \quad (74)$$

with a constant $\phi_{j\infty}$. One can see that when interactions are included, the relational time ϕ gets an upper bound, and the mode associated with the smallest asymptotic value $\phi_{j\infty}$ governs the late-time dynamics. Then, this leads to an asymptotic de Sitter-like phase. Contributions from other modes remain significant at intermediate volumes, and they are collectively driving the universe's expansion. As mentioned in Ref. [116], for vanishing high-order interaction couplings ($\mu_1 = \mu_2 = 0$) and small λ_1, λ_2 , the equation of state parameter w dominates by the free sector of the condensate at small enough volumes. In this regime, w approaches $w = 1$ from below as the volume increases, recovering the standard FLRW universe. However, at larger volumes, interaction terms progressively dictate the condensate dynamics. When the scale parameters ρ_j grow sufficiently large, w becomes dominated by these interaction terms. For the specific case $n_1 = n_2 = 6$, w in this interaction-dominated regime depends solely on the ratio $r = \rho_2/\rho_1$. Therefore, one has

$$w = -1 - \frac{4\mathcal{V}_1\mathcal{V}_2r^2(r^2 - \sqrt{\lambda_1/\lambda_2})}{(\sqrt{\lambda_1/\lambda_2}\mathcal{V}_1 + \mathcal{V}_2r^4)^2}. \quad (75)$$

Since the parameters are all real, and both couplings λ_1 and λ_2 are assumed to be negative, one can see that $w \leq -1$. Recall that when the volume is large enough, one of the two modes dominates over the other, and then $r \rightarrow 0$ or $r \rightarrow \infty$. In either case, w approaches -1 from below, which is a contrast with the discussion concerning the single mode.

When the universe volume gets larger, the ratio r gets smaller, and w can be expanded with respect to small r , such that the equation of state can be expressed by only the total volume $\mathcal{V}$:

$$w = -1 - \frac{b}{\mathcal{V}}, \quad (76)$$

where $b = 4\mathcal{V}_2\rho_2^2(\phi_{1\infty}) > 0$.

Before we state some results concerning the explicit form of the cosmological constant, it is instructive to discuss the effects of including more than two condensate modes. Let $\phi_{1\infty}$ denote the smallest asymptotic value of the relational time for mode ρ_1 . At very late times, the evolution should therefore be governed by this mode, according to the asymptotic solution (74). Introducing an additional mode with $\phi_{2\infty} > \phi_{1\infty}$ modifies the evolution by inducing a phantom phase before the onset of the de Sitter expansion and alters the asymptotic behavior. If a third mode ρ_3 is included, $\phi_{3\infty} > \phi_{2\infty}$ must be required, otherwise, the ρ_3 mode would dominate the phantom phase. This means that it is expected

that the inclusion of additional modes primarily affects the cosmological evolution before the phantom phase and should not significantly alter the main results discussed.

In this GFT model, a de Sitter phase with an effective cosmological constant emerges asymptotically from the microscopic quantum dynamics of spacetime constituents. The precise expression of such an effective cosmological constant can be determined as a function of the microscopic parameters of the model and of the underlying quantum state of the universe.

To investigate this, it is worth to note that in the asymptotically large volume region, one can ignore the contributions from other terms and only keep (order-6) interactions in the equation of motion (62). Further, it was shown above that in such a region only a single mode dominates, and one can use the relation (68) between Hubble constant H and the ratio V'/V . Therefore, in the asymptotic de Sitter regime, one obtains

$$H^2 = \frac{8}{9} Q_1^2 \left(-\frac{\lambda_1}{6V_1^2} \right) = \frac{1}{3} \left[\frac{4Q_1^2(-\lambda_1)}{3V_1^2} \right]. \quad (77)$$

Comparing Equation (77) with the Λ CDM at late time, one can find that the cosmological constant is determined by the microscopic parameters of the GFT model,

$$\Lambda = \frac{4Q_1^2(-\lambda_1)}{3V_1^2}. \quad (78)$$

Let us discuss the implications of this expression for Λ and emphasize several points.

1. Λ is determined by the parameters of a single mode despite two modes are considered.
2. Actually, for a non-vanishing Λ , one sees that $Q_1 = 0$ is necessary; from the theory, actually, one can consider the case where Q_j vanishes for all modes j [116], which may provide us with the Minkowski spacetime. However, Equation (78) shows that the observation of a non-vanishing cosmological constant at late times then requires that the Q_1 cannot be zero. This implies a quantum bounce [116], which resolves the Big Bang singularity. Being expressed differently, the non-vanishing cosmological constant Λ itself becomes a remnant of the expansion history of our universe in the far past. This indirect connection between very early and very late universe dynamics (and thus considerably small and considerably large scales is quite of interest, and it is only possible because of an emergent spacetime framework.

More generally, the value of Λ cannot be directly related to the bouncing scale, since Λ is determined by the parameters of a single mode, but the bounce depends on the collective contributions from all modes. The reason for this is that in the bouncing regime, the modulus ρ_j of each mode remains considerably small, and no single mode dominates the dynamics. This actually requires that the sum of all Q_j must vanish. If, instead a single mode, ρ_1 were to dominate the dynamics from the beginning of the bounce, then enforcing ϕ as a good clock would necessitate $Q_1 = 0$, and this leads to a vanishing cosmological constant. By contrast, when multiple modes contribute to the bounce while a single mode dominates the late-time dynamics, it becomes possible to satisfy $\sum_j Q_j = 0$ and still retain a nonzero Q_1 , which allows obtaining a nonvanishing cosmological constant in the model.

However, one can extract potentially relevant observational constraints, at least qualitatively. On the one hand, Q_1 can not be exceedingly large, otherwise, one gets quite a large cosmological constant Λ . On the other hand, Q_1 contributes to the critical energy density of the universe at the bounce [116]. Hence, it cannot be utterly small, or the established physics of the hot dense state of the universe in the very early time would be lost, with spacetime dynamics still governed by the Friedmann evolution (the

universe instead quickly enters into a QG bouncing regime). These two constraints can be actually used to narrow down the possible range of Q_1 using observations.

3. Λ does not depend on m_j , hence the mass renormalization of the GFT model does not change the value of the cosmological constant. One can note also that m_j is related to the effective Newton's constant emerging in the Friedmann phase of cosmic expansion. Therefore, the two key couplings of gravitational dynamics in classical GR are both emergent, and they are independent of each other in this QG model.

3.4. The Causal Set Theory

The causal set theory prediction for Λ [119–121] was startling in its simplicity, especially since it was made many years before the observation of accelerated expansion. One starts with the framework of unimodular gravity, in which the spacetime volume element will be fixed. The cosmological constant Λ then appears as a Lagrange multiplier in the action with $\Lambda \int dV = \text{const.}$ for any spacetime region of volume V . One sees that Λ and V are conjugate to each other in a canonical formulation of the theory, so one obtains the formula

$$\Delta V \Delta \Lambda \sim 1 \quad (79)$$

after the quantization of the theory. Even if one keeps N , which denotes the number of points in the discretization, constant in the volume V , V still fluctuates between $(N - \sqrt{N})$ and $(N + \sqrt{N})$. This signifies that $\Delta V \sim \sqrt{V}$. Therefore, there is the central result:

$$\Delta \Lambda \sim \frac{1}{\sqrt{V}}. \quad (80)$$

Let us presuppose now that, for some reason, Λ oscillates around zero, so $\langle \Lambda \rangle = 0$; then one obtains, with the help of the standard cosmological argument,

$$V \sim (H^{-1})^4 = H^{-4} \Rightarrow \Lambda \sim \frac{1}{\sqrt{V}} \sim H^2 \sim \rho_{\text{crit}}. \quad (81)$$

So,

$$\Lambda = \Delta \Lambda \sim 10^{-120}, \quad (82)$$

in Planck units, and this is quite close to the observed value of Λ . What is also significant is that the prediction states that under these assumptions, Λ always tracks the critical density, and so it is everpresent.

This argument is fully general and requires three key ingredients:

- (i) the assumption of conjugacy between Λ and V ,
- (ii) the number to volume correspondence $V \sim N$, and
- (iii) there are fluctuations in V , which are Poisson, with $\delta V = \sqrt{V} \sim \sqrt{N}$.

While the requirement (i) can be motivated by a wide range of theories of QG, the requirements (ii) and (iii) are both connected with causal set theory. No other discrete approach to QG sets the $N \sim V$ correspondence at the fundamental level and also takes into consideration Poisson fluctuations kinematically in the continuum approximation. Actually, according to Ref. [119], fluctuations in Λ arise as residual quantum effects of spacetime discreteness. What is of interest is that if spacetime admits relatively large extra dimensions, then the contribution to V highly differs and gives the wrong answer for $\Delta \Lambda$. This is shown in Ref. [122].

A significant question also arises in this quick calculation: why should one assume that $\langle \Lambda \rangle = 0$ holds? The answer to this may lie in the full and as-yet unknown quantum dynamics. However, this assumption leads to further predictions that can be well tested. The first implication is that a fluctuating Λ must violate conservation of the stress-energy tensor, and so the Einstein field equations (3).

A dynamical model for generating fluctuations of Λ was constructed in Ref. [79]. The model started with the flat $k = 0$ FLRW space-time and, in order to accommodate a fluctuating Λ , one of the Friedmann equations is dropped. However, the equation

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho + \rho_\Lambda \quad (83)$$

is kept, with

$$\rho_\Lambda = \Lambda, \quad p_\Lambda = -\Lambda - \frac{\dot{\Lambda}}{3H}, \quad (84)$$

and Λ is modeled as a stochastic function of V , such that

$$\Delta\Lambda \sim \frac{1}{\sqrt{V}}. \quad (85)$$

Dealing with the model more generally, Λ can be expressed as the action per unit volume, which for causal sets signifies that $\Lambda \sim S/V$. A quite simple set of stochastic dynamics is then generated by the assumption that every element contributes $\pm\hbar$ to S , so that

$$S = \sum_{\text{elements}} \pm\hbar \Rightarrow S/\hbar \sim \pm\sqrt{N} \sim \pm\sqrt{V/L_P^4} \Rightarrow \Lambda \sim \pm\frac{\hbar/L_P^2}{\sqrt{V}}, \quad (86)$$

where the discreteness scale was equated with the Planck length L_P . Then, one gets the integro-differential equations

$$\frac{da}{a} = \sqrt{\frac{\rho + \Lambda}{3}} d\tau, \quad (87)$$

$$Vd\Lambda = Vd(S/V) = dS - \Lambda\dot{V}d\tau, \quad (88)$$

where

$$V(\tau) = \frac{4\pi}{3} \int_0^\tau a(t')^3 \left(\int_0^{t'} \frac{1}{a(t'')} dt'' \right)^3 dt' \quad (89)$$

denotes the volume of the entire causal past of an event in the FLRW spacetime. The stochastic equation is subsequently generated as follows. At the i th step one obtains the variables a_i , N_i , S_i and Λ_i . The scale factor is then updated using the discrete Friedmann equation

$$a_{i+1} = a_i + a_i \sqrt{\frac{\rho + \Lambda}{3}} (\tau_{i+1} - \tau_i), \quad (90)$$

from which $V_i = V(\tau)$ can be calculated, and so $N_{i+1} = V_{i+1}/l^4$. The action is computed via

$$S_{i+1} = S_i + \alpha\zeta \sqrt{N_{i+1} - N_i}, \quad (91)$$

where ξ is a Gaussian random variable with $\Delta\xi = 1$, and α is a tunable free parameter that controls the magnitude of the fluctuations. Afterwards,

$$\Lambda_{i+1} = S_{i+1}/V_{i+1}, \quad (92)$$

with $S_0 = 0$. The value $0.01 < \alpha < 0.02$ is necessary in order to be consistent with astrophysical observations. The results of the observations also suggest that Λ is “everpresent” and tracks the energy density of the universe.

Let us comment further. This model assumes spatial homogeneity, and it is of importance to check how inhomogeneities may affect these results. In Refs. [123,124], the results were modeled by taking $\Lambda(x^\mu)$, such that $\Delta\Lambda(x)$ is dependent only on $\Lambda(y)$ for $y \in J^-(x)$. This means that well-separated patches in the CMB sky would contain uncorrelated fluctuations in Ω_Λ , which are strongly constrained to 10^{-6} by observations and, hence, insufficient to account for Λ . It was suggested in Refs. [79,125] that quantum Bell corrections may be a possible way to induce correlations in the CMB sky. However, incorporating inhomogeneities into the dynamics in a systematic way remains an essential open problem in this field.

A phenomenological model was adopted in Ref. [125], which uses the homogeneous temporal fluctuations in Λ to investigate models with a quintessence-type spatially inhomogeneous scalar field that contains a potential term that varies from realisation to realisation. Using Markov-chain Monte Carlo methods to sample the cosmological parameter space and generate different stochastic realisations, it was shown that these causal set theory-inspired models agree with the observations as well as the Λ CDM model. Indeed, they do better for baryonic acoustic observation measurements. The extensive and detailed analysis of Ref. [125] prepares the stage for direct comparisons with future observations, and it is a challenge for this field of QG phenomenology.

3.5. A Model of Tejinder Singh Based on String Theory and Non-Commutative Geometry

Let us now discuss an application of a QG model developed in Refs. [83,126–128], in which the universe is made of ‘atoms’ of space-time-matter, and we introduce a new DE particle.

The fundamental QG theory, which we use here, is a non-commutative matrix dynamics of Grassmann matrices, the so-called ‘atoms’ of space-time-matter which do not make a distinction between space-time and matter. These matrices evolve in the Hilbert space according to a time parameter characteristic of non-commutative geometry, which derives from the Tomita–Takesaki theory [129] and the Radon–Nykodym theorem [130]. We describe the space-time-matter atoms by a Lagrangian dynamics resulting from a well-defined action principle, and they interact via entanglement and ‘collisions’ [131]. The dynamics possess a unique conserved charge known as the Adler–Millard charge [132], responsible for the emergence of quantum theory [128].

This theory is assumed to operate at the Planck scale. Actually, there is no space-time, but one can define a Planck space foam of space-time and matter. If one is not observing dynamics at the Planck scale, a mean-field dynamics at lower energies is obtained. This is achieved just by averaging over time-scales much larger than Planck time, using the standard techniques of statistical thermodynamics. The two classes of these mean field dynamics can be distinguished.

1. If a length scale associated with the space-time-matter atoms is larger than Planck length, one then gets a quantum theory of gravity for the bosonic (gravity) and fermionic (matter) aspects of the space-time-matter atoms. One can see that these degrees of freedom evolve with respect to the characteristic time parameter of non-

commutative geometry. Then, QG is not exclusively a Planck scale phenomenon, but relevant even at much lower energies if the gravity associated with a space-time-matter atom cannot be neglected [126]. This can specifically happen if the Compton wavelength associated with the space-time-matter atom is of the order of the size of the observed universe.

2. In the other extreme limit, the entanglement of a considerably large number of space-time-matter atoms results in a rapid ‘spontaneous localisation’, giving rise to a classical space-time geometry driven by point matter sources. These are then obeying the laws of classical GR. Ordinary space-time is recovered, but the non-commutative time parameter are lost in the classical limit.

Consider the space-time-matter atoms which have not undergone the localisation. The standard QFT is recovered by taking their matter degrees of freedom from QG, ignoring their gravity, and taking space-time from the above classical limit of other space-time-matter atoms. Actually, in the framework of this theory, it is proposed that DE consists of about 10^{122} space-time-matter atoms, each having an associated mass of 10^{-33} eV/ c^2 . Hence, the Compton wavelength, $\hbar/mc$, of such a space-time-matter atom is of the order of 10^{28} cm, which is comparable to the size of the observed universe. A space-time-matter atom is nothing but an elementary particle whose gravitational aspect cannot be distinguished from the particle aspect. An atom labelled by the matrix/operator variable $q = q_B + q_F$, with q_B and q_F being its bosonic and fermionic parts, is described by the action principle

$$\frac{L_P \cdot S}{c \cdot C_0} = \frac{1}{2} \int \text{Tr} \left[\frac{L_P}{l^2 c^2} (\dot{q}_B + \beta_1 \dot{q}_F)(\dot{q}_B + \beta_2 \dot{q}_F) d\tau \right]. \quad (93)$$

$\kappa \equiv C_0/2$ is a constant chosen because it gives the correct dimensions of action and the correct numerical coefficient for recovery of the Einstein–Hilbert action. Further, the $\beta_{1,2}$ matrices are constant matrices, proportional to L_P^2/l^2 , where l is a fundamental length associated with the space-time-matter atom, subsequently identified with Compton wavelength [126].

For the action (93), the first integrals of the equations of motion, with evolution with respect to the Connes time parameter τ , reads

$$\begin{aligned} p_B &= \frac{\delta L}{\delta \dot{q}_B} = \frac{b}{2} [2\dot{q}_B + (\beta_1 + \beta_2)\dot{q}_F] \\ p_F &= \frac{\delta L}{\delta \dot{q}_F} = \frac{b}{2} [\dot{q}_B(\beta_1 + \beta_2) + \beta_1 \dot{q}_F \beta_2 + \beta_2 \dot{q}_F \beta_1], \end{aligned} \quad (94)$$

where L denotes the Lagrangian and $b \equiv L_P^2/(l^2 c^2)$. The first of these equations can be rewritten as an eigenvalue equation which, as is explained in Ref. [126], results from defining the modified Dirac operator:

$$\frac{1}{lc} \frac{dq}{d\tau} \sim D \equiv D_B + D_F \text{ with } D_B \equiv \frac{1}{lc} \frac{dq_B}{d\tau} \text{ and } D_F \equiv \frac{\beta_1 + \beta_2}{2lc} \frac{dq_F}{d\tau}. \quad (95)$$

Let us note that D is a constant operator and can be expressed as an eigenvalue equation:

$$[D_B + D_F] \psi \equiv \lambda \psi \equiv (\lambda_R + i\lambda_I) \psi \equiv \left(\frac{1}{l} + i\frac{1}{l_I} \right) \psi. \quad (96)$$

Since D is bosonic, it is assumed that the eigenvalues $\lambda_{R,I}$ are complex numbers and separate each eigenvalue into its real and imaginary parts, respectively. Further, this is taken as the definition of the length scale l . As was demonstrated in Ref. [126], $l^3 = L_P^2 l_I$, and since l is a Compton wavelength, this implies $l_I = \hbar^2/Gm^3$.

The two lengths, l and l_I , play a significant role as follows. The length l is a measure of the quantum dispersion, and the decoherence length l_I is a measure of the spontaneous localisation. If $l < l_I$, then the quantum dispersion dominates the classical behaviour, and the space-time-matter atom becomes a quantum elementary particle. If $l > l_I$, then the spontaneous localisation and classical behaviour dominate quantum dispersion, and the space-time-matter atom becomes a classical object. Actually, the quantum-to-classical transition takes place at $l = l_I$, in which case both the lengths are equal to the Planck lengths. This is the space-time-matter foam which exists at the Planck scale. Actually, it represents Planck-scale quantum dispersion and spontaneous localisation, which is equivalent to a stochastic oscillation of Planck scale curvature. In our theory, QG arises at time scales larger than Planck time, after coarse-graining over Planck scale foam. The above relation between l and l_I can be more reasonably written as $(l/L_P)^3 = (l_I/L_P)$ and is closely enough related to the Karolyhazy uncertainty relation [131]. It is worth to note that if $l_I < L_P$, then both lengths are smaller than Planck length—this is the classical limit. On the other hand, if $l_I > L_P$, then both lengths exceed Planck length, this being the quantum limit. The two lengths also highlight two crucial time scales: $t_q = l/c$, which is the quantum dispersion time scale, and $t_d = l_I/c$, which is the decoherence time scale. The classical behavior results if $t_d < t_q$, and the quantum behavior prevails if $t_q < t_d$. The possible regimes above are the defining equations for our DE particle. We propose to name these particles as mitrons. It is, naturally, a quantum gravitational entity, with its gravitational aspect described by the bosonic q_B and its matter aspect described by the fermionic q_F . The gravitational aspect of the mitron cannot be described classically, because its decoherence length l_I is exceptionally large. For the mass of 10^{-33} eV/ c^2 assumed, this length is 123 orders of magnitude larger than the size of the observed universe. Consequently, the decoherence time l_I/c is 10^{140} s, which is substantially greater than the current age of the universe. As a result, the mitrons never undergo the classicalisation process of spontaneous localisation. It is inherently quantum gravitational in nature. The essential assumption is made as well considering DE particles do not entangle with each other, nor with other particles.

On the other hand, as explained in the papers [128,133], ordinary matter undergoes spontaneous localisation because of entanglement and gives rise to the observed classical universe. Against the backdrop of the classical expanding universe, let us now understand why the mitrons are responsible for the acceleration of the universe. The contribution to the mass density of the universe, from a single DE particle, is of the order m_{DE}/R_H^3 , where the mass m_{DE} is of the order 10^{66} g, and R_H , the Hubble radius, is about 10^{28} cm. This gives us an exceptionally low mass density of the order of 10^{-150} g/cm³.

Let us now mention the recent discussion of the Karolyhazy uncertainty relation [131], and from the implied holography, that the universe has $(R_H/L_P)^2 \sim 10^{122}$ quanta of information [134]. Assuming there corresponds one unit of information per elementary particle, we realise that ordinary matter contributes only a considerably small fraction to the information content, there being some 10^{80} such particles. Thus we propose, following Ref. [134], that there are 10^{122} dark energy particles in the observed universe, and hence, their total contribution to the energy density of the universe is about 10^{28} g/cm³, which is a reasonable estimate for the inferred dark energy content of the universe. In summary, we are proposing that dark energy consists of 10^{122} mitrons, each of mass 10^{-33} eV/ c^2 . Further, these space-time-matter atoms are assumed to be unentangled, and hence these atoms do not undergo spontaneous localisation during the lifetime of the present universe. One might think of each such particle as a quantum gravitational wave of the size of the observed universe.

Such objects collectively behave like DE, and they are causing acceleration of the expanding universe because they have quite a long wavelength, and their kinetic energy K

is negligible compared to their potential energy U . The pressure is essentially $K - U$ and the energy density is $K + U$. With K being negligible, this implies an equation of state as for the cosmological constant $p = -\rho$. In this theory, one may argue for the dominance of the gravitational aspect over the matter aspect by examining the expression for the Hamiltonian for the space-time-matter atom given by [128]

$$\mathcal{H} = \text{Tr} \left[\frac{b}{2} (\dot{q}_B + \beta_1 \dot{q}_F) (\dot{q}_B + \beta_2 \dot{q}_F) \right] \quad (97)$$

and in terms of the momenta p_{FB} ,

$$\mathcal{H} = \text{Tr} \frac{2}{b} \left[(p_B \beta_1 - p_F) (\beta_2 - \beta_1)^{-1} (p_B \beta_2 - p_F) (\beta_1 - \beta_2)^{-1} \right]. \quad (98)$$

From Equation (98), one can see that since the β matrices scale as L_P^2/l^2 , and since $l \gg L_P$, the fermionic matter term is negligible, and the effective contribution to the Hamiltonian is from the gravitational term $\dot{q}_B^2$. This means that the kinetic energy term is negligible compared to the gravitational energy, and once again we see that the contribution of the mitrons behaves like a cosmological constant. The mass density estimate from above shows that it is a cosmological constant of the same magnitude as implied by observations.

One need to raise the key question: does the DE in the model behave like the cosmological constant, or is it dynamical? This can be explained as follows. The Compton wavelengths associated with a proton and a DM particle (within this Tejinder's theory it was proposed to be a dark matter candidate particle of mass about 10^{-12} eV/ c^2 , which is 10^{21} orders of magnitude lighter than the proton [126]) are 10^{-13} cm and 108 cm, respectively. Since these are both much smaller than the Hubble radius, it may be assumed that they are decoupled from the expansion of the universe and, hence, constant. On the other hand, the Compton wavelength for the mitron is of the order of the Hubble radius, so we assume it to stretch in linear proportion to the Hubble radius. Thus, it can be computed that the equivalent mass density of the DE particle falls as the inverse fourth power of the Hubble radius as the universe expands. Since the number of DE particles increases as H^2 , the net DE density falls as the inverse square of the Hubble radius, and, hence, the DE is dynamical. Further, because it falls more slowly than ordinary matter density as well as slower than DM density, the universe was matter-dominated in the past. Only in the present epoch does the DE predominate. It is worth to emphasize that in this approach, DE is entirely a quantum gravitational effect and could be properly described only in QG. Its description in the context of classical GR is only approximate. What is also remarkable is that such DE should be regarded as evidence for the quantum nature of gravity [126].

3.6. Causal Dynamical Triangulations and Baby Universe Dark Energy

Section 2.6 discussed the very basics of CDTs, and in this Section, we discuss CDT cosmological implications of perhaps highest interest. We provide an overview of the scenario where the late-time expansion of the universe arises through mergers with other universes, a framework situated within the context of CDTs [135–137]. Although the term “baby universe” is used here, it does not necessarily imply that these universes are significantly smaller than our universe. This terminology is adopted to align with prior studies on the subject, see [136,138–140] for extended studies on the cosmology of CDTs. By considering mergers with other universes, we inherently discuss a multi-universe theory. Analogous to a multi-particle theory, we define creation and annihilation operators, $\Psi^\dagger(v)$ and $\Psi(v)$, respectively, for individual universes characterized by spatial volume v . In a complete theory of four-dimensional QG, spatial volume alone is suffice to describe a universe's state at a given time t . However, within the minisuperspace approximation,

the spatial universe is described entirely by its spatial volume v , and its quantum state is represented as $|v\rangle$. The multi-universe Fock space, constructed from such single-universe states, includes the Fock vacuum state $|0\rangle$, governed by the following commutation relation and operator actions [141]

$$[\Psi(v), \Psi^\dagger(v')] = \delta(v - v'), \quad \Psi^\dagger(v)|0\rangle = |v\rangle, \quad \Psi(v)|0\rangle = 0, \quad (99)$$

where $\delta(a - b)$ is the Dirac delta function. The minisuperspace quantum Hamiltonian, incorporating the creation and annihilation of universes, takes the form

$$\begin{aligned} \hat{\mathcal{H}} = & \hat{\mathcal{H}}^{(0)} - g \int dv_1 \int dv_2 \Psi^\dagger(v_1) \Psi^\dagger(v_2) (v_1 + v_2) \Psi(v_1 + v_2) \\ & - g \int dv_1 \int dv_2 \Psi^\dagger(v_1 + v_2) v_2 \Psi(v_2) v_1 \Psi(v_1) - \int \frac{dv}{v} \rho(v) \Psi^\dagger(v), \end{aligned} \quad (100)$$

where g is the coupling constant and

$$\hat{\mathcal{H}}^{(0)} = \int_0^\infty \frac{dv}{v} \Psi^\dagger(v) \hat{h}^{(0)} v \Psi(v), \quad \hat{h}^{(0)} = v \left(-\frac{3}{4} \frac{d^2}{dv^2} + \lambda \right), \quad (101)$$

with λ serving as the potential term. The term $\hat{\mathcal{H}}^{(0)}$ (101) describes the propagation of a single universe, derived from the action associated with the minisuperspace approximation. The cubic second and third terms in Equation (100) represent the splitting of a universe into two and the merging of two universes into one. The last term indicates that a universe can emerge from the Fock vacuum $|0\rangle$ if its spatial volume is zero. Without this term, $\hat{\mathcal{H}}|0\rangle = 0$, rendering the Fock vacuum stable. While the details of merging and splitting processes are beyond the scope of this approximation, the primary focus lies on how these interactions influence the spatial volume of universes.

Even within the minisuperspace framework, the Hamiltonian $\hat{\mathcal{H}}$ is complex, as it allows for successive splitting, merging, and topology changes of universes. Notably, the dimension dependence of the Hamiltonian is absorbed into coupling constants, such as κ , λ , and g . In two-dimensional CDTs, a tractable truncation termed generalized CDTs exists, which is solvable analytically and holds particular cosmological significance. This theory models the evolution of the universe, incorporating mergers with other universes created over time. After detailed analysis, the Hamiltonian takes the form [139]

$$\mathcal{H} = v \left(-\frac{3}{4} (p^2 + \lambda - 2gF(p)) \right) = \frac{3}{4} v \left((p + \alpha) \sqrt{(p - \alpha)^2 + \frac{2g}{\alpha}} \right), \quad (102)$$

where $F(p)$ is the baby universe function, α is the asymptotic value of the conjugate momentum and p here is the classical momentum conjugate to v . This Hamiltonian demonstrates that cosmic expansion can occur without a cosmological constant, driven instead by mergers with other universes.

Introducing matter modifies the Hamiltonian to

$$\mathcal{H}[v, p] = v(-f(p) + \kappa \rho_m(v)), \quad v \rho_m(v) = v_{\text{pt}} \rho_m(v_{\text{pt}}), \quad (103)$$

where v_{pt} and $\rho_m(v_{\text{pt}})$ denote the spatial volume and matter density at the present time t_{pt} . The equations of motion for arbitrary $f(p)$ are derived as [139]

$$\dot{v} = \frac{\partial \mathcal{H}}{\partial p} = -v f'(p), \quad 3 \frac{\dot{a}}{a} = \frac{\dot{v}}{v} = -f'(p), \quad (104)$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial v} = f(p), \quad t - t_0 = \int_{p_0}^p \frac{dp}{f(p)}, \quad (105)$$

where the prime denotes the first derivative with respect to canonical momentum p and p_0 is the value of p at an initial time t_0 , not necessarily the universe's beginning. From Equations (104) and (105), the Hubble constant is expressed as

$$H = -\frac{f'(p)}{3}. \quad (106)$$

One can express various significant cosmological parameters, such as redshift and angular diameter, in terms of the variable p using the formalism discussed above. However, our focus here is on the so-called “formal pressure” and “formal energy,” which serve as the analogs of pressure and energy density in this cosmological framework. Any solution to the above equations must inherently satisfy the condition $\mathcal{H} = \text{const}$ by construction. For the “on-shell” solutions where $\mathcal{H} = 0$, one obtains the relation

$$f(p) = \kappa \rho_m(v) = \kappa \rho_m(v_{\text{tp}}) \frac{v_{\text{tp}}}{v} = f(t_{\text{tp}}) \frac{t_{\text{tp}}}{v}, \quad (107)$$

where p_0 represents the value of p at the present time t_{tp} . Equation (107) is often referred to as the “generalized Friedmann equation.”

The “formal density” $\rho_f(t)$ or equivalently $\rho_f(p)$, which is associated with the function $f(p)$, is determined by rewriting the generalized Friedmann equation (107) as

$$\left(\frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{\kappa \rho_m(v)}{3} + \frac{\kappa \rho_f(v)}{3}. \quad (108)$$

Using the equations of motion, one derives

$$\kappa \rho_f(p) = \frac{1}{3} (f'(p))^2 - f(p), \quad (109)$$

and its time evolution is governed by

$$\kappa \frac{d\rho_f}{dt} = f(p) f'(p) \left(\frac{2}{3} f''(p) - 1 \right). \quad (110)$$

The “formal pressure” P_f , associated with ρ_f , is introduced through the energy conservation equation

$$\frac{d}{dt} (v \rho_f) + P_f \frac{d}{dt} v = 0. \quad (111)$$

This relation leads to the expression

$$P_f = f(p) \left(\frac{2}{3} f''(p) - 1 \right) - \rho_f(v), \quad (112)$$

and the “formal equation of state parameter” w_f , defined for $\rho_f \neq 0$, is given by

$$w_f = \frac{P_f}{\rho_f} = \frac{f(p) \left(\frac{2}{3} f''(p) - 1 \right)}{\frac{1}{3} (f'(p))^2 - f(p)} - 1. \quad (113)$$

Furthermore, it can be noted that

$$P_f = \frac{1}{\kappa} \left[f(p) \left(\frac{2f''(p)}{3} - 1 \right) + f(p) - \frac{f'(p)^2}{3} \right]. \quad (114)$$

The definitions of ρ_f and P_f ensure that the equations of motion can be recast in the familiar form of GR, employing the variables a , ρ_m , ρ_f , and P . For various definitions of the function $f(p)$, there may be various forms of the evolution of the universe in this regime. The most simple de Sitter evolution can be recovered. Include matter contributions as well, and one can show that for certain polynomial forms of $f(p)$, one can have appropriate late-universe cosmology with expansion, in the scenario where the universe is continually merging with baby universes. For details and intriguing implications for the Hubble tension, see Refs. [138–140].

3.7. Asymptotically Safe Cosmology

The ability to construct gravitational renormalization group flow approximations beyond perturbation theory is essential for testing AS at a conceptual level. A particularly robust framework for carrying out such calculations is the functional renormalization group equation for the gravitational effective average action, given by [142]

$$\partial_k \Gamma_k[g, \bar{g}] = \frac{1}{2} \text{Tr} \left[(\Gamma_k^{(2)} + \mathcal{R}k)^{-1} \partial_k \mathcal{R}k \right], \quad (115)$$

where $\Gamma_k^{(2)}$ denotes the effective average action at scale k and $\mathcal{R}$ is an infrared regulator that suppresses low-energy fluctuations. Equation (115) is formulated using the background field method, where the metric is decomposed into a fixed background and fluctuations. The Hessian represents the second functional derivative with respect to the fluctuation field, evaluated in a given background. Additionally, $\mathcal{R}_k$ serves as a scale-dependent mass term for fluctuations with momenta satisfying, where the renormalization group scale is determined by the background metric. Since $\mathcal{R}_k$ is a mass term, it is typically assumed to be positive definite. The specific role of the regulator in both the numerator and denominator ensures that the trace term remains finite in both the infrared and ultraviolet regimes, thereby constraining the flow of Newton's coupling to fluctuations characterized by momenta.

A commonly used approach in studying AS involves projecting the functional renormalization group equation onto the Einstein–Hilbert action, thereby approximating scale-dependent effective average action as [98,143,144]

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{-g} [-R + 2\Lambda_k] + \text{gauge-fixing and ghost terms}, \quad (116)$$

where Λ_k and G_k denote the running cosmological constant and running Newtonian gravitational constant, respectively. Their scale dependence can be parameterized using dimensionless counterparts as

$$\Lambda_k = k^2 \lambda_k, \quad (117)$$

$$G_k = \frac{g_k}{k^2}, g_k/k^2, \quad (118)$$

where numerical values g_k and λ_k are obtained from AS studies.

To proceed, let us consider a flat FLRW background metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2). \quad (119)$$

Assuming a perfect fluid energy-momentum tensor, the modified Friedmann and continuity equations take the form

$$H^2 = \frac{8\pi G_k}{3} \rho + \frac{\Lambda_k}{3}, \quad (120)$$

$$\dot{\rho} + 3H(\rho + p) = -\frac{\dot{\Lambda}k + 8\pi\dot{G}k\rho}{8\pi G_k}. \quad (121)$$

The continuity equation arises from the Bianchi identity, which ensures that Einstein's equations are satisfied, signifying the conservation of the Einstein tensor's divergence. The additional terms on the right-hand side of Equation (121) can be interpreted as representing an energy exchange between gravitational and matter degrees of freedom.

To proceed further, an appropriate form for the cutoff scale is required. In the renormalization group improvement scheme, this is typically associated with a characteristic length scale of the system. In the cosmological context, several choices exist, as discussed in Ref. [98]. A commonly adopted choice is to take k to be proportional to the Hubble constant, as it naturally represents a time-dependent physical scale

$$k = \zeta H(t), \quad (122)$$

where ζ is a proportionality constant. This choice has been extensively used in asymptotically safe cosmology [98,145,146] and is motivated by its role as a dynamically relevant scale for cosmic evolution. However, this choice of cutoff is not unique, and there are four mainly used cutoff schemes in asymptotically safe cosmology.

- Type I cutoff is based on cosmic time t , given by

$$k^2 = \zeta^2 t^{-2}.$$

- Type II cutoff is the choice (122) proportional to the Hubble constant H , expressed as

$$k^2 = \zeta^2 H^2.$$

- Type III cutoff is related to spacetime curvature, using the Kretschmann scalar $R_{abcd}R^{abcd}$, given by

$$k^2 = \zeta^2 \sqrt{R_{abcd}R^{abcd}}.$$

- Type IV cutoff is tied to the temperature T of the cosmic plasma, where $T \propto \rho^{1/4}$, formulated as

$$k^2 = \zeta^2 T^2.$$

Here, ζ is a dimensionless proportionality constant. Each case reflects a different physical length scale governing the renormalization group evolution of gravitational couplings.

3.8. Holographic Dark Energy

The holographic principle has emerged as one of the most profound concepts in contemporary theoretical physics, playing a pivotal role in QG research. Deeply connected to the thermodynamics of black holes and the AdS/CFT correspondence, it naturally invites speculation about its consequences for cosmology, especially in explaining the late-time acceleration of the universe. Over the past two decades, this perspective has led to the formulation of holographic DE models, which attempt to encode DE dynamics within a holographic framework. For comprehensive treatments of these models, see [147–154].

The key premise of holographic DE theories lies in applying the holographic principle to the energy content of the universe. Within a spacetime region characterized by a length scale L , the total energy density of DE, ρ_{DE} , is assumed to be understood using parameters defined on the boundary surface. Since the reduced Planck mass M_{P} and L are the only relevant scales, dimensional arguments yield

$$\rho_{\text{DE}} = C_1 M_{\text{P}}^4 + C_2 M_{\text{P}}^2 L^{-2} + C_3 L^{-4} + \dots, \quad (123)$$

where C_1, C_2, C_3 are numerical constants. The first term, $C_1 M_{\text{P}}^4$, is associated with the vacuum energy expected from QFT, but its predicted magnitude exceeds the observed DE density by approximately 10^{120} times, which gives rise to the cosmological constant problem. Since this term violates holographic scaling, Andrew Cohen, David Kaplan, and Ann Nelson argued that the traditional field-theoretic estimate $\rho_{\text{DE}} \sim M_{\text{P}}^4$ becomes invalid as systems approach their Schwarzschild radius, implying that effective field theory must break down near gravitational collapse [147].

To address this issue, the holographic principle suggests a link between the UV and IR cutoffs of the theory. The constraint ensuring that the quantum vacuum energy does not lead to gravitational collapse is expressed as

$$L_{\text{IR}}^3 \Lambda_{\text{UV}}^3 \leq S^{3/4}, \quad (124)$$

where S is the entropy, L_{IR} denotes the IR scale, and Λ_{UV} is the UV cutoff. Assuming the Bekenstein–Hawking entropy relation $S \propto A$ and identifying the vacuum energy with $\rho_{\text{DE}} \sim \Lambda^4$, one obtains the characteristic holographic DE density

$$\rho_{\text{DE}} = 3b^2 L_{\text{IR}}^{-2}, \quad (125)$$

where b is a dimensionless constant determined observationally. Over time, a number of modified holographic scenarios have been developed by incorporating generalized entropies or non-standard geometrical effects, each yielding distinct cosmic evolution patterns.

For instance, in the Tsallis holographic framework, non-extensive Tsallis entropy modifies the standard black hole thermodynamics, giving

$$\rho_{\Lambda} = 3b^2 L_{\text{IR}}^{-(4-2\omega)}, \quad (126)$$

where ω represents the non-extensivity parameter, which is the standard holographic DE limit recovered (when $\omega \rightarrow 1$). Similarly, Barrow’s fractal deformation of horizon entropy, motivated by quantum gravitational microstructure, results in

$$\rho_{\Lambda} = 3b^2 L_{\text{IR}}^{\Delta-2}, \quad (127)$$

where $0 \leq \Delta \leq 1$, and $\Delta = 0$ reproduces the standard case. In models based on Rényi entropy, which is a further generalization of Boltzmann–Gibbs statistics, the DE density reads

$$\rho_{\Lambda} = \frac{3d^2}{8\pi L^2} (1 + \pi\delta L^2)^{-1}, \quad (128)$$

with δ acting as a deformation constant that vanishes in the standard limit. The Kaniadakis holographic DE model which is inspired by κ -deformed statistics, introduces a correction term giving

$$\rho_{\Lambda} = 3b^2 L_{\text{IR}}^{-2} + 3k^2 L_{\text{IR}}^2, \quad (129)$$

where the deformation parameter k satisfies $-1 < k < 1$. Finally, the fractional holographic DE approach incorporates fractional derivatives inspired by fractional calculus, giving

$$\rho_{\text{DE}} = 3b^2 L_{\text{IR}}^{\frac{2-3\zeta}{\zeta}}, \quad (130)$$

with $1 \leq \zeta < 2$; the canonical form follows when $\zeta = 2$. Together, the extensions (126)–(130) demonstrate the remarkable adaptability of the holographic framework in addressing the cosmological constant problem and describing diverse DE behaviors.

Different holographic prescriptions lead to distinct cosmological dynamics. For example, the equation-of-state parameter w in the standard holographic DE model typically evolves within the quintessence range, unlike the fixed $w = -1$ of the Λ CDM paradigm. This evolution allows holographic DE models to more flexibly accommodate cosmic observations such as CMB anisotropies, baryon acoustic oscillations, and Type Ia supernova data. Moreover, the inclusion of additional parameters such as ω , Δ , δ , k and ζ grants these models the ability to describe a variety of expansion histories, potentially offering insight into the coincidence problem, which is to say that it could tell one why DE domination emerges around the same epoch as matter dilution.

A central issue in all holographic DE constructions is the determination of the appropriate IR cutoff L . Since L effectively sets the energy density scale, its definition strongly influences the cosmological evolution predicted by the model. An early and natural proposal was to take $L = H^{-1}$, with H being the Hubble constant. While this choice directly ties DE to the expansion rate, it produces an equation of state approaching zero at late times, failing to drive acceleration.

A subsequent improvement considered the particle horizon,

$$L_{\text{par}} = a \int_0^t \frac{dt'}{a(t')}, \quad (131)$$

which captures the maximum comoving distance over which causal interactions could have occurred since the Big Bang. However, this choice results in $w > -1/3$, still insufficient to generate cosmic acceleration. The reason is that although it integrates over past causal structure, it lacks the negative pressure required for present-day acceleration.

To remedy this, Miao Li proposed using the future event horizon as the cutoff, defined as [148]

$$L_{\text{f}} = a \int_t^\infty \frac{dt'}{a(t')}. \quad (132)$$

This formulation successfully produces accelerated expansion but at the cost of a theoretical drawback, as L_{f} depends on the future evolution of the universe, leading to apparent causality violations since current dynamics would depend on unknown future conditions.

A more refined and locally defined alternative was introduced by Lizeth Granda and Antonio Oliveros, who proposed incorporating the time derivative of the Hubble constant into the cutoff definition [155]

$$L = (\alpha H^2 + \beta \dot{H})^{-\frac{1}{2}}, \quad (133)$$

where α and β are constants determined by observational constraints. This approach captures both the expansion rate and its rate of change, allowing for a more flexible description consistent with evolving cosmological data.

The most general form of this approach was proposed by Shin'ichi Nojiri and Sergei Odintsov [156], by extending the cutoff concept to include higher-order derivatives and other horizon scales. Then, the generalized definition can be expressed as

$$L = L(H, \dot{H}, \ddot{H}, \dots, L_{\text{par}}, L_{\text{f}}, \dot{L}_{\text{par}}, \dot{L}_{\text{f}}, \dots), \quad (134)$$

where L may depend on combinations of the Hubble constant and its derivatives as well as on particle and future horizons and their time derivatives. This framework unifies all earlier proposals and provides a powerful platform for constructing models capable of reconciling theoretical consistency with empirical viability.

3.9. Emergence of Dark Energy and Dark Matter in Entropic Gravity

The Verlinde's model of EG suggests that gravity is not a fundamental force but an emergent phenomenon which arises from thermodynamics and information on a holographic screen and can be used to explain DE. As was already stated above, the force of gravity is analogous to the entropic force that arises when information about the positions of material bodies changes. Here, we actually explore how DE emerges from this framework as a naturally small and positive cosmological constant, which potentially explains the accelerated expansion of the universe, and how the model can also account for DM. Let us be more definitive [157].

One may naively imagine that some sort of fluid is spread across the universe which is attributed to DE, and the deviation of the matter is responsible to the origin of a force connected with the existence of DM. The main hypothesis from which all the consequences to be derived is contained in the two statements [157].

1. There is a microscopic bulk perspective in which the area law for the entanglement entropy is due to the short distance entanglement of neighboring degrees of freedom that build the emergent bulk spacetime.
2. The de Sitter entropy is evenly divided over the same microscopic degrees of freedom that build the emergent spacetime through their entanglement, and is then caused by the long-range entanglement of part of these degrees of freedom.

We now depict the basic computations which are used to describe the entropy content of the de Sitter space for the static coordinate patch described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (135)$$

where the function $f(r)$ is prescribed as

$$f(r) = 1 - \frac{r^2}{L^2}. \quad (136)$$

Let us take now the perspective of an observer near the origin $r = 0$, so that the edge of his causal domain will coincide with the horizon at $r = L$. Then, the horizon entropy equals

$$S_{\text{DE}}(L) = \frac{A(L)}{4G\hbar} \quad \text{with} \quad A(L) = \Omega_{d-2} L^{d-2}, \quad (137)$$

where Ω_{d-2} is the volume of a $(d-2)$ -dimensional unit sphere. We hypothesize that this entropy is evenly distributed over microscopic degrees of freedom that make up the bulk spacetime. To determine the entropy density, it is to view the spatial section at $t = 0$ as a ball with radius L bounded by the horizon. The total de Sitter entropy is then divided over this volume so that a ball of radius r centered around the origin contains an entropy $S_{\text{DE}}(r)$ proportional to its volume

$$S_{\text{DE}}(r) = \frac{1}{V_0} V(r) \quad \text{with} \quad V(r) = \frac{\Omega_{d-2} r^{d-1}}{d-1}. \quad (138)$$

The subscript DE indicates that the entropy is carried by excitations of the microscopic degrees of freedom that lift the negative ground state energy to the positive value, which is associated with the DE.

We get the value of the volume V_0 per unit of entropy from the requirement that the total entropy $S_{\text{DE}}(L)$, where $r = L$, equals the Bekenstein–Hawking entropy associated

with the cosmological horizon. By comparing Equation (138) for $r = L$ with Equation (137), one computes V_0 as

$$V_0 = \frac{4G\hbar L}{d-1}, \quad (139)$$

where the factor $(d-1)/L$ arises from the relative normalization of the horizon area $A(L)$ and the volume $V(L)$. We therefore determine this entropy density by the Planck area and the Hubble scale. As a matter of fact, this value of the entropy density has already been proposed as a holographic upper bound in a cosmological setting.

An alternative way to write the entropy $S_{\text{DE}}(r)$ is in terms of the area $A(r)$

$$S_{\text{DE}}(r) = \frac{r}{L} \frac{A(r)}{4G\hbar} \quad \text{with} \quad A(r) = \Omega_{d-2} r^{d-2}. \quad (140)$$

From Equation (140), one immediately finds that for $r = L$, the Bekenstein–Hawking entropy (137) is recovered.

Let us now discuss how the “elasticity model” is used for the description of DM. The matter normally arises by adding excitations to the ground state. In our description of de Sitter space, there is, however, an alternative possibility, since it already contains delocalized excitations that constitute the DE. The matter particles then correspond to localized excitations. Therefore, it is natural to assume that at some moment in the cosmological evolution, these localized excitations appeared via some transition in the delocalized DE excitations. The string theoretic perspective described in Ref. [157] suggests that the DE excitations should be the basic constituents in our universe. Matter particles correspond to the bound states of these basic excitations that have escaped the DE medium. According to ST, these degrees of freedom have escaped from the Higgs branch onto the Coulomb branch.

The transition by which matter appeared has actually removed an amount of energy and entropy from the underlying microscopic state. The resulting redistribution of the entropy density with respect to its equilibrium position must then be described by some displacement vector. Since we study a system with a non-zero temperature, the displacement of entropy leads to a change in the free energy density. The effective theory that in general describes the response due to the displacement of the free energy density has already existed for a long time, and is actually older than GR. It is the known theory of elasticity.

Due to the competition between the area law and volume law entanglement, the microscopic de Sitter states exhibit glassy behavior, leading to slow relaxation and memory effects [157]. For our applications this means the displacement of the local entropy density due to matter is not immediately erased but leaves behind a memory imprint in the underlying quantum state. Afterwards, this results in a residual strain and stress in the DE medium, which can only relax highly slowly [157].

To us, the key aspect of this is that matter causes a displacement of the DE medium, implying that the medium also causes a reaction force on the matter. The magnitude of this elastic force should be determined in terms of the residual elastic strain and stress. It is proposed in Ref. [157] that this force leads to the excess gravity that is currently attributed to DM. Indeed, it was shown in Ref. [157] that the observed relationship between the surface mass densities of the apparent DM and the baryonic matter naturally follows by applying old elements of the linear theory of elasticity. Then, the main input that one needs to determine the residual strain and stress is the amount of entropy S_m that would be removed by matter. These observations are of high interest and the results, namely that one can also model DM with DE, make the EG a unique approach to QG. However,

it was shown, according to Ref. [158], that this method may have significant flaws in its foundations regarding thermodynamic considerations.

4. Ring Paradigm as a New Approach to Quantum Gravity

We have recently formulated a ring paradigm (RP) approach to QG [36–39]. To introduce this approach let us imagine a set of effectively 1-dimensional Hopf-linked rings S^1 in $\mathbb{R}^3$, where every two rings may be linked maximally once, the rings may not be knotted or twisted, and no any Brunnian type of links is considered in this set. This structure is stretched and attached to some boundary (whole bulk of the universe) and then we consider a projection of this set to a plane. The elementary particles and fields of the SM are placed at the points corresponding to the intersections of the projection of this set of rings to the plane.

The gravitational rings are modeled by these elongated one-dimensional objects. They should transfer the Newtonian interaction by some finite but exceptionally big superluminal velocity c_g and then decay to standard gravitons. Interestingly, the gravitons are modeled as phonons on the grid of matter in the universe. This also means that one is returning to an aether-like model in physics, because some medium which will enable the spreading of the vibrations in the crystal (this medium is spread in the space between galaxies) is needed. Note that this is created from some phantom field $\psi(x)$.

Let us now illustrate how the Newtonian limit is achieved in quite a simple toy model. We consider a row of galaxies connected by such rings. The standard quantization procedure is applied to the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^2 \frac{1}{2m} P_i^2 + \sum_{i,j=1}^2 V_{ij} Q_i Q_j, \quad (141)$$

where

$$V = \begin{pmatrix} \frac{1}{2}k + \frac{1}{2}k_3 & -\frac{1}{2}k_3 \\ -\frac{1}{2}k_3 & \frac{1}{2}k + \frac{1}{2}k_3 \end{pmatrix},$$

$k, k_3 > 0$ are spring constants, Q_i represents the generalized coordinate and P_i the momentum of the galaxies. After performing the standard computations [38], the Hamiltonian reads

$$\mathcal{H} = \sum_{\alpha=1}^2 \hbar \omega_{\alpha} a_{\alpha}^{\dagger} a_{\alpha}, \quad (142)$$

with the standard definition of the creation and annihilation operators (a_{α} and $a_{\alpha}^{\dagger}$). The eigenvalues are $\sum_{\alpha=1}^2 n_{\alpha} \hbar \omega_{\alpha}$. [38].

The next step is to apply the second quantization. We suppose further that there is a longitudinal rippling spreading inside every “connecting line” between massive objects (the decaying structure of the gravitational rings), and what is meant by that is that some material, which has very special properties, should be present in the space between the galaxies. (First of all, the gravitational rings are created among the most massive objects with the highest probability, so a probabilistic description is also included in the fundamentals of RP.) We start with a quantization of a real scalar field described by the Lagrangian [38],

$$\mathcal{L}(\psi, \psi) = \frac{1}{2} \int [\dot{\psi}(x)]^2 dx - \frac{1}{2} \iint K(x - x') \psi(x) \psi(x') dx dx', \quad (143)$$

where $K(x - x') = K(x' - x)$ denotes a potential, and we keep only one dimension. The Euler–Lagrange equations give:

$$0 = \frac{\partial}{\partial t} \frac{\delta \mathcal{L}}{\delta \dot{\psi}(x)} - \frac{\delta \mathcal{L}}{\delta \psi(x)} = \ddot{\psi}(x) + \int K(x - x') \psi(x') dx'. \quad (144)$$

We now perform this approach on an illustrative case, $K(x - x') = -c^2 \frac{\partial^2}{\partial x^2} \delta(x - x')$, for which the Lagrangian (143) reads

$$\mathcal{L} = \frac{1}{2} \int [\dot{\psi}(x)]^2 - c^2 \left[\frac{d}{dx} \psi(x) \right]^2 dx. \quad (145)$$

Therefore, we get a normal wave equation with the velocity of propagation c (to be identified with the velocity of light in a vacuum, so denoted)

$$\frac{d^2}{dx^2} \psi(x) - \frac{1}{c^2} \ddot{\psi}(x) = 0. \quad (146)$$

We construct the standard Hamiltonian, and therefore, the conjugate momentum $\Pi(x) = \delta \mathcal{L} / \delta \dot{\psi}(x) = \dot{\psi}(x)$ is needed.

One also needs to reproduce the Newtonian force with the bouncing of graviton-phonons, and so here, dP/dt , where P is the momentum, is computed in the Heisenberg representation, as one trivially obtains 0 in the Schrödinger representation. Since $dP/dt = \frac{1}{i} [P, \mathcal{H}]$, one finds $\frac{1}{i} [P, \mathcal{H}]$; see details in Ref. [38].

Let us now reveal the unknown function $K(x)$. To do so, we use the analogy of the modeling of photons and phonons. The photon is a quantum of the electromagnetic interaction, which to be described by the Maxwell equations in a vacuum (with the electric intensity $\vec{E}$, the magnetic induction $\vec{B}$, the current density $\vec{j}$, the charge density ρ and the permittivity of vacuum ϵ_0) [38]. We use source-free equations [38] and plug $\vec{E} = -\partial \vec{A} / \partial t$, with $\vec{A}$ being the vector potential of the last equation. After applying the rotation and using the gauge condition, $\nabla \cdot \vec{A} = 0$,

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0. \quad (147)$$

One may similarly deduce that

$$\nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0, \quad (148)$$

where one inserts $\vec{E} = -\nabla \varphi$ (φ is the electric potential) into the equations and conducts some modifications. Therefore, the wave equations are derived, and one recognizes the same form as Equation (146).

The most intriguing observation is that photons are the field particles for electromagnetic interaction, which we describe by Maxwell equations containing the Coulomb law, $F_{Q_1 Q_2} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{L^2}$ (this is a formula only with the magnitude of the vector). It has exactly the same form as the formula for the Newtonian force, $F_{m_1 m_2} = G \frac{m_1 m_2}{L^2}$, so, if one substitutes $K(x - y) = -c^2 \frac{d^2}{dx^2} \delta(x - y)$, graviton-phonons look indeed as the correct mediators of the Newtonian force.

Moreover, the equation for $\frac{1}{i} [P, \mathcal{H}]$ can be taken as a condition on the scalar field $\psi(x)$, and it can be immediately seen that all three constants of nature— $\hbar$, G and c —are in it.

We now state the true implications of these results. RP can give an explanation for two basic points in classical physics. First, RP tells us that the velocity of the gravitational waves c_{gw} is not almost of the value the velocity of light in a vacuum c (today, there is

good enough bound on this velocity: $|c - c_{gw}| < 10^{-15}$ [159]), but is actually exactly c . It is supposed (based on the Einstein's paper from 1915 [5]) that c_{gw} should be equal c , but there was actually no fundamental explanation for this. Further, the similarity between the Coulomb and Newton laws is not accidental, but is actually a necessary ingredient for the building of the parallelism between photons and graviton-phonons. It is worth emphasize that this duality between gravity and electromagnetism is of a quite deeper understanding than in the currently extensively studied double copy principle, which relates gauge theory amplitudes to gravity amplitudes [160].

It is of considerable interest to investigate the fundamental group of symmetries G in RP, under which the theory is invariant. Looking at the basic postulates, the elementary particles of the SM are traveling on the presupposed trajectories of the decaying gravitational rings, and the rings are created first. Therefore,

$$G = \mathbb{R}^{1,3} \rtimes \left(O(1,3)_c^+ \cup O(1,3)_{c_g}^+ \right). \quad (149)$$

The gravitational ring is then modeled as the “standard graviton” with the spin 2 and helicity ± 2 in the group $O(1,3)_{c_g}^+$. Furthermore, this group G (149) corresponds to the following transformations under which the ring paradigm should be invariant (special case in $(1+1)$ dimensions, based on generalization of the special Lorentz transformations in flat spacetime):

$$\begin{aligned} t' &= \frac{t - \frac{x}{v} \frac{v^2}{c^2} \epsilon - \frac{x}{v} \frac{v^2}{c_g^2}}{\sqrt{1 - \frac{v^2}{c^2} \epsilon - \frac{v^2}{c_g^2}}}, \\ x' &= \frac{x - tv}{\sqrt{1 - \frac{v^2}{c^2} \epsilon - \frac{v^2}{c_g^2}}}, \end{aligned} \quad (150)$$

where $c_g \gg c$ and $\epsilon = \epsilon(v)$ denotes some step function defined by the prescription

$$\epsilon(v) = \begin{cases} 1 & \text{for } v \leq c, \\ 3(v/\delta)^3 - 2(v/\delta)^2 & \text{for } c < v \leq c + \delta, \\ 0 & \text{for } v > c + \delta, \end{cases} \quad (151)$$

with the δ parameter supposed to be exceptionally small: $\delta \lesssim 10^{-100}$ m/s. Let us note that the group G (149) must be generalized to the BMS-like group in the case of a limit to GR in curved spacetime.

As it is understandable from the construction, the form of Einstein equations are not changed even in the high energy sector. What seems to be different is how quickly the gravity is mediated on the full non-perturbative level. One can obtain a bound on the spreading velocity based on the distance determined by the diameter of the observable universe, 94.10^9 pc, and the time, approximately Planck time, 10^{-43} s, during which the gravitational rings are created. So, the velocity of the spreading of the gravitational interaction is minimally 10^{70} m/s. This is a new limit to the maximal velocity of the spreading of the information in the universe.

Then, the new field equations reads

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} + \Lambda_r G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c_g^4} = \frac{8\pi G}{c_g^4} (T_{\mu\nu}^m + T_{\mu\nu}^r), \quad (152)$$

where $G_{\mu\nu}$ is the metric, and all the other quantities are also of similar sense as that in the GR. Specifically, the tensor $T_{\mu\nu}^m$ is the tensor of the energy-momentum of the casual matter, and

the tensor $T_{\mu\nu}^r$ corresponds to the tensor of energy-momentum of gravitational rings. The constant Λ_r can be computed from the QFT as the energy density of the vacuum, which is the sum of the zero-point energies of all quantum fields. We neglect the term corresponding to $T_{\mu\nu}$ with respect to the second term on the RHS of Equation (152). Therefore,

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} + \Lambda_r G_{\mu\nu} \approx \frac{8\pi G}{c_g^4} T_{\mu\nu}^r. \quad (153)$$

This picture is believed to be correct because the gravitational rings are considered only present at the beginning of the cosmological inflation. Afterwards, the rings decay quite quickly into the phantom fields (these create the mediator for the graviton-phonons); therefore, Equation (152) transforms to ($G_{\mu\nu}$, been unstable, decays to some $g_{\mu\nu}$ Planck time later):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu}^m + T_{\mu\nu}^{pf}), \quad (154)$$

where $T_{\mu\nu}$ is composed of the energy momentum tensor of casual matter $T_{\mu\nu}^m$ and the energy momentum tensor of the phantom field $T_{\mu\nu}^{pf} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\sigma \phi \partial_\sigma \phi + V(\phi) \right)$. Actually, the transformation happens in the universe repeatedly in the intervals as long as the Planck time. This reminds the cellular automaton models, where changes in the universe occur in quantized steps governed by fixed rules rather than smooth, continuous transitions [161]. If we were now to remove all the matter from the universe, the cosmological inflation would start again at an incredible speed.

It can be said that this approach to QG is, in some sense, highly conservative, because it does not change the nature of a quantum field. The only difference is here that the concept of QFT includes a field which mediates gravity superluminally. Naturally, this may have immediate consequences for the group of symmetries of this modified QFT (as was already shown above).

Actually, one needs to identify the full field Equation (152) of RP using the theory with the field Equation (154). This is necessary to equate the impulse of the phonon with the impulse of the graviton, which one obtains from the modified theory of gravity with the field Equation (154). We achieve this through the modeling of gravitons similarly as in ST. (However, one may get an approximate result from gravitational wave physics if the split of the metric $G_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$ is used, where $H_{\mu\nu}$ is a comparably small perturbation. Then, the impulse of the gravitational wave analogically is computed as done in Ref. [162]). The further step is to equate the impulse of the graviton in this theory with the impulse of the graviton in the modified theory with Einstein field equation (152).

If RP is a viable model of QG, then it has to lead to a perturbatively renormalizable theory. Therefore, one has to deal with the interaction of graviton-phonons with other particles. We again consider the crystal composed of rings. A macroscopic grid with quite large masses (typically super-massive black holes) must be created from it, so that the effect of the graviton-phonons on the elementary particle should be observable.

We use the model of electrons interacting with phonons [163], and as the first step, we develop the ground state for a fermion system. The concept of holes b with the defining relation

$$b_\alpha = a_\alpha^+ \quad (155)$$

is introduced. Then, we apply the creation and annihilation operator formalism to electrons placed on a vibrating regular grid of nuclei. However, one has to be accurate enough because in the case of the model in Ref. [163], the crystal approximated a continuum in

a limit. Yet, it follows directly from the computations in Ref. [163] that modeling the interactions of graviton-phonons with other particles would be doable. We actually obtain no infinities, in contrast to the “quantizing” pure GR. At first glance, it may be considered curious that graviton-phonons considered having no interaction, for example, with massless particles like photons. Nevertheless, there has to be an interaction on the fundamental non-perturbative level when gravitational rings “create a link” with all the particles.

5. Dark Energy Model Obtained by Ring Paradigm

The gravitational ring has two substantial functions [38]: First, it “creates a trajectory” for all particles and fields, which means that one needs to embed the Hopf-linked set of rings in the space $\mathbb{R}^3$. The second function is that there it should be the mediator of gravity. Therefore, we obtain a self-evident explanation for the problem of DE because we get some extra energy in our model, and this is identified with the missing energy.

Let us imagine two objects (for example, two quite massive objects such as galaxies) connected by gravitational rings, and they are continuously creating and decaying in Planck time. There remains some material between these two galaxies, and as soon as those rings start stretching, the gravitational resistance starts to be felt. This somewhat reminds some quasi-mechanical model wherein one obtains a Hook law for the dependence of the deformation on the stress. Then, the elasticity limit is reached, and finally, this spring breaks after crossing the yield point.

Actually, the decaying gravitational rings are made from some material, which we identified in the classical limit with the emergence of some sort of phantom field in the simplest case. The detailed quantum model of the bunch of creating and again decaying gravitational rings concerns condensed matter physics [164]. We characterize the “gravitational” material by the spring constants k, k_3 in Equation (141), and the yield point then corresponds to some moment in the history of the evolution of the universe when the galaxies were separated from each other by an average distance of 8.3 Mpc approximately five billion years ago.

One can find a solution to the old problem of the cosmological constant by using the application of RP. It was already stated that above we describe the evolution of the universe by the field equation (152) in high enough energies at the beginning of the cosmological inflation. The cosmological constant Λ_r should have exactly the value that one obtains in QFT. However, the effective cosmological “constant” $\Lambda = \Lambda(t)$ has begun to appear approximately eight billion years later, after the Big Bang, due to the phenomenon of QG:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}^m. \quad (156)$$

It is, certainly, not too much unexpected that the value of the cosmological constant from the cosmological inflation, Λ_r , possesses many orders of magnitude difference (approximately 10^{120}) from the value of the cosmological constant Λ responsible for the late-time cosmic acceleration, as those constants have a different origin. Λ_r to be connected with the energy in the false vacuum (corresponding to the pure “crystal” made of rings), known from QFT, and Λ is a manifestation of a limit of the firmness of the DE material [38]. However, the question of why the Λ nowadays is so small can be answered only with the detailed classical modified gravity model (154).

6. Comparison of Various Late-Time Acceleration Scenarios

We now discuss the various models considered so far and compare how each of the models approaches the problem of the late-time accelerated expansion of the universe. Although these models emerge from largely various mathematical and physical motiva-

tions, they are all united by the recognition that the cosmic acceleration might be a signal of underlying quantum gravitational phenomena, and the comparison here is not merely mathematical or historical, but is rather a critical view at all the scenarios discussed so far.

To begin with, the metastring formulation of ST offers one of the most mathematically mature descriptions of late-time acceleration, as here one finds that the cosmological constant and the Planck scale are related through a see-saw mechanism $M_\Lambda \sim M^2/M_P$. This goes towards implying that the smallness of DE is not accidental but a manifestation of T-duality between the large- and small-scale sectors of space-time, and the metastring model introduces a non-commutative doubled geometry in which space-time coordinates and their duals $(x, \tilde{x})$ coexist. This gives rise to an effective cosmological constant through their intrinsic curvature, and this approach thus replaces the fine-tuning problem with a geometric relation between curvature and vacuum energy. Nevertheless, its success critically depends on the non-perturbative stability of the vacuum and the precise implementation of dual space curvature, which are still open issues.

LQG and its cosmological reduction, which is loop quantum cosmology, pursue a quite another route, as here, by directly quantizing the canonical variables of GR, these approaches predict a modified Friedmann equation that prevents singularities and produces a quantum bounce. Interestingly, this same modification leads naturally to a late-time acceleration phase wherein their framework asserts that the acceleration is not attributed to a cosmological constant but arises dynamically from the underlying discreteness of space. Its main virtue is background independence and mathematical rigor, yet it lacks a distinct mechanism for the observed numerical value of the acceleration and cannot yet explain the coincidence problem in a predictive way.

The GFT condensate cosmology extends these results by translating the loop quantum gravity picture into a field-theoretic one and tells us that the universe itself is a condensate of quantum geometric “atoms of space” whose collective dynamics give rise to an effective Friedmann equation with an emergent phantom component that drives accelerated expansion. In contrast with ST, where DE arises from moduli or vacuum curvature, the group field theory explanation is entirely emergent and dynamical. Its success lies in connecting early and late time phenomena through the same microscopic principles, but still depends on phenomenological potentials and lacks direct observational signatures, which are essential in distinguishing it from other quantum cosmology models.

Discrete approaches such as the causal set program provide another viewpoint which is quite non-trivial, as in this model, the cosmological constant appears as an emergent quantity associated with fluctuations in the number of space-time elements, and its magnitude is naturally of the observed order when averaged over the cosmic volume. Moreover, relatively small violations of energy-momentum conservation due to spacetime discreteness can act as an effective source of DE. While this consideration is conceptually sophisticated its quantitative realization is still largely heuristic, and a complete dynamical law replacing Einstein’s equations remains missing as well.

CDTs share with the causal set program the assumption of discreteness, but CDTs preserve causal structure through a global foliation of spacetime by simplices, and remarkably, numerical simulations show that an ensemble average of such triangulations reproduces a semiclassical de Sitter universe at large scales. This signifies that acceleration in this model is not imposed but emerges statistically. This approach is one of the few to generate a cosmological constant dynamically without fine-tuning, also demonstrating that smooth 4-dimensional geometry can emerge from discrete microscopic building blocks. However, the issue here is the dependence on numerical results and the complication in coupling matter fields, which are quite significant limitations.

The AS scenario approaches the same question from the renormalization group perspective, and it posits that gravity is described by a non-trivial UV fixed point of the renormalization flow, making it a predictive and consistent QFT at all scales. Late-time acceleration arises naturally here from the scale-dependence of the gravitational coupling and cosmological constant. The virtue of this approach is its continuity with established QFT methods and its ability to interpolate smoothly between classical and quantum regimes. Its limitation lies in its perturbative formulation and the lack of a direct link between the renormalization group flow and observable cosmological data.

EG and NG approaches share a thermodynamic interpretation of gravity but also have some differences. Entropic gravity interprets the gravitational force as an emergent entropic tendency of microscopic degrees of freedom, while NG reformulates the SM and gravity as manifestations of algebraic spectral triples. Both naturally incorporate the holographic principle and thus hint that the cosmic acceleration could stem from an information-theoretic imbalance between bulk and boundary degrees of freedom. The main challenge here is empirical, which is that while these frameworks are conceptually unified and elegant, they do not have proper falsifiable predictions that differentiate them from more conventional theories.

Finally, the RP introduces a profoundly novel vision of spacetime and gravitation. Gravity is modeled not as curvature of a manifold but as the collective dynamics of “gravitational rings”, which are topologically linked excitations that mediate interactions between massive objects. The creation and decay of these rings release residual energy, identified with the observed DE component. In this picture, the cosmic acceleration is not a vacuum effect but a macroscopic outcome of quantum topological processes occurring at the Planck scale. The model succeeds in simultaneously addressing the cosmological constant and the coincidence problems while also offering a potential bridge between condensed matter analogies and cosmological behavior. However, its mathematical formalization is still in its early stages, and so, with time, we would be able to figure out proper observational tests for the theory.

However, let us pause considering the phenomenological implications of the RP because, if confirmed in the future, it may have significant consequences for the structure of other approaches to QG. First of all, according to the RP approach, gravity is considered a highly non-local interaction. There are many examples of approaches which have built-in non-locality: at the classical level, for example, non-local gravity [165,166] can be mentioned. String field theory (SFT) contains non-localities [10], and ST does as well [7]. Loop quantum gravity [59,60] is not inherently non-local, but group field theory is often non-local [117], particularly in its interaction terms. The causal set approach is non-local in its foundations [120], which follows from the combination of discreteness and Lorentz invariance. Further, NG [82] can be used to create non-local theories of gravity. Also, in CDTs [92], the overall geometries are not determined by local interactions alone. Non-local effects, which are not described by local operators, can arise in AS [93–96], particularly when dealing with the propagation of massless particles like gravitons. EG [99] is, then again, a theory where interactions are not limited to an immediate surroundings; therefore, it is significantly non-local.

Since the RP comes with the notion that the superluminal information transfer exists in nature, let us consider which models have already been studied with this phenomenon. Actually, some models in ST suggest the potential Lorentz breaking at the Planck scale or higher [167]. The same is true in loop quantum gravity [168,169], and such models are also known in group field theory [170]. On the other hand, the causal set approach does not allow Lorentz symmetry breaking. However, again, the NG is often discussed in the context of Lorentz symmetry violation [170–172]. CDT models do not in general violate Lorentz

symmetry [87], but that does not apply to AS [173]. Also, Lorentz symmetry violation may affect EG by altering thermodynamic properties like the entropy and Hawking temperature of black holes and by potentially leading to new astrophysical phenomena. So far, though, not much is known.

The existence of some sort of phantom field is also an integral part of the RP. Already some models with phantom fields were investigated inside AdS/CFT correspondence and SFT [174,175], further inside loop quantum gravity [176,177], and group field theory [116,117]. However, so far, not much is known about connections with the scalar field models in the causal set approach, NG, CDTS, AS, and EG.

It is actually highly intriguing that the RP is built on the $(1 + 1)$ -dimensional grid created from the gravitational rings in $\mathbb{R}^3$. These objects are effectively 1-dimensional, but the time coordinate on every such ring can be also introduced. A similar phenomenon—dimensional reduction mostly to the dimension two—was observed in many approaches to QG without any deeper explanation [178]: The first indication of dimensional reduction to $d = 2$ in QG came from the study of high-temperature ST. As was remarked by Joseph Atick and Edward Witten, the system that prevails under distances $\sqrt{\alpha'}$ behaves as if it were a $(1 + 1)$ -dimensional field theory [179]. Another place that one might look for evidence for dimensional reduction is loop quantum gravity [180], and further occurrence of this phenomenon is known in causal set theory [181], NG [182], causal dynamical triangulation [89], AS [183–185] and even the Wheeler–DeWitt approach [186]. Only the EG is not often associated with the dimensional reduction, but it can be said that this phenomenon appears there via the holographic principle.

The holographic principle is one of the key discoveries of modern QG research and cosmology, and it looks as though the RP is in concordance with that principle every effectively 1-dimensional ring can be wrapped inside a 2-dimensional surface made of rings. Therefore, it is to understand into which known approaches to QG we can incorporate it as well. Certainly to ST [44], but some studies with positive results have already been conducted in loop quantum gravity [187], and especially in group field theory [188]. Let us mention, however, the ongoing research in this area in NG [189] and causal dynamical triangulation [136], with positive results as well. The holographic principle is, certainly, firmly built in EG [157], but no evidence has so far been revealed in the CSA and AS, see Table 1.

One can find from Table 1 that highly preferable approaches to QG are modifications of ST/SFT, GFT (partly LQG), and NG. The RP occupies a completely different position in the modern approaches to QG because its confirmation may give the first hints as to how to solve a whole series of open problems in physics. Let us mention, for example, the old problem of the cosmological constant (problem of DE), the black hole information paradox, the problem of singularities or the problem of the universe's curvature. Since the RP has quite a special position among the modern approaches to QG, let us highlight a few overall observations concerning model building in cosmology by the applications of various approaches to QG and compare it with the RP.

The models based on the ST typically lead to the multiverse conjecture [190]. There are two quite a dispute for why the multiverse conjecture point to be rejected: first of all, this conjecture is commonly known that testability and falsifiability of the physical theory are lost when this conjecture point is accepted. However, one may argue that it is just a new phenomenon in modern physics that a theory has such a property. We claim that this is not the case because there is also a second philosophical reason why one should not trust that point. Imagine the collection of all universes in the multiverse theory. Now, add the energy-mass of all the objects in all these universes. the one distinctly obtains an infinity. How a real physical theory can be built while dealing with infinite energies? Unless

there is some unknown mechanism which may reduce this number to a finite value, no computations can be conducted in such a model of the multiverse.

Table 1. The occurrence (✓) or nonoccurrence (×) of the five phenomena in the known approaches to quantum gravity (QG) and its modifications (string theory/string field theory (ST/SFT), loop quantum gravity (LQG), group field theory (GFT), causal set approach (CSA), non-commutative geometry (NG), causal dynamical triangulation (CDT), asymptotic safety (AS), entropic gravity (EG) and ring paradigm (RP)).

Phenomenology\Approach to QG	ST	LQG	GFT	CSA	NG	CDT	AS	EG	RP
nonlocality	✓	×	✓	✓	✓	✓	✓	✓	✓
superluminal signalling	✓	✓	✓	×	✓	×	✓	✓	✓
phantom fields	✓	✓	✓	×	×	×	×	×	✓
dimensional reduction	✓	✓	✓	✓	✓	✓	✓	×	✓
holographic principle	✓	✓	✓	×	✓	✓	×	✓	✓

However, the ST played a key role in the development of the cyclic model of the universe of Paul Steinhardt and Neil Turok [40], and this is a quite an attractive model because provides an explanation of what happened before the Big Bang, what the evolution of the universe may look like in the far future (the universe is believed to collapse again to the Big Crunch) and what the DE represents. We have recently started to study a model inspired by the RP, where we claim that both happened in the universe, the cosmological inflation without multiverse and the big-trillion-year-lasting-cycles controlled by the modification of the cyclic model (there are two coupled scalar fields in the theory; one is controlling the cosmological inflation, and the other is responsible for the big cycles) [191]. These are all extra points for the ST paradigm, because it also has another quite rich phenomenology. Let us mention further the strings, branes and extra dimensions [7].

However, even the applications of the GFT approach provides models which connect the early and late-time universe cosmology (the group field cosmologies of Daniele Oriti and Xiankai Pang [116]). Yet, what is especially attractive is the model based on the introduction of mitrons of Tejinder Singh [83]. The mitrons show some similarity with gravitational rings (they are actually also other phenomenological objects based on a model of QG), and the origin of DM and DE is explained therein. In summary, precisely these paradigms—modifications of ST, GFT and modifications of NG—showed a remarkable similarity in the results of the test concerning phenomenology (nonlocality, super-luminal signalling, existence of phantom field, dimensional reduction and holographic conjecture). It is also worth mentioning that the modification of the cyclic model inspired by the RP by us [191] is based on the existence of some phantom fields, and phantom fields play a key role in the model of Daniele Oriti and Xiankai Pang [116].

The LQG models bring intriguing models in the early-universe cosmology as well via the modifications of the Friedmann equation [65], and therefore, one needs to consider the pre-Big Bang cosmologies. Let us note that we use the Einstein equations in the high-energy sector in the RP, paradigm, which also points to some connection with these models based on LQG. Actually, the RP may give a first hint of how to highlight paradigms, which are most probably only the study of the partial properties—even though those properties may be of high importance—of the resulting QG theory. Let us point out, for example, the CSA, CDT, AS or EG, which are not bringing such a rich phenomenology as, for example, the modifications of ST or the exactly novel RP (which comes from the mathematical machinery of ST, but as we already noted, the development of its mathematical formalism is still in its infancy). This does not at all mean that one should not study these other approaches to QG. We just point out what one should mainly focus on discerning the physical foundations of the new theory in the future. (We did not mention, for example,

the models of Petr Hořava [192] in this paper because the applications to cosmology are still under development. There are also attractive models based on the studies by Brett McNnes [193–196] and Aaron Trout [197], which, however, were not central to the discussion in this review.)

Altogether, a general summary of the main topic of this paper—cosmological constant and the problem of DE—is that all the presented theories share the conviction that the cosmological constant problem cannot be solved within just classical GR. Yet, the theories significantly differ in their mathematical tools and motivations. From a pedagogical standpoint, the comparative outlook of these models reveals an evolving notion that cosmic acceleration may encode a genuinely quantum aspect of spacetime rather than an additional classical field, which may be further exemplified with the advent of cosmic tensions in the current age. The crucial differences lie not in predictions by the theories for late-time acceleration but in how those theories reinterpret the physical meaning of that constant.

7. Conclusions and Future Outlook

Let us now make some concluding statements to summarize the extensive work discussed here.

- Our study here has examined a wide spectrum of QG paradigms, ranging from metatraining theory and its non-commutative phase-space geometry to loop quantum gravity, group field theory, causal set theory, asymptotic safety, entropic gravity, and the novel ring paradigm. We showed that the phenomenon of the accelerating expansion of the universe can naturally emerge within several independent theoretical frameworks, and each of these approaches replaces the traditional cosmological constant problem with a more fundamental quantum or geometric origin of cosmic acceleration.
- A unifying theme found across these diverse models is the concept of emergent spacetime, and it is essential. Whether through the condensation of quantum geometric degrees of freedom, holographic dualities, discrete causal structures, or topological excitations, all frameworks indicate that the universe's large-scale dynamics arises from microscopic quantum structures, suggesting that DE is a manifestation of inherent quantum geometry rather than quite a simple additional matter field as is often readily assumed.
- The consideration in this paper also emphasizes that the approaches discussed provide a new conceptual link between early and late-time cosmology. Quantum gravitational effects that are responsible for resolving the Big Bang singularity or initiating inflation appear in several models to emerge again at late times as drivers of acceleration. This dual appearance hints at a single underlying mechanism governing the universe's entire history, from its early beginnings to its late possible ends.
- Looking toward future developments, one of the most promising avenues lies in connecting these theoretical frameworks with high-precision cosmological data. This is of particular interest, as observational data for cosmology would here provide a testing bed for quantum gravitational theories. Upcoming observatories such as the Vera C. Rubin Observatory [198] and LSST [199], the Nancy Grace Roman Space Telescope [200], the Euclid mission, the Square Kilometre Array [201] and the Cosmic Explorer gravitational wave observatory [202] will deliver unprecedented constraints on the Hubble constant, structure growth, and DE equation of state. These facilities are believed to provide the first observational handles to distinguish QG imprints in the late-time universe from standard Λ CDM predictions.
- Nevertheless, there still remain profound challenges and open questions. The mathematical formulations of several QG paradigms are still incomplete, and their semiclassical or phenomenological limits are not yet derived in sufficient detail to allow for

particularly strong confrontation with data. The renormalization behavior of certain models, the exact nature of the emergent degrees of freedom and the translation between discrete and continuum formulations remain unsolved issues that must be addressed before a definitive picture can be achieved.

- The interplay between QG and cosmology also demands the development of new computational and analytical tools, and future research should hence aim to build consistent bridge theories that connect the microscopic quantum geometry with effective field-theoretic and thermodynamic descriptions of spacetime. Cross-disciplinary frameworks linking condensed matter analogies, information theory and algebraic geometry may also prove highly practical in constructing a coherent narrative that unifies all these approaches.
- Finally, the ultimate goal is to create a synergy between these results into a single, experimentally verifiable theory of QG capable of predicting both the cosmological constant and the time evolution of DE. As next-generation observations become more precise, the coming decade may well decide which of these paradigms most accurately captures the quantum fabric of our accelerating universe.

Author Contributions: Conceptualization, J.N.; formal analysis, J.N. and O.T.; writing—original draft preparation, J.N. and O.T.; writing—review and editing, J.N. and O.T. All authors have read and agreed to the published version of the manuscript.

Funding: O.T. is supported in part by the Vanderbilt University (TN, USA) Discovery Doctoral Fellowship.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors would like to thank Robert Scherrer, Avi Loeb, Maxim Khlopov, Paul Steinhardt, Sunny Vagnozzi, Sergei Odintsov, Paulo Moniz, Renate Loll, Claus Kiefer, Petr Hořava and Hermann Nicolai for various helpful discussions on dark energy and quantum gravity.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

AdS	anti-de Sitter
AS	asymptotic safety
BF	background field $\vec{B}$ and $\vec{F}$
BFSS	Banks–Fischler–Shenker–Susskind
BMS	Bondi–Metzner–Sachs
CDT	causal dynamical triangulation
CFT	conformal field theory
CMB	cosmic microwave background
CSA	causal set approach
DE	dark energy
DGP	Dvali–Gabadadze–Porrati
DM	dark matter
EG	entropic gravity
EPRL	Engle–Pereira–Rovelli–Livine
FLRW	Friedmann–Lemaître–Robertson–Walker
GFT	group field theory
GR	general relativity
IKKT	Ishibashi–Kawai–Kitazawa–Tsuchiya
IR	infrared
KAGRA	KAmioka GRAvitational wave detector

LHS	left-hand side
LIGO	Laser Interferometer Gravitational-wave Observatories
LQG	loop QG
LSST	Large Synoptic Survey Telescope
NG	non-commutative geometry
QFT	quantum field theory
QG	quantum gravity
RHS	right-hand side
RP	ring paradigm
SM	Standard Model
ST	string theory
SFT	string field theory
UV	ultra-violet
Λ CDM	Lambda cold dark matter

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