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Group $SU(2)$ Irreducible Representations and Probability Distributions Describing the Density Matrices of Qubit States

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Abstract

The general problem of describing quantum states not only by wave functions and density operators but also by probability distribution functions is discussed in this paper. The qubit-state density-matrix elements expressed in terms of probability distributions are connected with the irreducible representation of group $SU(2)$. The transform of the probability distributions corresponding to the unitary transform of the qubit state of the spin-1/2 system is obtained and expressed in terms of matrix elements of the group $SU(2)$. The possibility to extend the introduced formalism to other quantum states is suggested. For qubit systems, the notion of density operators algebra and the relation with the probability representation of quantum states are discussed. The Schrödinger equation for qubit states is obtained as an equation for probabilities.

Keywords: tomogram describing quantum system state; qubit spin system; density matrix in quantum mechanics; group probability representation

1. Introduction

Quantum states of physical systems are described by complex wave functions $\psi(x)$ [1] (English translation in Ref. [2]) and density matrices $\rho(x, x')$ [3] (English translation in Ref. [4]), which are, in the position representation, matrix elements of the Dirac density operator $\hat{\rho}$ [5] acting in the Hilbert space $\mathcal{H}$ such that, for pure states $|\psi\rangle$, one has $\psi(x) = \langle x | \psi \rangle$, and the density matrix reads $\rho_{x,x'} = \psi(x)\psi^*(x')$. In classical mechanics, the system (particle) states are described using the probability distribution $f(q, p)$ of random position q and momentum p . The possibility of constructing the analogous probability representation of quantum particles helps to provide the results, which for a long period of time (many decades) did not agree with the formalism of wave functions and the density description of operators of quantum system states. In classical mechanics, the states of a system are described by a probability function $f(q, p)$ of random position q and momentum p . However, several attempts to construct an analogous probability representation in the quantum domain initially yielded results inconsistent with both the wave-function and density-operator formalisms.

In quantum mechanics, any system always exhibits irreducible quantum fluctuations that cannot be described by a joint probability of position and momentum due to uncertainty relations. A classically uniformed trajectory of a particle in a phase space is undefined, and a joint probability distribution of position and momentum does not exist for a quantum system. However, there has been a number of attempts to translate the quantum picture into classical language.



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Different representations of quantum mechanics, where the quantum states were determined in terms of quasi-probability functions, were known before; for example, the Wigner function [6], the Husimi–Kano function [7,8], the Glauber–Sudarshan function [9,10], the Blokhintsev mixed density matrix [11], and the Wootters quantum mechanics without probability amplitudes [12].

Imre Fényes [13] proposed a probability-theoretic interpretation of quantum mechanics and showed that some processes considered in quantum mechanics are essentially Markov processes of a special type. José Moyal [14] introduced a quantum evolution equation for the quasi-distribution function, which is similar to the classical stochastic equation. A function that satisfies the Moyal equation is the quasi-distribution function, introduced by Eugene Wigner. It is analogous to the classical distribution function but can take negative values. This property indicates that this function cannot be a probability distribution function, which must be non-negative. Both the Wigner quasi-probability and the Glauber–Sudarshan quasi-probability, which define a complete quantum state, are related to each other and the density matrix in coordinate representation by invertible integral transformations. The Glauber–Sudarshan quasi-probability, which defines a quantum state, being although similar to positive distribution functions for some quantum states, like the Wigner function, takes negative values for many quantum states and thus does not describe probability in its standard definition. The Husimi–Kano quasi-probability function, which depends on two variables, q and p , is normalized and takes only positive values. Therefore, the Husimi–Kano function may be interpreted as a probability distribution. However, this function is not a distribution function in the system’s phase space, since its variables are not simultaneously measurable quantities (coordinate and momentum) due to the uncertainty principle.

In Refs. [15,16] the connection of the Wigner function [6] with a measurable marginal probability distribution for a homodyne observable, which is a coordinate in the phase space of the system rotated by a given angle, was found. The Wigner function of a one-dimensional system was expressed through this measurable normalized positive distribution function using the Radon transform [16], with the angle of rotation in the phase space used in conventional medical tomography. In this regard, the scheme of measurements of a quantum state for a continuous observable of the type of a coordinate or a momentum has been called the scheme of optical tomography, and this scheme has been applied in experiments on the reconstruction of the quantum state of a mode of electromagnetic radiation [17] and in molecular spectroscopy [18]. Experiments on the reproducible measurement of the compressed vacuum state of light generated by an optical parametric oscillator were considered in Ref. [19]. The reconstruction of the one-photon Fock state was experimentally realized in Ref. [20]. Resonance fluorescence was proposed for the reconstruction of the quantum–mechanical state of an ion in Ref. [21]. A review of new methods and experiments for measuring quantum states was given in Ref. [22].

The properties of classical probability distributions are described by conventional probability theory [23]. The different aspects of quantum system properties and methods of quantum–mechanical applications to different areas of science within the framework of the probability theory are discussed in Refs. [24–27].

This paper provides a detailed construction of the probability representation of quantum states and applies it to the specific case of qubit systems (spin-1/2 particles).

The properties of symmetry of the quantum system states are described by the symmetry groups [28] (see also the monograph [29]) and their irreducible representations [30]. As for spin properties of electrons, their wave functions are two-dimensional vectors $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and 2×2 density matrices $\rho(m, m') = \langle m | \hat{\rho} | m' \rangle$ of the density operator $\hat{\rho}$.

Recently, the probability representation of quantum states was constructed [31,32], where the density operators were invertibly mapped onto probability distribution functions, both for wave functions ψ of continuous variables and for the case of spin variables.

1.1. Quantizer–Dequantizer Operators

We briefly review the general method of constructing the map of the state density operator onto the probability (and quasi-probability) distributions, using the method of quantizer $\hat{D}(\vec{x})$ and dequantizer $\hat{U}(\vec{x})$ operators, where vector $\vec{x} = (x_1, x_2, \dots, x_n)$ contains both continuous and discrete parameters. Then, the map reads [33] $\hat{\rho} \rightleftharpoons w(\vec{x})$, where $w(\vec{x})$ is the conditional probability distribution given by the function

$$w(\vec{x}) = \text{Tr} \hat{\rho} \hat{U}(\vec{x}), \quad (1)$$

and the density operator reads

$$\hat{\rho} = \int w(\vec{x}) \hat{D}(\vec{x}) d\vec{x}. \quad (2)$$

The time-dependent density operator $\hat{\rho}(t)$ for a given Hamiltonian $\hat{H}(t)$ satisfies the evolution equation (with the reduced Planck constant $\hbar = 1$) of the form

$$\frac{\partial \hat{\rho}(t)}{\partial t} + i(\hat{H}(t)\hat{\rho}(t) - \hat{\rho}(t)\hat{H}(t)) = 0. \quad (3)$$

The relations (1) and (2) point that one can use the functions $w(\vec{x})$ (called symbols) of the density operators $\hat{\rho}$ to describe states of quantum systems. In the case of the Hermitian dequantizer operator $\hat{U}(\vec{x})$, the symbol $w(\vec{x})$ is the probability distribution; in other cases, the symbol $w(\vec{x})$ is called the quasi-probability distribution describing the system states, for example, the Wigner function, the Husimi–Kano function or the Glauber–Sudarshan function.

For discrete parameters x_j , the integral is replaced by a sum. Here we apply this method to spin variables and construct a probability representation of qubit states. We also examine how these probability distributions transform under unitary transformation generated by the $SU(2)$ group. More precisely, we consider the transformation $w(\vec{x}) \rightarrow w_u(\vec{x})$ due to the change in the density operator $\hat{\rho}$ of the form $\hat{\rho} \rightarrow \hat{u}\hat{\rho}\hat{u}^\dagger$ (where the dagger symbol $\dagger$ denotes the Hermitian conjugate) by unitary operators $\hat{u}^\dagger = \hat{u}^{-1}$ describing irreducible representation of a group like the $SU(2)$ group for spin states [34,35]. For continuous variables, the tomographic probability distribution was introduced, using the invertible map of wave functions $\psi(\vec{x})$ onto tomographic probability distribution $G_\psi(X|\mu, \nu)$, where $x_1 = X$, $x_2 = \mu$, $x_3 = \nu$ of the position X measured in the reference frame in the phase space determined by the parameters μ and ν such that $X = \mu q + \nu p$; the distribution then reads as follows [36]:

$$G_\psi(X|\mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(x) \exp\left(\frac{i\mu}{2\nu}x^2 - \frac{iXx}{\nu}\right) dx \right|^2. \quad (4)$$

The formula (4) corresponds to the choice of the dequantizer operator $\hat{U}(\vec{x})$ and quantizer operator $\hat{D}(\vec{x})$ in Equations (1) and (2) in the form

$$\hat{U}(\vec{x}) = \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}) \quad (5)$$

and

$$\hat{D}(\vec{x}) = \frac{1}{2\pi} \exp(i(X\hat{1} - \mu\hat{q} - \nu\hat{p})). \quad (6)$$

In this paper, we apply the analogous method to construct the invertible map of the density operator $\hat{\rho}$ onto probability distributions for discrete spin variables. This allows us to describe qubit states by probability distribution functions that contain the same information as the density operators. Such probability distributions are specific to quantum mechanics and are not part of classical probability theory, especially in the presence of quantum correlation and entanglement [37].

1.2. $SU(2)$ Symmetry of Qubit States Formalism

For spin-1/2 states, the density operator $\hat{\rho}$ was mapped onto probability distribution $w(X|j)$ with discrete spin parameters $X_1 = 1/2$, $X_2 = -1/2$ and $j = 1, 2$, and 3, describing three perpendicular directions in the three-dimensional space of coordinates x , y , and z respectively.

The development of the notion of quantum states using wave functions [1], density matrices [3] and quasi-distributions [6–10] was recently examined in the probability representation of quantum system states [31]. The tomograms (4) were constructed for oscillator states in explicit form as probability distributions contain complete information about the density matrices. A similar approach can be developed for discrete-variable systems to describe spin-like states, including entangled probability distributions [37] that are absent in classical probability theory.

This paper describes new symmetry properties of spin-1/2 states in probability representation. In particular, we construct the explicit transformation of the probability distributions associated with the standard spin representation and describe it in terms of the unitary matrix

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad (7)$$

and the transformed density operator

$$\hat{\rho}_u = u \hat{\rho} u^\dagger. \quad (8)$$

The matrix elements of $\rho(u)$ depend on probabilities describing the state and are quadratic functions of the matrix elements u_{jk} ($j, k = 1, 2$) of the fundamental representation of the $SU(2)$ group.

For $su(2)$ Lie algebra, the probability representation was introduced in Ref. [38]. The point to introduce new probability representation of the spin state is based on the known standard form of irreducible representation of a group, where one has the Hilbert space $\mathcal{H}$ with vectors $|\psi\rangle$, and vectors are acted on by group of symmetry elements $u(g)$, such that one obtains

$$|\psi_i\rangle \Rightarrow |\psi_{gi}\rangle = u|\psi_i\rangle. \quad (9)$$

We assume that there are orthogonal basis vectors $|\psi_j\rangle$, $j = 1, \dots, N$ such that

$$|\psi_j\rangle = \sum_k u_{jk}(g) |\psi_k\rangle, \quad (10)$$

where u_{jk} is the matrix of the group's irreducible unitary representation. Since the group properties are valid for any group elements g_i , $i = 1, 2$, namely,

$$g_1 g_2 = g, \quad (11)$$

we have

$$u(g) = u(g_1)u(g_2). \quad (12)$$

This signifies that there is a group representation, and let us consider it as a unitary one. Since in quantum mechanics there are density operator states

$$\hat{\rho} = |\psi\rangle\langle\psi|, \quad (13)$$

their transform corresponding to Equation (12) reads

$$\hat{\rho} \rightarrow \hat{\rho}_u = u|\psi\rangle\langle\psi|u^\dagger, \quad (14)$$

which signifies that the matrix element of density matrix is given by the formula

$$(\rho_u)_{jk} = \sum_{m,m'=0}^N u_{jm} \rho_{mm'}(u^\dagger)_{m'k}. \quad (15)$$

Therefore, if the matrix elements ρ_{mn} are linear on the probability distribution $G_j(X)$, the group action induces a corresponding transformation of those distributions. Formally, one can write

$$G_n(X) = M_{XX'}^{(n)} G(X'), \quad (16)$$

where matrix $M_{XX'}^{(n)}$ quadratically depends on standard matrix elements u_{jk} of the standard known group representation of the group of symmetry elements. We consider the probability representation of qubit density operators developed in Refs. [31,39]. Different kinds of problems were solved in the probability representation in Refs. [40–52]. A remarkable analysis of the foundation of quantum-mechanics papers from the viewpoint of teaching aspects was done in [53]. Some mathematical aspects of tomography and the probability representation of quantum and classical states were considered in [54]. The Green function in probability representation was considered in [55]. The method with time-dependent integrals of motion and its connection to the Green function $\mathcal{G}(x, x', t)$ was considered in [56–58]. We point out that the formalism of the quantizer–dequantizer approach can be used for all quasi-probability distributions like the Wigner, the Glauber–Sudarshan and the Husimi–Kano quasi-distributions.

2. Spin-1/2 Density Matrix in the Probability Representation

In this Section, we construct dequantizer operators (matrices) for spin-1/2 system states using the same general method employed for harmonic oscillators. Using this method one can obtain the tomogram (4), which describes the harmonic oscillator state. Now we consider the state vectors for the qubit system

$$|\psi_{x+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\psi_{x-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (17)$$

One can also consider a general pure state $|\psi\rangle$ of the form $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$. The vectors (17) are eigenvectors for the spin projection operator

$$s_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (18)$$

onto the x -axis.

Also, vectors

$$|\psi_{y+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |\psi_{y-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad (19)$$

are eigenvectors of the spin projection operator

$$s_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (20)$$

onto the y -axis.

Finally, the vectors

$$|\psi_{z+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_{z-}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (21)$$

are eigenvectors of the spin projection operator

$$s_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (22)$$

onto the z -axis.

The commutation relations of the spin projection operators are

$$[s_x, s_y] = i s_z. \quad (23)$$

The matrices of the spin-1/2 pure states with spin projection on the x -axes read

$$\rho_{x+} = |\psi_{x+}\rangle \langle \psi_{x+}| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad (24)$$

$$\rho_{x-} = |\psi_{x-}\rangle \langle \psi_{x-}| = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}. \quad (25)$$

The matrices of the spin-1/2 pure states with spin projection on the y -axes are

$$\rho_{y+} = |\psi_{y+}\rangle \langle \psi_{y+}| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \times \frac{1}{\sqrt{2}} (1, -i) = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad (26)$$

and

$$\rho_{y-} = |\psi_{y-}\rangle \langle \psi_{y-}| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \times \frac{1}{\sqrt{2}} (1, i) = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \quad (27)$$

For states corresponding to spin projections on the z -axes, we have the density matrices

$$\rho_{z+} = |\psi_{z+}\rangle \langle \psi_{z+}| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (28)$$

and

$$\rho_{z-} = |\psi_{z-}\rangle \langle \psi_{z-}| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times (0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (29)$$

Also,

$$w(-1/2|z) = \text{Tr} \rho_{\rho_{z-}} = \rho_{22}. \quad (30)$$

Thus, one can see that six matrices $\rho_{x,y,z\pm}$ play the role of dequantizers for qubit states given by Equations (24)–(29). These six density matrices (dequantizers) provide the possibility to construct probability distributions $w(x|j)$, where $x = \pm 1/2$ and $j = 1, 2$, and 3. The conditional probability distributions contain six non-negative numbers, namely, probabilities of spin projection $\pm 1/2$ onto axes x , y , and z , respectively, in three-dimensional space. The density matrices $\rho_{x,y,z\pm}$ can determine the dequantizer operators $U(x, j)$. This signifies that six probabilities $w(x|j)$ are defined using the formula for the density matrix $\rho_{x,y,z\pm}$:

$$w(x|j) = \text{Tr}(\rho U(x, j)) = \text{Tr} \rho \rho_{x,y,z\pm}. \quad (31)$$

Since

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad (32)$$

one has the formulas for probabilities; for example,

$$w(+1/2|3) = \text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \rho_{11}. \quad (33)$$

Also, one has relations

$$w(+1/2|1) = \text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \frac{\rho_{11} + \rho_{21} + \rho_{12} + \rho_{22}}{2} \quad (34)$$

and

$$w(-1/2|1) = \text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \frac{\rho_{11} - \rho_{21} - \rho_{12} + \rho_{22}}{2}. \quad (35)$$

One can check that

$$w(+1/2|1) + w(-1/2|1) = \rho_{11} + \rho_{22}. \quad (36)$$

Analogous relation

$$w(+1/2|1) + w(-1/2|1) = 1 \quad (37)$$

takes place due to Equations (34) and (35). Finally, using matrix ρ_{y+} (26), one has

$$w(+1/2|2) = \frac{1}{2} \text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{2} [\rho_{11} + i\rho_{12} - i\rho_{21} + \rho_{22}] = \frac{1}{2} + i(\rho_{12} - \rho_{12}^*). \quad (38)$$

The last probability $w(-1/2|y)$ reads

$$w(-1/2|2) = \frac{1}{2} \text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2} - i(\rho_{12} - \rho_{12}^*). \quad (39)$$

The six probabilities obtained make it possible to write the density matrix ρ (32) of the qubit state in the form

$$\rho = \frac{1}{2} \begin{pmatrix} 2w(+\frac{1}{2}|3) & w(+\frac{1}{2}|1) - w(-\frac{1}{2}|1) - i(w(+\frac{1}{2}|2) - w(-\frac{1}{2}|2)) \\ w(+\frac{1}{2}|1) - w(-\frac{1}{2}|1) + i(w(+\frac{1}{2}|2) - w(-\frac{1}{2}|2)) & 1 - 2w(+\frac{1}{2}|3) \end{pmatrix}. \quad (40)$$

One can introduce the probability form using the numbers

$$p_1 = w(+1/2|1), \quad p_2 = w(+1/2|2), \quad \text{and} \quad p_3 = w(+1/2|3). \quad (41)$$

One can experimentally check that for the mentioned $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, the probability $p_3 = |a|^2$ and the probability $p_1 = \text{Re}(ab^*) + \frac{1}{2}$. Then, the density matrix (40) takes the form

$$\rho = \begin{pmatrix} p_3 & p_1 - 1/2 - i(p_2 - 1/2) \\ p_1 - 1/2 + i(p_2 - 1/2) & 1 - p_3 \end{pmatrix}, \quad (42)$$

where p_1 , p_2 , and p_3 are probabilities of spin projections $+1/2$ onto axes x , y , and z , respectively. We introduce six quantizer matrices $D(x|j)$ to obtain the density matrix of the form

$$\rho = \sum_x \sum_j w(x|j) D(x|j). \quad (43)$$

One has six quantizer matrices

$$D(+1/2|3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D(-1/2|3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (44)$$

corresponding to spin projections $+1/2$ and $-1/2$. Also, one has

$$D(+1/2|1) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (45)$$

$$D(-1/2|1) = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (46)$$

For the spin states with spin projections onto the y axes, one has the quantizer matrices

$$D(+1/2|2) = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (47)$$

and

$$D(-1/2|2) = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (48)$$

Thus, one has the relations

$$\rho = \sum_x \sum_j w(x|j) D(x|j) \quad (49)$$

and

$$w(x|j) = \text{Tr} \rho U(x, j), \quad (50)$$

where the quantizer and dequantizer matrices are Hermitian.

Thus, we have identified the quantizer and dequantizer operators for qubit states in a form analogous to the construction used for the harmonic oscillator (5) and (6).

3. Unitary Transform of the Probability Distribution Describing the Qubit State

In this Section, we introduce the unitary transform of density matrices expressed in terms of probability distributions. One can express this transform as transformation of

the probability distributions by describing the density matrices in an explicit form. Thus, irreducible representation of the $SU(2)$ Lie group can be considered as the representation of the probability distributions with probability $w(x|j)$ for the spin-1/2 system. Now we write this explicitly. For the given density matrix (32) we find the matrix $u\rho u^\dagger$, where u is a unitary 2×2 -matrix. We have $\rho_u = u\rho u^\dagger$, $u^\dagger = u^{-1}$. So, one obtains

$$\begin{pmatrix} \rho_{11}^u & \rho_{12}^u \\ \rho_{21}^u & \rho_{22}^u \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix}. \quad (51)$$

Thus, one has matrix elements ρ_{jk}^u of matrix ρ_u

$$(\rho^u)_{11} = |u_{11}|^2 \rho_{11} + u_{11}^* u_{12} \rho_{21} + u_{12}^* u_{11} \rho_{12} + |u_{12}|^2 \rho_{22}. \quad (52)$$

Since the matrix element of the density matrix ρ_{11} is equal to probability $w(x = +1/2|j = 3) = p_3$ and $(\rho_u)_{11}$ to the transformed probability $w_u(x = +1/2|j = 3) = p_{3,u}$, we obtain the transformed probability as the linear combination

$$p_{3,u} = |u_{11}|^2 p_3 + u_{11}^* u_{12} [p_1 - 1/2 + i(p_2 - 1/2)] + u_{12}^* u_{11} [p_1 - 1/2 - i(p_2 - 1/2)]. \quad (53)$$

Analogous relations can be derived for other probabilities. Therefore, the action of $SU(2)$ on the density matrix induces a linear transformation of the probabilities associated with spin projections along the three axes.

4. Density Matrices and Irreducible Representation of the Spin-1/2 $SU(2)$ Group

There is the possibility to find other relations of the probability representations with unitary matrices. Now we consider the example of the density matrix $\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ describing the qubit state with spin projection $+1/2$ onto the z -axis and the density matrix $\rho_u = u\rho_0 u^\dagger$ considered just above. This density matrix reads

$$\rho_u = \begin{pmatrix} |u_{11}|^2 & u_{12} u_{21}^* \\ u_{12}^* u_{21} & |u_{21}|^2 \end{pmatrix}. \quad (54)$$

The matrix (54) is the density matrix of the qubit state with spin projection probability onto the z -axis $w(-1/2|z) = |u_{12}|^2$ and probability onto the x -axis $w(+1/2|x) = \text{Re}(u_{12} u_{21}^*)$. Then, the matrix $R = u\rho_u u^\dagger$ is also the density matrix with probabilities determined by the quadratic form of the unitary 2×2 matrices u .

5. Density Matrices in the Probability Representation and Unitary Matrix

In this Section, we construct density matrices using the unitary 2×2 -matrices u . One can use the properties of density matrix ρ to determine the other density matrix due to the transform $\rho \rightarrow \rho_u$, which reads (similar to Equation (8))

$$\rho_u = u\rho_0 u^\dagger, \quad (55)$$

Taking ρ to be the density matrix corresponding to spin projection $+1/2$ along the z -axis, one obtains a transformed density matrix whose elements are determined by the matrix elements of u . According to Equation (55),

$$\rho_u = \begin{pmatrix} |u_{11}|^2 & u_{21}^* \\ u_{12}^* & |u_{21}|^2 \end{pmatrix}. \quad (56)$$

According to the construction of the probabilities describing the density matrices, we have the connection between the probabilities $w(x|j)$ describing the density matrix (55) in the form

$$w(+1/2|3) = |u_{11}|^2, \quad (57)$$

$$w(-1/2|3) = |u_{12}|^2, \quad (58)$$

$$\frac{1}{2}(w(+1/2|1) - w(-1/2|1)) = \text{Re}(u_{12}u_{21}^*), \quad (59)$$

$$\frac{1}{2}(w(+1/2|2) - w(-1/2|2)) = \text{Im}(u_{12}u_{21}^*). \quad (60)$$

Thus, the density matrix ρ contains matrix elements determined by matrix elements of the unitary matrix u . The density matrix

$$u\rho_u u^\dagger = u^2 \rho u^{\dagger 2} \quad (61)$$

contains the matrix elements of polynomials that are quadratically dependent on matrix elements u_{jk} . Introducing the set of the density matrices

$$\rho_u^N = u^N \rho u^{\dagger N},$$

we construct the set of probability distributions determining the qubit states. The latter determine the relation of the probability distributions given by numbers $w_u^N(x|j)$, $N = 0, 1, 2, \dots$ with spin-1/2 irreducible representation of group $SU(2)$.

Let us take $\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $R = u\rho_0 u^\dagger$. Namely,

$$R = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} |u_{11}|^2 & u_{12}u_{21}^* \\ u_{21}u_{12}^* & |u_{21}|^2 \end{pmatrix} \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix}. \quad (62)$$

Another way to relate the matrices u to probabilities is through pairs of dequantizers; for example,

$$R_1 = u\rho u^\dagger, \quad R_2 = u^2 \rho (u^\dagger)^2.$$

Quantizers (six self-adjoint matrices) $D_{jk}(x|j)$ read Equations (44)–(48). There is Equation (49), in which the density operator ρ is constructed from any number of the other density operators ρ_j with coefficients w_j like

$$\hat{\rho} = \sum_i w_i \hat{\rho}_i, \quad (63)$$

where w_i is the probability $w_i \geq 0$, such that $\sum_i w_i = 1$ is also the probability. Since in the case considered of the spin-1/2 system we described the density operators as probability distributions, the obtained sum belongs to the same set. The density operator ρ_u is known to be another density operator after unitary transform $\hat{u}$, defined by the formula (8). Thus, we can consider all the density operators ρ as the set of probability distributions under the action of unitary irreducible representation of the group $SU(2)$. Thus, expressing the probabilities describing the density operator $\hat{u}\hat{\rho}\hat{u}^\dagger$ as a linear sum of probabilities describing each of the operators in the sum

$$\hat{R} = \hat{u}\hat{\rho}\hat{u}^\dagger = \sum_k w_k \hat{\rho}_k \quad \text{with} \quad \hat{\rho}_k = \hat{u}_k \hat{\rho} \hat{u}_k^\dagger,$$

where u_k are unitary matrices, we get the probability representation of the final density operator $\hat{R}$. This is the construction of the probabilistic density representation of the group g , for which the matrix form of the group irreducible representation is known. In our example, we have a system of qubit density operators described by probability distributions $0 \leq w(x|j) \leq 1$.

The introduced representation (49) signifies that we take the initial state

$$\hat{\rho} = \sum_{x,j} w_0^{(k)}(x|j) \hat{D}(x, j)$$

and obtain the transformed density operator

$$\hat{\rho}_u^{(k)} = \hat{u} \hat{\rho}^{(k)} \hat{u}^\dagger.$$

Then, we get the linear transform of the operator $\hat{\rho}$ in the form of the sum of probability distributions multiplied by probabilities depending on the matrix elements of a unitary matrix describing probabilities, which determine the change in the state density matrices related to the quantizer $\hat{D}(x, j)$. Then, the exact transform reads

$$\rho_u = \begin{pmatrix} p_{3,u} & p_{12,u} \\ p_{12,u}^* & 1 - p_{3,u} \end{pmatrix}, \quad (64)$$

where

$$p_{12,u} = (p_{1,u} - 1/2) - i(p_{2,u} - 1/2).$$

The transformed probabilities are expressed through the probabilities p_1 , p_2 , and p_3 (41) in the form

$$p_{12,u} = u_{11}u_{21}^*p_3 + u_{12}u_{21}^*p_{12}^* + u_{11}u_{22}^*p_{12} + u_{12}u_{22}^*(1 - p_3) \quad (65)$$

and

$$p_{3,u} = |u_{11}|^2p_3 + 2\text{Re}(u_{12}u_{11}^*p_{12}^*) + |u_{12}|^2(1 - p_3). \quad (66)$$

The transformed probability $p_{3,u}$ is defined by a formula analogous to Equation (53).

6. Density Operators Algebra

In this Section, we discuss the algebraic properties of density operators $\hat{\rho}$ for all the quantum systems and describe them as properties of probability distributions, which determine the density operators. Also, we consider the examples of the density operators of qubit states and their algebraic properties. Any density operator of the quantum state $\hat{\rho}$ has the following properties: $\hat{\rho} = \hat{\rho}^\dagger$, $\text{Tr} \hat{\rho} = 1$. The diagonal elements of density matrices are non-negative numbers. The density operators of qubit states demonstrate all these properties. Now we consider two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ and the operator $\hat{\rho}$, which is the normalized anti-commutator $\hat{\rho}_1\hat{\rho}_2 + \hat{\rho}_2\hat{\rho}_1$ of these two density operators. This operator is Hermitian and has non-negative diagonal elements.

We introduce and consider the operator

$$\hat{\rho}_3 = \frac{1}{2\text{Tr}(\hat{\rho}_1\hat{\rho}_2)}(\hat{\rho}_1\hat{\rho}_2 + \hat{\rho}_2\hat{\rho}_1). \quad (67)$$

We consider the case where $\text{Tr}(\hat{\rho}_1\hat{\rho}_2) \neq 0$. The operator $\hat{\rho}_3$ has the property $\text{Tr} \hat{\rho}_3 = 1$, since for the density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ one has the relation $\text{Tr} \hat{\rho}_1\hat{\rho}_2 = \text{Tr} \hat{\rho}_2\hat{\rho}_1$. The relation (67) demonstrates the algebraic property of density operators of quantum states, which points that the density operators form the algebra. For the algebra $su(2)$, the spin-projection

operators satisfy the commutation relations $\hat{s}_x\hat{s}_y - \hat{s}_y\hat{s}_x = i\hat{s}_z$. The density matrices satisfy the anti-commutation relations. These qubit state density operators satisfying the anti-commutation relation (67) and expressed in the probability representation in terms of the considered probability distribution functions form the algebra of the density operators. This signifies that probability distributions describing the qubit states satisfy the relations for the probability distributions corresponding to the formula (67). The Lie group $SU(2)$'s irreducible representation realized by unitary 2×2 -matrices also makes it possible to obtain the transformation of probabilities satisfying the properties of the algebra of the density operators, given by the relation (67).

Examples of the algebra's behaviour can be demonstrated if one takes

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\rho}_2 = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad (68)$$

and

$$\rho_3 = \frac{1}{2\rho_{11}} \begin{pmatrix} 2\rho_{11} & \rho_{12} \\ \rho_{21} & 0 \end{pmatrix}. \quad (69)$$

Using the probability representation of these qubit states' density matrices, we get the probabilities p'_1 , p'_2 , and p'_3 and determine the density matrix ρ_3 as follows:

$$p'_1 = \frac{p_1 - 1/2}{2p_3} + \frac{1}{2}, \quad (70)$$

$$p'_2 = \frac{p_2 - 1/2}{2p_3} + \frac{1}{2}, \quad (71)$$

$$p'_3 = 1. \quad (72)$$

These equalities determine a new density matrix ρ_3 in terms of probabilities $w(+1/2|1)$, $w(+1/2|2)$, and $w(+1/2|3)$, respectively. All the numbers must satisfy the properties of the probability distributions.

7. Conclusions

To conclude, we considered the general problem of introducing a new notion of the probability representation of groups, using the example of the $SU(2)$ group representation of qubit quantum states. In quantum mechanics, the general states are associated with density matrices (operators acting in a Hilbert space) or with vectors describing the pure quantum states. The set of states (density operator $\hat{\rho}$) is the density operators' algebra.

The example of the qubit state demonstrated a general scheme of constructing the density matrix of spin state dependence on the probability distributions. Thus, if one has an irreducible representation of any spin $SU(2)$ group with an arbitrary value of spin j , where vectors of spin $|\psi_m\rangle$ are eigenvectors of spin projections onto axes x , y , and z , namely,

$$\hat{S}_x|\psi_m^j\rangle = m|\psi_m^j\rangle, \quad m = -j, -j+1, \dots, j,$$

one has the density matrices of these states

$$\hat{\rho}_{mm'}^j = |\psi_m^j\rangle\langle\psi_{m'}^j|.$$

An arbitrary density matrix (operator) determines the probabilities of spin projections

$$w^c(x|j, m) = \text{Tr}(\hat{\rho}|\psi_m^j\rangle\langle\psi_m^j|).$$

Experimental verification of the proposed method may be achieved by measuring spin projections $\pm 1/2$ along the three axes x , y and z and reconstructing the probability $w(x|j)$ describing qubit states.

These probabilities make it possible to express matrix elements of the spin state density operator $\hat{\rho}^j$ in terms of positive probability distributions and apply unitary matrices u to consider their transform, which depends on the u_{jk} matrix elements. The qutrit and ququart states will be studied in our future papers. Also, the Jordan–Schwinger map of the spin representation based on the study of tomograms of the oscillator states is considered for future papers.

7.1. The Schrödinger Equation in Terms of Probabilities

We emphasize that writing the density matrix of qubit states in terms of the probabilities $w(x|j)$ or using three probabilities p_1 , p_2 , and p_3 given in Equation (42) provides us with a method to write the Schrödinger equation for a given Hamiltonian matrix

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

in the form of the equation for probabilities p_1 , p_2 , and p_3 , i.e.,

$$\begin{aligned} & \frac{\partial}{\partial t} \begin{pmatrix} p_3(t) & p_1(t) - \frac{1}{2} - i(p_2(t) - \frac{1}{2}) \\ p_1(t) - \frac{1}{2} + i(p_2(t) - \frac{1}{2}) & 1 - p_3(t) \end{pmatrix} \\ & + \left[\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \begin{pmatrix} p_3(t) & p_1(t) - \frac{1}{2} - i(p_2(t) - \frac{1}{2}) \\ p_1(t) - \frac{1}{2} + i(p_2(t) - \frac{1}{2}) & 1 - p_3(t) \end{pmatrix} \right] = 0. \end{aligned} \quad (73)$$

This gives the eigenvalue equation for energies of qubit states in the form

$$\begin{pmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} \begin{pmatrix} p_3(t) & p_1(t) - \frac{1}{2} - i(p_2(t) - \frac{1}{2}) \\ p_1(t) - \frac{1}{2} + i(p_2(t) - \frac{1}{2}) & 1 - p_3(t) \end{pmatrix} = 0$$

which is the form of the equation for the probabilities p_1 , p_2 , and p_3 .

7.2. Quantizer–Dequantizer Evolution Equation

Here, we express the new equation for quantum evolution using quantizer–dequantizer operators $\hat{U}(\vec{x})$ and $\hat{D}(\vec{x})$.

Actually, Equation (73) can be rewritten in the form of Equations (1) and (2)

$$\frac{\partial}{\partial t} \int w(\vec{x}) \hat{D}(\vec{x}) d\vec{x} + i \left[\hat{H}, \int w(\vec{x}) \hat{D}(\vec{x}) d\vec{x} \right] = 0, \quad (74)$$

where

$$w(\vec{x}, t) = \text{Tr} \hat{\rho}(t) \hat{U}(\vec{x}).$$

Equation (74) coincides with Equation (73) for qubit states, but in the case of other kinds of quantizer–dequantizer operators, this equation is also correct for quasi-probability distributions (like the Wigner function).

Thus, the evolution equation for the density operator of a qubit state can be reformulated as an evolution equation for the corresponding probability distributions. This approach provides a probability-based description tailored to quantum states. Experimental validation of this formulation could be achieved by measuring spin projections in qubit systems. Additionally, experimental tests of Bell inequalities for qubit states using

tomography, that is, the probability representation of spin-1/2 states [59], may highlight the connection between measurements and formulas analogous to Equation (73). Recently, several new inequalities for quantum states, analogous to the Heisenberg uncertainty relations for position and momentum, were derived by Victor V. Dodonov [60]. Experimental verification of these inequalities would represent further progress in the field.

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