

Article

Infinite Horizon H_2/H_∞ Control for Discrete-Time Mean-Field Stochastic Systems

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Abstract: In this paper, we deal with the H_2/H_∞ control problem in the infinite horizon for discrete-time mean-field stochastic systems with (x, u, v) -dependent noise. First of all, a stochastic-bounded real lemma, which is the core of H_∞ analysis, is derived. Secondly, a sufficient condition in terms of the solution of coupled difference Riccati equations (CDREs) is obtained for solving the H_2/H_∞ control problem above. In addition, an iterative algorithm for solving CDREs is proposed and a numerical example is given for verification of the feasibility of the developed results.

Keywords: mean-field systems; stochastic-bounded real lemma; exact detectability; H_2/H_∞ control

1. Introduction

Mixed H_2/H_∞ control is a crucial robust control research subject that has been thoroughly investigated by numerous researchers over the past decades. Robust control, as a control design method that can effectively eliminate the effect of disturbances, has been refined since the 1960s. However, in engineering practice, it is desired that the controller not only minimizes performance targets, but also eliminates the influence of disturbances when the worst disturbance is applied. Therefore, mixed H_2/H_∞ control is better suited to the needs of engineering practice. Mixed H_2/H_∞ control problems regarding deterministic systems have been discussed in [1,2] and others. H_∞ and H_2/H_∞ control problems of stochastic systems and their applications have also been extensively studied by many researchers in recent years. For example, stochastic H_2/H_∞ control for continuous-time systems with state-dependent noise has been discussed in [3]. In [4], the H_∞ control problem for a type of nonlinear system with noise that is dependent on the state and disturbance has been discussed by the authors. Ref. [5] considered H_∞ control for discrete-time linear systems with their states and inputs affected by random perturbations. Ref. [6] proposed the notion and stochastic PBH criterion of exactly observable linear autonomous systems, which provided great convenience for the analysis and synthesis of stochastic systems. Under the presumption that the system is exactly observable, an infinite horizon H_2/H_∞ controller of discrete-time stochastic systems with state- and disturbance-dependent noise has been designed in [7]. As the research on stochastic H_2/H_∞ control continues, many researchers have begun to focus on systems with Markov jumps. H_2/H_∞ control for stochastic systems with Markov jumps not only contributes to improving system performance but also holds significant practical significance for areas such as economics and technology. Ref. [8] investigated the robust control problem for continuous-time systems that were subject to multiplicative noise and Markovian parameter jumps. And a necessary/sufficient condition was presented for the existence of a controller using two coupled algebraic Riccati equations. Ref. [9] developed a robust control theory for discrete-time stochastic systems subject to independent stochastic perturbations and Markov jumps.



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The origins of the mean-field theory can be traced back to the early 20th century, when physicists began to focus on complex systems in quantum mechanics and statistical physics. As an approximate method, mean-field theory can effectively solve problems in complex systems. In recent years, the application of mean-field theory has become increasingly widespread in fields such as power systems and biology. Many researchers have conducted numerous studies in this area [10–14]. For example, the LQ control problems of discrete-time stochastic mean-field systems in the finite and infinite horizon are discussed in [10] and [11], respectively. Ref. [13] discussed the H_2/H_∞ control problem of discrete-time mean-field stochastic systems in the finite horizon, while [14] addressed the H_2/H_∞ control problem of continuous-time systems in the finite horizon. The problem of mean-field games has been studied in [15] to illustrate the feasibility of the mean-field modeling approach used in economics, and a number of issues worthy of further research were mentioned at the end of the article. With the gradual maturation of theoretical studies, mean-field theory has been widely applied, for instance, in [16]; mean-field theory has provided great convenience for solving the behavior of node interaction behavior in complex networks, and [17] has used mean-field theory to deal with the problem of large population stochastic differential games connected to the optimal and robust decentralized control of large-scale multiagent systems.

Ref. [18] has addressed finite and infinite horizon stochastic H_2/H_∞ control problems for stochastic systems with (x, u, v) -dependent noise. The infinite horizon H_2/H_∞ control problem for discrete-time mean-field stochastic systems with (x, u, v) -dependent noise is the topic of this paper. The difference with the considered stochastic systems in [18] is that the system equation considered in this paper includes both the state x , control u , and perturbation v , as well as the expectation of the state, control, and perturbation, i.e., Ex, Eu, Ev , respectively. Expectation terms can reduce the sensitivity of the system to random events; hence, mean-field theory simplifies complex problems and has attracted a great deal of attention. In [19–21], the mean-field stochastic maximum principle for dealing with the mean-field linear–quadratic (LQ) optimal control issue have been discussed. Separately, there are discussions of mean-field maximum principles for the mean-field stochastic differential equations (MFSDEs), backward stochastic differential equations (BSDEs), and forward–backward stochastic differential equations (FBSDEs). Later, [22] extended the results in [21] to the case of an infinite horizon. For discrete-time mean-field systems, ref. [10] investigated the finite horizon LQ optimal control problem. Based on [6,11,23], these works gave the notion of exact detectability of mean-field systems and derived the existence and uniqueness of the solution of the infinite horizon optimal control problem. This research has enriched the existing theory about stochastic algebraic equations. By providing a deeper understanding of the relationships between l^2 stability and optimal control problems, it contributes to the development of more efficient control strategies for linear mean-field stochastic systems. Ref. [24] investigated the stabilization and optimal LQ control problems for infinite horizon discrete-time mean-field systems. Unlike previous works, the authors showed for the first time that, under the exact detectability (exact observability) assumption, the mean-field system is stabilizable in the mean square sense with the optimal controller if and only if coupled algebraic Riccati equations have a unique positive semidefinite (or positive definite) solution. Ref. [12] considered the LQ optimal control problem for a class of mean-field systems with Markov jump parameters, and the unique optimal control could be given by the solution to generalized difference Riccati equations. Currently, driven by the research on mean-field games and mean-field-type control problems [15], many researchers have contributed to MFSDEs and their applications. By using a purely random approach, ref. [25] worked on solutions to McKean–Vlasov-type forward–backward stochastic differential equations (FBSDEs). Ref. [26] investigated the Markov framework for mean-field backward stochastic differential equations (MFBSDEs). The authors in [27] investigated the existence and uniqueness of the McKean–Vlasov FBSDE solution.

Nonlinear systems and time-delay systems pose inherent difficulties in control due to their complex behavior. Ref. [4] is an example of extending control strategies to address nonlinearities in systems, thus further broadening the scope of control theory. Ref. [28] examined the H_2/H_∞ control problem for nonlinear stochastic systems with time delay and state-dependent noise. The work explored the sufficient condition for the existence of nonlinear stochastic H_2/H_∞ control and the design method for stochastic H_2/H_∞ controllers. In [29], a fuzzy method was used to design multiobjective H_2/H_∞ control for a class of nonlinear mean-field random jump diffusion systems with uncertainty, and the stability, robustness, and performance optimization were deeply studied. Ref. [30] discussed noncooperative and cooperative multiplayer minmax H_∞ mean-field target tracking game strategies for nonlinear mean-field stochastic systems, with applications in the realm of cyberfinancial systems. Stochastic control problems with time delays are highly challenging in practice, because delays and uncertainties may lead to the degradation of the system performance. Ref. [31] focused on the mixed H_2/H_∞ control problem under an open-loop information pattern with input delay. In [32], the authors investigated the state feedback control laws for Markov jump linear systems with state and mode observation delays. Ref. [33] addressed the robust stability problem and mixed H_2/H_∞ control for Markovian jump time-delay systems with uncertain transition probabilities in the discrete-time domain. The obtained results have generalized several results in the previous literature that consider Markov transition probabilities as either a priori known or partially unknown.

Recently, robust controls for stochastic mean-field systems have received a great deal of attention. For instance, ref. [34] treated the discrete-time mean-field systems H_∞ output feedback control problem. Ref. [13] discussed the finite horizon H_2/H_∞ control problem for a class of stochastic mean-field systems. Ref. [14] studied a continuous-time stochastic control problem for mean-field stochastic differential systems with random initial values and diffusion coefficients that depend on the state, control, and disturbance explicitly, together with their expectations. The work first established a mean-field stochastic-bounded real lemma for continuous-time mean field stochastic systems, thereby revealing the equivalence between robust stability and the solvability of two indefinite differential Riccati equations, which provided a theoretical basis for H_∞ control. Based on this significant result, an equivalent condition for the existence of a controller was proposed by utilizing the solution of two crosscoupled Riccati equations. In contrast, there are relatively few studies on the robust control of discrete-time mean-field systems in the infinite horizon. Compared with the finite horizon case, it is required for the infinite horizon feedback controller to ensure that the closed-loop system is stable. The primary accomplishments of this paper are the following: (i) A mean-field stochastic bounded real lemma has been obtained. This lemma reveals the equivalence between robust stability and the solvability of algebraic Riccati equations, thereby providing a useful tool for the stability analysis of H_∞ control; (ii) By means of exact detectability, a sufficient condition for the solvability of the H_2/H_∞ control problem has been obtained. This condition offers a theoretical basis for designing robust controllers and ensures the stability of closed-loop systems; (iii) An iterative algorithm has been proposed to solve the coupled SDREs. This algorithm reduces computational complexity and enhances the practicality of controller design.

This paper is structured as follows: Section 2 recalls the notions of l^2 stability and exact detectability; The mean-field SBRL is presented in Section 3. Section 4 discusses the infinite horizon H_2/H_∞ control problem for the considered system. An iterative algorithm and a numerical example are given in Section 5.

2. Preliminaries

Consider the following discrete-time mean-field stochastic system:

$$\begin{cases} x_{s+1} = Ax_s + \bar{A}Ex_s + B_1u_s + \bar{B}_1Eu_s + B_2v_s + \bar{B}_2Ev_s \\ \quad + [Cx_s + \bar{C}Ex_s + D_1u_s + \bar{D}_1Eu_s + D_2v_s + \bar{D}_2Ev_s]\omega_s, \\ z_s = \begin{bmatrix} Kx_s \\ Fu_s \end{bmatrix}, F^T F = I, \\ x_0 = \zeta_0, s \in \mathbb{N} := \{0, 1, 2, \dots\}. \end{cases} \tag{1}$$

In (1), R^n denotes the n -dimensional real space, $R^{n \times m}$ is the set of all real $n \times m$ matrices, and I is the identity matrix. $x_s \in R^n$, $u_s \in R^{n_u}$, and $v_s \in R^{n_v}$ are, respectively, the system state, control input, and external disturbance; $\zeta_0 \in R^n$ is the initial condition. $\{\omega_s, s \in \mathbb{N}\}$ indicates the stochastic disturbance, which is a second-order process with $E\{\omega_{s+1}|\omega_t, t = 0, 1, \dots, s\} = 0$ and $E\{(\omega_{s+1})^2|\omega_t, t = 0, 1, \dots, s\} = 1$. We assume that ζ_0 and $\{\omega_s, s \in \mathbb{N}\}$ are mutually independent. $A, \bar{A}, B_1, \bar{B}_1, B_2, \bar{B}_2, C, \bar{C}, D_1, \bar{D}_1, D_2, \bar{D}_2, K$, and F are adequate dimensional constant matrices. Let \mathcal{F}_s be the σ algebra generated by $\{\omega_t, t = 0, 1, \dots, s\}$. $L^2(\Omega, R^k)$ represents the space of R^k -valued square integrable random vectors, while $l^2_\omega(\mathbb{N}, R^k)$ is the collection of nonanticipative square summable stochastic processes $z = \{z_s : z_s \in R^k, s \in \mathbb{N}\}$, with $z_s \in l^2(\Omega, R^k)$ being F_{s-1} -measurable, in which $F_{s-1} = \{\phi, \Omega\}$. The norm of $l^2_\omega(\mathbb{N}, R^k)$ is expressed by $\|z\|_{l^2_\omega(\mathbb{N}, R^k)} = (\sum_{s=0}^\infty E\|z_s\|^2)^{\frac{1}{2}}$. $H_n(R)$ denotes the set of all $n \times n$ real symmetric matrices. $H_n^{0,+}(R)$ is the subset of all nonnegative definite matrices of $H_n(R)$. We define $\mathbb{N} := \{0, 1, 2, \dots\}$. For $\forall S \in \mathbb{N}, N_S := \{0, 1, \dots, S\}$.

Next, we recall the concept of stability for the discrete-time mean-field system and present some results that will be needed in the following study. Consider the following unforced stochastic system:

$$\begin{cases} x_{s+1} = Ax_s + \bar{A}Ex_s + B_2v_s + \bar{B}_2Ev_s \\ \quad + [Cx_s + \bar{C}Ex_s + D_2v_s + \bar{D}_2Ev_s]\omega_s, \\ z_s = Kx_s, \\ x_0 = \zeta_0. \end{cases} \tag{2}$$

For simplicity, we denote (2) with $[A, \bar{A}, B_2, \bar{B}_2; C, \bar{C}, D_2, \bar{D}_2]$. Particularly, $[A, \bar{A}; C, \bar{C}]$ indicates $[A, \bar{A}, 0, 0; C, \bar{C}, 0, 0]$.

Definition 1 ([11]). $[A, \bar{A}; C, \bar{C}]$ is considered to be l^2 -asymptotically stable (l^2 -stable for short), if $\lim_{s \rightarrow \infty} E|x_s|^2 = 0$.

Taking the mathematical expectation in (2), the system equation of Ex_s can be written as

$$\begin{cases} Ex_{s+1} = (A + \bar{A})Ex_s + (B_2 + \bar{B}_2)Ev_s, \\ Ex_0 = E\zeta_0. \end{cases} \tag{3}$$

By a simple calculation, we obtain the following system equation of $x_s - Ex_s$:

$$\begin{cases} x_{s+1} - Ex_{s+1} = A(x_s - Ex_s) + B_2(v_s - Ev_s) + [C(x_s - Ex_s) + (C + \bar{C})Ex_s \\ \quad + D_2(v_s - Ev_s) + (D_2 + \bar{D}_2)Ev_s]\omega_s, \\ x_0 - Ex_0 = \zeta_0 - E\zeta_0. \end{cases} \tag{4}$$

Theorem 1 ([11]). *The following statements are equivalent:*

- (i) $[A, \bar{A}; C, \bar{C}]$ is l^2 -stable;
- (ii) For any $R, R + \bar{R} > 0$; the Lyapunov equations

$$\begin{cases} P = A^T P A + C^T P C + R, \\ Q = (A + \bar{A})^T Q (A + \bar{A}) + (C + \bar{C})^T P (C + \bar{C}) + R + \bar{R}, \end{cases} \tag{5}$$

admit a unique solution (P, Q) with $P, Q > 0$;

- (iii) There exist $R, R + \bar{R} > 0$, and the Lyapunov equations (5) to allow a unique solution (P, Q) with $P, Q > 0$.

When $v_s \equiv 0, s \in \mathbb{N}$ in (1), l^2 stabilizability can be formulated as follows:

Definition 2 ([11]). $[A, \bar{A}, B_1, \bar{B}_1; C, \bar{C}, D_1, \bar{D}_1]$ is said to be closed-loop l^2 -stabilizable if there exists a pair (U, \bar{U}) such that for any $\zeta_0 \in R^n$ and under control $\{u_s = Ux_s + \bar{U}Ex_s, s \in \mathbb{N}\}$, the closed-loop system

$$\begin{cases} x_{s+1} = [(A + B_1 U)x_s + (\bar{A} + B_1 \bar{U} + \bar{B}_1 U + \bar{B}_1 \bar{U})Ex_s] \\ \quad + [(C + D_1 U)x_s + (\bar{C} + D_1 \bar{U} + \bar{D}_1 U + \bar{D}_1 \bar{U})Ex_s]\omega_s, \\ x_0 = \zeta_0, \quad s \in \mathbb{N}, \end{cases}$$

is l^2 -stable. In this instance, we refer to (U, \bar{U}) as a closed-loop l^2 -stabilizer.

Below, we present the following linear operator:

$$\mathcal{L}(X) = AXA^T + CXC^T + \hat{C}X\hat{C}^T, \quad X \in H_{2n}(R),$$

where $H_{2n}(R) = \left\{ X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \mid X_1, X_2 \in H_n(R) \right\}$, and

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & A + \bar{A} \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathcal{C}} = \begin{bmatrix} 0 & C + \bar{C} \\ 0 & 0 \end{bmatrix}.$$

It is obvious that \mathcal{L} is a linear positive operator, i.e., if $X \in H_{2n}^+(R)$, we have $\mathcal{L}(X) \in H_{2n}^+(R)$, where $H_{2n}^+(R)$ indicates the set of nonnegative definite real matrices of $H_{2n}(R)$. For any $H_1, H_2 \in H_{2n}(R)$, define the inner product $\langle H_1, H_2 \rangle = Tr(H_1 H_2)$. Therefore, the adjoint operator of \mathcal{L} is expressed as follows:

$$\mathcal{L}^*(X) = \mathcal{A}^T X \mathcal{A} + \mathcal{C}^T X \mathcal{C} + \hat{\mathcal{C}}^T X \hat{\mathcal{C}}, \quad X \in H_{2n}(R).$$

Furthermore, the spectrum of \mathcal{L} is $\sigma(\mathcal{L}) := \{\lambda \mid \mathcal{L}(X) = \lambda X, X^T = X, X \neq 0\}$.

The following definition is about the exact detectability of discrete-time mean-field systems.

Definition 3 ([11]). In (2), let $v_s \equiv 0, s \in \mathbb{N}$. $[A, \bar{A}; C, \bar{C} \mid K]$ is considered to be exactly detectable if, for any $S \in \mathbb{N}$ and $s \in N_S, z_s \equiv 0$ (a.s.) implies that $\lim_{s \rightarrow \infty} E|x_s|^2 = 0$.

Let

$$Y_s = \begin{bmatrix} E[(x_s - Ex_s)(x_s - Ex_s)]^T & 0 \\ 0 & Ex_s Ex_s^T \end{bmatrix}.$$

It can be seen that $\lim_{s \rightarrow \infty} E|x_s|^2 = 0$ is equivalent to $\lim_{s \rightarrow \infty} Y_s = 0$. By calculation, one has that

$$E[z_s z_s^T] = E[K(x_s - Ex_s)(x_s - Ex_s)^T K^T + K(Ex_s)(Ex_s)^T K^T] = \mathcal{K} Y_s \mathcal{K}^T = 0$$

with $\mathcal{K} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$. Now, we introduce the following dynamics:

$$\begin{cases} Y_{s+1} = \mathcal{L}Y_s, \\ Y_0 = \begin{bmatrix} E[(\zeta_0 - E\zeta_0)(\zeta_0 - E\zeta_0)]^T & 0 \\ 0 & E\zeta_0 E\zeta_0^T \end{bmatrix}, \\ Z_s = \mathcal{K}Y_s \mathcal{K}^T. \end{cases} \quad (6)$$

The following lemma extends the result of Theorem 5.6 in [11], and the complete proof is omitted here because we could easily demonstrate the following result using Theorem 3 of [35].

Lemma 1. $[A, \bar{A}; C, \bar{C}|K]$ is exactly detectable if and only if, for each $X \geq 0$, $\mathcal{K}X\mathcal{K}^T \neq 0$, where $\mathcal{L}(X) = \lambda X$, and $\lambda \geq 1$.

The following lemma links the l^2 -stability with the exactly detectability of the uncontrolled system, and the detailed proof process can be referred to in [23].

Lemma 2. $[A, \bar{A}; C, \bar{C}]$ is l^2 stable if and only if $[A, \bar{A}; C, \bar{C}|K]$ is exactly detectable and the following generalized Lyapunov equation holds:

$$X = \mathcal{L}^*(X) + \mathcal{K}^T \mathcal{K} \quad (7)$$

admits a unique solution $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \geq 0$.

3. Stochastic-Bounded Real Lemma

In this section, we will derive a mean-field stochastic-bounded real lemma that plays a crucial role in the analysis of the stochastic disturbance attenuation problem and H_2/H_∞ control problems. By utilizing two coupled Riccati difference equations, this lemma establishes an equivalent condition guaranteeing the stability of a mean-field stochastic system with an H_∞ norm that is less than a given disturbance attenuation level γ .

Definition 4. In system (2), assume that the disturbance input is $v_s \in l_\omega^2(\mathbb{N}, R^{n_v})$ and the controlled output is $z_s \in l_\omega^2(\mathbb{N}, R^{n_z})$. The perturbed operator $L : l_\omega^2(\mathbb{N}, R^{n_v}) \rightarrow l_\omega^2(\mathbb{N}, R^{n_z})$ is defined by

$$Lv_s := Kx_{(s;0,v)}, \quad \forall v_s \in l_\omega^2(\mathbb{N}, R^{n_v}), \quad x_0 = 0$$

with its norm

$$\begin{aligned} \|L\| &= \sup_{v \in l_\omega^2(\mathbb{N}, R^{n_v}), v_s \neq 0, x_0=0} \frac{\|z_s\|_{l_\omega^2(\mathbb{N}, R^{n_z})}}{\|v_s\|_{l_\omega^2(\mathbb{N}, R^{n_v})}} \\ &= \sup_{v \in l_\omega^2(\mathbb{N}, R^{n_v}), v_s \neq 0, x_0=0} \frac{(\sum_{s=0}^{\infty} E\|Kx_s\|^2)^{\frac{1}{2}}}{(\sum_{s=0}^{\infty} E\|v_s\|^2)^{\frac{1}{2}}}. \end{aligned}$$

Remark 1. The infinite horizon H_∞ gain is given by the norm of L . However, it is rather hard to compute the H_∞ gain using the formula directly. Hence, the objective of H_∞ analysis is toward presenting a sufficient and necessary condition in terms of the solution to the CDREs to ensure that the norm of L is below a prescribed level $\gamma > 0$, namely, the bounded real lemma.

The following is the definition of the performance function for the infinite horizon H_∞ control:

$$J_\infty^{\gamma^2}(\zeta_0, v) = \sum_{s=0}^{\infty} E[\gamma^2 \|v_s\|^2 - \|z_s\|^2]. \quad (8)$$

Lemma 3. In system (2), assume that $S \in \mathbb{N}$ is given and that P_s and Q_s are arbitrary matrices in $H_n(\mathbb{R})$ with $s = 0, 1, 2, \dots, S + 1$. Thus, for any $\zeta_0 \in \mathbb{R}^n$ and $v_s \in l_w^2(N_S, \mathbb{R}^n)$, we have that

$$\begin{aligned} J_S^{\gamma^2}(\zeta_0, v) &= \sum_{s=0}^S E \left(\gamma^2 \|v_s\|^2 - \|z_s\|^2 \right) \\ &= \sum_{s=0}^S E \left\{ \begin{bmatrix} x_s - Ex_s \\ v_s - Ev_s \end{bmatrix}^T M_s^{\gamma^2}(P) \begin{bmatrix} x_s - Ex_s \\ v_s - Ev_s \end{bmatrix} \right\} \\ &\quad + \sum_{s=0}^S \begin{bmatrix} Ex_s \\ Ev_s \end{bmatrix}^T G_s^{\gamma^2}(P, Q) \begin{bmatrix} Ex_s \\ Ev_s \end{bmatrix} \\ &\quad + \zeta_0^T Q_0 \zeta_0 - (Ex_{S+1})^T Q_{S+1} (Ex_{S+1}) \\ &\quad - E \left[(x_{S+1} - Ex_{S+1})^T P_{S+1} (x_{S+1} - Ex_{S+1}) \right], \end{aligned} \quad (9)$$

where

$$\begin{aligned} \tilde{A} &= A + \bar{A}, \quad \tilde{B}_2 = B_2 + \bar{B}_2, \quad \tilde{C} = C + \bar{C}, \quad \tilde{D}_2 = D_2 + \bar{D}_2, \\ M_s^{\gamma^2}(P) &= \begin{bmatrix} -P_s + A^T P_{s+1} A + C^T P_{s+1} C - K^T K & A^T P_{s+1} B_2 + C^T P_{s+1} D_2 \\ B_2^T P_{s+1} A + D_2^T P_{s+1} C & \gamma^2 I + B_2^T P_{s+1} B_2 + D_2^T P_{s+1} D_2 \end{bmatrix}, \end{aligned}$$

and

$$G_s^{\gamma^2}(P, Q) = \begin{bmatrix} -Q_s + \tilde{A}^T Q_{s+1} \tilde{A} + \tilde{C}^T P_{s+1} \tilde{C} - K^T K & \tilde{A}^T Q_{s+1} \tilde{B}_2 + \tilde{C}^T P_{s+1} \tilde{D}_2 \\ \tilde{B}_2^T Q_{s+1} \tilde{A} + \tilde{D}_2^T P_{s+1} \tilde{C} & \gamma^2 I + \tilde{B}_2^T Q_{s+1} \tilde{B}_2 + \tilde{D}_2^T P_{s+1} \tilde{D}_2 \end{bmatrix}.$$

Proof of Lemma 3. This is identical to the proof of Lemma 3.1 of [34], so the details are omitted. \square

For convenience, we define some notations as follows:

$$\begin{aligned} M^{\gamma^2}(P) &= \begin{bmatrix} \mathbf{L}_s(P) & \mathbf{K}_s(P) \\ \mathbf{K}_s(P)^T & \mathbf{H}_s(P) \end{bmatrix}, \\ G^{\gamma^2}(P, Q) &= \begin{bmatrix} \tilde{\mathbf{L}}_s(P, Q) & \tilde{\mathbf{K}}_s(P, Q) \\ \tilde{\mathbf{K}}_s(P, Q)^T & \tilde{\mathbf{H}}_s(P, Q) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{L}_s(P) &= -P + A^T P A + C^T P C - K^T K, \\ \mathbf{K}_s(P) &= A^T P B_2 + C^T P D_2, \\ \mathbf{H}_s(P) &= \gamma^2 I + B_2^T P B_2 + D_2^T P D_2, \\ \tilde{\mathbf{L}}_s(P, Q) &= -Q + \tilde{A}^T Q \tilde{A} + \tilde{C}^T P \tilde{C} - K^T K, \\ \tilde{\mathbf{K}}_s(P, Q) &= \tilde{A}^T Q \tilde{B}_2 + \tilde{C}^T P \tilde{D}_2, \\ \tilde{\mathbf{H}}_s(P, Q) &= \gamma^2 I + \tilde{B}_2^T Q \tilde{B}_2 + \tilde{D}_2^T P \tilde{D}_2. \end{aligned}$$

Theorem 2. Assume that $[A, \bar{A}; C, \bar{C}]$ is l^2 stable. The following generalized difference Riccati equation (GDRE) is defined as follows:

$$\begin{cases} P = A^T P A + C^T P C - K^T K \\ \quad - (A^T P B_2 + C^T P D_2)(\gamma^2 I + B_2^T P B_2 + D_2^T P D_2)^{-1} [\dots]^T, \\ Q = \tilde{A}^T Q \tilde{A} + \tilde{C}^T P \tilde{C} - K^T K \\ \quad - (\tilde{A}^T Q \tilde{B}_2 + \tilde{C}^T P \tilde{D}_2)(\gamma^2 I + \tilde{B}_2^T Q \tilde{B}_2 + \tilde{D}_2^T P \tilde{D}_2)^{-1} [\dots]^T, \\ \mathbf{H}_s(P) = \gamma^2 I + B_2^T P B_2 + D_2^T P D_2 > 0, \\ \tilde{\mathbf{H}}_s(P, Q) = \gamma^2 I + \tilde{B}_2^T Q \tilde{B}_2 + \tilde{D}_2^T P \tilde{D}_2 > 0 \end{cases} \quad (10)$$

exists as a stabilizing solution (P, Q) with $P \leq 0$ and $Q \leq 0$ if and only if for a given $\gamma > 0$ and $\|L\| < \gamma$ in which $[\dots]$ denotes the M in MGM^T ; in the following, the symbol represents the same meaning.

Proof of Theorem 2. Necessity. Consider the following associated finite horizon GDRE:

$$\begin{cases} P_s = A^T P_{s+1} A + C^T P_{s+1} C - K^T K - \mathbf{K}_s(P_{s+1}) \mathbf{H}_s(P_{s+1})^{-1} [\dots]^T, \\ Q_s = \tilde{A}^T Q_{s+1} \tilde{A} + \tilde{C}^T P_{s+1} \tilde{C} - K^T K - \tilde{\mathbf{K}}_s(P_{s+1}, Q_{s+1}) \tilde{\mathbf{H}}_s(P_{s+1}, Q_{s+1})^{-1} [\dots]^T, \\ \mathbf{H}_s(P_{s+1}) = \gamma^2 I + B_2^T P_{s+1} B_2 + D_2^T P_{s+1} D_2 > 0, \\ \tilde{\mathbf{H}}_s(P_s + 1, Q_s + 1) = \gamma^2 I + \tilde{B}_2^T Q_{s+1} \tilde{B}_2 + \tilde{D}_2^T P_{s+1} \tilde{D}_2 > 0, \\ P_{S+1} = 0, Q_{S+1} = 0, s \in N_S, \end{cases} \quad (11)$$

and the corresponding cost functional

$$J_S^{\gamma^2}(\zeta_0, v) = \sum_{s=0}^S E\left(\gamma^2 \|v_s\|^2 - \|z_s\|^2\right) = \sum_{s=0}^S E\left(\gamma^2 v_s^T v_s - x_s^T K^T K x_s\right).$$

According to Lemma 3 and [13], we have that

$$\min_{v_s \in l_{\omega}^2(N_S, R^{n_v})} J_S^{\gamma^2}(\zeta_0, v) = \zeta_0^T Q_S(0) \zeta_0,$$

and the worst disturbance can be denoted by

$$v_S^*(s) = -\tilde{\mathbf{H}}_S(P_S)^{-1} \tilde{\mathbf{K}}_S(P_S)^T E x_S(s) - \mathbf{H}_S(P_S)^{-1} \mathbf{K}_S(P_S)^T [x_S(s) - E x_S(s)], \quad (12)$$

where $\{x_S(s), s = 0, 1, \dots, S, S+1\}$ is the solution to (11). From [7], one can obtain that $\lim_{S \rightarrow \infty} P_S(s) = P \leq 0$. Moreover, because $Q_S(s)$ is bounded from below and decreases as S increases, we can derive that

$$\lim_{S \rightarrow \infty} Q_S(s) = \lim_{S \rightarrow \infty} Q_{S-s}(0) = Q.$$

Therefore, GDRE (10) has a solution (P, Q) with $P \leq 0, Q \leq 0$. By replacing K with $K_{\delta} = \begin{bmatrix} K \\ \delta I \end{bmatrix}$ and z_s with $z_{s,\delta} = K_{\delta} x_s$, we obtain the associated perturbation operator L_{δ} and the cost functional

$$J_{S,\delta}^{\gamma^2}(\zeta_0, v) = \sum_{s=0}^S E\left(\gamma^2 \|v_s\|^2 - \|z_{s,\delta}\|^2\right) = \sum_{s=0}^S E\left(\gamma^2 v_s^T v_s - x_s^T K^T K x_s - \delta^2 x_s^T x_s\right).$$

Since (2) is l^2 -stabilizable, $x_s \in l_{\omega}^2(\mathbb{N}, R^n)$ can be concluded for every $v_s \in l_{\omega}^2(\mathbb{N}, R^{n_v})$; thus, $\|L_{\delta}\| < \gamma$ for sufficiently small $\delta > 0$. Similarly, it can be inferred that the GDRE

$$\begin{cases} P_s = A^T P_{s+1} A + C^T P_{s+1} C - \delta^2 I - K^T K \\ \quad - \mathbf{K}_s(P_s + 1) \mathbf{H}_s(P_s + 1)^{-1} [\dots]^T, \\ Q_s = \tilde{A}^T Q_{s+1} \tilde{A} + \tilde{C}^T P_{s+1} \tilde{C} - K^T K \\ \quad - \delta^2 I - \tilde{\mathbf{K}}_s(P_{s+1}, Q_{s+1}) \tilde{\mathbf{H}}_s(P_{s+1}, Q_{s+1})^{-1} [\dots]^T, \\ \mathbf{H}_s(P_{s+1}) = \gamma^2 I + B_2^T P_{s+1} B_2 + D_2^T P_{s+1} D_2 > 0, \\ \tilde{\mathbf{H}}_s(P_{s+1}, Q_{s+1}) = \gamma^2 I + \tilde{B}_2^T Q_{s+1} \tilde{B}_2 + \tilde{D}_2^T P_{s+1} \tilde{D}_2 > 0, \\ P_{S+1} = 0, Q_{S+1} = 0, s \in N_S \end{cases} \quad (13)$$

admits a unique solution $(P_{S,\delta}, Q_{S,\delta})$, and

$$\min_{v_s \in l_{\tilde{w}}^2(N_S, R^{nv})} J_{S,\delta}^{\gamma^2}(x_0, v) = \zeta_0^T Q_{S,\delta}(0) \zeta_0.$$

According to the time invariance of $P_S(s)$, $Q_S(s)$, $P_{S,\delta}(s)$, and $Q_{S,\delta}(s)$ on N_{S+1} , i.e., $P_S(s) = P_{S-s}(0)$, $Q_S(s) = Q_{S-s}(0)$, $P_{S,\delta}(s) = P_{S-s,\delta}(0)$, and $Q_{S,\delta}(s) = Q_{S-s,\delta}(0)$, $0 \leq s \leq S$, it can be inferred that, for any $\zeta_0 \in R^n$,

$$\begin{aligned} \zeta_0^T Q_{S,\delta}(s) \zeta_0 &= \zeta_0^T Q_{S-s,\delta}(0) \zeta_0 = \min_{v_s \in l_{\tilde{w}}^2(N_{S-s}, R^{nv})} J_{S-s,\delta}^{\gamma^2}(x_0, v) \\ &\leq \min_{v_s \in l_{\tilde{w}}^2(N_S, R^{nv})} J_{S-s}^{\gamma^2}(x_0, v) = \zeta_0^T Q_{S-s}(0) \zeta_0 = \zeta_0^T Q_S(s) \zeta_0. \end{aligned}$$

As ζ_0 is arbitrary, we have that $P_{S,\delta}(s) \leq P_S(s)$ and $Q_{S,\delta}(s) \leq Q_S(s)$, where $s \in N_S$. By applying the time invariance of $P_{S,\delta}(s)$ and $Q_{S,\delta}(s)$ on N_{S+1} , one can imply that

$$\lim_{S \rightarrow \infty} P_{S,\delta}(s) = \lim_{S \rightarrow \infty} P_{S-s,\delta}(0) = P_\delta, \quad \lim_{S \rightarrow \infty} Q_{S,\delta}(s) = \lim_{S \rightarrow \infty} Q_{S-s,\delta}(0) = Q_\delta,$$

$P \geq P_\delta$, and $Q \geq Q_\delta$. Moreover, (P, Q) and (P_δ, Q_δ) satisfy GDRE (10) and the following GDRE:

$$\begin{cases} P_\delta = A^T P_\delta A + C^T P_\delta C - K^T K \\ \quad - (A^T P_\delta B_2 + C^T P_\delta D_2)(\gamma^2 I_l + B_2^T P_\delta B_2 + D_2^T P_\delta D_2)^{-1} [\dots]^T, \\ Q_\delta = \tilde{A}^T Q_\delta \tilde{A} + \tilde{C}^T P_\delta \tilde{C} - K^T K \\ \quad - (\tilde{A}^T Q_\delta \tilde{B}_2 + \tilde{C}^T P_\delta \tilde{D}_2)(\gamma^2 I_l + \tilde{B}_2^T Q_\delta \tilde{B}_2 + \tilde{D}_2^T P_\delta \tilde{D}_2)^{-1} [\dots]^T, \\ \mathbf{H}_s(P) = \gamma^2 I + B_2^T P_\delta B_2 + D_2^T P_\delta D_2 > 0, \\ \tilde{\mathbf{H}}_s(P_\delta, Q_\delta) = \gamma^2 I + \tilde{B}_2^T Q_\delta \tilde{B}_2 + \tilde{D}_2^T P_\delta \tilde{D}_2 > 0, \end{cases} \quad (14)$$

respectively. Next, we shall prove that $[A + B_2 V, \tilde{A} + B_2 \tilde{V} + \tilde{B}_2 V + \tilde{B}_2 \tilde{V}; C + D_2 V, \tilde{C} + D_2 \tilde{V} + \tilde{D}_2 V + \tilde{D}_2 \tilde{V}]$ is l^2 stable. To this end, it should be noted that that (10) and (14) can be rewritten as

$$\begin{cases} P = (A + B_2 V)^T P (A + B_2 V) + (C + D_2 V)^T P (C + D_2 V) - K^T K + \gamma^2 V^T V, \\ Q = (\tilde{A} + \tilde{B}_2 \tilde{V})^T Q (\tilde{A} + \tilde{B}_2 \tilde{V}) + (\tilde{C} + \tilde{D}_2 \tilde{V})^T P (\tilde{C} + \tilde{D}_2 \tilde{V}) - K^T K - \gamma^2 \tilde{V}^T \tilde{V}, \end{cases} \quad (15)$$

and

$$\begin{cases} P_\delta = (A + B_2 V)^T P_\delta (A + B_2 V) + (C + D_2 V)^T P_\delta (C + D_2 V) - K^T K + \gamma^2 V_\delta^T V_\delta, \\ Q_\delta = (\tilde{A} + \tilde{B}_2 \tilde{V})^T Q_\delta (\tilde{A} + \tilde{B}_2 \tilde{V}) + (\tilde{C} + \tilde{D}_2 \tilde{V})^T P_\delta (\tilde{C} + \tilde{D}_2 \tilde{V}) - K^T K - \gamma^2 \tilde{V}_\delta^T \tilde{V}_\delta. \end{cases} \quad (16)$$

By subtracting (16) from (15), one obtains

$$\begin{aligned} &-(P - P_\delta) + (A + B_2 V)^T (P - P_\delta) (A + B_2 V) + (C + D_2 V)^T (P - P_\delta) (C + D_2 V) \\ &= -(V - V_\delta)^T (\gamma^2 I + B_2^T P_\delta B_2 + D_2^T P_\delta D_2) (V - V_\delta) - \delta^2 I. \end{aligned} \quad (17)$$

We propose that $P - P_\delta$ must be strictly positive. If not, there exists $\zeta_0 \neq 0$ such that $(P - P_\delta)\zeta_0 = 0$ with $P \geq P_\delta$. Multiplying the left side by ζ_0^T and the right side by ζ_0 , (17) yields

$$\begin{aligned} 0 &\leq \zeta_0^T (A + B_2 V)^T (P - P_\delta) (A + B_2 V) \zeta_0 + \zeta_0^T (C + D_2 V)^T (P - P_\delta) (C + D_2 V) \zeta_0 \\ &= -\zeta_0^T (V - V_\delta)^T (\gamma^2 I + B_2^T P_\delta B_2 + D_2^T P_\delta D_2) (V - V_\delta) \zeta_0 - \delta^2 \zeta_0^T \zeta_0 < 0, \end{aligned}$$

which is a contradiction. Hence, $P - P_\delta > 0$. And according to a similar discussion, it can be proven that $Q - Q_\delta > 0$. Let $R_1 = P - P_\delta$ and $R_2 = Q - Q_\delta$. Since

$$\begin{cases} -R_1 + (A + B_2V)^T R_1 (A + B_2V) + (C + D_2V)^T R_1 (C + D_2V) < 0, \\ -R_2 + (\bar{A} + \bar{B}_2\bar{V})^T R_1 (\bar{A} + \bar{B}_2\bar{V}) + (\bar{C} + \bar{D}_2\bar{V})^T R_2 (\bar{C} + \bar{D}_2\bar{V}) < 0, \end{cases}$$

then, by Theorem 1, we deduced that $[A + B_2V, \bar{A} + B_2\bar{V} + \bar{B}_2V + \bar{B}_2\bar{V}; C + D_2V, \bar{C} + D_2\bar{V} + \bar{D}_2V + \bar{D}_2\bar{V}]$ is l^2 -stable. The necessity is proven.

Sufficiency. Considering (11), (13), and Lemma 3, we have that

$$\begin{aligned} J_S^{\gamma^2}(\zeta_0, v) &= \sum_{s=0}^S E(\gamma^2 \|v_s\|^2 - \|z_s\|^2) \\ &= \sum_{s=0}^S E \left\{ \begin{bmatrix} x_s - Ex_s \\ v_s - Ev_s \end{bmatrix}^T M^{\gamma^2}(P) \begin{bmatrix} x_s - Ex_s \\ v_s - Ev_s \end{bmatrix} \right\} + \sum_{s=0}^S \begin{bmatrix} Ex_s \\ Ev_s \end{bmatrix}^T G^{\gamma^2}(P, Q) \begin{bmatrix} Ex_s \\ Ev_s \end{bmatrix} \\ &\quad + \zeta_0^T Q \zeta_0 - [Ex_{S+1}]^T Q [Ex_{S+1}] - E[(x_{S+1} - Ex_{S+1})^T P (x_{S+1} - Ex_{S+1})] \\ &= \zeta_0^T Q \zeta_0 - [Ex_{S+1}]^T Q [Ex_{S+1}] + \sum_{s=0}^{\infty} [Ev_s - Ev_s^*]^T \tilde{H}_s(P, Q) [Ev_s - Ev_s^*] \\ &\quad + \sum_{s=0}^{\infty} E[(v_s - Ev_s) - (v_s^* - Ev_s^*)]^T H_s(P) [(v_s - Ev_s) - (v_s^* - Ev_s^*)], \end{aligned}$$

where

$$v_s^* - Ev_s^* = -H_s^{-1} K_s^T (x_s - Ex_s), \quad Ev_s^* = -\tilde{H}_s(P, Q)^{-1} \tilde{K}_s(P, Q)^T Ex_s.$$

Because $[A, \bar{A}; C, \bar{C}]$ is l^2 -stable, it can be inferred that $x_s \in l_\omega^2(\mathbb{N}, R^n)$ when $v_s \in l_\omega^2(\mathbb{N}, R^{n_v})$. Letting $S \rightarrow \infty$, then $\lim_{S \rightarrow \infty} E|x(S)|^2 = 0$ is obtained. So

$$\begin{aligned} J_\infty^{\gamma^2}(\zeta_0, v) &= \zeta_0^T Q \zeta_0 + \sum_{s=0}^S [Ev_s - Ev_s^*]^T \tilde{H}_s(P, Q) [Ev_s - Ev_s^*] \\ &\quad + \sum_{s=0}^S E[(v_s - Ev_s) - (v_s^* - Ev_s^*)]^T H_s(P) [(v_s - Ev_s) - (v_s^* - Ev_s^*)] \\ &\geq J_\infty^{\gamma^2}(\zeta_0, v = v^*) = \zeta_0^T Q \zeta_0. \end{aligned}$$

When $\zeta_0 = 0$, $\|L\| \leq \gamma$ can be derived from $J_\infty^{\gamma^2}(0, v) \geq 0$. Next, in order to prove that $\|L\| < \gamma$, we define the following operators:

$$\begin{aligned} L_1 : l_\omega^2(\mathbb{N}, R^{n_v}) &\rightarrow l_\omega^2(\mathbb{N}, R^{n_v}), \quad L_1(v_s - Ev_s) = (v_s - Ev_s) - (v_s^* - Ev_s^*), \\ \bar{L}_1 : l_\omega^2(\mathbb{N}, R^{n_v}) &\rightarrow l_\omega^2(\mathbb{N}, R^{n_v}), \quad \bar{L}_1(Ev_s) = Ev_s - Ev_s^* \end{aligned}$$

According to (3) and (4), it can be seen that

$$Ev_s - Ev_s^* = Ev_s + \tilde{H}_s(P, Q)^{-1} \tilde{K}_s(P, Q)^T Ex_s$$

and

$$(v_s - Ev_s) - (v_s^* - Ev_s^*) = (v_s - Ev_s) + H_s(P)^{-1} K_s(P)^T (x_s - Ex_s).$$

Then, L_1^{-1} and \bar{L}_1^{-1} exist. Therefore, it holds that

$$\begin{cases} Ex_{s+1} = [(A + \bar{A}) - (B_2 + \bar{B}_2)\tilde{H}_s(P, Q)^{-1} \tilde{K}_s(P, Q)^T Ex_s] Ex_s + (B_2 + \bar{B}_2)(Ev_s - Ev_s^*), \\ Ex_0 = 0, \end{cases}$$

with $Ev_s = -\tilde{\mathbf{H}}_s(P, Q)^{-1}\tilde{\mathbf{K}}_s(P, Q)^T Ex_s + (Ev_s - Ev_s^*)$, and

$$\begin{cases} x_{s+1} - Ex_{s+1} = [A - B_2\mathbf{H}_s(P)^{-1}\mathbf{K}_s(P)^T](x_s - Ex_s) + B_2[(v_s - Ev_s) - (v_s^* - Ev_s^*)] \\ \quad + \{[C - D_2\mathbf{H}_s(P)^{-1}\mathbf{K}_s(P)^T](x_s - Ex_s) + D_2[(v_s - Ev_s) - (v_s^* - Ev_s^*)] \\ \quad + [\tilde{C} - \tilde{D}_2\tilde{\mathbf{H}}_s(P, Q)^{-1}\tilde{\mathbf{K}}_s(P, Q)^T]Ex_s + \tilde{D}_2[Ev_s - Ev_s^*]\}\omega_s, \\ x_0 - Ex_0 = 0, \end{cases}$$

with $v_s - Ev_s = -\mathbf{H}_s(P)^{-1}\mathbf{K}_s(P)^T(x_s - Ev_s) + [(v_s - Ev_s) - (v_s^* - Ev_s^*)]$. Let us assume that $\mathbf{H}_s(P) \geq \epsilon I$ and $\tilde{\mathbf{H}}_s(P, Q) \geq \epsilon I$ for some $\epsilon > 0$. Since $\mathbf{H}_s(P) > 0$, $\tilde{\mathbf{H}}_s(P, Q) > 0$. Because L_1^{-1} and \tilde{L}_1^{-1} exist, there exists a constant $c > 0$ such that

$$\begin{aligned} J_\infty^{\gamma^2}(0, v) &= \sum_{s=0}^{\infty} [Ev_s - Ev_s^*]^T \tilde{\mathbf{H}}_s(P, Q) [Ev_s - Ev_s^*] \\ &\quad + \sum_{s=0}^{\infty} E[(v_s - Ev_s) - (v_s^* - Ev_s^*)]^T \mathbf{H}_s(P) [(v_s - Ev_s) - (v_s^* - Ev_s^*)] \\ &\geq \epsilon [\|L_1(v_s - Ev_s)\|_{l_\omega^2(\mathbb{N}, R^{n_v})}^2 + \|\tilde{L}_1(Ev_s)\|_{l_\omega^2(\mathbb{N}, R^{n_v})}^2] \\ &\geq c [\|v_s - Ev_s\|_{l_\omega^2(\mathbb{N}, R^{n_v})}^2 + \|Ev_s\|_{l_\omega^2(\mathbb{N}, R^{n_v})}^2] \\ &\geq c \|v_s\|_{l_\omega^2(\mathbb{N}, R^{n_v})}^2 > 0, \end{aligned}$$

which shows that $\|L\| < \gamma$. The proof is completed. \square

4. Infinite Horizon Mean-Field Stochastic H_2/H_∞ Control

In this section, for (1), we shall handle the infinite horizon H_2/H_∞ control problem. The design objective is to find a controller that not only can eliminate the effect of disturbance, but also can minimize the output energy when the worst case disturbance is implemented, as well as can ensure the internal stability. Hence, the mixed H_2/H_∞ control design, as one of multiobjective design methods, is better suited to the needs of engineering practice.

Given the disturbance attenuation level of $\gamma > 0$, the corresponding performances are characterized by

$$J_1^\infty(u, v) = \sum_{s=0}^{\infty} E[\gamma^2 \|v_s\|^2 - \|z_s\|^2] \quad (18)$$

and

$$J_2^\infty(u, v) = \sum_{s=0}^{\infty} E\|z_s\|^2. \quad (19)$$

The infinite horizon H_2/H_∞ control problem of system (1) can be expressed as follows. Considering $\gamma > 0$, find a state feedback control $u_s^* \in l_\omega^2(\mathbb{N}, R^{n_u})$, such that the following are achieved:

- (i) u_s^* stabilizes system (1), i.e., when $v_s \equiv 0$, $u_s = u_s^*$; the trajectory of (1) with any initial value $x_0 = \zeta_0$ satisfies $\lim_{s \rightarrow \infty} E|x_s|^2 = 0$;
- (ii) $\|L_{u^*}\| = \sup_{v_s \in l_\omega^2(\mathbb{N}, R^{n_v}), v_s \neq 0, x_0=0} \frac{(\sum_{s=0}^{\infty} E\|Kx_s\|^2 + \|u^*(s)\|^2)^{\frac{1}{2}}}{(\sum_{s=0}^{\infty} E\|v_s\|^2)^{\frac{1}{2}}} < \gamma$;
- (iii) When the worst case disturbance $v_s^* \in l_\omega^2(\mathbb{N}, R^{n_v})$ is applied to (1), u_s^* minimizes the output energy to $J_2^\infty(u, v^*) = \sum_{s=0}^{\infty} E\|z_s\|^2$.

The infinite horizon H_2/H_∞ control problem for the stochastic mean-field system (1) is said to be solvable if the previously mentioned (u^*, v^*) exist. Clearly, (u^*, v^*) are the Nash equilibria of (18) and (19), which satisfy

$$J_1^\infty(u^*, v^*) \leq J_1^\infty(u^*, v), J_2^\infty(u^*, v^*) \leq J_2^\infty(u, v^*).$$

The following four coupled matrix-valued equations are introduced before the main result is presented:

$$\begin{cases} P_1 = (A + B_1U)^T P_1(A + B_1U) + (C + D_1U)^T P_1(C + D_1U) - K^T K - U^T U \\ \quad - \mathbf{K}_u(P_1) \mathbf{H}_u(P_1)^{-1} \mathbf{K}_u(P_1)^T, \\ Q_1 = (\tilde{A} + \tilde{B}_1 \tilde{U})^T Q_1(\tilde{A} + \tilde{B}_1 \tilde{U}) + (\tilde{C} + \tilde{D}_1 \tilde{U})^T P_1(\tilde{C} + \tilde{D}_1 \tilde{U}) - K^T K - \tilde{U}^T \tilde{U} \\ \quad - \tilde{\mathbf{K}}_u(P_1, Q_1) \tilde{\mathbf{H}}_u(P_1, Q_1)^{-1} \tilde{\mathbf{K}}_u(P_1, Q_1)^T, \\ \mathbf{H}_u(P_1) > 0, \tilde{\mathbf{H}}_u(P_1, Q_1) > 0, s \in \mathbb{N}, \\ V = -\mathbf{H}_u(P_1)^{-1} \mathbf{K}_u(P_1)^T, \\ \tilde{V} = V + \tilde{V} = -\tilde{\mathbf{H}}_u(P_1, Q_1)^{-1} \tilde{\mathbf{K}}_u(P_1, Q_1)^T, \end{cases} \quad (20)$$

$$\begin{cases} P_2 = (A + B_2V)^T P_2(A + B_2V) + (C + D_2V)^T P_2(C + D_2V) + K^T K + I \\ \quad - \mathbf{K}_v(P_2) \mathbf{H}_v(P_2)^{-1} \mathbf{K}_v(P_2)^T, \\ Q_2 = (\tilde{A} + \tilde{B}_2 \tilde{V})^T Q_2(\tilde{A} + \tilde{B}_2 \tilde{V}) + (\tilde{C} + \tilde{D}_2 \tilde{V})^T P_2(\tilde{C} + \tilde{D}_2 \tilde{V}) + K^T K + I \\ \quad - \tilde{\mathbf{K}}_v(P_2, Q_2) \tilde{\mathbf{H}}_v(P_2, Q_2)^{-1} \tilde{\mathbf{K}}_v(P_2, Q_2)^T, \\ \mathbf{H}_v(P_2) > 0, \tilde{\mathbf{H}}_v(P_2, Q_2) > 0, s \in \mathbb{N} \\ U = -\mathbf{H}_v(P_2)^{-1} \mathbf{K}_v(P_2)^T, \\ \tilde{U} = U + \tilde{U} = -\tilde{\mathbf{H}}_v(P_2, Q_2)^{-1} \tilde{\mathbf{K}}_v(P_2, Q_2)^T, \end{cases} \quad (21)$$

where

$$\begin{cases} \mathbf{H}_u(P_1) = \gamma^2 I + B_2^T P_1 B_2 + D_2^T P_1 D_2, \\ \mathbf{K}_u(P_1) = (A + B_1U)^T P_1 B_2 + (C + D_1U)^T P_1 D_2, \\ \tilde{\mathbf{H}}_u(P_1, Q_1) = \gamma^2 I + \tilde{B}_2^T Q_1 \tilde{B}_2 + \tilde{D}_2^T P_1 \tilde{D}_2, \\ \tilde{\mathbf{K}}_u(P_1, Q_1) = (\tilde{A} + \tilde{B}_1 \tilde{U})^T Q_1 \tilde{B}_2 + (\tilde{C} + \tilde{D}_1 \tilde{U})^T P_1 \tilde{D}_2, \end{cases} \quad (22)$$

and

$$\begin{cases} \mathbf{H}_v(P_2) = I + B_1^T P_2 B_1 + D_1^T P_2 D_1, \\ \mathbf{K}_v(P_2) = (A + B_2V)^T P_2 B_1 + (C + D_2V)^T P_2 D_1, \\ \tilde{\mathbf{H}}_v(P_2, Q_2) = I + \tilde{B}_1^T Q_2 \tilde{B}_1 + \tilde{D}_1^T P_2 \tilde{D}_1, \\ \tilde{\mathbf{K}}_v(P_2, Q_2) = (\tilde{A} + \tilde{B}_2 \tilde{V})^T Q_2 \tilde{B}_1 + (\tilde{C} + \tilde{D}_2 \tilde{V})^T P_2 \tilde{D}_1. \end{cases} \quad (23)$$

The following lemma will be used in the proof of our main result.

Lemma 4. Define the following matrices:

$$\tilde{\mathcal{K}}_1 = \begin{bmatrix} K & 0 \\ U & 0 \\ I & 0 \end{bmatrix}, \tilde{\mathcal{K}}_2 = \begin{bmatrix} K & U + \tilde{U} & I \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\tilde{\mathcal{K}}_2 = \begin{bmatrix} K & 0 \\ U & 0 \\ \mathbf{H}_u(P_1)^{-\frac{1}{2}} \mathbf{K}_u(P_1)^T & 0 \end{bmatrix}, \tilde{\mathcal{K}}_2 = \begin{bmatrix} K & U + \tilde{U} & \tilde{\mathbf{H}}_u(P_1, Q_1)^{-\frac{1}{2}} \tilde{\mathbf{K}}_u(P_1, Q_1)^T \\ 0 & 0 & 0 \end{bmatrix}.$$

Let

$$\mathcal{K}_1 = \begin{bmatrix} \tilde{\mathcal{K}}_1 & 0 \\ 0 & \tilde{\mathcal{K}}_1 \end{bmatrix}, \mathcal{K}_2 = \begin{bmatrix} \tilde{\mathcal{K}}_2 & 0 \\ 0 & \tilde{\mathcal{K}}_2 \end{bmatrix}.$$

Then, the following statements hold:

- (i) If $[A, \bar{A}; C, \bar{C}|K]$ is exactly detectable, so is $[A + B_1U, \bar{A} + \bar{B}_1\bar{U} + B_1\bar{U}; C + D_1U, \bar{C} + \bar{D}_1\bar{U} + D_1\bar{U}|K_2]$;
- (ii) If $[A + B_2V, \bar{A} + \bar{D}_1\bar{V} + D_1\bar{V}; C + D_2V, \bar{C} + \bar{D}_2\bar{V} + D_2\bar{V}|K]$ is exactly detectable, so is $[A + B_1U + B_2V, \bar{A} + B_1\bar{U} + B_2\bar{V} + \bar{B}_1\bar{U} + \bar{B}_2\bar{V}; C + D_1U + D_2V, \bar{C} + D_1\bar{U} + D_2\bar{V} + \bar{D}_1\bar{U} + \bar{D}_2\bar{V}|K_1]$.

Proof of Lemma 4. Suppose that $[A, \bar{A}; C, \bar{C}|K]$ is exactly detectable, but $[A + B_1U, \bar{A} + \bar{B}_1\bar{U} + B_1\bar{U}; C + D_1U, \bar{C} + \bar{D}_1\bar{U} + D_1\bar{U}|K_2]$ is not. According to Lemma 3, there exists $X \geq 0$ such that $K_2XK_2^T = 0$. Since $K_2XK_2^T = 0$ means that $KXK^T = 0$, this contradicts that $[A, \bar{A}; C, \bar{C}|K]$ is exactly detectable. Along the same line of the proof of (i), (ii) could be demonstrated. \square

Theorem 3. For (1), suppose that the above coupled matrix-valued Equations (20)–(23) have a solution $(P_1, Q_1; P_2, Q_2; U, \bar{U}; V, \bar{V})$ with $P_1 < 0, Q_1 < 0, P_2 > 0$, and $Q_2 > 0$. If $[A, \bar{A}; C, \bar{C}|K]$ and $[A + B_2V, \bar{A} + \bar{D}_1\bar{V} + D_1\bar{V}; C + D_2V, \bar{C} + \bar{D}_2\bar{V} + D_2\bar{V}|K]$ are exactly detectable, then the H_2/H_∞ control problem has a pair of solutions (u_s^*, v_s^*) with

$$u_s^* = Ux_s + \bar{U}Ex_s, v_s^* = Vx_s + \bar{V}Ex_s.$$

Proof of Theorem 3. We first define the following matrices:

$$\begin{aligned} \hat{P}_1 &= \begin{bmatrix} P_1 & 0 \\ 0 & Q_1 \end{bmatrix}, \mathcal{A}_1 = \begin{bmatrix} A + B_1U & 0 \\ 0 & \bar{A} + \bar{B}_1\bar{U} \end{bmatrix}, \mathcal{C}_1 = \begin{bmatrix} C + D_1U & 0 \\ 0 & 0 \end{bmatrix}, \\ \hat{\mathcal{C}}_1 &= \begin{bmatrix} 0 & \bar{C} + \bar{D}_1\bar{U} \\ 0 & 0 \end{bmatrix}, \mathbf{U} = \begin{bmatrix} U & 0 \\ 0 & \bar{U} \end{bmatrix}, \mathbf{H}_1 = \begin{bmatrix} \mathbf{H}_u(P_1)^{-\frac{1}{2}}\mathbf{K}_u(P_1)^T & 0 \\ 0 & \bar{\mathbf{H}}_u(P_1)^{-\frac{1}{2}}\bar{\mathbf{K}}_u(P_1)^T \end{bmatrix}, \\ \hat{P}_2 &= \begin{bmatrix} P_2 & 0 \\ 0 & Q_2 \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} A + B_1U + B_2V & 0 \\ 0 & \bar{A} + \bar{B}_1\bar{U} + \bar{B}_2\bar{V} \end{bmatrix}, \mathcal{C}_2 = \begin{bmatrix} C + D_1U + D_2V & 0 \\ 0 & 0 \end{bmatrix}, \\ \hat{\mathcal{C}}_2 &= \begin{bmatrix} 0 & \bar{C} + \bar{D}_1\bar{U} + \bar{D}_2\bar{V} \\ 0 & 0 \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} \mathbf{H}_v(P_2)^{-\frac{1}{2}}\mathbf{K}_v(P_2)^T & 0 \\ 0 & \bar{\mathbf{H}}_u(P_1)^{-\frac{1}{2}}\bar{\mathbf{K}}_u(P_1)^T \end{bmatrix}. \end{aligned}$$

Notice that (20) and (22) can be written as

$$\hat{P}_1 = \mathcal{A}_1^T \hat{P}_1 \mathcal{A}_1 + \mathcal{C}_1^T \hat{P}_1 \mathcal{C}_1 + \hat{\mathcal{C}}_1^T \hat{P}_1 \hat{\mathcal{C}}_1 - \mathcal{K}_2^T \mathcal{K}_2 \quad (26)$$

and

$$\hat{P}_2 = \mathcal{A}_2^T \hat{P}_2 \mathcal{A}_2 + \mathcal{C}_2^T \hat{P}_2 \mathcal{C}_2 + \hat{\mathcal{C}}_2^T \hat{P}_2 \hat{\mathcal{C}}_2 + \mathcal{K}_1^T \mathcal{K}_1 \quad (27)$$

respectively, where \mathcal{K}_1 and \mathcal{K}_2 are defined in Lemma 4. Since $[A + B_2V, \bar{A} + \bar{D}_1\bar{V} + D_1\bar{V}; C + D_2V, \bar{C} + \bar{D}_2\bar{V} + D_2\bar{V}|K]$ is exactly detectable, according to Lemma 4, $[A + B_1U + B_2V, \bar{A} + B_1\bar{U} + B_2\bar{V} + \bar{B}_1\bar{U} + \bar{B}_2\bar{V}; C + D_1U + D_2V, \bar{C} + D_1\bar{U} + D_2\bar{V} + \bar{D}_1\bar{U} + \bar{D}_2\bar{V}|K_1]$ is also exactly detectable, which implies from Lemma 3 and (27) that $[A + B_1U + B_2V, \bar{A} + B_1\bar{U} + B_2\bar{V} + \bar{B}_1\bar{U} + \bar{B}_2\bar{V}; C + D_1U + D_2V, \bar{C} + D_1\bar{U} + D_2\bar{V} + \bar{D}_1\bar{U} + \bar{D}_2\bar{V}]$ is l^2 -stable. Hence,

$$u_s^* = Ux_s + \bar{U}Ex_s \in l_\omega^2(\mathbb{N}, R^{n_u}), v_s^* = Vx_s + \bar{V}Ex_s \in l_\omega^2(\mathbb{N}, R^{n_v}).$$

From Lemma 3, Lemma 4, and (20), $[A + B_1U, \bar{A} + B_1\bar{U} + \bar{B}_1\bar{U}; C + D_1U, \bar{C} + D_1\bar{U} + \bar{D}_1\bar{U}]$ is l^2 -stable, i.e., $u_s^* = Ux_s + \bar{U}Ex_s$ stabilizes system (1).

Substituting $u_s = u_s^* = Ux_s + \bar{U}Ex_s$ into (1), one yields that

$$\begin{cases} x_{s+1} = (A + B_1U)x_s + (\bar{A} + B_1\bar{U} + \bar{B}_1\bar{U})Ex_s + B_2v_s + \bar{B}_2Ev_s \\ \quad + [(C + D_1U)x_s + (\bar{C} + D_1\bar{U} + \bar{D}_1\bar{U})Ex_s + D_2v_s + \bar{D}_2Ev_s]\omega_s, \\ z_s = \begin{bmatrix} Kx_s \\ F(Ux_s + \bar{U}Ex_s) \end{bmatrix}, F^T F = I, \\ x(0) = \zeta_0 \in R^n. \end{cases} \tag{28}$$

Because $[A + B_1U, \bar{A} + B_1\bar{U} + \bar{B}_1\bar{U}; C + D_1U, \bar{C} + D_1\bar{U} + \bar{D}_1\bar{U}]$ is l^2 stable, for every $v_s \in l^2_\omega(\mathbb{N}, R^{n_v})$, we have that $x_s \in l^2_\omega(\mathbb{N}, R^n)$. Therefore, it is directly obtained from Theorem 2 that $\|L_{u^*}\| < \gamma$.

Next, we prove that v^* exists and $v_s^* = Vx_s + \bar{V}Ex_s$. Being identical to the proof of Lemma 2 and by taking (20), (28), and Lemma 3 into consideration, it is clear that

$$\begin{aligned} J_1^S(u^*, v) &= \sum_{s=0}^S E(\gamma^2 \|v_s\|^2 - \|z_s\|^2) \\ &= \sum_{s=0}^S E \left\{ \begin{bmatrix} x_s - Ex_s \\ v_s - Ev_s \end{bmatrix}^T M_1^{\gamma^2}(P_1) \begin{bmatrix} x_s - Ex_s \\ v_s - Ev_s \end{bmatrix} \right\} + \sum_{s=0}^S \begin{bmatrix} Ex_s \\ Ev_s \end{bmatrix}^T G_1^{\gamma^2}(P_1, Q_1) \begin{bmatrix} Ex_s \\ Ev_s \end{bmatrix} \\ &\quad + \zeta_0^T Q_1 \zeta_0 - [Ex_{S+1}]^T Q_1 [Ex_{S+1}] - E[(x_{S+1} - Ex_{S+1})^T P_1 (x_{S+1} - Ex_{S+1})] \\ &= \sum_{s=0}^S E[(v_s - Ev_s) - (v_s^* - Ev_s^*)]^T \mathbf{H}_u(P) [(v_s - Ev_s) - (v_s^* - Ev_s^*)] \\ &\quad + \zeta_0^T Q_1 \zeta_0 + \sum_{s=0}^S [Ev_s - Ev_s^*]^T \tilde{\mathbf{H}}_u(P_1, Q_1) [Ev_s - Ev_s^*], \end{aligned}$$

where

$$M_1^{\gamma^2}(P_1) = \begin{bmatrix} \mathbf{L}_u^1(P_1) & \mathbf{K}_u^1(P_1) \\ \mathbf{K}_u(P_1)^T & \mathbf{H}_u(P_1) \end{bmatrix}, G_1^{\gamma^2}(P_1, Q_1) = \begin{bmatrix} \tilde{\mathbf{L}}_u(P_1, Q_1) & \tilde{\mathbf{K}}_u(P_1, Q_1) \\ \tilde{\mathbf{K}}_u(P_1, Q_1)^T & \tilde{\mathbf{H}}_u(P_1, Q_1) \end{bmatrix},$$

with

$$\begin{aligned} \mathbf{L}_u^1(P_1) &= -P_1 + (A + B_1U)^T P_1 (A + B_1U) + (C + D_1U)^T P_1 (C + D_1U) - K^T K - U^T U, \\ \tilde{\mathbf{L}}_u(P_1, Q_1) &= -Q_1 + (\bar{A} + \bar{B}_1\bar{U})^T Q_1 (\bar{A} + \bar{B}_1\bar{U}) + (\bar{C} + \bar{D}_1\bar{U})^T Q_1 (\bar{C} + \bar{D}_1\bar{U}) - K^T K - \bar{U}^T \bar{U}. \end{aligned}$$

Since $\lim_{S \rightarrow \infty} E\|x_S\|^2 = 0$, one can infer that

$$\begin{aligned} J_1^\infty(u^*, v) &= \sum_{s=0}^\infty E[(v_s - Ev_s) - (v_s^* - Ev_s^*)]^T \mathbf{H}_u(P) [(v_s - Ev_s) - (v_s^* - Ev_s^*)] \\ &\quad + \zeta_0^T Q_1 \zeta_0 + \sum_{s=0}^\infty [Ev_s - Ev_s^*]^T \tilde{\mathbf{H}}_u(P_1, Q_1) [Ev_s - Ev_s^*] \\ &\geq J_1^\infty(u^*, v^*) = \zeta_0^T Q_1 \zeta_0. \end{aligned}$$

From the above, it can be seen that v_s^* is the worst case disturbance related to u^* that exists and is denoted by $v_s^* = Vx_s + \bar{V}Ex_s \in l^2_\omega(\mathbb{N}, R^{n_v})$. When $v_s = v_s^*$ is applied to system (1), we obtain

$$\begin{cases} x_{s+1} = (A + B_2V)x_s + (\bar{A} + B_2\bar{V} + \bar{B}_2\bar{V})Ex_s + B_1u_s + \bar{B}_1Eu_s \\ \quad + [(C + D_2V)x_s + (\bar{C} + D_2\bar{V} + \bar{D}_2\bar{V})Ex_s + D_1u_s + \bar{D}_1Eu_s]\omega_s, \\ z_s = \begin{bmatrix} Kx_s \\ Fu_s \end{bmatrix}, F^T F = I, \\ x(0) = \zeta_0 \in R^n. \end{cases} \tag{29}$$

Due to the fact that $[A + B_1U + B_2V, \bar{A} + B_1\bar{U} + B_2\bar{V} + \bar{B}_1\bar{U} + \bar{B}_2\bar{V}; C + D_1U + D_2V, \bar{C} + D_1\bar{U} + D_2\bar{V} + \bar{D}_1\bar{U} + \bar{D}_2\bar{V}]$ is l^2 stable, $[A + B_2V, \bar{A} + \bar{B}_2\bar{V} + B_2\bar{V}, B_1, \bar{B}_1; C + D_2V, \bar{C} + \bar{D}_2\bar{V} + D_2\bar{V}, D_1, \bar{D}_1]$ is l^2 stable. By applying Theorem 4.1 of [11], we assert that

$$\min_{u \in l^2_0(\mathbb{N}, \mathbb{R}^{nu})} J_2^\infty(u, v^*) = J_2^\infty(u^*, v^*) = \zeta_0^T Q_2 \zeta_0.$$

The proof is completed. \square

Remark 2. According to the sufficient condition stated in Theorem 3, the solvability of the H_2/H_∞ control problem is transformed into the existence of solutions to the coupled matrix-valued Equations (20)–(23). Since system (1) contains mean-field terms, the Riccati equations in (20)–(23) with respect to P_1 and P_2 are inscribed as Ex_s , while the Riccati equations with respect to Q_1 and Q_2 are used to characterize $x_s - Ex_s$. If system (1) does not contain mean-field terms and the noise does not depend on the control, then Theorem 3 degenerates to Theorem 1 of [7].

5. Iterative Algorithm and Example

The following iterative algorithm is applicable to solve GDREs (20)–(23):

- (i) Given S and the initial conditions $P_1(S + 1) = 0, Q_1(S + 1) = 0, P_2(S + 1) = 0$ and $Q_2(S + 1) = 0$, we can obtain $K_u(P_1(S + 1)) = 0, \tilde{K}_u(P_1(S + 1), Q_1(S + 1)) = 0, K_v(P_2(S + 1)) = 0$, and $\tilde{K}_v(P_2(S + 1), Q_2(S + 1)) = 0$, as well as $H_u(P_1(S + 1)) = \tilde{H}_u(P_1(S + 1), Q_1(S + 1)) = \gamma^2 I, H_v(P_2(S + 1)) = \tilde{H}_v(P_2(S + 1), Q_2(S + 1)) = I$.
- (ii) Given $P_1(s + 1), Q_1(s + 1), P_2(s + 1)$, and $Q_2(s + 1)$, as well as $K_u(P_1(s + 1)), \tilde{K}_u(P_1(s + 1), Q_1(s + 1)), K_v(P_2(s + 1)),$ and $\tilde{K}_v(P_2(s + 1), Q_2(s + 1))$, then $H_u(P_1(s + 1)), \tilde{H}_u(P_1(s + 1), Q_1(s + 1)), H_v(P_2(s + 1)),$ and $\tilde{H}_v(P_2(s + 1), Q_2(s + 1))$; as well, $U(s), \tilde{U}(s), V(s), \tilde{V}(s)$ can be computed using (23) and (21). Thus, $P_1(s), Q_1(s), P_2(s),$ and $Q_2(s)$ can be obtained.
- (iii) Once $(P_1(s), Q_1(s); P_2(s), Q_2(s); U(s), \tilde{U}(s); V(s), \tilde{V}(s))$ is obtained, we continue to calculate $K_u(P_1(s - 1)), \tilde{K}_u(P_1(s - 1), Q_1(s - 1)), K_v(P_2(s - 1)),$ and $\tilde{K}_v(P_2(s - 1), Q_2(s - 1))$, respectively.
- (iv) Repeat steps (ii)–(iii) when $s = s - 1$ until $\max\{\|P_1(s + 1) - P_1(s)\|_\infty, \|P_2(s + 1) - P_2(s)\|_\infty, \|Q_1(s + 1) - Q_1(s)\|_\infty, \|Q_2(s + 1) - Q_2(s)\|_\infty\} \leq e$, where $e = 10^{-5}$; then, we can obtain the solutions of (20)–(23).

Example 1. Take into account the one-dimensional discrete-time mean-field stochastic system below:

$$\begin{cases} x_{s+1} = ax_s + \bar{a}Ex_s + b_1u_s + \bar{b}_1Eu_s + b_2v_s + \bar{b}_2Ev_s \\ \quad + [cx_s + \bar{c}Ex_s + d_1u_s + \bar{d}_1Eu_s + d_2v_s + \bar{d}_2Ev_s]w_s, \\ z_s = \begin{bmatrix} kx_s \\ u_s \end{bmatrix}, \\ x_0 = \zeta_0, s \in N_S. \end{cases} \tag{30}$$

Given $a = 0.4, \bar{a} = 0.45, b_1 = 0.6, \bar{b}_1 = 0.7, b_2 = 0.65, \bar{b}_2 = 0.7, c = 0.45, \bar{c} = 0.55, d_1 = 0.3, \bar{d}_1 = 0.3, d_2 = 0.5, \bar{d}_2 = 0.55,$ and $k = 0.65,$ thus, $\gamma = 3$.

Consider systems $[a, \bar{a}; c, \bar{c}|k]$ and $[a + b_2V, \bar{a} + \bar{d}_1\bar{V} + d_1\bar{V}; c + d_2V, \bar{c} + \bar{d}_2\bar{V} + d_2\bar{V}|k]$. According to the definition of \mathcal{L} , we can convert $\mathcal{L}(X) = \lambda X$ into $\begin{bmatrix} a^2 + c^2 & (c + \bar{c})^2 \\ 0 & (a + \bar{a})^2 \end{bmatrix} X = \lambda X$ and

$\begin{bmatrix} (a + b_2V)^2 + (c + d_2V)^2 & (c + d_2V + \bar{c} + \bar{d}_2\bar{V} + d_2\bar{V})^2 \\ 0 & (a + b_2V + \bar{a} + \bar{d}_1\bar{V} + d_1\bar{V})^2 \end{bmatrix} X = \lambda X$ using algebraic operations.

Since the eigenvalues of both of the matrices are less than one, by applying Lemma 1, the exact detectability of the systems can be obtained.

By letting $S = 24$ and using the iterative algorithm, we can obtain the solutions of the Riccati equations; see Table 1. The evolutions of the solutions have been shown in Figure 1.

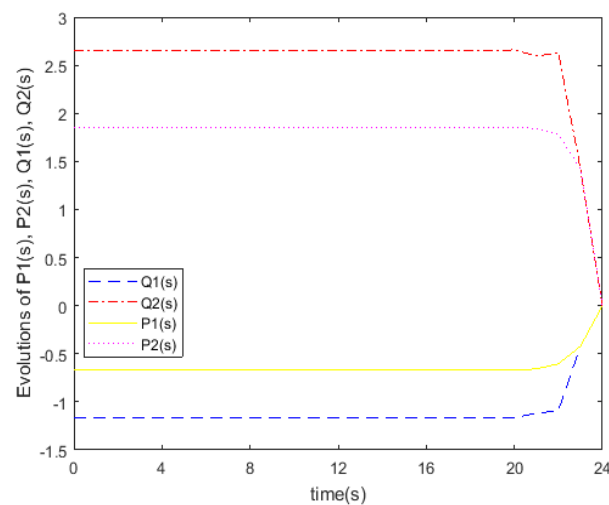
Table 1. The solutions of Riccati equations.

P_1	Q_1	P_2	Q_2
−0.6720	−1.1627	1.8537	2.6530

By applying Theorem 3, we can obtain the coefficients of $u_s = U_s x_s + \bar{U}_s E x_s$ and $v_s = V_s x_s + \bar{V}_s E x_s$ respectively; see Table 2.

Table 2. The coefficients of u_s and v_s .

U_s	\bar{U}_s	V_s	\bar{V}_s
−0.3908	−0.7044	0.0215	0.0496

**Figure 1.** The evolutions of the solutions of Riccati equations.

6. Conclusions

This paper has investigated the infinite horizon H_2/H_∞ control problem for the discrete-time mean-field stochastic system with (x, u, v) -dependent noise. Firstly, a stochastic-bounded real lemma has been derived for the considered system. Under the condition that the system is exactly detectable, a sufficient condition for the presence of an H_2/H_∞ controller has been obtained, which relates to the existence of the solution for the coupled Riccati equations. An iterative algorithm for solving the coupled Riccati equations has been proposed, and a numerical example has been presented. Note that Theorem 3 in this paper has only given a sufficient condition for the solvability of H_2/H_∞ control. In future studies, we will try our best to explore the necessity of Theorem 3 and draw a necessary condition.

As stated in the introduction, nonlinear systems and time-delay systems are applied extensively and thoroughly in engineering practice. Therefore, it is undoubtedly valuable to extend the design method obtained in this paper to nonlinear mean-field systems and time-delay mean-field systems. However, nonlinearities and time delays will cause a lot of difficulty in analysis and design. How to set up the stochastic-bounded real lemma is the key issue in H_∞ control. Moreover, H_2/H_∞ control problems for discrete-time mean-field systems with Markov jumps or Poisson jumps could be further investigated based on the research in this paper.

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