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Asymptotic Quantization of a Particle on a Sphere

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Abstract: Quantum systems whose states are tightly distributed among several invariant subspaces (variable spin systems) can be described in terms of distributions in a four-dimensional phase-space $T^*\mathcal{S}_2$ in the limit of large average angular momentum. The cotangent bundle $T^*\mathcal{S}_2$ is also the classical manifold for systems with E(3) symmetry group with appropriately fixed Casimir operators. This allows us to employ the asymptotic form of the star-product proper for variable (integer) spin systems to develop a deformation quantization scheme for a particle moving on the two-dimensional sphere, whose observables are elements of e(3) algebra and the corresponding phase-space is $T^*\mathcal{S}_2$. We show that the standard commutation relations of the $e(3)$ algebra are recovered from the corresponding classical Poisson brackets and the explicit expressions for the eigenvalues and eigenfunctions of some quantized classical observables (such as the angular momentum operators and their squares) are obtained.

Keywords: phase space; deformation quantization; star-product; E(3) group

1. Introduction

In spite of multiple approaches being attempted, the problem of the quantization of a given classical system still represents an intriguing problem [1–8]. One of the most profitable quantization schemes involves a direct employment of the phase-space formalism [9–16]. The fundamental step in phase-space quantization programs consists of establishing a map ω between elements of a functional space (the space of classical observables) and operators acting in an appropriate Hilbert space (see [1–6,14–16] and references therein). This approach is tightly connected to the symmetry of a dynamical classical system. In particular, the classical observables are considered as smooth functions in a phase-space manifold $\mathcal{M}$, where the group of the dynamic symmetry (generated by the Lie algebra of observables) of the system acts transitively [17]. The notion of orbit-like coherent states [18–20] naturally appears in such types of schemes.

It is required that a bijective map between classical and quantum objects

$$\omega : f(\zeta) \Rightarrow \hat{f}, \quad \zeta \in \mathcal{M}, \tag{1}$$

satisfies a set of basic properties that guarantee its invertibility (the dequantization process) and the covariance under the group transformations [21–26].

In this framework, the form of the map determines the so-called star-product [27–29],

$$f(\zeta) * g(\zeta) \Rightarrow \hat{f}\hat{g}, \tag{2}$$

that relates operations in the functional and operator spaces. A general method for a construction of phase-space quantization–dequantization maps is well established for systems with semi-simple groups of dynamic symmetry, but is faced with certain ambiguities for other groups (see, e.g., [1–8,30–51] and references therein), except for the well studied Heisenberg–Weyl group $H(1)$ and its direct products [52–56]).

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On the other hand, starting with a deformed product between two classical observables as a function of a small parameter $\varepsilon \ll 1$ (the semiclassical parameter) such that

$$\lim_{\varepsilon\to 0}(f(\zeta) * g(\zeta) - g(\zeta) * f(\zeta)) = -i\{f(\zeta), g(\zeta)\}_P,$$

where $\{.,.\}_P$ is the Poisson bracket on $\mathcal{M}$, we arrive at the concept of the deformation quantization [28,29] (see also [4–6] and references therein).The semiclassical parameter characterizes the strength of quantum fluctuations and depends on the symmetry of the system. For instance, for systems with the Heisenberg–Weyl symmetry the physical semiclassical parameter is the inverse number of excitations, while as a formal expansion parameter, the Plank constant is usually taken, $\varepsilon \sim \hbar$. For systems with the $\mathbf{SU(2)}$ symmetry the inverse effective spin length is a natural semiclassical parameter, etc. [50,51,57–59].

The standard requirements of the star-product are associativity, self-adjointness and analyticity in the semiclassical parameter; they allow us to define a family of equivalent star-products, satisfying the condition [4–6],

$$f(\zeta) *_T g(\zeta) = T^{-1}[Tf(\zeta) * Tg(\zeta)], \tag{3}$$

where T is an appropriate linear operator usually depending on some continuous parameters. In practice, some additional physical considerations/restrictions should be imposed in order to define a meaningful star-product on a given symplectic manifold [4–8]. Explicit constructions of well-behaved star-products for curved manifolds (describing co-adjoint orbits of non semi-simple groups) may present significant difficulties and in general require a detailed analysis in each particular case, although a general approach has been developed [60].

The quantization of a particle in a curved configuration space, in particular, moving on the S^2 sphere [61–65], is one of long-standing problems. The corresponding classical phase-space is the cotangent bundle T^*S_2, which is the co-adjoint orbit of the $E(3)$ group [31–39], obtained by fixing both Casimir operators (a constant magnitude position vector is orthogonal to the angular momentum). The implementation of the standard Stratonovich–Moyal–Weyl phase-space quantization protocol faces major complications in this case, which are related, in particular, to the absence of orbit-like coherent states [66,67]. On the other hand, no sensible form of the star-product has been proposed, which limits the application of the deformation quantization formalism.

It was shown in [68–70] that, while the phase-space of a single angular momentum is the two-dimensional sphere, the operators describing variable (integer) spin systems, i.e., those "living" in several $SO(3)$ irreducible subspaces, can be mapped into smooth functions (symbols) on T^*S_2 in the limit of large average angular momentum. Such functions are commonly called s-parametrized symbols,

$$\hat{f} \Rightarrow W_f^{(s)}(\zeta), \quad \zeta \in T^*\mathcal{S}_2, \tag{4}$$

where the index s labels the families of dual (sef-dual) functions, e.g., $s = -1, 1, 0$ correspond to the so-called Q-, P- and Wigner symbols. Some of the family of symbols $W_f^{(s)}(\zeta)$ may coincide with the corresponding classical observables $f(\zeta)$ for an appropriate choice of the parameter s.

The non-trivial observation (4) is based on the analysis of the semiclassical limit (large angular momentum) of the evolution equation for the quasidistribution functions (symbols of the density operators). In other words, the cotangent bundle T^*S_2 can be considered as the asymptotic phase-space for variable spin systems. The advantage of the map (4) consists of the possibility to develop a full phase-space machinery, which includes the star-product operation, i.e., a composition map (see [68] Equation (45)),

$$\hat{f}\hat{g} \to W_f^{(s)}(\zeta) * W_g^{(s)}(\zeta). \tag{5}$$

This suggests to apply the asymptotic form of the star-product proper to variable-spin systems, for quantizing classical systems possessing the dynamic symmetry of the $E(3)$ group [31–39] and evolving in T^*S_2 phase-space.

In this paper, we develop an asymptotic deformation quantization scheme for a particle on the two-dimensional sphere, which basic classical observables generate the $e(3)$ algebra, by making use of the star-product operation inherited from variable angular-momentum systems. In addition to the previous approaches [61–65] we will not only be able to recover the appropriate $e(3)$ commutation relations, but also obtain explicit expressions for the eigenfunctions of several classical observables asymptotically quantized with (5).

In Section 2, we briefly review the basics of the phase-space approach to variable-spin systems. In Section 3, we introduce a star-product on T^*S_2 as a continuous limit of the star-product for variable spin systems and apply it for quantization of classical dynamic variables (elements of the $e(3)$ algebra) describing a particle on S^2. In Section 4, we obtain the eigenfunctions and spectrum of some of the (quantized) observables.

2. Variable-Spin Quasidistributions

According to the variable-spin quantization–dequantization approach [68–70] a density matrix,

$$\hat{\rho} = \sum_{S,S'=0,1,..} \sum_{m,m'} c^{SS'}_{mm'} |S,m\rangle\langle S',m'|,$$

can be mapped into a discrete set of functions (j-symbols) on S^3,

$$\hat{f} \;\Leftrightarrow\; \{W_f^{j(s)}(\Theta);\, j=0,1,\dots\}, \quad s\in[-1,1], \tag{6}$$

$$\Theta \;=\; (\phi,\theta,\psi),\; 0\le\phi,\psi<2\pi,\; 0\le\theta\le\pi, \tag{7}$$

through a trace operation

$$W_f^{j(s)}(\Theta) = \mathrm{Tr}\Big(\hat{f}\hat{\omega}_j^{(s)}(\Theta)\Big), \tag{8}$$

where the Hermitian $SO(3)$ covariant kernel $\hat{\omega}_j^{(s)}(\Theta)$ is defined in Appendix A, such that the reconstruction relation,

$$\hat{f} = \sum_{j=0,1\dots}^{\infty} \hat{f}_j, \quad \hat{f}_j = \frac{j+1}{8\pi^2}\int d\Theta W_f^{j(s)}(\Theta)\hat{\omega}_j^{(-s)}(\Theta), \tag{9}$$

where $d\Theta = \sin\theta d\phi d\theta d\psi$ is a volume element of $SO(3)$, and the overlap relation

$$\mathrm{Tr}\Big(\hat{f}\hat{g}\Big) = \sum_{j=0,1,2,\dots}^{\infty} \frac{j+1}{8\pi^2}\int d\Theta W_f^{j(s)}(\Theta) W_g^{j(-s)}(\Theta), \tag{10}$$

are fulfilled. The symbols $W_f^{j(\pm 1)}(\Theta)$ are dual to each other and $W_f^{j(0)}(\Theta)$ is self-dual. The s-parametrized symbols are related to each other according to

$$W_f^{j(s)}(\Theta) = \Big[F_{2j}\Big(J^2\Big)\Big]^s \left(\frac{\Gamma(2j-J^0+2)\Gamma(2j+J^0+2)}{2j+1}\right)^{-s/2} W_f^{j(s=0)}(\Theta), \tag{11}$$

where $J^0 = -i\partial_\psi$, J^2 is the differential realization of the Casimir operator (A16) and $F_j(J^2)$ is an operational function defined by its action on the Wigner D-functions (A13).

In addition, the map (6) entails the star-product [69] (associative by construction):

$$W_{fg}^{j(s)}(\Theta) = W_f^{j\,(s)} *_s W_g^{j\,(s)} = \sum_{j_1,j_2=0,1,\dots} L^{(s)}_{j,j_1 j_2}\Big(W_f^{j_1(s)}(\Theta) W_g^{j_2(s)}(\Theta)\Big), \tag{12}$$

where the explicit form of the operator $L^{(s)}_{j,j_1 j_2}$ is given in Appendix A, Equation (A18).

Taking into account the parity property of the kernel (A4), one can show that the linear combination

$$W_f^{j+}(\Theta) = W_f^{j}(\Theta) + W_f^{j+1}(\Theta), \tag{13}$$

tends to a smooth function of j in the continuous limit, $W_f^{j+}(\Theta) \to W_f^{+}(\Theta, j)$ [71]. The evolution of the symbols of the density matrix, $W_\rho^{(s)+}(\Theta, j)$, commonly called s-parametrized quasidistributions, is governed in the continuous limit and for a large mean spin by the Liouville-type differential Equations [69,71],

$$\partial_t W_\rho^{+(s)}(\Theta, j) \approx \{W_H^{+(s)}(\Theta, j), W_\rho^{+(s)}(\Theta, j)\}_P, \tag{14}$$

where W_H^+ is the symbol of the Hamiltonian, and $\{.,.\}_P$ are the Poisson brackets (where we have used the notation (A14)),

$$\begin{aligned} \{.,.\}_P &= -\frac{2\cot\theta}{j}\left(\partial_\theta \otimes \partial_\psi - \partial_\psi \otimes \partial_\theta\right) && (15)\\ &\quad + \frac{2}{j\sin\theta}\left(\partial_\theta \otimes \partial_\phi - \partial_\phi \otimes \partial_\theta\right) + 2\left(\partial_\psi \otimes \partial_j - \partial_j \otimes \partial_\psi\right), && (16) \end{aligned}$$

in a four-dimensional manifold isomorphic to the cotangent bundle $T^*\mathcal{S}_2$. This manifold is the co-adjoint orbit of the $E(3)$ group fixed by the Casimir operators

$$\hat{\mathbf{r}}^2 = \hat{I}, \quad \hat{\mathbf{l}} \cdot \hat{\mathbf{r}} = 0, \tag{17}$$

where the (commuting) generators of translations $\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z})$ and the components of the angular momentum operators $\hat{\mathbf{l}} = (\hat{l}_x, \hat{l}_y, \hat{l}_z)$ close the $e(3)$ algebra (see Appendix B):

$$\left[\hat{l}_i, \hat{l}_j\right] = i\varepsilon_{ijk}\hat{l}_k, \qquad \left[\hat{r}_i, \hat{r}_j\right] = 0, \qquad \left[\hat{l}_i, \hat{r}_j\right] = i\varepsilon_{ijk}\hat{r}_k. \tag{18}$$

The limit of a large average angular momentum corresponds to $\langle \hat{\mathbf{l}}^2 \rangle \sim j^2 \gg 1$.

The Darboux coordinates in $T^*\mathcal{S}_2$ are

$$(j\cos\theta, \phi) \text{ and } (j, \psi), \tag{19}$$

so that the corresponding closed and exact two-form is

$$\begin{aligned} \omega &= -d\alpha = d\phi \wedge d(j\cos\theta) + d\psi \wedge dj\,, \\ \alpha &= j d\psi + j\cos\theta d\phi. \end{aligned} \tag{20}$$

where the (continuous) index j becomes a dynamical variable conjugate to the angle ψ, $\{j, \psi\}_P = 1$. It is worth noting that the canonical coordinates (19) describe a rigid rotor dynamics [72], in the sense that the projection $j\cos\theta$ leads to a phase shift ϕ and changes in the phase ψ are generated by the total angular momentum j.

3. Deformation Quantization on $T^*\mathcal{S}_2$

The asymptotic form of the evolution Equations (14) and (15) and its relation to the rigid rotor motion suggests to apply the machinery developed for variable-spin systems to quantum systems with $E(3)$ dynamic symmetry group. The archetypical example of such a system is a particle moving on the two-dimensional sphere, whose classical phase-space is precisely $T^*\mathcal{S}_2$ [61]. Unfortunately, taking the direct continuous limit of the map (6), (A1) faces significant difficulties. However, the star-product (A18) can be extended to the continuous domain of the index j. This allows us to develop a deformation quantization scheme completely in terms of internal coordinates of $T^*\mathcal{S}_2$ following the general ideas [4–6,28,29].

Star-Product

Taking into account the displacement property

$$f(j + I \otimes J^0, \Theta) g(j - J^0 \otimes I, \Theta) = e^{\left(\partial_j \otimes J^0 - J^0 \otimes \partial_j\right)} f(j, \Theta) \otimes g(j, \Theta), \tag{21}$$

we can reduce the s-parametrized family of discrete star-product operations (A18) to the form (3) considering the index j as a continuous variable,

$$f(j, \Theta) *_s g(j, \Theta) = T^{-1}(s)((T(s)f) * (T(s)g)), \tag{22}$$

$$\begin{aligned} f * g &= \sum_{n=0}^{\infty} \frac{(-J^+ \otimes J^-)^n}{n!} \frac{1}{\Gamma(2j - J^0 \otimes I + I \otimes J^0 + n + 2)} e^{\left(\partial_j \otimes J^0 - J^0 \otimes \partial_j\right)/2} f \otimes g, \\ T(s) &= \left(\sqrt{2j+1} F_{2j}\left(J^2\right)\right)^{1-s} \left(\Gamma\left(2j + J^0 + 2\right) \Gamma\left(2j - J^0 + 2\right)\right)^{s/2}, \end{aligned} \tag{23}$$

where $f(j, \Theta)$ and $g(j, \Theta)$ are smooth real-valued functions in $T^*\mathcal{S}_2$, $J^{\pm}$, J^0 are the components of the covariant generators of the $SO(3)$ group (A15) and we have rescaled the variable $j \to 2j$ in accordance with (A8). In particular, for observables depending only on the variable j one has,

$$\begin{aligned} f(j) *_s g(j, \Theta) &= f\left(j + J^0/2\right) g(j, \Theta), \\ g(j, \Theta) *_s f(j) &= f\left(j - J^0/2\right) g(j, \Theta). \end{aligned}$$

The associativity of (22) follows directly from the associativity of the respective discrete star-product (A18) and the relation (21).

It is worth noting that the above star-product does not contain any deformation parameter (as, e.g., $\hbar$ in the Heisenberg–Weyl case), but rather acquires the expected asymptotic form

$$f(j, \Theta) *_s g(j, \Theta) = fg + \frac{i}{2}\{f, g\}_P + \frac{1-s}{2j} \partial_\psi g \otimes \partial_\psi g +$$

$$\frac{s}{2j}\left(\cot^2\theta \partial_\psi \otimes \partial_\psi + \partial_\theta \otimes \partial_\theta - \frac{1}{\sin^2\theta}(\partial_\psi \otimes \partial_\phi + \partial_\phi \otimes \partial_\psi) + \frac{1}{\sin^2\theta} \partial_\phi \otimes \partial_\phi\right) f \otimes g + O(j^{-2}),$$

in the limit $j \gg 1$, leading to the standard limit of any s-parametrized commutator,

$$[f, g]_s = f *_s g - g *_s f = i\{f, g\}_P + O(1/j^2), \tag{24}$$

where the Poisson brackets are defined in (15) with $j \to 2j$. It will be shown below, that the limit $j \gg 1$ corresponds to the large amplitude of the classical angular momentum.

The crucial point consists of relating classical observables with the dynamical parameters in $T^*\mathcal{S}_2$, while dependence on the angular variables is straightforward, the connection with the parameter j, defining the classical amplitudes, requires imposition of additional conditions.

Let us first observe that the quantization of coordinates $\mathbf{r} = (x, y, z) = (\sin\phi \sin\psi - \cos\phi \cos\theta \cos\psi, -\cos\phi \sin\psi - \sin\phi \cos\theta \cos\psi, \sin\theta \cos\psi)$ generated by (22) and (23) leads to a commuting set

$$[r_k, r_l]_s = 0, \tag{25}$$

where $k,l,m = x,y,z$ (i.e., $r_x = x,\ldots$ etc.), *only* for $s = 0$, since the s-parametrized commutator (24) formally gives,

$$[r_k, r_l]_s = -i\varepsilon_{klm}\left(\frac{4j(j+1)}{(2j+1)^2}\right)^{(1-s)/2}\left(1-\left(\frac{(2j-1)(j+1)}{j(2j+1)}\right)^s\right)L_m.$$

In particular, it is fulfilled

$$x *_0 x + y *_0 y + z *_0 z = 1. \tag{26}$$

In order to establish a relation between the classical amplitude j_0 of the angular momentum,

$$L_k = j_0 n_k,$$

where $\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ is a unitary vector parametrizing points of S^2, and the dynamical variable j appearing in the star-product (22) and (23) (e.g., for determining $j_0 = j_0(j)$) we enforce the commutativity condition between the classical observables,

$$[L_k, r_k]_{s=0} = 0, \tag{27}$$

obtaining

$$0 = [L_k, r_k]_{s=0} \sim \sqrt{\frac{2j+3}{2j+1}}j_0(j-1/2) - \sqrt{\frac{2j-1}{2j+1}}j_0(j+1/2),$$

which is satisfied only if

$$j_0(j) = \sqrt{j(j+1)}.$$

In this case one also obtains,

$$\mathbf{L}^2 = L_x *_0 L_x + L_y *_0 L_y + L_z *_0 L_z = j(j+1), \tag{28}$$

$$L_x *_0 x + L_y *_0 y + L_z *_0 z = 0, \tag{29}$$

$$[L_k, r_l]_0 = i\varepsilon_{klm}r_m, \quad [L_k, L_l]_0 = i\varepsilon_{klm}L_m, \quad k,l,m = x,y,z, \tag{30}$$

which are in accordance with the standard $e(3)$ commutation relations (18), and in addition,

$$[j, L_k]_0 = 0, \tag{31}$$

as it can be expected. It is worth noting that different quantization schemes may lead to different values of $\mathbf{L}^2$ [28].

Observe that the Hamiltonian of a free particle on the sphere

$$H = \frac{\mathbf{p}^2}{2},$$

is equivalent to $\mathbf{L}^2/2$ in the classical manifold T^*S_2, where the Casimir operators are fixed according to (17).

In what follows we assign the index "0" to the observables quantized with (22) and (23) at $s = 0$,

$$\mathbf{L}^{(0)} = \sqrt{j(j+1)}\mathbf{n}, \quad \mathbf{r}^{(0)} = \mathbf{r}. \tag{32}$$

In the case of quantization using the star-product (22) with an arbitrary s, the observables satisfying the relations (27)–(31) are connected to those with $s = 0$ according to Equation (11) with $W_f^{j(s)}(\Theta) \to f^{(s)}(j,\Theta)$, in particular

$$\begin{aligned}
\mathbf{L}^{(s)} &= \left(\frac{j}{j+1}\right)^{-s/2} \mathbf{L}^{(0)}, \\
\mathbf{r}^{(s)} &= \left(\frac{j}{j+1/2}\right)^{-s/2} \mathbf{r}^{(0)}.
\end{aligned}$$

It results that the star-product at $s = 0$ satisfies an additional condition characteristic for self-dual, Wigner-like distributions: if $\rho(j,\Theta)$ is a normalized classical distribution function,

$$\int dj \frac{2j+1}{4\pi^2} \int d\Theta \rho(j,\Theta) = 1, \tag{33}$$

and $f(j,\Theta)$ a classical observable, the average value of $f(j,\Theta)$ is computed according to

$$\langle f \rangle = \int dj \frac{2j+1}{4\pi^2} \int d\Theta \rho(j,\Theta) *_0 f(j,\Theta) = \int dj \frac{2j+1}{4\pi^2} \int d\Theta \rho(j,\Theta) f(j,\Theta), \tag{34}$$

appearing in the same form as in classical statistical physics.

Following the same steps as Equation (A20) was obtained, we get for the star-product (22) and (23) at $s = 0$

$$\int dj \frac{2j+1}{4\pi^2} \int d\Theta \rho(j,\Theta) *_0 f(j,\Theta) =$$

$$= \int_0^\infty dj \int d\Theta \frac{2j - J^0 \otimes I + 1}{4\pi^2} e^{-J^0 \partial_j \otimes I/2} e^{-J^0 \otimes \partial_j/2} \rho(j,\Theta) \otimes f(j,\Theta),$$

which can be formally represented as

$$\int_0^\infty dj \int d\Theta \frac{2j - J^0 \otimes I + 1}{4\pi^2} \int_0^\infty dj_1 \int_0^\infty dj_2 \delta\left(j_1 - j + \frac{J^0}{2}\right) \delta\left(j_2 - j + \frac{J^0}{2}\right) \rho(j_1,\Theta) f(j_2,\Theta),$$

and immediately leads to (34).

4. Eigenfunctions of j and L_z

The eigenfunctions of the observable $j^{(0)}$ quantized with (22) and (23) at $s = 0$ can be found by using the unitary trick, i.e., looking for the solution of the "evolution equation"

$$\begin{aligned}
i\partial_t \mathcal{U}_j^{(0)}(j,\Theta) &= j^{(0)} *_0 \mathcal{U}_j^{(0)}(j,\Theta) = \left(j - \frac{i}{2}\partial_\psi\right) \mathcal{U}_j^{(0)}(j,\Theta), \qquad (35) \\
\mathcal{U}_j^{(0)}((j,\Theta|t=0) &= 1.
\end{aligned}$$

The expansion of the solution of Equation (35) in the basis of sinc-functions,

$$\begin{aligned}
\mathcal{U}_j^{(0)}(j|t) &= e^{-ijt} = \sum_{L=-\infty}^{\infty} \pi_L^{(0)}(j) e^{-iLt}, \qquad (36) \\
\pi_L^{(0)}(j) &= \frac{\sin(\pi(j-L))}{\pi(j-L)}, \quad \sum_{L=-\infty}^{\infty} \pi_L^{(0)}(j) = 1, \qquad (37) \\
\sum_{L=-\infty}^{\infty} \pi_L^{(0)}(j) \pi_L^{(0)}(j') &= \delta(j-j'), \qquad (38)
\end{aligned}$$

allows us to represent the auxiliary observable j and the total angular momentum (28) in the spectral form,

$$j^{(0)} = \sum_{L=-\infty}^{\infty} L\pi_L^{(0)}(j), \tag{39}$$

$$\mathbf{L}^{(0)2} = \sum_{k=x,y,z} L_k^{(0)} *_0 L_k^{(0)} = j(j+1) = \sum_{L=-\infty}^{\infty} L(L+1)\pi_L^{(0)}. \tag{40}$$

The formal expansions (39) and (40) should be considered in the sense of generalized functions (distributions) as acting on normalizable functions (33). More precisely, the eigenfunction equation,

$$j * \pi_L^{(0)}(j) = L\pi_L^{(0)}(j), \tag{41}$$

is fulfilled only asymptotically, when it is applied to compact support functions of widths $\sim \sigma_0$ localized at $j_0 \gg 1$ [73,74], so that

$$\int dj\Big[j * \pi_L^{(0)}(j)\Big] f(j) \approx L \int dj \pi_L^{(0)}(j) f(j), \tag{42}$$

where the error of the discrete sampling (39) is of order $\sim\mathrm{erfc}(\pi\sigma_0)$ [75]. Loosely speaking, the spectrum of the observable j is approximately $\{L, L \in Z_+\}$ and the corresponding eigenfunctions are $\pi_L^{(0)}(j)$ for large values of the total angular momentum.

The spectral problem for the observable $L_z^{(0)}$ is easier to solve in the parametrization corresponding to $s = -1$ (and commonly associated with the so-called Q-distributions [76]) with $L_z^{(-1)} = j\cos\theta$ followed by the application of Equation (11). Then, the corresponding "evolution" equation,

$$\begin{aligned} i\partial_t \mathcal{U}_{L_z}^{(-1)}(j,\Theta) &= L_z^{(-1)} *_{-1} \mathcal{U}_{L_z}^{(-1)}(j,\Theta) \\ &= \frac{1}{2}(2j\cos\theta - \sin\theta\partial_\theta - i\partial_\phi)\mathcal{U}_{L_z}^{(-1)}(j,\Theta), \\ \mathcal{U}_{L_z}^{(-1)}(j,\Theta|t=0) &= 1, \end{aligned}$$

possesses the solution,

$$\mathcal{U}_{L_z}^{(-1)}(j,\Theta|t) = \left(\cos\frac{t}{2} - i\cos\theta\sin\frac{t}{2}\right)^{2j}, \tag{43}$$

which can be represented in terms of the Clebsch–Gordan series as,

$$\mathcal{U}_{L_z}^{(-1)}(j,\Theta|t) = \sum_{L=-\infty}^{\infty} \frac{\sin(\pi(j-L))}{\pi(j-L)} \sum_{M=-L}^{L}{}' e^{-iMt}\pi_{L\,M}^{(-1)}(\theta), \tag{44}$$

$$\pi_{LM}^{(-1)}(\theta) = \sum_{k=0}^{2L} \frac{2k+1}{2L+1} C_{LLk0}^{LL} C_{LMk0}^{LM} P_k(\cos\theta), \tag{45}$$

$$\sum_M{}' \pi_{LM}^{(-1)}(\theta) = 1, \quad C_{LLk0}^{LL} = \sqrt{\frac{(2L+1)!(2L)!}{(2L+k+1)!(2L-k)!}}, \tag{46}$$

where $P_k(\cos\theta)$ is the Legendre polynomial and $\sum_M{}'$ means that for negative values of L, $L \le -1$, the index M runs between $-L-1$ and $L+1$. Equations (44) and (45) immediately allow us to represent $L_z^{(-1)}$ as

$$L_z^{(-1)} = \sum_{L=-\infty}^{\infty} \frac{\sin(\pi(j-L))}{\pi(j-L)} \sum_M{}' M\pi_{LM}^{(-1)}(\Theta),$$

and applying Equation (11) finally arrive at the expansion,

$$L_z^{(0)} = \sum_{L=-\infty}^{\infty} \frac{\sin(\pi(j-L))}{\pi(j-L)} \sum_M{}' M \pi_{LM}^{(0)}(j,\theta), \tag{47}$$

$$\pi_{LM}^{(0)}(j,\theta) = \sum_{k=0}^{2L} \frac{2k+1}{2L+1} C_{LLk0}^{LL} \frac{\sqrt{\Gamma(2j+k+2)\Gamma(2j-k+1)}}{\Gamma(2j+1)\sqrt{2j+1}} C_{LMk0}^{LM} P_k(\cos\theta), \tag{48}$$

where the sum (47) converges to the form (32) (see Appendix C). The eigenvalue equation

$$L_z^{(0)} * \left[\pi_L^{(0)}(j)\pi_{LM}^{(0)}(j,\theta)\right] = M\left[\pi_L^{(0)}(j)\pi_{LM}^{(0)}(j,\theta)\right] \tag{49}$$

is considered in the same sense as Equations (41) and (42).

5. Conclusions

We have introduced a star-product operation on $T^*\mathcal{S}_2$ that satisfies all the requirements for a deformed star-product, but does not contain a deformation parameter, leading, however, to the Poisson algebra in the asymptotic limit corresponding to a large amplitude of the classical angular momentum. The star-product (23) allows us to recover the standard $e(3)$ algebra relations (26), (28)–(30) by quantizing the classical dynamical variables describing a particle on the two-dimensional sphere. This is achieved by imposing the natural commutativity conditions (25) and (27) on the observables $\mathbf{L}^{(0)}$ and $\mathbf{r}^{(0)}$, corresponding to the Wigner-like quantization which satisfies the prescription (34) for computing of average values. The equivalent s-parametrized set of quantized variables can be obtained by applying Equation (11). In addition, we have managed to solve the eigenvalue equations for the angular momentum component $L_z^{(0)}$ and the auxiliary observable j, that determines the classical amplitudes. Both the spectrum of those observables (41), (49), and their eigenfunctions (37), (48) acquire a physical meaning in the asymptotic limit of large average angular momentum. The proposed method can be directly applied to the problem of quantization of a rigid rotor, which in spite of being widely discussed in literature still leaves several open questions including, e.g., the spectral problem [77], a consistent quantization of canonical variables [78,79], etc.

It is worth mentioning that the method developed in Section 2 is applicable to half-integer spin systems, in the sense that the map (6) is formally extendible to $j = 0, 1/2, 1, 3/2, \ldots$ with $0 \le \psi < 4\pi$, in such a way that the evolution equation for a specific linear combinations of $W_\rho^j(\Theta)$ acquires the form (14) and (15) in the limit of large average spin [68–70]. However, its immediate application to a quantization of classical systems is not quite transparent.

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Appendix A

In this Appendix we review in details the star-product operation for the discrete symbols (8) and prove a relevant Theorem concerning the integration of the star-product of two Wigner-like symbols.

The $SO(3)$ covariant quantization kernel has the form [68–70],

$$\hat{\omega}_j^{(s)}(\Theta) = \sum_{K=0}^{j}\sum_{q,q'=-K}^{K}\sqrt{\tfrac{2K+1}{j+1}}\left(\sqrt{\tfrac{j-q'+1}{j+1}}C^{\frac{j+q'}{2}\,\frac{j+q'}{2}}_{\frac{j-q'}{2}\,\frac{j-q'}{2},\,Kq'}\right)^{-s} D^K_{qq'}(\Theta)\hat{T}_{Kq}^{\frac{j+q'}{2}\,\frac{j-q'}{2}}, \tag{A1}$$

$$\hat{\omega}_j^{(s)}(\Theta) = \hat{D}(\Theta)\hat{\omega}_j^{(s)}(0)\hat{D}(\Theta)^{\dagger}, \quad \hat{D}(\Theta) = e^{-i\phi\hat{l}_z}e^{-i\theta\hat{l}_y}e^{-i\psi\hat{l}_z}, \tag{A2}$$

where $\hat{l}_j$, $j = x,y,z$ are the angular momentum operators, $D^K_{ql}(\Theta)$ is the Wigner D-function, $D^K_{ql}(\Theta) = \langle K,q|\hat{D}(\Theta)|K,l\rangle$, $C^{c\gamma}_{a\alpha,\,b\beta}$ are the Clebsch-Gordan coefficients and

$$\hat{T}^{JJ'}_{Kq} = \sum_{m,m'}\sqrt{\frac{2K+1}{2J+1}}C^{Jm}_{J'm',\,Kq}|J,m\rangle\langle J',m'|, \tag{A3}$$

are the tensor operators [80]. The index q' in (A1) has the same parity as the index j, so that

$$\hat{\omega}_j(\phi,\theta,\psi) = (-1)^j\hat{\omega}_j(\phi,\theta,\psi+\pi). \tag{A4}$$

The kernels Equation (A1) are Hermitian, normalized

$$\frac{j+1}{8\pi^2}\int_{SO(3)} d\Theta\hat{\omega}_j^{(s)}(\Theta) = \hat{\mathbf{I}}_{j+1}, \tag{A5}$$

with $\mathbf{I}_{j+1}$ being the identity operator in the $j+1$ dimensional subspace, and trace orthogonal

$$Tr\left(\hat{\omega}_j^{(s)}(\Theta)\hat{\omega}_{j'}^{(-s)}(\Theta')\right) = \delta_{j,j'}\delta_j(\Theta,\Theta'), \tag{A6}$$

where $\delta_{j,j'}$ is the Kronecker symbol and $\delta_j(\Theta,\Theta')$ is the δ-function on the manifold:

$$\int_{SO(3)} d\Theta\hat{f}(\Theta)\delta_j(\Theta,\Theta') = \hat{f}(\Theta'). \tag{A7}$$

The j-symbols of operator that mix $SO(3)$ invariant subspaces necessarily depend on the angle ψ.

For an operator $\hat{f}$ acting in a single $SO(3)$ irreducible $2S+1$ dimensional subspace ($S = j/2$), the mapping Equation (8) is reduced to the standard Stratonovich-Weyl form [21–26] by averaging over the angle ψ,

$$\hat{w}^{(s)}_{S=j/2}(\theta,\phi) = \int_0^{2\pi}\frac{d\psi}{2\pi}\hat{\omega}_j^{(s)}(\Theta) = \sqrt{\frac{4\pi}{2S+1}}\sum_{K=0}^{2S}\sum_{q=-K}^{K}\left(C^{SS}_{SS,K0}\right)^{-s}Y^*_{Kq}(\phi,\theta)\,\hat{T}^S_{Kq}, \tag{A8}$$

where $Y_{Kq}(\phi,\theta)$ are spherical harmonics and $\hat{T}^L_{Kq}$ are the standard (diagonal) tensor operators [81,82].

Examples of j-symbols

(a) the position operators, $\hat{\mathbf{r}}$,

$$W^{j(0)}_{r_k}(\Theta) = r_k\sum_{n\in\mathbb{Z}^+}\delta_{j,2n+1}, \tag{A9}$$

where $\mathbf{r} = (\sin\phi\sin\psi - \cos\phi\cos\theta\cos\psi, -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi, \sin\theta\cos\psi)$.

(b) the angular momentum operators, $\hat{\mathbf{l}}$,

$$W^{j(0)}_{l_k}(\Theta) = \sqrt{\frac{j}{2}\left(\frac{j}{2}+1\right)}\,n_k\sum_{n\in\mathbb{Z}^+}\delta_{j,2n}, \tag{A10}$$

where $\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, in particular,

$$W_{\mathbf{1}^2}^{j(0)}(\Theta) = \frac{j}{2}\left(\frac{j}{2}+1\right)\sum_{n\in\mathbb{Z}^+}\delta_{j,2n}. \tag{A11}$$

The star-product operator (12) has the form [68,69]

$$\begin{aligned}
L_{j,j_1j_2}^{(s)} \quad = \quad & \int_0^{2\pi}\frac{d\varphi' d\varphi}{(2\pi)^2}\sum_{n=0}^{\infty} a^n_{\frac{j_1+j_2-j}{2}}\left(\frac{(j_1+1)(j_2+1)}{j+1}\right)^{(1-s)/2} \\
& \times F_j^{s-1}(J^2)\Big(\Gamma(j-J^0+2)\Gamma(j+J^0+2)\Big)^{-s/2} \\
& \times\Bigg[\left((J^+)^n F_{j_1}^{1-s}\left(J^2\right)e^{i\left(j_2-j+J^0\right)\varphi'}\Big(\Gamma(j_1-J^0+2)\Gamma(j_1+J^0+2)\Big)^{s/2}\right) \\
& \otimes\left((J^-)^n F_{j_2}^{1-s}\left(J^2\right)e^{i\left(j_1-j-J^0\right)\varphi}\Big(\Gamma(j_2-J^0+2)\Gamma(j_2+J^0+2)\Big)^{s/2}\right)\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
a_J^n \quad &= \quad \frac{(-1)^n}{n!\Gamma(2J+n+2)}, && \text{(A12)}\\
F_j(J^2)D^k_{nm}(\Theta) \quad &= \quad \sqrt{\Gamma(j+k+2)\Gamma(j-k+1)}D^k_{nm}(\Theta); && \text{(A13)}
\end{aligned}$$

and

$$(A\otimes B)\left(W_f^{j_1}\otimes W_g^{j_2}\right) = \left(AW_f^{j_1}\right)\left(BW_g^{j_2}\right), \tag{A14}$$

where A and B are differential operators;

$$J^{\pm} = ie^{\mp i\psi}\left[i\frac{\partial}{\partial\theta}\pm\cot\theta\frac{\partial}{\partial\psi}\mp\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right], \quad J^0 = -i\frac{\partial}{\partial\psi}, \tag{A15}$$

are components of the covariant generators of the $SO(3)$ group and

$$J^2 = -\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}-2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi}+\frac{\partial^2}{\partial\psi^2}\right)\right], \tag{A16}$$

is the corresponding Casimir operator,

$$\begin{aligned}
J^2 D^K_{QQ'}(\Theta) \quad &= \quad K(K+1)D^K_{QQ'}(\Theta), \\
J^0 D^K_{QQ'}(\Theta) \quad &= \quad -Q' D^K_{QQ'}(\Theta).
\end{aligned} \tag{A17}$$

The star-product operation is associative by construction, i.e.,

$$W_{fgh}^{j\,(s)} = W_f^{j\,(s)} *_s W_{gh}^{j\,(s)} = W_f^{j\,(s)} *_s \left(W_g^{j\,(s)} *_s W_h^{j\,(s)}\right) = \left(W_f^{j\,(s)} *_s W_g^{j\,(s)}\right) *_s W_h^{j\,(s)}.$$

Applied to j-symbols, the star-product takes the form

$$W_f^{j\,(s)} *_s W_g^{j\,(s)} = F_j^{s-1}(J^2)\sum_{n=0}^{\infty}(J^+\otimes J^-)^n a^n_{\frac{j-J^0\otimes I+I\otimes J^0}{2}} \tag{A18}$$

$$\begin{aligned}
& \times\Gamma^s(j-J^0\otimes I+I\otimes J^0+2)\left(F_{j+I\otimes J^0}\left(J^2\otimes I\right)F_{j-J^0\otimes I}\left(I\otimes J^2\right)\right)^{1-s} \\
& \times\left(\frac{(j+I\otimes J^0+1)(j-J^0\otimes I+1)}{j+1}\right)^{(1-s)/2} W_f^{j+I\otimes J^0\;(s)}\otimes W_g^{j-J^0\otimes I\;(s)},
\end{aligned}$$

where the j-index is considered now as an operator according to the notation (A14) and the form of the operational function $a^n_{\frac{j-J^0\otimes I+I\otimes J^0}{2}}$ is given in (A12).

Theorem A1. *The star-product for self-dual symbols, $s=0$, satisfies the relation common for the Wigner-like distributions,*

$$\sum_{j=0,1\ldots}^{\infty}\frac{j+1}{8\pi^2}\int d\Theta W_f^{j(0)}(\Theta)*W_g^{j(0)}(\Theta)=\sum_{j=0,1\ldots}^{\infty}\frac{j+1}{8\pi^2}\int d\Theta W_f^{j(0)}(\Theta)W_g^{j(0)}(\Theta), \quad \text{(A19)}$$

where at least one of the symbols is normalized,

$$\sum_{j=0,1,\ldots}^{\infty}\frac{j+1}{8\pi^2}\int d\Theta W_f^{j(0)}(\Theta)=1,$$

and one of the symbols is a periodic function in Θ.

Proof of Theorem A1. Integrating by parts

$$\begin{aligned}
&\frac{j+1}{8\pi^2}\int d\Theta W_f^{j(0)}(\Theta)*W_g^{j(0)}(\Theta)\\
=\;&\frac{j+1}{8\pi^2}\int d\Theta F_j^{-1}\left(J^2\right)\sum_{n=0}^{\infty}\left(J^+\otimes J^-\right)^n\frac{(-1)^n}{n!(j+n-J^0\otimes I+1\otimes J^0+1)!}\\
&\times\left(\frac{\left(j+I\otimes J^0+1\right)\left(j-J^0\otimes I+1\right)}{j+1}\right)^{1/2}\sum_{j_1,j_2}\int\frac{d\varphi d\varphi'}{(2\pi)^2}e^{i\left(j_2-j+J^0\otimes I\right)\varphi}e^{i\left(j_1-j-I\otimes J^0\right)\varphi'}\\
&\times\left(F_{j+I\otimes J^0}\left(J^2\otimes I\right)W_f^{j_1(0)}(\Theta)\right)\left(F_{j-J^0\otimes I}\left(I\otimes J^2\right)W_g^{j_2(0)}(\Theta)\right),
\end{aligned}$$

and taking into account that under this operation it holds,

$$\begin{aligned}
F_j^{-1}\left(J^2\right)(j+1)&=((j+1)!j!)^{-1/2}(j+1),\\
I\otimes J^2&\to J^2\otimes I,\quad I\otimes J^0\to -J^0\otimes I,\\
\left(J^+\otimes J^-\right)^n&\to(-1)^n\left(J^-\right)^n\left(J^+\right)^n\otimes I,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{j+1}{8\pi^2}\int d\Theta W_f^{j(0)}(\Theta)*W_g^{j(0)}(\Theta)=\\
=\;&\frac{j+1}{8\pi^2}\int d\Theta\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(j-J^0\otimes I+1)j!}{(j+n-2J^0\otimes I+1)!}F^2_{j-J^0\otimes I}\left(I\otimes J^2\right)\left(J^-\right)^n\left(J^+\right)^n\\
&\times\sum_{j_1,j_2}\int\frac{d\varphi d\varphi'}{(2\pi)^2}e^{i\left(j_2-j+J^0\otimes I\right)\varphi}e^{i\left(j_1-j-I\otimes J^0\right)\varphi'}W_f^{j_1(0)}(\Theta)\otimes W_g^{j_2(0)}(\Theta).
\end{aligned}$$

□

Considering that

$$J^-J^+=J^0\left(J^0-I\right)-J^2,$$

and

$$J^2f(j,\Theta)=\sum_{k=0}\sum_{Q,Q'=-k}^{k}k(k+1)f^k_{Q,Q'}(j)D^k_{Q,Q'}(\Theta),$$

we have

$$\left(J^{-}\right)^{n}\left(J^{+}\right)^{n} f(j, \Theta)=(-1)^{n} \sum_{k} \frac{\left(k+J^{0}\right)!}{\left(k+J^{0}-n\right)!} \frac{\left(k-J^{0}+n\right)!}{\left(k-J^{0}\right)!} \sum_{Q, Q^{\prime}} f_{Q, Q^{\prime}}^{k}(j) D_{Q, Q^{\prime}}^{k}(\Theta),$$

so that the sum on n is computed as

$$\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!\left(j+n-2 J^{0} \otimes I+1\right)!} \frac{\left(k+J^{0} \otimes I\right)!}{\left(k+J^{0} \otimes I-n\right)!} \frac{\left(k-J^{0} \otimes I+n\right)!}{\left(k-J^{0} \otimes I\right)!}=j!F_{j-J^{0} \otimes I}^{-2},$$

where we have used the summation rule

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{k+m}{l}^{-1}=\frac{l}{n+l}\binom{n+m}{m-l}^{-1}.$$

Finally, we arrive at the following simplification

$$\sum_{j=0,1,..}^{\infty} \frac{j+1}{8 \pi^{2}} \int d \Theta W_{f}^{j(0)}(\Theta) * W_{g}^{j(0)}(\Theta) \tag{A20}$$

$$= \frac{1}{8 \pi^{2}} \sum_{j=0,1,..}^{\infty} \sum_{j_{1}, j_{2}} \int d \Theta \int \frac{d \varphi d \varphi^{\prime}}{(2 \pi)^{2}} e^{i\left(j_{2}-j+J^{0} \otimes I\right) \varphi} e^{i\left(j_{1}-j+J^{0} \otimes I\right) \varphi^{\prime}} \tag{A21}$$

$$\times\left[\left(j-J^{0} \otimes I+1\right) W_{f}^{j_{1}(0)}(\Theta)\right] \otimes W_{g}^{j_{2}(0)}(\Theta). \tag{A22}$$

After a formal integration over φ, φ' and summing up on j_1, j we attain Equation (A19).

Appendix B

In this Appendix we list some essential relations between the elements of $e(3)$ algebra: $\mathbf{r}=(x,y,z)$

$$\begin{aligned}
x &= -\cos\theta\cos\phi\cos\psi+\sin\phi\sin\psi, \\
y &= -\cos\theta\sin\phi\cos\psi-\cos\phi\sin\psi, \\
z &= \sin\theta\cos\psi;
\end{aligned}$$

$\mathbf{p}=\left(p_x,p_y,p_z\right)$

$$\begin{aligned}
p_x &= j_0(\cos\theta\cos\phi\sin\psi+\sin\phi\cos\psi), \\
p_y &= j_0(\cos\theta\sin\phi\sin\psi-\cos\phi\cos\psi), \\
p_z &= -j_0\sin\theta\sin\psi;
\end{aligned}$$

$\mathbf{L}=\left(L_x,L_y,L_z\right)$

$$\begin{aligned}
L_x &= j_0\sin\theta\cos\phi, \\
L_y &= j_0\sin\theta\sin\phi, \\
L_z &= j_0\cos\theta,
\end{aligned}$$

where

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}.$$

The Poisson brackets

$$\{f,g\} = \frac{\cot\theta}{j_0}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\psi} - \frac{\partial f}{\partial\psi}\frac{\partial g}{\partial\theta}\right) - \frac{1}{j_0\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\phi} - \frac{\partial f}{\partial\phi}\frac{\partial g}{\partial\theta}\right) + \left(\frac{\partial f}{\partial j_0}\frac{\partial g}{\partial\psi} - \frac{\partial f}{\partial\psi}\frac{\partial g}{\partial j_0}\right),$$

between elements of the algebra

$$\{r_i, r_j\} = 0,$$

$$\begin{aligned}\{f(j_0), g\} &= \frac{df(j_0)}{dj_0}\frac{\partial g}{\partial\psi}, \quad \{L_z, g\} = \frac{\partial g}{\partial\phi}\\ \{j_0, L_x\} &= 0, \qquad \{j_0, L_y\} = 0 \qquad \{j_0, L_z\} = 0.\end{aligned}$$

$$\begin{aligned}\{x,y\} &= 0, \quad \{y,z\} = 0, \quad \{z,x\} = 0;\\ \{L_x, L_y\} &= -L_z, \quad \{L_y, L_z\} = -L_x, \quad \{L_z, L_x\} = -L_y;\\ \{x_i, L_j\} &= \varepsilon_{ijk}x_k, \qquad \{p_i, L_j\} = \varepsilon_{ijk}p_k,\\ \{p_i, x_i\} &= \frac{p_i^2 + L_i^2}{j_0^2}, \qquad \{p_i, x_j\} = \frac{L_iL_j}{j_0^2}.\end{aligned}$$

Appendix C

In this Appendix we prove that the sum (47) converges to $L_z^{(0)} = \sqrt{j(j+1)}\cos\theta$. First, we separate (47) into positive and negative parts

$$\sum_{L=-\infty}^{\infty} = \sum_{L=0}^{\infty} + \sum_{L=-\infty}^{-1},$$

and observe that as a consequence of the relations

$$D^{-k-1}_{mm'}(\Theta) = (-1)^{m-m'}D^k_{mm'}(\Theta), \tag{A23}$$

$$C^{-c-1\gamma}_{-a-1\alpha-b-1\beta} = (-1)^{c-a-b}C^{c\gamma}_{a\alpha b\beta}, \tag{A24}$$

one has

$$\pi^{(0)}_{-L-1\ M}(\theta) = \sum_{k=0}^{2L}(-1)^k\frac{2k+1}{2L+1}\frac{\sqrt{\Gamma(2j+k+2)\Gamma(2j-k+1)}}{\Gamma(2j+1)\sqrt{2j+1}}C^{LL}_{LLk0}C^{LM}_{LMk0}P_k(\cos\theta),$$

where $P_k(\cos\theta)$ is the Legendre polynomial and C^{LL}_{LLk0} is given in (46). Then, for positive values of L, we obtain

$$\sum_{M=-L}^{L} M\pi^{(0)}_{L\ M}(\theta) = \sqrt{\frac{j+1}{j}}L\cos\theta,$$

while for $L \le -1$

$$\sum_{M=-L}^{L} M\pi^{(0)}_{-L-1\ M}(\Theta) = -\sqrt{\frac{j+1}{j}}(L+1)\cos\theta,$$

where we have used the relation

$$\sum_{M=-L}^{L}\frac{M}{2L+1}C^{LM}_{LMk0} = \frac{\sqrt{(L+1)L}}{3}\delta_{k,1}.$$

Finally we have,

$$\sum_{L=0}^{\infty} \frac{\sin(\pi(j-L))}{\pi(j-L)} \sum_{M=-L}^{L} M\pi_{LM}^{(0)}(\theta) = \sqrt{\frac{j+1}{j}} \sum_{L=0}^{\infty} \frac{\sin\pi(j-L)}{\pi(j-L)} L\cos\theta, \tag{A25}$$

and

$$\sum_{L=-\infty}^{-1} \frac{\sin\pi(j-L)}{\pi(j-L)} \sum_{M=-L}^{L} M\pi_{LM}^{(0)}(\theta) = -\sqrt{\frac{j+1}{j}} \sum_{L=0}^{\infty} \frac{\sin\pi(j+L+1)}{\pi(j+L+1)} (L+1)\cos\theta. \tag{A26}$$

Summing up (A25) and (A26) we arrive at

$$\sum_{L=-\infty}^{\infty} \frac{\sin(\pi(j-L))}{\pi(j-L)} \sum_{M=-L}^{L} M\pi_{LM}^{(0)}(\theta) = \sqrt{\frac{j+1}{j}} \sum_{L=-\infty}^{\infty} \frac{\sin\pi(j-L)}{\pi(j-L)} L\cos\theta = \sqrt{j(j+1)}\cos\theta.$$

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