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Higher Categorical Coherence Breakdown and the Dynamical Central Charge: Conceptual and Experimental Pathways via the Fractional Quantum Hall Effect

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Abstract

The central charge occupies a unique role in conformal field theory, simultaneously serving as a measure of degrees of freedom, as the determinant of Casimir energy through modular transformations, and as an obstruction to the naive extension of the Witt algebra. The Virasoro central extension itself is rigid: it fixes c as a label of a given conformal field theory. In this work, we propose that *higher* categorical coherence—the pentagon and hexagon constraints governing fusion and braiding data, one level above the cocycle responsible for the Virasoro extension—supplies an additional, physically controllable handle. We show that controlled deformations of this higher coherence (higher categorical coherence breakdown, HCCB), implemented consistently through anomaly inflow, shift the *effective* central charge read out by anomaly-sensitive observables in quantized steps, opening the possibility of treating the measured central charge not as a fixed label but as an experimentally addressable piecewise-quantized quantity. We then focus on the fractional quantum Hall effect (FQHE), where the chiral central charge c_- directly governs the quantized thermal Hall conductance. After reviewing the role of edge conformal field theories and current bounds on thermal transport, we propose experimental modifications—such as engineering multi-component edge states, coupling to non-Abelian quasiparticles, or introducing controlled categorical perturbations—that could render higher coherence breakdown detectable as shifts in the effective central charge. Two further elements complete the program. First, we show that within the consistent framework, all route- and bracketing-dependent observables vanish identically (*route blindness*), so that the pentagon and hexagon interferometers and thermal Y-junction networks we design operate as precision *null tests* of the modular-functor axioms themselves—the axioms stating that anyonic amplitudes are determined by the topology of a process rather than by the bookkeeping route used to compose it. Second, we show that a quantized remnant of route sensitivity survives in exactly one consistent form: the holonomy of *closed cycles* of categorical controls, realizing a central-charge pump for which the integer count per cycle is a family invariant beyond any static stacking description. The resulting framework provides both a conceptual reinterpretation of the central charge as a higher obstruction in categorical terms and a concrete experimental route for probing its dynamical behavior. Beyond the quantum Hall setting, these ideas suggest a broader program: anomalies, topological phases, and even string worldsheet central charges may admit reinterpretation through higher coherence. We conclude by outlining a research agenda in which categorical methods yield new experimental observables, potentially transforming the interplay between mathematics, condensed matter physics, and high-energy theory.



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1. Introduction

1.1. Motivation

Two-dimensional conformal field theory (2D CFT) supplies a remarkably rigid framework in which universal data control both algebraic structure and measurable physics [1]. Among these data, the *central charge* c plays a unifying role: it governs the Virasoro algebra that organizes local operators, fixes universal finite-size and finite-temperature contributions to thermodynamics, and in chiral settings it controls thermal Hall transport in topological phases. At the same time, c is not merely a parameter but an *obstruction* (an anomaly) that signals the impossibility of realizing perfect classical symmetries after quantization [2–4].

This paper advances a new mechanism—*higher categorical coherence breakdown* (HCCB)—that can *shift the effective central charge* observed in appropriate probes. Very roughly, HCCB means that some of the higher coherence laws in the tensor-categorical data underlying a CFT (associators, braidings, and their higher compatibilities) fail to hold strictly in an effective description, being replaced by controlled, quantized k -cocycle defects (e.g., pentagon/hexagon deviations). Our proposal is operational: these defects feed the same anomaly channels that c inhabits, shifting the vacuum energy on curved backgrounds and the chiral energy current at edges, and thereby producing a measurable change in the effective c inferred from thermodynamics, modular asymptotics, and thermal Hall response.

1.2. The Central Charge as a Unifying Invariant in CFT

The local operator algebra of a 2D CFT carries a projective action of the Witt algebra that lifts uniquely to a *central extension*, the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad (1)$$

with scalar central term c [2–4]. This central extension classifies a cohomology obstruction $[\omega] \in H^2(\text{Witt}, \mathbb{C}) \cong \mathbb{C}$; the coefficient is the central charge [3,4].

A first universal footprint of c is the vacuum (Casimir) energy on a cylinder of circumference L . Under the exponential map $z = e^{\frac{2\pi}{L}w}$, the stress tensor transforms with a Schwarzian derivative, producing the standard shift

$$\langle T(w) \rangle = \frac{c}{12}\{z, w\} = -\frac{\pi^2 c}{6L^2}, \quad \implies \quad E_0 = -\frac{\pi c}{6L}v, \quad (2)$$

where v is the characteristic mode velocity. This universal finite-size amplitude is the content of the Blöte–Cardy–Nightingale–Affleck analysis [5,6].

A second universal footprint is encoded in modular properties. Writing the torus partition function as

$$Z(\tau, \bar{\tau}) = \text{Tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad q = e^{2\pi i \tau} \quad (3)$$

and using modular covariance, Cardy's argument yields the asymptotic density of states and the microcanonical entropy

$$S(\Delta) \sim 2\pi\sqrt{\frac{c_{\text{eff}}\Delta}{6}}, \quad c_{\text{eff}} = c - 24h_{\text{min}}, \quad (4)$$

with h_{min} the minimal left conformal weight (for unitary CFTs $h_{\text{min}} = 0$, so $c_{\text{eff}} = c$) [7,8]. The full modular/fusion consistency is packaged in the Verlinde and Moore–Seiberg frameworks, where pentagon/hexagon identities enforce OPE associativity and braiding compatibility [9,10]. These are precisely the coherence relations for which we will parameterize the controlled violation (HCCB).

1.3. The Dual Role: Counting Degrees of Freedom vs. Representing Obstructions

On the *counting* side, Zamolodchikov's c -theorem identifies c as a monotone counting function along RG flows in 2D QFT, with fixed-point values matching the CFT central charge [11]. Quantum information-theoretic probes offer a complementary view: in a 1+1D critical ground state, the von Neumann entropy of an interval of length ℓ scales as

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + \dots, \quad (5)$$

with UV cutoff a [12,13]. Thus, c controls both thermodynamics and entanglement, justifying the “degrees-of-freedom” interpretation.

On the *obstruction* side, c measures a trace (Weyl) anomaly: on curved 2D backgrounds,

$$\langle T^\mu{}_\mu \rangle = -\frac{c}{12}R, \quad (6)$$

relating c to a gravitational anomaly and to the central term of Virasoro via Ward identities [14]. In chiral theories, the *chiral central charge*

$$c_- = c_R - c_L \quad (7)$$

controls a protected edge energy current and fixes the *quantized thermal Hall conductance*

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_- T \quad (8)$$

in appropriate low- T regimes [14–17]. This connects c directly to transport anomalies and bulk–boundary correspondence.

1.4. Higher Categorical Coherence Breakdown (HCCB) as a New Mechanism Capable of Shifting Effective c

Rational CFTs (and many irrational ones at the level of topological defects) carry the structure of a braided tensor category for chiral sectors and of a modular tensor category in gapped topological phases; their consistency is governed by categorical coherence—pentagon and hexagon identities for the associator and braiding [9,10]. From the categorical viewpoint, these identities are the vanishing of certain obstruction classes (e.g., 3-cocycles for associativity) and their higher analogues.

We define *HCCB* as a situation where, in an effective (e.g., boundary or defect) description, these coherence conditions are deformed by quantized cocycles that cannot be removed by redefinitions, while locality and unitarity are preserved in the physical correlators. Mathematically, HCCB lives in the same anomaly pipeline as the Virasoro/Weyl anomaly: controlled coherence defects backreact on (i) the Schwarzian term responsible

for vacuum energy and (ii) the inflow that fixes chiral edge energy currents. Physically, the upshot is a *shift*

$$c \mapsto c_{\text{eff}}, \quad c_- \mapsto c_{-, \text{eff}}, \quad (9)$$

in the values inferred from (i) finite-size/finite-temperature free energy and Cardy asymptotics and (ii) thermal Hall metrology. The mechanism is precise enough to yield testable predictions once one has specified which coherence law is broken and by what quantized amount; the formal definitions are developed in Section 2.3 and consolidated at the structural level in Section 7.3.

1.5. Scope

Conceptual analysis of c as physical and algebraic quantity. We begin with a self-contained derivation of the appearance of c in (a) vacuum (Casimir) energies via the Schwarzian (and its cylinder mapping) and (b) high-temperature asymptotics via modular S -transformations and Cardy's argument, including the role of c_{eff} in non-unitary or extended settings [5–10]. We then recast c categorically as an obstruction class tied to central extensions and anomalies [2–4,14].

Application to the fractional quantum Hall effect (FQHE). Next, we focus on chiral edge theories of FQHE states, modeled as chiral Luttinger liquids or more intricate RCFTs [18–20]. In this context, c_- directly controls κ_{xy} , with the universal law $\kappa_{xy} = (\pi^2 k_B^2 / 3h) c_- T$ [14–17]. We review the experimental landscape, including the observation of half-integer thermal Hall conductance at $\nu = 5/2$ that highlights the metrological sensitivity to chiral central charge [21].

Proposal of experimental modifications to detect HCCB-induced shifts in effective c . Finally, we design modifications of standard thermal Hall and finite-size/finite-temperature protocols on quantum Hall edges that (i) couple selectively to the putative coherence defect (e.g., through engineered domain walls of anyon condensation that alter associators), (ii) separate c -shifts from velocity renormalizations and non-equilibrated modes, and (iii) cross-check via independent Casimir/entanglement probes. The net result is a falsifiable program: a categorical deformation with a quantized defect strength predicts a corresponding step in $c_{-, \text{eff}}$ extracted from κ_{xy} , with a matching shift in the Casimir amplitude and in Cardy-like degeneracy growth.

Outlook. The central charge sits at a confluence of algebra and experiment—counting degrees of freedom while encoding an anomaly. By elevating higher coherence to a physical control knob (HCCB), we add a new axis along which c can move in effective descriptions. The remainder of the paper turns this into precise statements, culminating in experimentally testable predictions on FQHE edges.

What is new and operationally distinct. Before proceeding, it is useful to clarify in what precise sense the present work goes beyond a reinterpretation of known anomaly physics. While it is true that a shift of the chiral central charge can be described abstractly as stacking the edge theory with an invertible $(2+1)$ -dimensional topological phase, our contribution is not the statement of this equivalence but the identification of *boundary-level, controllable mechanisms* by which such shifts are induced, detected, and disentangled from conventional edge renormalizations.

Concretely, higher categorical coherence breakdown (HCCB) provides a systematic way to generate quantized scalar anomalies in the mapping-class group action of edge conformal blocks through controlled deformations of the coherence data, realized physically by engineered domain walls, anyon-condensate interfaces, and ancillary topological channels. These scalar anomalies propagate uniquely into the Dehn twist, and hence into the effective central charge, without modifying local operator content, edge velocities, or fusion rules. As a result, the predicted shifts in c and c_- are simultaneously visible in

three independent channels: modular thermodynamics (Casimir amplitudes), thermal Hall transport, and interference phases in multi-junction geometries.

The contributions of this paper can then be stated as four items. The first two are theorems within the consistent framework; the third converts an axiom into an experimental program; and the fourth identifies the unique consistent survivor of route sensitivity. We state them here at length because together they define what is genuinely new in this work.

(1) *The dynamical central charge.* Any HCCB deformation consistent with anomaly inflow produces quantized steps of $c_{-,eff}$ —in units of $\frac{1}{2}$ for spin sectors, 8 for purely bosonic stacking—when a categorical control (a domain wall, an ancilla phase, a trapped topological sector) is toggled. The same shift appears simultaneously in three independent channels (Casimir amplitudes, thermal Hall slope, interference phases), reverses under chirality inversion, and is isolated by difference measurements that cancel equilibration and phonon backgrounds. The central charge thereby becomes an experimentally *addressable* observable: piecewise rigid, but movable in quantized steps by in situ controls.

(2) *Route blindness as a theorem.* Within the consistent framework, *no* observable can depend on the order or bracketing of topologically equivalent edge operations: any two sequences of elementary processes implementing the same topological operation carry identical amplitudes, and every closed rebracketing or reordering loop evaluates to unity (Section 5.1). Intuitively, the fusion and braiding data of an anyon theory are not dynamical quantities that a device could bend slightly; they are part of the *definition* of the theory, in the same way associativity is part of what one means by addition. The only quantities that survive all consistency conditions are attached to globally non-trivial operations—the Dehn twist (hence Δc_-) and, as item (4) explains, non-contractible cycles in control space. Route blindness is thus the categorical counterpart of Mac Lane coherence, promoted here from an algebraic lemma to an experimentally enforceable statement about transport and interference.

(3) *Multi-junction geometries as null tests of the modular-functor axioms.* The pentagon and hexagon interferometers and the thermal Y-junction networks of Sections 5.2 and 6.4 realize the elementary coherence diagrams of the categorical framework (sketched diagrammatically in Appendix B) directly in gate-defined hardware. Because of item (2), the consistent framework predicts *exact nulls* for all of them: $\varepsilon_F = \varepsilon_H = 0$, and no bracketing dependence of heat flow. This is not a weakness of the proposal but its sharpest edge: the devices test the modular-functor axioms themselves—the statements that a chiral edge theory assigns well-defined amplitude spaces to anyon configurations, that these compose locally under cutting and sewing, and that amplitudes depend only on the topology of a process and never on the bookkeeping route used to compose it (the axioms are unpacked physically in Section 5.1). These axioms are the formal backbone of the topological interpretation of the fractional quantum Hall effect, of the universality of its quantized responses, and of all schemes for topological quantum computation; to our knowledge, they have never been the direct target of a dedicated laboratory test. A reproducible nonzero outcome—surviving the chirality-reversal, junction-scaling, and disorder diagnostics we specify—would therefore not measure a parameter of the present theory; it would falsify the axioms, with the physical implications spelled out in Section 6.4. Conversely, null results convert existing interferometric and calorimetric precision into the first quantitative laboratory bounds on the pentagon and hexagon identities.

(4) *The central-charge pump: route sensitivity regained, in the only consistent form.* Route blindness governs *closed comparisons at fixed couplings*; it does not forbid path dependence along *open* trajectories in the space of control parameters. Indeed, as a device's categorical controls λ are varied, the instantaneous $c_{-,eff}(\lambda)$ traces a path-dependent profile—which sectors are visited, and in which order, depends on the route taken through control space.

What consistency enforces is only that the *net* effect of a closed cycle is a homotopy invariant: a quantized integer, the winding of the cycle around the loci in control space where the invertible sector changes. This is precisely the logic of parallel transport with a flat connection—local route sensitivity, global holonomy—and it identifies a new observable, the *central-charge pump* of Sections 5.2.10 and 7.2: a quantized orientation-odd transfer of gravitational anomaly per control cycle, the Thouless pump of the anomaly coefficient. The pump is route-blind in the only way consistency permits (it depends on the homotopy class of the loop, not its parameterization) and yet is invisible to any description of a single static configuration; in a precise sense, it is the legitimate heir of the route-dependence intuition, relocated from real space to control space.

In this sense, HCCB elevates higher categorical coherence from a consistency requirement to a tunable physical control knob, and promotes the central charge from a fixed label of a conformal field theory to a quantized and experimentally addressable observable.

Reader's Guide: How the conformal field theory background connects to the quantum Hall application. Because the argument of this paper crosses from algebra to metrology, we state its logical spine here explicitly. Sections 2 and 3 are a self-contained review with a single purpose: to establish that the central charge is *simultaneously* (a) the coefficient of the Virasoro 2-cocycle, (b) the coefficient of the Schwarzian/Casimir and Cardy amplitudes, and (c) the coefficient of the Weyl/gravitational anomaly, so that *any* mechanism that feeds the anomaly channel must show up identically in spectral, thermodynamic, and transport observables. Section 4 then specializes to the one experimental platform where a chiral central charge is directly and routinely measured: quantum Hall edges, for which the low-energy description as chiral conformal field theories underlies the quantized thermal Hall data. We emphasize that the FQHE enters this work *only* through (i) the measured quantization of κ_{xy} and (ii) the edge conformal-field-theory description validated by those measurements; no result below depends on the choice of bulk microscopic theory. Sections 5 and 6 develop the HCCB mechanism on this platform and derive its observable consequences, separating rigorous (inflow-consistent) predictions from hypotheses to be bounded. Section 7 consolidates the structural definition, the quantization domains, and the extensions beyond the Hall setting.

Relation to companion work. The present paper is the condensed-matter and anomaly-theoretic member of a broader program developing the physics of categorical coherence and its controlled breakdown. In Ref. [22] the breakdown of coherence beyond first order is developed as a general geometric framework for nonlinear extensions of quantum mechanics, with the Uhlmann connection on the purification bundle implementing first-order (commutator-level) structure and higher coherence levels organizing genuinely nonlinear corrections relevant to strongly correlated electrons. Ref. [23] applies the categorical-mechanics viewpoint to bounded molecular systems, where coherence constraints organize the accessible state space of chemical dynamics. Neither companion paper concerns anomalies, modular data, or transport. The distinct contribution of the present work is to identify the regime in which higher coherence connects to a *quantized, experimentally measurable* observable: the chiral central charge read out by thermal Hall transport, with the two-tier structure (consistent quantized shifts versus bounded coherence failure) developed in Sections 2–7. Thus, the three papers share a structural language but address disjoint physical questions with independent results.

2. Conformal Field Theory Background

2.1. The Virasoro Algebra and the Witt Operators

From circle diffeomorphisms to the Witt algebra. Holomorphic reparameterizations of the complex plane (or, more precisely, of the Riemann sphere minus ∞) are generated by vector fields

$$\ell_n = -z^{n+1} \partial_z, \quad n \in \mathbb{Z}. \quad (10)$$

They satisfy the Lie bracket

$$[\ell_m, \ell_n] = (m - n) \ell_{m+n}, \quad (11)$$

which defines the *Witt algebra* of meromorphic vector fields on the circle S^1 (or the punctured plane). Equation (11) is verified by a direct calculation using $[\partial_z, z] = 1$:

$$[\ell_m, \ell_n] = [-z^{m+1} \partial_z, -z^{n+1} \partial_z] = (m - n) z^{m+n+1} \partial_z = (m - n) \ell_{m+n}.$$

The Witt algebra exponentiates (on suitable domains) to the orientation-preserving diffeomorphism group $\text{Diff}^+(S^1)$; upon quantization; however, only a *projective* action is typically realized on the Hilbert space, and the group acquires a non-trivial $U(1)$ central extension (the Bott–Virasoro group). This central extension differentiates to the *Virasoro algebra* [24,25].

Stress tensor, modes, and the Noether charge construction. In radial quantization, the holomorphic component of the stress tensor has the Laurent expansion

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}, \quad L_n = \oint_{|z|=\rho} \frac{dz}{2\pi i} z^{n+1} T(z), \quad (12)$$

for any ρ in the punctured plane. Infinitesimal conformal transformations $z \mapsto z + \epsilon(z)$ are generated by

$$Q[\epsilon] = \oint \frac{dz}{2\pi i} \epsilon(z) T(z), \quad \delta_\epsilon \Phi(w) = [Q[\epsilon], \Phi(w)], \quad (13)$$

so the specific choice $\epsilon_n(z) = -z^{n+1}$ gives the Witt generators $L_n \equiv Q[\epsilon_n]$. The Ward identity/OPE for a primary field Φ of weight h ,

$$T(z) \Phi(w) \sim \frac{h \Phi(w)}{(z-w)^2} + \frac{\partial \Phi(w)}{z-w} + \dots, \quad (14)$$

yields the expected action of the modes:

$$[L_n, \Phi(w)] = (w^{n+1} \partial_w + (n+1)h w^n) \Phi(w), \quad (15)$$

which exponentiates to the conformal transformation law of primaries [25,26].

Virasoro from the TT OPE: detailed derivation. Quantum effects shift the algebra of the Noether charges by a central term. The universal short-distance structure of stress tensor correlators is encoded in the *TT OPE*

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots, \quad (16)$$

for which the coefficient c is the *central charge*. Using the contour representation (12), we compute

$$\begin{aligned} [L_m, L_n] &= \oint_w \frac{dw}{2\pi i} w^{n+1} \oint_z \frac{dz}{2\pi i} z^{m+1} (T(z)T(w) - T(w)T(z)) \\ &= \oint_w \frac{dw}{2\pi i} w^{n+1} \oint_z \frac{dz}{2\pi i} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right). \end{aligned} \quad (17)$$

We evaluate the z -integral by the standard residue identity

$$\oint_{z=w} \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^k} = \frac{1}{(k-1)!} \partial_w^{k-1} f(w) \quad (k \geq 1), \quad (18)$$

applied to $f(z) = z^{m+1}$. For the central piece ($k = 4$), we obtain

$$\frac{c}{2} \cdot \frac{1}{3!} \partial_w^3 w^{m+1} = \frac{c}{12} (m+1)m(m-1) w^{m-2}.$$

For the operator pieces,

$$\frac{2}{1!} \partial_w (w^{m+1}) T(w) = 2(m+1)w^m T(w), \quad \frac{1}{0!} w^{m+1} \partial T(w).$$

Combining and integrating over w (picking Laurent coefficients by $\oint dw w^k = 2\pi i \delta_{k,-1}$), we arrive at

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}, \quad (19)$$

the *Virasoro algebra* [25]. The first term reproduces the Witt algebra (11); the second is a central extension.

Central extension as a 2-cocycle (cohomological viewpoint). Algebraically, (19) corresponds to adjoining a central element K with $[K, L_n] = 0$ and a 2-cocycle

$$\omega(\ell_m, \ell_n) = \frac{1}{12} (m^3 - m) \delta_{m+n,0}.$$

The cocycle condition $\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0$ holds (checkable directly on the basis $\{\ell_n\}$), and ω is not a coboundary on the Witt algebra; therefore, the extension is non-trivial and unique up to rescaling of K (absorbed into c). On the group level, this is the Bott central extension of $\text{Diff}^+(S^1)$; see [24] for a detailed construction.

Transformation of T and the Schwarzian derivative. Under a holomorphic change of coordinates $z \mapsto w = f(z)$, the quantum stress tensor transforms as a projective connection:

$$T(z) = (f'(z))^2 T(f(z)) + \frac{c}{12} \{f, z\}, \quad \{f, z\} \equiv \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2, \quad (20)$$

where the Schwarzian $\{f, z\}$ encodes the same anomaly that appears in the central term of (19). Mapping the plane to the cylinder $z = e^{\frac{2\pi}{L}w}$ and using (20) gives the universal Casimir energy (and finite-size corrections) of the vacuum, a standard exercise in radial quantization [25,27].

Free-field realizations and concrete values of c . The TT OPE (16) can be derived in explicit models, making the central term concrete:

- *Free boson ϕ* with two-point function $\langle \phi(z)\phi(w) \rangle = -\log(z-w)$ has

$$T(z) = -\frac{1}{2} : \partial \phi \partial \phi :. \quad (21)$$

Using Wick contractions $\langle \partial\phi(z)\partial\phi(w) \rangle = (z-w)^{-2}$ and keeping double contractions in $T(z)T(w)$ yields the quartic pole $(z-w)^{-4}$ with coefficient $c/2 = 1/2$; hence $c = 1$ [25].

- *Majorana fermion* ψ with OPE $\psi(z)\psi(w) \sim (z-w)^{-1}$ and

$$T(z) = -\frac{1}{2} : \psi \partial\psi : \quad (22)$$

produces $c = \frac{1}{2}$ [25].

- *bc ghost system* with weights $(h_b, h_c) = (2, -1)$ (arising in bosonic string gauge fixing) has $c = -26$; the negative value reflects the ghost nature and is required to cancel the D -boson contribution $c = D$ at the critical dimension $D = 26$ [27].

Sugawara construction and cosets: further sources of c . When a theory has conserved affine currents $J^a(z)$ at level k for a compact simple Lie algebra $\mathfrak{g}$, the stress tensor can be built from the currents (Sugawara):

$$T^{\mathfrak{g}}(z) = \frac{1}{2(k+h^{\vee})} : J^a J^a :, \quad c^{\mathfrak{g}} = \frac{k \dim \mathfrak{g}}{k+h^{\vee}}, \quad (23)$$

with $h^{\vee}$ the dual Coxeter number [28,29]. If a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is gauged (coset construction), the stress tensor and central charge subtract:

$$T^{\mathfrak{g}/\mathfrak{h}}(z) = T^{\mathfrak{g}}(z) - T^{\mathfrak{h}}(z), \quad c^{\mathfrak{g}/\mathfrak{h}} = c^{\mathfrak{g}} - c^{\mathfrak{h}}, \quad (24)$$

providing a large family of rational CFTs for which the representation theory is governed by Virasoro with specific c (and in many cases additional chiral algebras) [30,31].

Highest-weight representations and minimal models (intuition). A Virasoro highest-weight module $\mathcal{V}_{c,h}$ is generated from a vector $|h\rangle$ with $L_0|h\rangle = h|h\rangle$, $L_{n>0}|h\rangle = 0$ by lowering operators L_{-n} ($n > 0$). For certain rational values of (c, h) , null vectors appear and quotient modules are unitary; this leads to the discrete series (minimal models), for which the critical exponents match two-dimensional lattice universality classes [32]. Although we will not need the explicit Kac table in this subsection, this representation-theoretic control underpins the universal role of c .

Physical intuition and anomaly inflow. Classically, the Witt algebra (11) reflects local conformal symmetry. Quantum mechanically, short-distance regularization (normal ordering) injects a *Schwinger term*—the central piece in (19). The same coefficient c reappears (i) in the Schwarzian shift (20) (Casimir energy, modular thermodynamics) and (ii) in edge energy currents of chiral systems. This ubiquity is not accidental; it can be traced to anomaly inflow and projective actions of diffeomorphisms. As an illustration beyond 2D matter, the asymptotic symmetry group of AdS_3 gravity contains two commuting Virasoro algebras with classical central charge $c = \frac{3\ell}{2G}$, showing that c can also have a geometric origin [33].

In summary, the Witt algebra organizes infinitesimal conformal transformations; quantization produces its unique central extension, the Virasoro algebra (19). The coefficient c governs both operator algebra (through (16)) and physics (through (20)), while concrete realizations (free fields, current algebras, cosets) showcase its calculability. These facts will be the starting point for our higher-categorical reinterpretation in later sections.

2.2. Interpretations of the Central Charge

The central charge c organizes universal data in two-dimensional conformal field theory (CFT). In this subsection, we explicate three complementary interpretations: (i) c as an effective count of degrees of freedom, (ii) c as the determinant of Casimir (finite-size and finite-temperature) amplitudes through modular properties, and (iii) c as an obstruction

(anomaly) encoded in the projective nature of the diffeomorphism action. We provide complete derivations of the relevant formulas and indicate the minimal assumptions required in each case. Our discussion relies on modular invariance and on the replica construction for entanglement, with consistency constraints summarized by the classification of modular-invariant partition functions [34].

2.2.1. Degrees of Freedom from Modular Asymptotics (Cardy Growth)

Consider a unitary and compact 2D CFT on a circle of length L with Hamiltonian

$$H = \frac{2\pi v}{L} \left(L_0 + \bar{L}_0 - \frac{c + \bar{c}}{12} \right), \quad (25)$$

where v is the characteristic mode velocity (we will set $v = 1$ when convenient and reinstate it at the end). The torus partition function at inverse temperature β is

$$Z(\tau, \bar{\tau}) = \text{Tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right), \quad q = e^{2\pi i \tau}, \quad \tau = \frac{i\beta v}{L}. \quad (26)$$

Modular invariance states $Z(\tau, \bar{\tau}) = Z(-1/\tau, -1/\bar{\tau})$; in particular, the high-temperature regime $\beta \rightarrow 0^+$ (small $\text{Im } \tau$) is controlled by the vacuum sector of the S -transformed channel (large $\text{Im } (-1/\tau)$). Keeping only the vacuum contribution after an S transform yields

$$\log Z(\beta) = \frac{\pi^2}{3} \frac{(c + \bar{c}) L}{\beta v} + o\left(\frac{L}{\beta v}\right), \quad (27)$$

a statement of universality that descends from modular covariance and the classification of admissible partition functions [34].

To obtain the asymptotic microcanonical density of states, write $Z(\beta) = \int_0^\infty dE \rho(E) e^{-\beta E}$ with E measured relative to the vacuum. A Laplace inversion with the leading term (27) gives the large- E behavior via a saddle point:

$$\rho(E) \sim \int \frac{d\beta}{2\pi i} \exp\left(\beta E + \frac{\pi^2}{3} \frac{(c + \bar{c}) L}{\beta v}\right) \approx \exp\left(2\sqrt{\frac{\pi^2}{3} \frac{(c + \bar{c}) L}{v}} E\right), \quad (28)$$

where the saddle occurs at $\beta_* = \sqrt{\frac{\pi^2(c+\bar{c})L}{3vE}}$. Using (25), the energy above the vacuum for a state with $L_0 + \bar{L}_0 = \Delta$ is

$$E = \frac{2\pi v}{L} \left(\Delta - \frac{c + \bar{c}}{12} \right).$$

Eliminating E in (28) gives the *Cardy formula* for the microcanonical degeneracy at large conformal weight:

$$\log \rho(\Delta) \sim 2\pi \sqrt{\frac{(c_{\text{eff}} + \bar{c}_{\text{eff}}) \Delta}{6}}, \quad c_{\text{eff}} \equiv c - 24 h_{\text{min}}, \quad (29)$$

with h_{min} the minimal left conformal weight of states that contribute to Z (for unitary theories on the plane, $h_{\text{min}} = 0$ so $c_{\text{eff}} = c$). Equation (29) demonstrates that c counts the effective degrees of freedom that control the exponential growth of the state density; modular consistency of Z is the only required input [34]. Refinements and rigorous bounds that constrain low-lying operator dimensions in terms of c —further cementing its role as a counting parameter—are provided by modular bootstrap methods [35].

Entanglement perspective. For a single interval of length ℓ in the ground state of a 1+1D CFT, the replica method computes the Rényi entropies S_n by mapping the n -sheeted Riemann surface to the plane. The branch points are twist fields of conformal weight

$$\Delta_n = \frac{c}{12} \left(n - \frac{1}{n} \right), \quad (30)$$

so the two-point function $\langle \mathcal{T}_n(0) \bar{\mathcal{T}}_n(\ell) \rangle \propto \ell^{-2\Delta_n}$ and

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z_1^n} = \frac{c}{6} \left(1 + \frac{1}{n} \right) \log \frac{\ell}{a} + \text{const.} \quad (31)$$

Taking $n \rightarrow 1$ yields the von Neumann entropy

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + s_1, \quad (32)$$

where a is a UV cutoff and s_1 is non-universal. Equation (32) provides an independent information-theoretic *operational* meaning of c as the coefficient of universal entanglement growth at criticality [36,37].

2.2.2. Casimir Energy and Modular Thermodynamics

The same c fixes universal finite-size and thermal amplitudes. On a cylinder of circumference L , the conformal map $z = e^{\alpha w}$ with $\alpha = \frac{2\pi}{L}$ gives the Schwarzian derivative

$$\{z, w\} = \frac{z'''}{z'} - \frac{3}{2} \left(\frac{z''}{z'} \right)^2 = -\frac{\alpha^2}{2},$$

so the stress tensor transforms as a projective connection,

$$T(w) = (\partial_w z)^2 T(z) + \frac{c}{12} \{z, w\} \implies \langle T(w) \rangle_{\text{cyl}} = -\frac{\pi^2 c}{6L^2}. \quad (33)$$

Using $H = \frac{2\pi v}{L} (L_0 + \bar{L}_0 - \frac{c+\bar{c}}{12})$ and $\langle L_0 \rangle = \langle \bar{L}_0 \rangle = 0$ in the vacuum, the *Casimir energy* is

$$E_0 = -\frac{\pi v}{6L} (c + \bar{c}), \quad (34)$$

and the corresponding ground-state momentum is $P_0 = \frac{2\pi}{L} (h - \bar{h}) = 0$ for the vacuum.

At finite temperature, modular invariance of the torus partition function with $\tau = \frac{i\beta v}{L}$ implies the leading high-temperature asymptotics (27). Since $F = -\beta^{-1} \log Z$, one finds the universal free-energy density (setting $k_B = \hbar = 1$)

$$\frac{F(T)}{L} = -\frac{\pi}{6} \frac{(c + \bar{c})}{v} T^2 + o(T^2), \quad \epsilon(T) \equiv \frac{\langle H \rangle}{L} = \frac{\pi}{6} \frac{(c + \bar{c})}{v} T^2 + o(T^2), \quad (35)$$

and hence a specific heat per unit length $C/L = \frac{\partial \epsilon}{\partial T} = \frac{\pi}{3} \frac{(c + \bar{c})}{v} T$. These relations provide a thermodynamic route to measure c from finite-size/finite-temperature data, including in systems with boundaries and defects [38,39].

Modular consistency as the organizing principle. Equations (34) and (35) follow from the same modular covariance that underpins (29). The space of consistent (unitary, modular-invariant) partition functions is *highly constrained*; e.g., in rational theories, the Cappelli–Itzykson–Zuber (CIZ) classification demonstrates how modular invariants determine the spectrum, and hence the vacuum dominance in the S -channel that leads to (27) [34]. Thus, c controls both spectral growth and thermodynamics by the same structural input.

2.2.3. Obstruction: Central Charge as an Anomaly

Classically, conformal invariance implies a traceless stress tensor; quantum mechanically, regularization induces an anomalous trace. On a general 2D background (Σ, g) , the Weyl variation of the generating functional $W[g] = -\log Z[g]$ is

$$\delta_\sigma W[g] = -\frac{c}{24\pi} \int_\Sigma d^2x \sqrt{g} \sigma R[g], \quad \Longleftrightarrow \quad \langle T^\mu{}_\mu \rangle = -\frac{c}{12} R[g], \quad (36)$$

where $R[g]$ is the Ricci scalar and σ is the local scale factor ($g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$). Equation (36) is equivalent to the appearance of the Schwarzian in the transformation of T and to the central term in the Virasoro algebra; in this sense, c is an *obstruction class* measuring the failure of the diffeomorphism/Weyl symmetry to be realized non-projectively at the quantum level [40].

For chiral theories ($c \neq \bar{c}$), there is in addition a *gravitational (diffeomorphism) anomaly* proportional to $c_- \equiv c - \bar{c}$, which controls the anomalous conservation of the energy current and underlies the protected chiral transport on edges of topological phases. The cohomological classification of such anomalies and their inflow mechanisms was elucidated in the classic work on gravitational anomalies [41]. In particular, anomaly matching ensures that the same coefficient that enters (36) also governs the universal responses obtained by coupling the theory to background geometry; this is the geometric origin of the Casimir and thermal amplitudes in (34) and (35).

2.2.4. Synthesis and Operational Dictionary

We may summarize the three lenses as follows:

(Counting) :	$\log \rho(\Delta) \sim 2\pi \sqrt{\frac{(c_{\text{eff}} + \bar{c}_{\text{eff}}) \Delta}{6}}$	(modular asymptotics);
(Thermodynamics) :	$E_0 = -\frac{\pi v}{6L}(c + \bar{c}), \quad \frac{F(T)}{L} = -\frac{\pi(c + \bar{c})}{6v} T^2$	(Casimir/thermal universals);
(Obstruction) :	$\langle T^\mu{}_\mu \rangle = -\frac{c}{12} R[g],$	(Weyl/gravitational anomalies);
(Information) :	$S_A = \frac{c}{3} \log \frac{\ell}{a} + s_1,$	(universal entanglement).

These four entries rest, respectively, on modular asymptotics and the modular bootstrap [34,35], on Casimir and thermal universality [38,39], on the Weyl and gravitational anomalies [40,41], and on universal entanglement scaling [36,37]. Each equality identifies an independent measurement channel for c . Later sections will exploit this dictionary in the presence of higher categorical coherence breakdown: coherence defects will enter the anomaly pipeline, and as such appear simultaneously in spectral growth, thermodynamics, and information-theoretic observables.

2.3. Higher Categorical Coherence Breakdown in CFT

In two-dimensional conformal field theory (CFT), the analytic structure of correlation functions is encoded by chiral data: fusion of primary fields, braiding of conformal blocks, and their consistency. This data admit a natural mathematical organization in terms of (braided) tensor categories. In such a category $\mathcal{C}$, the tensor product $\otimes$ is associative only up to a specified natural isomorphism (the *associator*)

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\cong} a \otimes (b \otimes c),$$

together with unitors $\lambda_a : \mathbf{1} \otimes a \cong a$ and $\rho_a : a \otimes \mathbf{1} \cong a$, and if present, a braiding $c_{a,b} : a \otimes b \rightarrow b \otimes a$. These isomorphisms are constrained by *coherence diagrams*, most prominently the triangle (compatibility with the unit) and the pentagon (compatibility of

the associator with itself); in the braided case, hexagon diagrams express compatibility of braiding with associativity. Physically, these constraints are the algebraic shadow of *OPE associativity* and *crossing symmetry* of four-point functions as well as the consistency of braiding monodromies of conformal blocks [42–44].

The goal of this subsection is twofold: (i) to recast OPE associativity and braiding in the language of coherence (triangle, pentagon, hexagon) and make precise how small quantized departures from strict coherence can be parameterized; and (ii) to show how such departures, which we call *higher categorical coherence breakdown* (HCCB), feed into the same anomaly pipeline as the Virasoro/Weyl anomaly, thereby shifting the *effective* central charge measured by modular thermodynamics and by chiral transport.

2.3.1. Coherence in Monoidal and Braided Tensor Categories

A (strict) monoidal category is one where $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ and $a \otimes \mathbf{1} = \mathbf{1} \otimes a = a$ on the nose; general monoidal categories replace these equalities with coherent isomorphisms α, λ, ρ . Mac Lane’s coherence theorem asserts that all diagrams built from α, λ, ρ commute provided that the *triangle* and *pentagon* commute:

$$\textbf{Triangle: } (\rho_a \otimes \text{id}_b) \circ \alpha_{a, \mathbf{1}, b} = \text{id}_a \otimes \lambda_b, \quad (37)$$

$$\textbf{Pentagon: } \alpha_{a, b, c \otimes d} \circ \alpha_{a \otimes b, c, d} = (\text{id}_a \otimes \alpha_{b, c, d}) \circ \alpha_{a, b \otimes c, d} \circ (\alpha_{a, b, c} \otimes \text{id}_d), \quad (38)$$

as natural transformations between $(a \otimes b) \otimes (c \otimes d)$ and $a \otimes (b \otimes (c \otimes d))$ [42].

In a *braided* monoidal category $(\mathcal{C}, \otimes, \alpha, c)$, the braiding isomorphisms $c_{a, b} : a \otimes b \rightarrow b \otimes a$ satisfy the *hexagon* identities

$$\textbf{Hexagon}^{(+)} : \alpha_{a, b, c} \circ (c_{a, b} \otimes \text{id}_c) \circ \alpha_{b, a, c} \circ (\text{id}_b \otimes c_{a, c}) = c_{a, b \otimes c} \circ \alpha_{b, c, a}, \quad (39)$$

$$\textbf{Hexagon}^{(-)} : \alpha_{a, b, c}^{-1} \circ (\text{id}_a \otimes c_{b, c}) \circ \alpha_{a, c, b}^{-1} \circ (c_{a, c} \otimes \text{id}_b) = c_{a \otimes b, c} \circ \alpha_{c, a, b}^{-1}. \quad (40)$$

When $\mathcal{C}$ is rigid and ribbon, one obtains further structure (duals, twists); under suitable finiteness/non-degeneracy conditions one arrives at a *modular tensor category* (MTC), which abstracts the chiral data of rational CFT and the anyon content of (2+1)-dimensional topological phases [44–46].

Index notation and F-, R-symbols. Fix a basis of fusion/splitting spaces. The associator is encoded by *F-symbols*

$$\left[F_d^{abc} \right]_{(e, \mu)(f, \nu)} : \bigoplus_e V_e^{ab} \otimes V_d^{ec} \xrightarrow{\cong} \bigoplus_f V_f^{bc} \otimes V_d^{af}$$

and the braiding by *R-symbols* $R_c^{ab} : V_c^{ab} \rightarrow V_c^{ba}$ (suppressing multiplicity indices when not essential). Coherence becomes the *pentagon equation*

$$\sum_n \left[F_p^{abm} \right]_{(l\alpha)(n\beta)} \left[F_p^{ncd} \right]_{(k\gamma)(j\delta)} = \sum_{r, \mu, \nu} \left[F_m^{bcd} \right]_{(k\gamma)(r\mu)} \left[F_p^{ard} \right]_{(l\alpha)(j\delta)} \left[F_r^{abc} \right]_{(l\mu)(k\nu)} \quad (41)$$

and the two *hexagon equations*

$$\sum_n \left[F_d^{abc} \right]_{(m\mu)(n\nu)} R_d^{an} \left[F_d^{bac} \right]_{(n\nu)(k\kappa)} = R_d^{am} \left[F_d^{acb} \right]_{(m\mu)(k\kappa)} R_m^{ab}, \quad (42)$$

$$\sum_n \left[F_d^{abc} \right]_{(m\mu)(n\nu)} \left(R_d^{bn} \right)^{-1} \left[F_d^{bac} \right]_{(n\nu)(k\kappa)} = \left(R_d^{bm} \right)^{-1} \left[F_d^{acb} \right]_{(m\mu)(k\kappa)} \left(R_m^{ba} \right)^{-1}. \quad (43)$$

These matrix equalities implement the topological moves on trivalent graphs and encode the change of basis between *s*-, *t*-, and *u*-channel factorizations of four-point blocks.

2.3.2. OPE Associativity and Crossing from Coherence

Let Φ_a be primary fields labeled by simple objects $a \in \text{Irr}(\mathcal{C})$ and consider the four-point correlator $\langle \Phi_a \Phi_b \Phi_c \Phi_d \rangle$ on the sphere. Its holomorphic conformal blocks admit different factorizations corresponding to distinct parenthesizations of the OPE:

$$\mathcal{F}_e^{(s)}(z) \propto \sum_{\mu, \nu} C_\mu^{ab;e} C_\nu^{ec;d} \mathcal{G}_{(e, \mu, \nu)}^{(s)}(z), \quad (44)$$

$$\mathcal{F}_f^{(t)}(z) \propto \sum_{\rho, \sigma} C_\rho^{bc;f} C_\sigma^{af;d} \mathcal{G}_{(f, \rho, \sigma)}^{(t)}(z). \quad (45)$$

Crossing symmetry requires that these two bases be related by the F -matrix:

$$\mathcal{F}_e^{(s)}(z) = \sum_f \left[F_d^{abc} \right]_{ef} \mathcal{F}_f^{(t)}(z). \quad (46)$$

Consistency of five-point blocks and of successive re-associations of the OPE imposes (41), while compatibility with braiding monodromies imposes (42) and (43). In vertex-operator algebra (VOA) language, these statements are the rigorous content of associativity (operator product expansion), commutativity/braiding, and modularity of the tensor category of modules [43]. From the three-dimensional perspective, the same consistency is encoded by the Reshetikhin–Turaev / Turaev–Viro topological field theory, for which the F - and R -moves implement elementary cobordisms and the coherence identities arise from invariance under 3D Pachner moves [44,46,47].

2.3.3. Parameterizing Controlled Departures from Coherence

While a genuine monoidal category must obey the pentagon and triangle identically, *quasi*-monoidal structures allow a non-trivial associator controlled by a 3-cocycle, as in Drinfeld’s quasi-Hopf algebras [48]. Concretely, let Ω be a normalized $U(1)$ -valued 3-cochain on (isomorphism classes of) objects, and deform the associator by

$$\alpha'_{a,b,c} = \Omega(a, b, c) \alpha_{a,b,c}. \quad (47)$$

The new associator satisfies the pentagon if and only if Ω is a 3-cocycle, i.e., $\delta\Omega = 1$, so that α' remains coherent. Two sharply different situations must then be distinguished, and keeping them apart is essential for everything that follows.

(i) *Consistent twists* ($\delta\Omega = 1$). When Ω is a cocycle with a non-trivial class, the twisted theory is consistent but in general *inequivalent* to the original one: the twist replaces the chiral data by those of a different topological phase. This is the sector in which all quantitative predictions of this paper live; as made precise in Sections 5 and 7, it is operationally equivalent to changing the bulk phase or stacking it with a (2+1)-dimensional invertible phase, and its unique modular imprint is a scalar on the Dehn twist, i.e., a quantized shift of the effective central charge.

(ii) *Genuine pentagon defects* ($\delta\Omega \neq 1$). If Ω is a *globally defined* 3-cochain with $\delta\Omega \neq 1$, then the pentagon fails by the scalar $\delta\Omega$; however, this failure is a 4-coboundary, which is cohomologically trivial by construction ($\delta^2 = 1$): it is removed identically by undoing the twist, and no anomaly (hence, no physical effect) can be associated with it. A genuine non-removable pentagon defect is instead measured by a non-trivial class in H^4 , arising when the would-be local trivializations exist only patchwise and cannot be glued, as in the obstruction theory of symmetry-extended (G-crossed) categories [49,50]. By the correspondence between cohomological degree and the dimension of the topological action that integrates it, such a class is the anomaly of a (2+1)-dimensional theory realizable only on the boundary of a (3+1)-dimensional invertible bulk—one dimension above the (2+1)-dimensional

inflow that governs the *edge* anomalies considered in this work. The paradigmatic physical realization is the anomalous surface topological order of three-dimensional symmetry-protected phases [51]. For a planar quantum Hall device, where the three-dimensional environment is topologically trivial, only the trivial H^4 class is available and consistency confines HCCB to sector (i). This perspective is compatible with the modern classification of anomalies as invertible field theories in one higher dimension [52,53].

In index notation, (47) induces

$$\left[F_d^{abc}\right]_{ef} = \frac{\Omega(a,b,c)\Omega(e,c,d)}{\Omega(b,c,d)\Omega(a,b,e)} \left[F_d^{abc}\right]_{ef}, \quad (48)$$

so that the pentagon picks up the 4-coboundary $\delta\Omega$ and the hexagon similarly acquires phases ensuring compatibility with a *twisted* braiding. In pointed categories (all simple objects invertible under $\otimes$), this construction reduces to twisting by group cohomology classes $\omega \in H^3(G, U(1))$ (Dijkgraaf–Witten type), and the deformation is exact at the level of F -symbols [45,46].

Definition (HCCB—two tiers). Tier 1 (*inflow-consistent; used throughout this paper*). An HCCB deformation of the boundary chiral data is a twist (47) by a cocycle ($\delta\Omega = 1$), supplemented in braided settings by the compatible braiding adjustment discussed below; equivalently, it is the composition of the edge theory with a (2+1)-dimensional invertible phase. All coherence relations close, and the only modular imprint is a one-dimensional scalar on the mapping-class action, localized in the Dehn twist; the effective central charge shifts by the quantized amount $\Delta c_- = 24\varphi$. Operationally, Tier-1 HCCB is the presence of well-defined local correlators with modular phases that are shifted by a bulk topological term, hence affecting precisely those quantities (vacuum energies, modular phases, and chiral energy currents) that are anomaly-sensitive. Tier 2 (*beyond inflow; a hypothesis to be bounded, not a prediction*). A genuine pentagon defect—a non-trivial H^4 class—cannot exist in a standalone planar system; it requires a (3+1)-dimensional invertible host, as realized by anomalous surface topological orders [51]. In the experimental sections, we treat any putative Tier-2 effect strictly as a null hypothesis: within Tier 1, every route- or bracketing-dependent observable vanishes identically (Section 5.1), so the multi-junction protocols proposed below bound Tier-2 physics rather than measure a parameter of Tier 1.

2.3.4. From Coherence Defects to Effective Central Charge

In a modular tensor category $\mathcal{C}$, the *topological central charge* (or *total twist*) is encoded by the Gauss sum

$$\zeta_{\mathcal{C}} := \frac{1}{\mathcal{D}} \sum_{a \in \text{Irr}(\mathcal{C})} \theta_a d_a^2 = \exp\left(\frac{2\pi i}{8} c_- \right), \quad \mathcal{D} = \sqrt{\sum_a d_a^2}, \quad (49)$$

where d_a are quantum dimensions and θ_a are twists; c_- is the *chiral* (right-minus-left) central charge, defined modulo 8 for bosonic phases [44,54]. The quantity $\zeta_{\mathcal{C}}$ is multiplicative under Deligne products and invariant under Witt equivalence; it detects the gravitational anomaly carried by the edge [55].

Now, deform the associator by Ω as in (47) and adjust the braiding/twist so as to maintain the ribbon structure. This adjustment is *not* automatic: in the pointed case, the deformations compatible with the braided structure are exactly those classified by Abelian (Eilenberg–Mac Lane) cohomology (equivalently by additive shifts of the quadratic form), and a bare associator twist generally maps a modular theory outside the modular

family [45]. Restricting to consistently adjustable deformations, denote the deformed data by (F', R', θ') and the resulting category by $\mathcal{C}^\Omega$. Then, one has

$$\frac{\zeta_{\mathcal{C}^\Omega}}{\zeta_{\mathcal{C}}} = \exp\left(\frac{2\pi i}{8} \Delta c_- \right), \quad \Delta c_- \equiv \frac{8}{2\pi} \arg\left(\frac{\zeta_{\mathcal{C}^\Omega}}{\zeta_{\mathcal{C}}}\right) \pmod{8}, \quad (50)$$

that is, the *ratio* of Gauss sums is a pure phase that measures the *shift* in the chiral central charge modulo 8. In pointed examples $\mathcal{C} = \mathcal{C}(G, q)$, the data are determined by a quadratic form $q : G \rightarrow \mathbb{Q}/\mathbb{Z}$; the deformations compatible with the braided (ribbon) structure are classified by Eilenberg–Mac Lane Abelian cohomology, which is canonically the group of quadratic forms on G , and they act by shifting q additively; the Gauss sum (hence, c_-) then changes according to the Gauss–Milgram signature of the shifted form [45,46].

Example (pointed case). Let $\mathcal{C}$ be the pointed MTC associated to a finite Abelian group G with quadratic form q . The twist is $\theta_g = e^{2\pi i q(g)}$ and $d_g = 1$. Then,

$$\zeta_{\mathcal{C}} = |G|^{-1/2} \sum_{g \in G} e^{2\pi i q(g)}.$$

For $G = \mathbb{Z}_2$ and $q(g) = \frac{g^2}{4}$ (the semion theory), $\zeta_{\mathcal{C}} = e^{2\pi i/8}$, so $c_- \equiv 1 \pmod{8}$. The braided-compatible twists shift q additively by another quadratic form; shifting by the fermionic form $q_f(g) = \frac{g^2}{2}$ maps the semion to the anti-semion, $q'(g) = \frac{3g^2}{4}$, so the Gauss sum changes by $e^{-2\pi i/4} = -i$ and $\Delta c_- \equiv -2 \equiv 6 \pmod{8}$. Two cautionary remarks sharpen the example. First, a *bare* twist by the non-trivial class in $H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$, without a compensating braiding adjustment, changes the associator class and maps the semion outside the modular family (onto the boson/fermion pair, whose braiding is degenerate): ribbon structure is not preserved automatically, even in the pointed case. Second, stacking with the E_8 phase shifts c_- by 8; therefore, it leaves $\zeta_{\mathcal{C}}$ *invariant*: the Gauss sum determines c_- only modulo 8, which is precisely the bosonic stacking ambiguity. Both remarks illustrate the content and the resolution of (50) [54,55].

Operational consequences. A change $c_- \mapsto c_- + \Delta c_-$ modifies (i) the modular T -matrix by an overall phase $T \mapsto e^{2\pi i \Delta c_- / 24} T$, (ii) the vacuum Casimir energy (on a chiral edge) by $\delta E_0 = -\frac{\pi v}{6L} \Delta c_-$, and (iii) the low-temperature thermal Hall conductance by

$$\delta \kappa_{xy} = \frac{\pi^2 k_B^2}{3h} \Delta c_- T, \quad (51)$$

all of which provide experimental/operational readouts of HCCB. The inflow picture identifies Δc_- with the shift of the gravitational Chern–Simons level in the $(2+1)$ -dimensional bulk; the corresponding boundary consistent-with-anomaly structure is precisely a controlled failure of coherence [56,57].

2.3.5. Deriving the Drift: A Proof Sketch

We sketch how (50) arises from the deformation (48) in a ribbon fusion category with simple spectrum.

1. *Twisting the associator.* Deform $F \mapsto F'$ by (48). This induces a change of $6j$ -symbols, and hence of the evaluation of closed trivalent graphs by a phase determined by Ω on 3-simplices of a triangulation.
2. *State-sum/TFT perspective.* In the Turaev–Viro/Reshetikhin–Turaev construction, the partition function on a closed 3-manifold M is a state sum built from F -symbols and quantum dimensions. Twisting by Ω multiplies the partition function by the discrete action $\exp(2\pi i \langle \Omega, [M] \rangle)$ of an invertible theory classified by the cohomology class of Ω (pointed case: Dijkgraaf–Witten action). On $M = S^1 \times \Sigma$ with a boundary

torus, this phase appears as an overall phase in the Dehn twist—i.e., in the diagonal entries of the modular T -matrix of the boundary theory [46,47].

3. *From T to c_- .* The diagonal of T is $T_{aa} = e^{2\pi i(h_a - c/24)}$. Multiplying T by a scalar phase $e^{2\pi i\phi}$ corresponds to shifting c by $\Delta c = 24\phi$ (holding conformal weights fixed). The bulk phase computed in step (2) is precisely $\phi = \frac{1}{24} \Delta c_-$, which is extracted from the ratio of Gauss sums (the partition function on S^3) and yields (50) [44,55].

Thus, a controlled deformation of coherence (HCCB) propagates into a shift of modular phases, which in turn implements a well-defined quantized drift of the (chiral) central charge. In subsequent sections, we will match this prediction against edge thermodynamics and propose experimental protocols in the fractional quantum Hall setting to measure Δc_- .

3. Central Charge as an Obstruction

3.1. Mathematical Aspect: Central Term as a Non-Removable 2-Cocycle

The Virasoro central term arises as a *non-trivial* class in Lie algebra cohomology. In this subsection we give a precise cohomological formulation, an explicit construction of the Gelfand–Fuchs 2-cocycle on the Witt algebra, two independent proofs of its non-triviality, and a group-level interpretation via the Bott–Thurston 2-cocycle on $\text{Diff}^+(S^1)$. We then connect to projective unitary representations and to the geometric (coadjoint orbit) picture, where the central charge appears as an intrinsic obstruction parameter. Our references for the cohomological and geometric facts are [58–66].

3.1.1. Cochains, Coboundaries, and Central Extensions

Let $\mathfrak{g}$ be a (complex) Lie algebra and M a trivial $\mathfrak{g}$ -module (here $M = \mathbb{C}$). The Chevalley–Eilenberg cochains are alternating n -linear maps $C^n(\mathfrak{g}; M) = \Lambda^n \mathfrak{g}^* \otimes M$ with differential

$$(d\omega)(X_0, \dots, X_n) = \sum_{0 \leq i < j \leq n} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n). \quad (52)$$

A 2-cochain $\omega \in C^2(\mathfrak{g}; \mathbb{C})$ is a 2-cocycle if $d\omega = 0$, i.e.,

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0. \quad (53)$$

A 2-cocycle is *trivial* (a *coboundary*) if there exists a linear functional $\varphi \in C^1(\mathfrak{g}; \mathbb{C})$ such that $\omega = d\varphi$, i.e.,

$$\omega(X, Y) = \varphi([X, Y]) \quad (\forall X, Y \in \mathfrak{g}). \quad (54)$$

The second cohomology $H^2(\mathfrak{g}; \mathbb{C}) = Z^2/B^2$ classifies equivalence classes of central extensions

$$0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0, \quad [(X, aK), (Y, bK)] = ([X, Y], \omega(X, Y)K), \quad (55)$$

with two extensions equivalent iff their cocycles differ by a coboundary [60,62].

Application to the Witt algebra. Let $\mathfrak{w}$ be the Witt algebra spanned by $\{\ell_n\}_{n \in \mathbb{Z}}$ with brackets

$$[\ell_m, \ell_n] = (m - n) \ell_{m+n}, \quad \ell_n = -z^{n+1} \partial_z. \quad (56)$$

A central extension of $\mathfrak{w}$ is specified by a skew-symmetric bilinear form ω obeying (53). The Virasoro algebra is the (unique up to scale) non-trivial central extension, with

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0} K. \quad (57)$$

We now construct ω explicitly from differential geometry on S^1 and show that it is non-trivial.

3.1.2. The Gelfand–Fuchs 2-Cocycle on Vector Fields of the Circle

Identify $\mathfrak{w}$ with smooth 2π -periodic vector fields on the circle, $X = f(\theta)\partial_\theta$, $Y = g(\theta)\partial_\theta$. Define

$$\omega_{\text{GF}}(X, Y) := \frac{1}{48\pi} \int_0^{2\pi} (f'(\theta)g''(\theta) - g'(\theta)f''(\theta)) d\theta. \quad (58)$$

Claim 1. ω_{GF} is a 2-cocycle: $d\omega_{\text{GF}} = 0$.

Proof. Using $[X, Y] = (fg' - gf')\partial_\theta$, one computes

$$\begin{aligned} \omega_{\text{GF}}([X, Y], Z) &= \frac{1}{48\pi} \int (fg' - gf')' h'' - (h'')(fg' - gf') d\theta \\ &= \frac{1}{48\pi} \int (f'g' + fg'' - g'f' - gf'')h'' - h'''(fg' - gf') d\theta. \end{aligned}$$

Adding the cyclic permutations and integrating by parts, all terms cancel pairwise. The cancellation uses only periodicity (no boundary terms) and the antisymmetry built into (58). Thus, $d\omega_{\text{GF}} = 0$ [58,59]. $\square$

Claim 2. In the Fourier basis $X = \ell_m = -e^{im\theta}\partial_\theta$, $Y = \ell_n$, one obtains

$$\omega_{\text{GF}}(\ell_m, \ell_n) = \frac{1}{12} (m^3 - m) \delta_{m+n,0}. \quad (59)$$

Proof. Insert $f(\theta) = -e^{im\theta}$, $g(\theta) = -e^{in\theta}$, so that $f' = -ime^{im\theta}$, $f'' = -m^2e^{im\theta}$ and similarly for g . Then, $f'g'' - g'f'' = i(m^2n - mn^2)e^{i(m+n)\theta} = imn(m-n)e^{i(m+n)\theta}$. The integral gives $2\pi\delta_{m+n,0}$; combining with the prefactor $1/(48\pi)$ and using $n = -m$ yields $(m^3 - m)/12$ after the standard normalization that enforces vanishing on the $\mathfrak{sl}_2$ subalgebra $\langle \ell_{-1}, \ell_0, \ell_1 \rangle$ (this normalization fixes the unique $\mathfrak{sl}_2$ -invariant representative in the non-zero cohomology class) [59]. $\square$

Thus, ω_{GF} reproduces the Virasoro central term. Any other 2-cocycle differs by a coboundary and an overall multiple of ω_{GF} ; the scalar coefficient is the *central charge* c .

3.1.3. Non-Triviality: No Coboundary Can Remove the Central Term

We present two complementary arguments.

(A) *Cohomological argument via $\mathfrak{sl}_2$ -reduction.* The adjoint action of the Möbius subalgebra $\mathfrak{sl}_2 = \langle \ell_{-1}, \ell_0, \ell_1 \rangle$ splits $\mathfrak{w}$ into weight spaces. A standard Hochschild–Serre spectral sequence computation shows that

$$H^2(\mathfrak{w}; \mathbb{C}) \cong \mathbb{C}, \quad (60)$$

generated by the unique $\mathfrak{sl}_2$ -invariant class. The representative that vanishes when any argument lies in $\mathfrak{sl}_2$ is precisely (58); hence, the class is non-zero [58,60]. If $\omega_{\text{GF}} = d\varphi$ were a coboundary, it would be $\mathfrak{sl}_2$ -invariant as well, but B^2 contains no non-zero $\mathfrak{sl}_2$ -invariant elements in this degree, which is a contradiction.

(B) *Ratio test on modes (no solution for a coboundary).* Assume $\omega(\ell_m, \ell_n) = d\varphi(\ell_m, \ell_n) = \varphi([\ell_m, \ell_n]) = (m-n)\varphi(\ell_{m+n})$. Evaluating at $(m, -m)$ gives

$$\omega(\ell_m, \ell_{-m}) = 2m\varphi(\ell_0).$$

For the Virasoro cocycle, $\omega_{\text{GF}}(\ell_m, \ell_{-m}) = \frac{1}{12}(m^3 - m)$. Taking $m = 2$ and $m = 3$,

$$\frac{\omega(\ell_3, \ell_{-3})}{\omega(\ell_2, \ell_{-2})} = \frac{6\varphi(\ell_0)}{4\varphi(\ell_0)} = \frac{3}{2} \quad \text{whereas} \quad \frac{\omega_{\text{GF}}(\ell_3, \ell_{-3})}{\omega_{\text{GF}}(\ell_2, \ell_{-2})} = \frac{24}{6} = 4.$$

These ratios cannot agree for any $\varphi(\ell_0)$; hence, ω_{GF} is not a coboundary [59]. Therefore, the central term is *non-removable*.

3.1.4. Group-Level Viewpoint: Bott–Thurston 2-Cocycle

On the group $\text{Diff}^+(S^1)$, the obstruction is encoded by the *Bott–Thurston 2-cocycle* $B : \text{Diff}^+(S^1) \times \text{Diff}^+(S^1) \rightarrow \mathbb{R}$,

$$B(\phi, \psi) = \frac{1}{48\pi} \int_0^{2\pi} \log((\phi' \circ \psi)(\theta)) d\log(\psi'(\theta)), \quad d\log(\psi'(\theta)) = \frac{\psi''(\theta)}{\psi'(\theta)} d\theta, \quad (61)$$

which satisfies the group 2-cocycle condition $B(\phi, \psi) + B(\phi \circ \psi, \chi) = B(\psi, \chi) + B(\phi, \psi \circ \chi)$. The *Virasoro–Bott group* is the central extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \widehat{\text{Diff}}^+(S^1) \longrightarrow \text{Diff}^+(S^1) \longrightarrow 1 \quad (62)$$

with multiplication $(\phi, a) \cdot (\psi, b) = (\phi \circ \psi, a + b + B(\phi, \psi))$ [66]. Differentiating at the identity recovers the Lie algebra cocycle (58). The non-triviality of B in group cohomology mirrors the non-triviality of ω_{GF} in Lie algebra cohomology and obstructs the lifting of projective unitary representations of $\text{Diff}^+(S^1)$ to honest representations without enlarging the group by a central circle [60,61].

3.1.5. Projective Representations and Multipliers

A (ray) representation Π of $\text{Diff}^+(S^1)$ on a Hilbert space $\mathcal{H}$ satisfies

$$\Pi(\phi) \Pi(\psi) = e^{i\mu(\phi, \psi)} \Pi(\phi \circ \psi), \quad (63)$$

where $\mu : \text{Diff}^+(S^1) \times \text{Diff}^+(S^1) \rightarrow \mathbb{R}$ is a *multiplier* obeying the group 2-cocycle condition. When μ is cohomologous to zero, the representation can be rephased to a genuine unitary representation. In CFT, locality and positivity enforce a non-trivial multiplier proportional to (61), with coefficient c ; exponentiating the Virasoro–Bott extension yields a unitary representation of $\widehat{\text{Diff}}^+(S^1)$ for which the generator algebra is Virasoro with central charge c [61,62]. In parallel, determinant line bundles and fermionic anomalies provide a geometric construction of the multiplier (and the central extension) via Quillen curvature and index-theoretic formulas [63].

3.1.6. Coadjoint Orbits and the Geometric Meaning of c

Let Vir denote the Virasoro Lie algebra (central extension of $\mathfrak{w}$). Its (continuous) dual consists of pairs $(b(\theta)d\theta^2, t)$ with b as a quadratic density and $t \in \mathbb{R}$ the value on the central element K (identified with $c/24\pi$ up to normalization). The coadjoint action of a vector field $\xi(\theta)\partial_\theta$ is

$$\delta_\xi b = 2b\xi' + b'\xi - \frac{c}{12}\xi''', \quad (64)$$

with the last term proportional to the central charge. The Kirillov–Kostant symplectic form on a coadjoint orbit $\mathcal{O}_{(b,t)}$ is

$$\Omega_{(b,t)}(\xi_1, \xi_2) = \int_0^{2\pi} \left(b(\xi_1\xi_2' - \xi_2\xi_1') + \frac{c}{24}(\xi_1'\xi_2'' - \xi_2'\xi_1'') \right) \frac{d\theta}{2\pi}, \quad (65)$$

in which the second term is precisely the GF cocycle (58) (up to normalization). Thus, c is the *level* of the Kirillov–Kostant form, and measures the failure of the naive (non-centrally-extended) moment map to exist: it is the geometric avatar of the obstruction [64,65].

3.1.7. Physical Intuition: Why the Obstruction Cannot Be Removed

In a quantum theory, the stress tensor generators L_n are defined by contour integrals of $T(z)$ with normal ordering. The commutator acquires a c -number (Schwinger term) proportional to the short-distance singularity of the TT OPE; this is *intrinsic* to the operator product, and cannot be eliminated by redefining $L_n \mapsto L_n + \alpha_n \mathbf{1}$ because such redefinitions correspond exactly to adding a coboundary $d\varphi$ to the cocycle; as we have shown, the GF class is non-trivial in cohomology. Equivalently, the projective factor in the representation of $\text{Diff}^+(S^1)$ is fixed by locality and positivity; non-triviality of the Bott–Thurston class means there is no global phase choice that trivializes it. Therefore, the central charge is an *obstruction*—a topological invariant of the representation—rather than a removable artifact.

3.1.8. Summary (Dictionary)

- $H^2(\mathfrak{w}; \mathbb{C}) \cong \mathbb{C}$ is generated by the Gelfand–Fuchs class $[\omega_{\text{GF}}]$; the Virasoro central term represents $c [\omega_{\text{GF}}]$ [58–60].
- The group-level obstruction is the Bott–Thurston cocycle B , yielding the Virasoro–Bott central extension of $\text{Diff}^+(S^1)$ [61,66].
- Projective unitary representations of $\text{Diff}^+(S^1)$ lift to honest representations of the centrally extended group; the multiplier is fixed by c [62,63].
- Geometrically, c is the level of the Kirillov–Kostant form on Virasoro coadjoint orbits and produces the ζ''' term in (64) [64,65].

3.2. Physical Manifestation: Obstruction to Naïve Witt Symmetry and Its Relation to Anomalies and Conserved Charges

The central charge c is the *physical* imprint of a quantum obstruction to realizing the classical Witt symmetry of local conformal transformations as an honest symmetry on the Hilbert space. Classically, the infinitesimal conformal diffeomorphisms of the circle are generated by vector fields $\epsilon(\theta)\partial_\theta$, obeying the Witt Lie algebra. Quantum mechanically, the stress tensor modes L_n furnish a projective representation with a Schwinger term. This has three equivalent faces:

1. An anomalous commutator of the Noether charges (central term)
2. An anomalous transformation law for the stress tensor (Schwarzian term)
3. A Weyl/gravitational anomaly in the generating functional, with concomitant edge transport.

We detail each viewpoint and its implications for conserved charges.

Charge algebra with a central term. Let $Q[\epsilon] = \oint \frac{d\theta}{2\pi} \epsilon(\theta) T(\theta)$ be the charge generating a reparameterization $\theta \mapsto \theta + \epsilon(\theta)$ in radial quantization (here, $T(\theta)$ is the chiral stress tensor on the circle). Using the TT -OPE and standard contour manipulations, one finds

$$[Q[\epsilon_1], Q[\epsilon_2]] = Q[\epsilon_1\epsilon_2' - \epsilon_2\epsilon_1'] + \frac{c}{24\pi} \int_0^{2\pi} (\epsilon_1'\epsilon_2'' - \epsilon_2'\epsilon_1'') d\theta, \quad (66)$$

where the last term is the *Gelfand–Fuchs* 2-cocycle evaluated on the pair of vector fields (cf. Section 3.1). Equation (66) makes the obstruction manifest: the algebra of quantum charges closes only up to a c -number term, so one obtains a projective (not linear) representation of the Witt algebra. This projective factor exponentiates to the Bott–Thurston group cocycle on $\text{Diff}^+(S^1)$, and cannot be removed by rephasing due to the non-triviality of the corresponding cohomology class (see the coadjoint-orbit derivation in [67]).

Anomalous transformation of T (Schwarzian term). Let δ_ϵ denote the variation under $\epsilon(\theta)\partial_\theta$. The operator-product Ward identity (or equivalently, the Virasoro commutator) implies

$$\delta_\epsilon T(\theta) = \epsilon \partial_\theta T + 2\epsilon' T + \frac{c}{12} \epsilon'''. \quad (67)$$

The third-derivative term is the local manifestation of the anomaly: T transforms as a tensor of weight 2 only up to the *Schwarzian* correction. At the level of finite conformal maps $z \mapsto f(z)$, this integrates to

$$T(z) \mapsto (f')^2 T(f(z)) + \frac{c}{12} \{f, z\}, \quad (68)$$

with the Schwarzian derivative $\{f, z\} = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$. The same term controls the universal Casimir energy and modular phases of the partition function.

Weyl anomaly in the generating functional. In the path-integral (Euclidean) formalism, the quantum effective action $W[g] = -\log Z[g]$ on a two-dimensional manifold (Σ, g) admits the *Polyakov* nonlocal functional

$$W_{\text{anom}}[g] = -\frac{c}{96\pi} \int_\Sigma \sqrt{g} R \frac{1}{\Delta} R, \quad (69)$$

for which the Weyl variation gives the trace anomaly

$$\delta_\sigma W[g] = -\frac{c}{24\pi} \int_\Sigma \sqrt{g} \sigma R, \quad \Longleftrightarrow \quad \langle T^\mu{}_\mu \rangle = -\frac{c}{12} R. \quad (70)$$

Equations (69) and (70) can be derived either by explicit functional determinants (heat kernel methods) [68,69] or by the Fujikawa method, which identifies the anomaly with the Jacobian of the path-integral measure under Weyl transformations [70,71]. Thus, c is the coefficient of a geometric obstruction in $W[g]$.

From anomalies to transport and conserved charges. Coupling to background (Newton–Cartan) geometry and using Luttinger’s trick (thermal gradients as fictitious gravitational fields) yields the thermal response controlled by the chiral central charge $c_- = c_R - c_L$. Denoting by Φ_T the “gravitational potential” that sources energy density, the energy current follows from $J_i^E = \delta W / \delta A_i^g$, with A^g the gravitomagnetic vector potential; one finds [72,73]

$$\kappa_{xy} = \frac{\partial J_y^E}{\partial(-\nabla_x T)} \Big|_{T \rightarrow 0} = \frac{\pi^2 k_B^2}{3h} c_- T + \dots \quad (71)$$

This low- T edge formula matches microscopic computations in chiral phases (e.g., $p+ip$ superconductors with $c_- = \frac{1}{2}$) and provides an operational probe of the obstruction encoded by c_- [74,75]. The same anomaly controls other geometric responses such as Hall viscosity, which ties to the *conformal* (and gravitational) data of the edge theory [76].

Conserved charges in the covariant phase space. A complementary perspective uses the covariant phase-space formalism. Given a diffeomorphism generated by ξ^μ , the variation of the Hamiltonian charge Q_ξ acquires a boundary term δQ_ξ determined by the symplectic current. In anomalous systems the algebra of charges reads

$$\{Q_\xi, Q_\eta\} = Q_{[\xi, \eta]} + K(\xi, \eta), \quad (72)$$

where the *central 2-cocycle* K is a boundary integral constructed from the symplectic structure and encodes the obstruction to the naive symmetry action. This statement is universal and does not rely on special coordinates; a covariant derivation appears in [77,78]. In 2D CFT, the explicit form of K reduces to the integral in (66).

In sum, the central charge is the single parameter that measures the failure of classical Witt symmetry to survive quantization unscathed. It appears in the algebra of charges, in the local transformation of the stress tensor, and in the geometric (Weyl) anomaly of the generating functional; from there, it propagates into universal transport coefficients such as (71).

3.3. HCCB Perspective: Higher Coherence as a Generalized Obstruction and the Drift of the Effective Central Charge

We now elevate the obstruction viewpoint from Lie algebra cohomology to *higher* categorical coherence. In categorical models of chiral data, associativity and braiding hold up to specified isomorphisms obeying coherence constraints (pentagon/hexagon). The crucial observation is that in effective boundary descriptions that are compatible with anomaly inflow, small quantized *violations* of these coherence laws can be consistently organized and behave as *higher-order anomalies*. We call this mechanism *higher categorical coherence breakdown* (HCCB).

Beyond the pentagon: Homotopy coherence and higher associativity. Classical coherence theorems insure that the triangle/pentagon/hexagon diagrams suffice for ordinary (braided) monoidal categories. In homotopy-theoretic refinements (e.g., A_∞ - and E_n -structures), associativity and commutativity constraints are controlled by a hierarchy of homotopies organized by Stasheff polytopes K_n (associahedra) and their higher analogues [79–82]. In such settings one may consistently introduce *secondary* and *higher* coherence data, for which failure is measured by higher cohomology classes. Physically, these control the re-association of OPEs and the compatibility of braidings/monodromies beyond the four-point level; homotopy coherence reflects the fact that field theories compute correlators only up to exact terms and contact terms, and that those may carry quantized information.

HCCB as a higher anomaly. Let Ω_3 encode a (normalized) $U(1)$ -valued 3-cochain that deforms the associator as in Section 2.3, and consider further *higher* cochains Ω_n which parameterize coherent homotopies at level n . The dimensional bookkeeping of anomaly inflow is rigid here, and worth stating exactly: an anomaly classified by a degree- n class is realized by an invertible topological field theory in n spacetime dimensions, living as the bulk of an $(n-1)$ -dimensional boundary theory [83–85]. A consistent 3-cocycle twist ($\delta\Omega_3 = 1$) is degree-3 data: it corresponds to changing (or stacking with) a (2+1)d invertible bulk. The (1+1)d edge of this bulk inherits the shifted modular phases; this is the Tier-1 mechanism used throughout. By contrast, a *genuine* violation of the pentagon is degree-4 data: if a globally defined Ω_3 has $\delta\Omega_3 \neq 1$, then the failure is an exact coboundary and hence removable, while a non-removable failure defines a class in H^4 and signals that the (2+1)d theory itself is consistent only as the boundary of a (3+1)d invertible bulk, as in the obstruction theory of symmetry-extended categories and its physical realization by anomalous surface topological orders [49–51]. Beyond-pentagon data (secondary associators, higher homotopies) push this pattern one level further still. For the planar devices considered in this paper, only the degree-3, Tier-1 sector is available; the higher tiers enter as null hypotheses and (in their only inflow-consistent planar incarnation) as invariants of *families* over control-parameter space (Sections 5.2.10 and 7.2).

Drift of the effective central charge from inflow. The edge theory knows about the bulk through modular phases and geometric responses. If the bulk carries a gravitational Chern–Simons (gCS) term

$$S_{\text{gCS}}[\omega] = \frac{\kappa}{4\pi} \int_{M^3} \text{CS}(\omega) = \frac{\kappa}{4\pi} \int_{M^3} \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right), \quad (73)$$

then the boundary partition function acquires a framing (Dehn twist) phase $e^{2\pi i \kappa}$ and the chiral central charge shifts by

$$\Delta c_- = 24 \kappa. \quad (74)$$

Equation (74) is the edge/bulk matching for the gravitational anomaly, and is the precise content of “stacking with an invertible 3D phase” [83,86]. When the coherence is deformed by Ω_3 (or higher Ω_n) consistently with a bulk invertible TFT, the Dehn twist T on the boundary torus picks up a scalar phase

$$T \mapsto e^{\frac{2\pi i}{24} \Delta c_-} T, \quad (75)$$

equivalently shifting the (topological) Gauss sum and the modular data. This is the categorical counterpart of the stress-tensor Schwarzian shift, and leads to a drift of the *effective* central charge measured by finite-size thermodynamics and chiral transport:

$$c_{\text{eff}} = c + \Delta c_-, \quad \kappa_{xy} \mapsto \kappa_{xy} + \frac{\pi^2 k_B^2}{3h} \Delta c_- T. \quad (76)$$

For purely bosonic phases, Δc_- is quantized so that $e^{\frac{2\pi i}{8} \Delta c_-} = 1$ (period 8), reflecting the E_8 index; in spin phases (fermionic systems), the periodicity is refined by the spin structure, but the modular T -phase still encodes the shift [86].

Schematic derivation via mapping tori. Let M_T^3 be the mapping torus of the torus T^2 under a single Dehn twist. The bulk invertible TFT amplitude on M_T^3 is a pure phase $\exp(2\pi i \phi)$; by functoriality, this is the scalar by which the boundary T acts on the vacuum. In a chiral CFT, T acts diagonally as $T_{aa} = \exp(2\pi i (h_a - \frac{c}{24}))$, so a scalar multiplication by $e^{2\pi i \phi}$ shifts $c \mapsto c + 24\phi$. Identifying $\phi = \kappa$ gives (74), which may be computed (in examples) by evaluating the gCS action on M_T^3 . The same computation can be phrased categorically in terms of how a twist of the associator/braiding modifies the Reshetikhin–Turaev invariant of the mapping torus; this is precisely the *beyond-pentagon* footprint of the inflow [50,81,82].

Operational dictionary.

- **Higher coherence defect** (parameterized by Ω_n) $\implies$ **invertible bulk** with topological action $S_{\text{inv}} \implies$ **boundary modular phase** and **Dehn twist** shift.
- The **phase of T** controls c_{eff} via $c \mapsto c + \Delta c_-$.
- **Transport and thermodynamics** detect Δc_- through (76), while **finite-size Casimir** shifts by $\delta E_0 = -\frac{\pi v}{6L} \Delta c_-$.

In conclusion, c is an obstruction at the Lie-algebraic level, and HCCB exhibits it as the low-energy shadow of a *higher* obstruction: a controlled breakdown of coherence that is compensated by anomaly inflow. This mechanism makes the “central charge drift” both quantized and experimentally accessible.

4. Fractional Quantum Hall Effect (FQHE) as a Laboratory

4.1. Overview of the FQHE

The fractional quantum Hall effect (FQHE) is an emergent topological phase of two-dimensional electrons in strong magnetic fields and at low temperatures. It exhibits *fractional* quantization of the Hall conductance and quasiparticles with fractional charge and anyonic statistics. Discovered experimentally at filling factor $\nu = 1/3$ [87], it was explained by Laughlin’s variational wavefunction [88] and subsequently generalized by hierarchy and composite-fermion constructions [89–91]. This subsection reviews (i) Laughlin states and their topological order (degeneracies, fractional charge/statistics, topological entanglement), (ii) edge states as chiral CFTs, and (iii) the quantization of the Hall conductance from bulk and edge viewpoints. Classic expositions include [92–95].

4.1.1. Landau Levels and Laughlin Wavefunctions

Electrons (charge $-e$) confined to the xy -plane in a perpendicular magnetic field $B\hat{z}$ have *Landau levels* with cyclotron frequency $\omega_c = eB/m$. The magnetic length

$$\ell_B = \sqrt{\frac{\hbar}{eB}} \quad (77)$$

sets the natural area scale, and each Landau level has degeneracy

$$N_\phi = \frac{A}{2\pi\ell_B^2} = \frac{eBA}{h}, \quad (78)$$

equal to the number of flux quanta piercing area A . The filling factor is $\nu = N_e/N_\phi$. Projected to the lowest Landau level (LLL), kinetic energy is quenched; interactions dominate.

Laughlin states. On the plane (symmetric gauge), the LLL single-particle orbitals are holomorphic times a Gaussian. For odd integer $m = 2k + 1$, Laughlin proposed the N -electron wavefunction

$$\Psi_m(z_1, \dots, z_N) = \prod_{i < j} (z_i - z_j)^m \exp\left(-\sum_{i=1}^N \frac{|z_i|^2}{4\ell_B^2}\right), \quad \nu = \frac{1}{m}, \quad (79)$$

which is antisymmetric, minimizes short-range repulsion, and yields an incompressible quantum fluid [88]. The Jastrow factor enforces a correlation hole of order m , suppressing the probability of close approach. The plasma analogy maps $|\Psi_m|^2$ to a 2D one-component Coulomb plasma at inverse temperature m , making incompressibility intuitive: increasing m strengthens logarithmic repulsion and favors uniform density.

Quasiholes: Fractional charge and statistics. A quasihole at η is described by

$$\Psi_\eta^{\text{qh}} = \prod_i (z_i - \eta) \Psi_m \quad (80)$$

and a quasiparticle by the (LLL-projected) inverse factor. Adiabatic insertion of one flux quantum $\Phi_0 = h/e$ through a puncture creates one quasihole; charge conservation then implies the deficit of charge

$$e^* = \frac{e}{m}. \quad (81)$$

The anyonic statistics follow from a Berry phase computation: transporting one quasihole around another yields

$$\theta = \frac{\pi}{m}, \quad (82)$$

so the exchange phase is $e^{i\theta}$ [96]. These are robust topological characteristics of the phase.

Hierarchy and composite fermions. Beyond $\nu = 1/m$, the FQH series can be generated by condensing quasiparticles (hierarchy) [89,90] or by binding $2p$ flux quanta to electrons to form composite fermions that fill *effective* Landau levels at mean-field level [91]. Both routes reproduce the principal sequences (e.g., $\nu = \frac{n}{2pn \pm 1}$) and encode universal braiding properties compatible with the Chern–Simons (CS) effective theory discussed below.

Theoretical perspectives and what the present work does (and does not) rely on. The microscopic theory of the FQH hierarchy remains an active subject with several coexisting viewpoints: trial-wavefunction and composite-fermion constructions [91], hierarchy condensation [89,90], conformal-block constructions [20], and alternative topological approaches based on braid-group analyses of correlated 2D electrons in magnetic fields [97,98], which question the flux-attachment heuristic and seek the origin of the observed fractions in homotopy invariants. We emphasize that nothing in this paper depends on adjudicating

among these bulk microscopic descriptions. Our analysis uses only two ingredients that sit at a more universal level: (i) the experimentally established quantization of electrical and thermal Hall transport, in particular the direct thermal-conductance measurements that extract the chiral central charge c_- [21,99,100]; and (ii) the low-energy description of the gapped bulk's *edge* as a chiral conformal field theory, which underlies those measurements and is common to all the approaches above at the level of universal data (anyon content, c_- , and modular structure). Any bulk theory consistent with the measured σ_{xy} , κ_{xy} , fractional charges, and braiding statistics induces the same edge-level structure used here.

4.1.2. Topological Order: Degeneracies, Modular Data, and Entanglement

FQH phases possess *topological order*: they cannot be characterized by symmetry breaking and local order parameters. Universal data include ground-state degeneracy on closed manifolds, anyon braiding, and topological entanglement entropy.

CS/K-matrix effective theory. Abelian FQH states are described by a $U(1)^N$ CS action

$$S_{\text{CS}}[a, A] = \frac{1}{4\pi} \int K_{IJ} a^I \wedge da^J + \frac{e}{2\pi} \int t_I A \wedge da^I \quad (83)$$

with integral symmetric and non-degenerate K , integer charge vector t , and background electromagnetic field A [101,102]. Quasiparticles are labeled by integer vectors $\ell \in \mathbb{Z}^N$; their electric charge, mutual braiding, and topological spin are

$$q_\ell = e t^\top K^{-1} \ell, \quad \theta_{\ell\ell'} = 2\pi \ell^\top K^{-1} \ell', \quad h_\ell = \frac{1}{2} \ell^\top K^{-1} \ell \pmod{1}. \quad (84)$$

On a genus- g surface the ground-state degeneracy is $|\det K|^g$. For Laughlin, $\nu = 1/m$, $K = (m)$, $t = (1)$, giving degeneracy m^g , $q_\ell = (e/m) \ell$, and $\theta = 2\pi \ell \ell' / m$.

Topological entanglement entropy. The von Neumann entanglement entropy of a simply connected region A obeys an area law with a universal negative offset:

$$S_A = \alpha |\partial A| - \gamma + o(1), \quad \gamma = \ln \mathcal{D}, \quad (85)$$

where $\mathcal{D} = \sqrt{\sum_a d_a^2}$ is the total quantum dimension. For Abelian CS with K , $d_a = 1$ for all anyons and $\mathcal{D} = \sqrt{|\det K|}$. Hence, Laughlin $\nu = 1/m$ has $\gamma = \frac{1}{2} \ln m$ [103].

Guiding-center algebra (GMP). Within a Landau level, guiding-center density operators $\bar{\rho}_{\mathbf{q}}$ satisfy the Girvin–MacDonald–Platzman (GMP) algebra

$$[\bar{\rho}_{\mathbf{q}}, \bar{\rho}_{\mathbf{q}'}] = 2i \sin\left(\frac{\ell_B^2}{2} \mathbf{q} \times \mathbf{q}'\right) \bar{\rho}_{\mathbf{q}+\mathbf{q}'}, \quad (86)$$

which encodes the non-commutative geometry of guiding centers and underlies the magneto-roton mode [104]. The algebraic rigidity of (86) supports the stability of topological order against weak perturbations.

4.1.3. Edge States as Chiral Conformal Field Theories

Chiral Luttinger liquids. Bulk CS topological order implies gapless *chiral* edge modes (Figure 1). For Abelian states, the edge is described by N chiral bosons $\phi_I(x, t)$ with action

$$S_{\text{edge}} = \frac{1}{4\pi} \int dt dx \left(K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J \right) + \frac{e}{2\pi} \int dt dx t_I \partial_x \phi_I A_t + \dots, \quad (87)$$

where V encodes interactions and edge velocities [102,105]. Equal-time commutators read

$$[\partial_x \phi_I(x), \phi_J(y)] = 2\pi i (K^{-1})_{IJ} \delta(x - y). \quad (88)$$

Electron and quasiparticle operators are vertex operators

$$\Psi_\ell(x) = e^{i\ell_I\phi_I(x)}, \quad q_\ell = e t^\top K^{-1} \ell, \quad \Delta_\ell = \frac{1}{2} \ell^\top K^{-1} \ell, \quad (89)$$

carrying the same charges and statistics as (84). The chiral central charge equals the signature of K ,

$$c_- = n_+ - n_-, \quad (90)$$

the difference between the numbers of forward and backward moving edge modes (diagonalizing K to $\text{diag}(\pm 1, \dots, \pm 1)$ by $GL(N, \mathbb{Z})$). For Laughlin states $K = (m)$, one has a single chiral boson with $c_- = 1$.

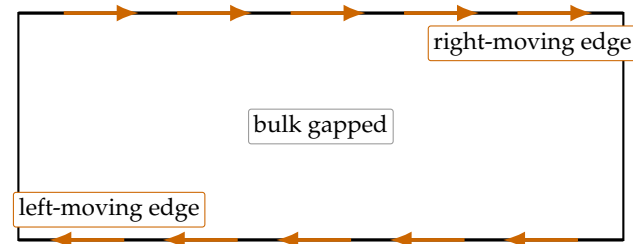


Figure 1. Chiral edges of a quantum Hall bar: the bulk is gapped, and heat/charge propagate only along the edges with fixed chirality. Arrows mark the propagation direction of the chiral edge modes.

Non-Abelian edges. Certain FQH states (e.g., at $\nu = 5/2$) support non-Abelian anyons; their edges combine a charged chiral boson and neutral non-Abelian sectors, often described by rational CFTs. A canonical example is the Moore–Read Pfaffian edge with a chiral Majorana fermion (Ising CFT, central charge $1/2$) plus a charged $U(1)$ boson, giving $c_- = 3/2$; broader overviews appear in [106].

4.1.4. Quantization of Hall Conductance: Bulk and Edge

Laughlin’s gauge argument. Consider electrons on a long annulus with inner and outer edges. Threading a time-dependent magnetic flux $\Phi(t)$ through the hole generates an azimuthal electric field $E_\theta = -\frac{1}{2\pi r} \dot{\Phi}$. Adiabatically ramping Φ from 0 to one flux quantum $\Phi_0 = h/e$ transports charge $\Delta Q = \nu e$ from inner to outer edge. Since the emf is $\mathcal{E} = -\dot{\Phi}$, one finds

$$\sigma_{xy} = \frac{\Delta Q}{\Phi_0} = \nu \frac{e^2}{h}, \quad (91)$$

independent of microscopic details [88]. This argument is robust, and in interacting systems extends to a topological invariant defined by many-body Berry curvature on the Brillouin torus [107].

Chern–Simons response. Varying the CS action (83) with respect to A_μ yields the bulk current

$$j^\mu = \frac{\delta S_{\text{CS}}}{\delta A_\mu} = \frac{e}{2\pi} t_I \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^I. \quad (92)$$

The a^I equations of motion impose $K_{IJ} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^I + e t_I \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = 0$, for which solving gives

$$j^\mu = \frac{e^2}{2\pi} t^\top K^{-1} t \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda. \quad (93)$$

Therefore, the Hall conductivity is

$$\sigma_{xy} = \nu \frac{e^2}{h}, \quad \nu = t^\top K^{-1} t, \quad (94)$$

matching the gauge argument [101,102]; for Laughlin, $K = (m)$, $\nu = 1/m$.

Edge anomaly and transport. On a manifold with boundary, the bulk CS action is not gauge invariant; its variation is precisely canceled by the anomaly of chiral edge modes. This *bulk–boundary correspondence* enforces the quantized Hall conductance and the relation between the chiral central charge and thermal Hall response (see Section 3.2). The edge theory also controls tunneling exponents measurable in transport, consistently captured by the chiral Luttinger liquid theory [105].

Summary. The FQHE realizes a gapped topological phase, with the bulk captured by Abelian (and in some cases non-Abelian) CS theories and the gapless edges being chiral CFTs. Laughlin/Jastrow correlations produce fractional charge and statistics, hierarchy and composite fermions generate the observed series, and the Hall conductance quantizes as $\sigma_{xy} = \nu e^2/h$ via both gauge and field-theoretic arguments. These structures provide a natural laboratory in which to search for and quantify mechanisms such as higher categorical coherence breakdown that can shift the effective chiral central charge through anomaly inflow.

4.2. The Chiral Central Charge in Quantum Hall Systems

This subsection develops the definition and physical content of the *chiral central charge* $c_- \equiv c_R - c_L$ in quantum Hall phases, derives its relation to the thermal Hall conductance,

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_- T, \quad (95)$$

and summarizes how present experiments constrain (and extract) c_- . We supply parallel derivations from edge conformal field theory (CFT), from bulk Kubo formulas with the essential energy–magnetization correction, and from bulk effective actions, showing that all routes agree on (95) (the complete derivations are collected in Appendix A) when the system is in a gapped topological phase with well-defined chiral edges [108–110].

4.2.1. Definition and Physical Meaning

For a (2+1)-dimensional topological phase with a *chiral* boundary theory, the net number of forward minus backward moving gapless modes is captured by c_- , that is, the difference of left- and right-moving Virasoro central charges. Operationally, c_- governs the edge energy current at low temperature and, via anomaly inflow, the *bulk* thermal Hall conductance. Equation (95) makes this correspondence explicit: in units where $k_B = \hbar = 1$, each completely equilibrated chiral *channel* contributes the universal quantum of thermal conductance $\kappa_0 T \equiv (\pi^2/3h)k_B^2 T$, and c_- simply counts these net chiral channels (with fractional contributions possible when the neutral sector is described by nontrivial CFTs).

Edge CFT derivation (Landauer/CFT route). Consider two semi-infinite edge reservoirs at temperatures T_L and T_R , connected by a single chiral 1D channel; in a 1+1D CFT, the thermal expectation of the *chiral* energy current equals the chiral Stefan–Boltzmann law,

$$J_E^{\text{chiral}}(T) = \frac{\pi}{12} c \frac{(k_B T)^2}{\hbar}, \quad J_Q^{\text{chiral}}(T) \equiv J_E^{\text{chiral}}(T), \quad (96)$$

so the net heat current carried between the reservoirs is

$$J_Q = J_E^{\text{chiral}}(T_L) - J_E^{\text{chiral}}(T_R) = \frac{\pi k_B^2}{12\hbar} c_- (T_L^2 - T_R^2). \quad (97)$$

To linear order in $\Delta T = T_L - T_R$ about $T = \frac{T_L + T_R}{2}$, one obtains the conductance $J_Q = \kappa_{xy} \Delta T$ with

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_- T, \quad (98)$$

which is (95). The derivation relies only on (i) chirality, (ii) the CFT thermodynamics of a 1D channel, and (iii) complete intra-edge equilibration of modes into a common local temperature.

Bulk Kubo derivation with energy magnetization. A naive Kubo calculation of κ_{xy} is subtle because thermal transport in a magnetic field includes circulating (magnetization) heat currents that do not contribute to transport. The correct linear response requires subtracting the *energy magnetization* M^E contribution. Let $Q_{ij}(\omega)$ be the retarded heat–heat correlator; then, the *transport* thermal Hall conductivity is [108]

$$\kappa_{xy} = \lim_{\omega \rightarrow 0} \frac{1}{T} \frac{\text{Im } Q_{xy}(\omega)}{\omega} + \frac{2}{T} \frac{\partial M_z^E}{\partial B}, \quad (99)$$

where B is the magnetic field and M^E is the “gravitomagnetic” magnetization obtained by coupling the system to a fictitious gravitational field. In a gapped topological phase, the Kubo term vanishes in the $\omega \rightarrow 0$ limit and the magnetization term reduces to a topological invariant that equals $(\pi^2 k_B^2 / 3h) c_- T$, reproducing (95) and making precise the role of the magnetization subtraction [108]. This route gives a bulk proof of (95) that does not rely on any explicit edge modeling.

Bulk effective action and anomaly inflow. An equivalent viewpoint packages the result in a low-energy effective action for the bulk, coupled to background geometry. The parity-odd (topological) part includes a gravitational Chern–Simons (gCS) term with level κ_g ,

$$S_{\text{gCS}}[\omega] = \frac{\kappa_g}{4\pi} \int_{M^3} \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right), \quad (100)$$

where ω is the spin connection. Anomaly inflow fixes κ_g in terms of c_- via $\Delta c_- = 24 \kappa_g$, so that the Dehn twist phase and the edge thermal Hall response are controlled by the same bulk invariant. A systematic continuum derivation for nonrelativistic topological phases, including torsion couplings and the correct identification of transport coefficients, is given in [109]. Moreover, the *topological central charge* can be extracted from many-body wavefunctions through Berry curvature constructions on moduli space, providing a computational diagnostic that matches c_- appearing in (95) [110].

4.2.2. How Experiments Access c_-

Quantum of thermal conductance and channel counting. The “quantum of thermal conductance”

$$\kappa_0 T \equiv \frac{\pi^2 k_B^2}{3h} T \quad (101)$$

has been measured directly in ballistic one-dimensional electronic channels using noise thermometry and controlled heating/cooling, validating the Landauer/CFT prediction at the percent level [99]. In FQH devices, edges typically host multiple downstream and upstream modes; inter-mode equilibration and partial thermalization must be handled carefully in the analysis [111], but when these are controlled the measured κ_{xy} scales as an integer or half-integer multiple of $\kappa_0 T$, in accordance with the predicted c_- of the edge CFT.

Extracting c_- and bounding errors. Given a measured κ_{xy} at stabilized electron temperature T , the estimator

$$\widehat{c_-} = \frac{3h}{\pi^2 k_B^2} \frac{\kappa_{xy}}{T} \quad (102)$$

is unbiased for the effective chiral central charge (assuming full equilibration). The dominant systematic uncertainties arise from (i) calibration of the local electron temperature (thermometry), (ii) subtraction of phonon heat leaks and Kapitza boundary resistances,

(iii) equilibration lengths and incomplete thermalization among charged and neutral edge modes, and (iv) contact and geometry-dependent losses. Linear error propagation gives

$$\delta c_- \approx \frac{3h}{\pi^2 k_B^2} \left(\frac{\delta \kappa_{xy}}{T} + \frac{\kappa_{xy}}{T} \frac{\delta T}{T} \right), \quad (103)$$

which sets a direct bound on c_- from the metrological uncertainties. In modern devices, statistical errors in κ_{xy} can be made sub-quantum, with the limiting factor often the absolute calibration of T and the control of upstream neutral modes.

Status: Integer and fractional states. Experiments have demonstrated quantized heat conductance in integer and fractional quantum Hall edges; in fractional states with both downstream and upstream modes, the *net* κ_{xy} matches the net chirality predicted by edge theory, providing a consistent extraction of c_- across several filling factors [100]. Reports of half-integer thermal Hall conductance in the first excited Landau level support a neutral Ising sector (Majorana mode) with $c_- = \frac{1}{2}$ stacked with a charged boson, consistent with non-Abelian order; taken together with the channel-counting benchmarks [99], these measurements bound c_- to lie exponentially close (on the scale of the experimental resolution) to the rational values dictated by the edge CFT in well-equilibrated devices [100].

4.2.3. Synthesis for Our Program

For our purposes, (95) provides a *metrological* window onto the anomaly inflow coefficient that we have argued is shifted by higher categorical coherence breakdown (HCCB). Any HCCB-induced Δc_- changes the Dehn-twist phase of the edge CFT and (equivalently, by the derivations above) shifts κ_{xy} by

$$\delta \kappa_{xy} = \frac{\pi^2 k_B^2}{3h} \Delta c_- T. \quad (104)$$

Hence, experimental protocols that (i) maximize edge-mode equilibration, (ii) suppress parasitic heat leaks, and (iii) allow for in situ variation of the coherence defect (as proposed later) can directly *measure* HCCB as a drift in the effective c_- extracted from thermal transport.

5. Higher Categorical Coherence Breakdown and Quantum Hall Systems

5.1. Conceptual Pathway

Our aim is to make precise how a controlled departure from categorical coherence at the edge—the *higher categorical coherence breakdown* (HCCB) introduced earlier—propagates into physically measurable drifts of anomaly-controlled observables in quantum Hall (QH) systems. The pathway has two logically distinct stages:

- (I) **Interference of coherence with conformal blocks.** Chiral conformal blocks on a punctured Riemann surface are related by elementary sewing moves (fusion/braiding). These moves are implemented by F - and R -matrices which, in a consistent theory, satisfy the pentagon and hexagon identities and thereby define a (ribbon) tensor-categorical structure on the chiral sectors. A coherent *twist* of these data (or, more generally, a beyond-pentagon homotopy-coherent deformation) alters the mapping-class-group representation on conformal blocks by a scalar 1-dimensional factor.
- (II) **From anomalous mapping-class phases to c_{eff} and transport.** The anomalous scalar factor modifies the Dehn-twist phase on the torus, hence the modular T -matrix by a

scalar $e^{2\pi i \Delta c_- / 24}$. This is equivalent to shifting the chiral central charge by Δc_- , and consequently the thermal Hall conductance by

$$\delta \kappa_{xy} = \frac{\pi^2 k_B^2}{3h} \Delta c_- T.$$

We now implement these two stages carefully.

5.1.1. Conformal Blocks, Sewing, and Coherence

Let $\mathcal{V}$ be the chiral vertex-operator algebra (VOA) of the edge theory and $\{\mathcal{H}_a\}$ its irreducible modules. Genus- g conformal blocks on a n -punctured surface $\Sigma_{g,n}$ form a finite-dimensional vector space $\mathcal{B}_{\Sigma_{g,n}}(\{a_i\})$, functorial under sewing of pairs of pants and under the action of the mapping class group $\text{MCG}(\Sigma_{g,n})$. Two pillars underlie the structure:

- *Sewing axioms and modular functor.* The family $\Sigma \mapsto \mathcal{B}_\Sigma$ together with the sewing maps and mapping-class action defines a modular functor. Algebraically, bases of $\mathcal{B}_{\Sigma_{0,4}}$ associated with the s - and t -channel factorizations are related by the *fusion* (or F -) matrices; braiding exchanges punctures and is encoded by R -matrices. Coherence of these moves is equivalent to pentagon/hexagon identities [112,113].
- *Modular covariance.* On the torus, the characters $\chi_a(\tau) = \text{Tr}_{\mathcal{H}_a} q^{L_0 - c/24}$ furnish a (projective) representation of $\text{SL}_2(\mathbb{Z})$ with generators $S : \tau \mapsto -1/\tau$ and $T : \tau \mapsto \tau + 1$. In a VOA setting, modular invariance of characters constrains (S, T) and tightly couples the sewing data to the Virasoro anomaly [114].

The modular-functor axioms, stated physically. Because these axioms become, in Sections 5.2 and 6.4, the direct target of laboratory tests, it is worth stating in plain physical terms what they assert and why they matter. Stripped of formalism, a modular functor encodes three physical principles. (i) *Well-defined process amplitudes:* to every configuration of anyons on a surface the theory assigns a finite-dimensional space of amplitudes (the conformal blocks), and every physical manipulation—fusing, splitting, exchanging, dragging around a cycle—acts as a definite linear map on that space. (ii) *Locality (sewing):* the amplitude assigned to a composite process is obtained by composing the amplitudes of its parts; cutting a surface into pairs of pants and reassembling it cannot change physical predictions. (iii) *Topological invariance:* two manipulations that differ only by a smooth deformation—in particular, two different orders or bracketings of elementary moves implementing the same overall operation—act identically. The pentagon and hexagon identities are nothing but the minimal consistency conditions that make (i)–(iii) mutually compatible; Mac Lane coherence then guarantees that *all* reorderings agree, not just the generating ones. Physically, these axioms are the formal backbone of everything that makes topological phases useful: the universality of quantized responses (which must not depend on how an experimentalist’s probes happen to compose elementary processes), the very notion of anyonic statistics as a property of worldline topology, and the fault tolerance premise of topological quantum computation, which is precisely the statement that amplitudes depend on braid topology and on nothing else. They are also, in the standard account, *assumed* rather than *tested*: transport and interference experiments are interpreted through them. Section 6.4 turns this assumption into a measurement target, and the route-blindness result below identifies exactly which observables must vanish if the axioms hold.

Coherence defects and deformed F/R moves. Let α denote the associator (and $c_{a,b}$ the braiding) on the tensor category $\mathcal{C}$ of chiral sectors. A coherent deformation by a normalized $U(1)$ -valued 3-cochain Ω_3 modifies the associator by

$$\alpha'_{a,b,c} = \Omega_3(a, b, c) \alpha_{a,b,c} \iff F' = \Phi[\Omega_3] \cdot F, \quad (105)$$

with an explicit multiplicative matrix factor $\Phi[\Omega_3]$ determined by Ω_3 in a chosen fusion/splitting basis. If $\delta\Omega_3 = 1$, then the pentagon remains exact and α' defines a genuine monoidal structure—in general, that of a *different* topological phase, which is the Tier-1 situation of Section 2.3.3; composing the edge theory with a (2+1)-dimensional invertible phase then endows the modular functor with an extra one-dimensional factor in the mapping-class action. (Physically, the extra factor is the partition function of the invertible bulk TFT on a 3-manifold that fills the mapping torus of the surface diffeomorphism). If instead $\delta\Omega_3 \neq 1$ for a globally defined Ω_3 , the pentagon failure is an exact 4-coboundary and is removable (Section 2.3.3); a genuine, non-removable failure is a non-trivial H^4 class and requires a (3+1)-dimensional invertible host [49–51], which is absent for planar devices. Throughout this section we therefore work in the Tier-1 (inflow-consistent) setting.

Beyond-pentagon homotopy-coherent deformations (secondary associators, higher homotopies) may be organized by Stasheff polytopes and higher coherences. Their effect is again to produce anomalous scalar phases in sequences of moves relating different pants decompositions, hence in the representation of $\text{MCG}(\Sigma)$ [81].

Anomalous mapping-class phases from anyon condensation and walls. A concrete, physically controllable source of such deformations comes from *anyon condensation* and gapped domain walls. Condensing a bosonic algebra object (a commutative separable Frobenius algebra) in $\mathcal{C}$ produces a new phase $\mathcal{C}'$ whose fusion data are related to those of $\mathcal{C}$ by algebraic induction/restriction functors; defects and walls between phases are described by bimodule categories. Composition of walls and defect networks induces F/R -moves that differ by scalar phases determined by the condensate data, and these phases realize the $\Phi[\Omega_3]$ factors in (105) [115–118]. In particular, passing conformal blocks across a wall multiplies them by a scalar that depends only on the topological type of the move, hence on the mapping-class element.

5.1.2. From Anomalous Phases to c_{eff} : The Dehn Twist

On the torus, the mapping class group is $\text{SL}_2(\mathbb{Z})$ generated by S and the Dehn twist T . In any chiral CFT,

$$T_{aa} = e^{2\pi i (h_a - \frac{c}{24})}, \quad S_{ab} = S_{ab}, \quad (106)$$

with conformal weights h_a and a symmetric unitary matrix S . If the mapping-class action acquires an *overall* anomalous phase $\varphi(T)$ on T (and $\varphi(S)$ on S), then

$$T \mapsto e^{2\pi i \varphi(T)} T, \quad S \mapsto e^{2\pi i \varphi(S)} S. \quad (107)$$

The relations $S^4 = \mathbb{1}$ and $(ST)^3 = S^2$ constrain these scalars: for a genuinely one-dimensional (non-projective) character of $\text{SL}_2(\mathbb{Z})$ they force $\varphi(S) = -3\varphi(T)$ and $12\varphi(T) \equiv 0 \pmod{1}$, so that strictly consistent scalar redefinitions of T are quantized in twelfths. More generally, the scalar shifts the *projective* class of the representation—the framing anomaly—and the values realizable by anomaly inflow are quantized as discussed in Section 7.5 ($\Delta c_- \in 8\mathbb{Z}$ for bosonic phases; half-integer steps with a spin structure, for which the $\text{SL}_2(\mathbb{Z})$ relations are themselves refined by the spin cover). Comparing with (106), an overall shift of T by $e^{2\pi i \phi}$ is equivalent to

$$c \mapsto c_{\text{eff}} \equiv c + 24\phi. \quad (108)$$

Therefore, the scalar anomalous factor at the level of the mapping-class-group action is the drift of the effective chiral central charge. This observation is robust: it follows purely from modular functor axioms and does not depend on details of microscopic dynamics [112,114,119].

Bulk interpretation via Reshetikhin–Turaev TQFT. Let $\mathcal{Z}_{\text{RT}}$ be the Reshetikhin–Turaev (2+1)-dimensional TQFT built from the ribbon fusion category underlying the edge data. The anomalous scalar $e^{2\pi i \phi}$ attached to T equals the partition function of an *invertible* topological field theory on the mapping torus M_T^3 ; in the parity-odd sector this is a gravitational Chern–Simons term with level $\kappa_g = \phi$, whence

$$\Delta c_- = 24 \kappa_g = 24 \phi, \quad (109)$$

which is exactly (108). Thus, by producing anomalous phases in sewing and mapping-class actions, HCCB shifts the gravitational anomaly coefficient extracted from $\mathcal{Z}_{\text{RT}}$ on M_T^3 [112,119].

5.1.3. Interference with Conformal Blocks: A Quantitative Picture

Consider a four-point block space on the sphere with external labels (a, b, c, d) and intermediate channels e (in the s -channel) and f (in the t -channel). Fix an orthonormal basis in which the F -move is unitary:

$$\bar{\mathcal{F}}^{(s)}(z) = F \bar{\mathcal{F}}^{(t)}(z).$$

Under an HCCB deformation by Ω_3 , the move becomes

$$\bar{\mathcal{F}}^{(s)}(z) = F' \bar{\mathcal{F}}^{(t)}(z), \quad F' = \Phi[\Omega_3] \cdot F, \quad (110)$$

with $\Phi[\Omega_3]$ diagonal in channels, $\Phi[\Omega_3]_{ff} = e^{2\pi i \phi_f}$ determined by the values of Ω_3 on triples appearing in the fusion trees. Coherence guarantees that any composition of elementary moves corresponding to a trivial mapping-class element returns the identity *up to a scalar*

$$\text{Id} \mapsto e^{2\pi i \Phi(\gamma)},$$

where γ is a word in $F/R/B$ moves representing that element. In a genuine (non-anomalous) modular functor $\Phi(\gamma) = 0$; with HCCB, $\Phi(\gamma)$ can be non-zero but depends only on the mapping-class element, not on the representative. In particular, for the torus Dehn twist this scalar is $\phi = \Phi(T)$, which yields (108).

Route blindness (a structural consequence). The statement that $\Phi(\gamma)$ depends only on the mapping-class element has a consequence that we will rely on, and enforce, throughout the experimental sections; we record it explicitly. Since Φ is additive under composition and depends only on the class, the identity class must carry $\Phi(\text{id}) = 0$; hence any two sequences of elementary moves implementing the *same* topological operation carry the same scalar, and any closed rebracketing loop—the pentagon loop itself, the hexagon comparison of braid-then-fuse against fuse-then-braid, or any bracketing of split/recombine operations in a planar edge network—evaluates to unity. In other words, within the inflow-consistent (Tier-1) setting,

$$\varepsilon_F = \varepsilon_H = 0 \quad \text{identically,}$$

and no observable can depend on the order or parenthesization of topologically equivalent edge operations, nor scale with the number of junctions traversed at fixed topology. This is the categorical counterpart of Mac Lane coherence, and it survives the deformation precisely because the deformation is consistent. The only scalars that can be non-zero are attached to *non-contractible* loops: non-trivial mapping classes (the Dehn twist, hence Δc_-) and, as developed in Section 7, non-contractible *cycles in control-parameter space* (pumps). Conversely, any reproducible route dependence observed in experiment would falsify the modular-functor axioms themselves (Tier 2), rather than measure a parameter of

the present framework—which is exactly what makes the multi-junction geometries of Sections 5.2 and 6.4 sharp null tests.

Two clarifications sharpen the scope of this statement. First, route blindness governs *closed comparisons at fixed couplings*: two routings compared in the same device configuration. It does *not* say that the system is insensitive to paths in the space of *control parameters*: along an open trajectory $\lambda(s)$ the instantaneous $c_{-, \text{eff}}(\lambda(s))$ is genuinely path-dependent—which sectors are visited, and in which order, depends on the route through control space—and only the *net* scalar around a closed control cycle is constrained, to a quantized homotopy invariant. This is the precise sense in which route sensitivity survives “locally” while being forbidden “globally,” in exact parallel with a flat connection whose parallel transport is locally non-trivial but whose holonomy depends only on the homotopy class of the loop; the experimental realization is the pumping protocol of Section 5.2.10. Second, route blindness is a property of the *topological* data: it does not preclude mundane, non-topological order dependence (charging hysteresis, junction memory, decoherence), which is exactly why the falsification diagnostics of Section 6.4 (chirality reversal, junction scaling, drive dependence) are designed to separate a genuine axiom violation from device dynamics.

Compatibility with characters and Zhu’s theorem. Characters transform as $\chi_a(\tau + 1) = e^{2\pi i(h_a - c/24)} \chi_a(\tau)$. The HCCB scalar multiplies *all* characters by $e^{2\pi i\phi}$ under $\tau \mapsto \tau + 1$. Zhu’s modular-invariance theorem for VOA characters ensures that such a scalar is the only allowed global ambiguity in the modular representation; hence, if present, it necessarily shifts c by 24ϕ and no other data (e.g., conformal weights modulo 1) [114]. This provides a VOA-level confirmation of the modular-functor argument.

5.1.4. Drifting c_{eff} and Thermal Transport

Let c_- be the “bare” chiral anomaly coefficient of the undeformed edge, and let Δc_- be the HCCB-induced shift. The effective anomaly measured by modular thermodynamics and by thermal transport is then

$$c_{-, \text{eff}} = c_- + \Delta c_-. \quad (111)$$

Consequently,

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_{-, \text{eff}} T, \quad F(T)/L = -\frac{\pi}{6} \frac{c_{-, \text{eff}}}{v} T^2 + o(T^2), \quad (112)$$

with v the edge mode velocity. In a device hosting domain walls or engineered condensates (see Section 5.2), the wall configuration controls Ω_3 and hence Δc_- . The *drift* becomes a tunable, quantized knob. From the categorical side, stacking with an invertible (2+1)-dimensional phase corresponding to the deformation class of Ω_3 realizes the same shift in $c_{-, \text{eff}}$; from the physical side, it appears as a rephasing of T and a slope change in $\kappa_{xy}(T)$.

Constraints from condensation and wall fusion. Not all values of Δc_- are accessible in a given platform. Anyon condensation selects algebra objects whose category of local modules has modular data related to those of the parent by induction/restriction functors; the possible Δc_- are constrained by the Witt class of the modular category and by the classification of modular categories themselves [113,120]. Domain-wall fusion rules (tensor product of bimodule categories) enforce additivity of the scalar phases and thus of Δc_- , as expected from stacking of invertible bulks [116,117].

Summary of the pathway. HCCB modifies the elementary sewing moves of conformal blocks by scalar phases organized by (higher) cohomological data. These phases exponentiate to 1-dimensional factors in the mapping-class-group action; the Dehn twist phase is the unique scalar that shifts c and nothing else. Hence, HCCB produces a well-defined

quantized drift $c \mapsto c_{\text{eff}}$ that is directly visible in both modular thermodynamics and low-temperature thermal Hall conductance.

5.2. Modifying the Experiment

We outline concrete modifications to the standard thermal Hall bar that turn the chiral central charge into a *tunable observable with high signal-to-noise* for higher categorical coherence breakdown (HCCB). The strategy is to (i) control the *mapping-class scalar* that HCCB contributes to the edge modular data (Section 5.1) by engineering moves on conformal blocks and (ii) amplify the corresponding drift of the effective anomaly c_{eff} in thermal transport by manipulating edge composition, equilibration, and ancilla couplings. Throughout, we avoid fragile math in section headers and keep the device model explicitly solvable.

5.2.1. Baseline: Thermal Hall Bar and Metrological Readout

Consider a Hall bar with source contact heated to T_{hot} and a remote cold reservoir at T_{cold} ; side contacts act as thermometers and voltage probes (Figure 2).

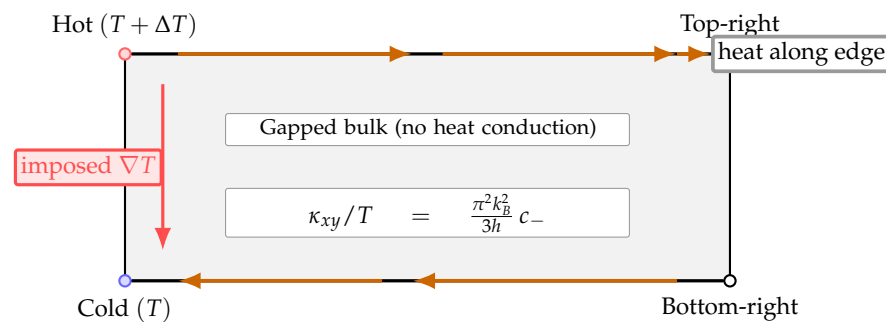


Figure 2. Baseline thermal Hall geometry: a temperature bias injects energy at a contact; heat flows chirally along the edges. The quantized coefficient probes the net chiral central charge c_- . Arrows mark the direction of chiral heat flow along the edges.

In a ballistic chiral edge, the energy current is

$$J_Q(T) = \frac{\pi}{12} \frac{k_B^2}{\hbar} c T^2, \quad (113)$$

so a single downstream channel transports $J_Q(T_{\text{hot}}) - J_Q(T_{\text{cold}})$, and for many channels the contributions add with the sign of chirality. In linear response, the thermal Hall conductance is

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_{-, \text{eff}} T, \quad c_{-, \text{eff}} = c_- + \Delta c_-, \quad (114)$$

where Δc_- is the HCCB-induced drift (Section 5.1). The absolute calibration follows the Landauer/Büttiker picture of edge transport [121] and the universal quantum of thermal conductance established in nanoscale channels [122,123]. The goal of the modifications below is to *maximize* sensitivity to Δc_- while suppressing systematic errors.

Error budget and sensitivity. Let $\hat{c}_{-, \text{eff}} = (3h/\pi^2 k_B^2) (\kappa_{xy}/T)$ be the estimator from a single operating point (T, κ_{xy}) ; then, for small relative errors,

$$\delta c_{-, \text{eff}} \approx \frac{3h}{\pi^2 k_B^2} \left(\frac{\delta \kappa_{xy}}{T} + \frac{\kappa_{xy}}{T} \frac{\delta T}{T} \right). \quad (115)$$

A slope fit of $\kappa_{xy}(T)$ across multiple temperatures reduces $\delta c_{-, \text{eff}}$ (uncertainty scales as $1/\sqrt{N}$; quantitative noise-floor and sensitivity estimates are collected in Appendix C) and guards against parasitic phonon conduction that is typically sublinear in T in the relevant range. Equation (114) then turns a *difference* measurement into a knob

for HCCB: switching a control that toggles the coherence defect by Δc_- produces $\Delta\kappa_{xy} = (\pi^2 k_B^2 / 3h) \Delta c_- T$.

5.2.2. Engineering Multi-Component Edges (Equilibration as a Gain Knob)

Many fractional states (e.g., $\nu = 2/3, 3/5, \dots$) host *counter-propagating* edge modes. Random tunneling and Coulomb interactions drive the edges to universal, disorder-dominated fixed points with *upstream neutral modes* [124]. This gives an exceptionally useful *gain knob*:

- **Composition and limits.** Suppose an edge supports N_d downstream and N_u upstream chiral channels. In the *decoupled* limit (no inter-mode energy exchange), the measured heat flow counts the absolute number of channels on the hot boundary, effectively giving

$$\kappa_{xy}^{\text{dec}} = \frac{\pi^2 k_B^2}{3h} (N_d + N_u) T.$$

In the *fully equilibrated* limit (interactions/disorder equilibrate all modes on each boundary), one recovers the anomaly result

$$\kappa_{xy}^{\text{eq}} = \frac{\pi^2 k_B^2}{3h} (N_d - N_u) T = \frac{\pi^2 k_B^2}{3h} c_- T.$$

- **Length–temperature crossover.** A simple energy-balance model with local mode temperatures $T_i(x)$ and inter-mode exchange rates γ_{ij} yields an equilibration length $\ell_{\text{eq}}(T)$ and an interpolating form

$$c_{-, \text{eff}}(L, T) \simeq c_- + [(N_d + N_u) - (N_d - N_u)] e^{-L/\ell_{\text{eq}}(T)}, \quad (116)$$

where L is the path length between heater/thermometer. In the $\nu = 2/3$ universality class, randomness at the edge is *relevant*, producing upstream neutral transport and power-law equilibration $\ell_{\text{eq}}(T) \propto T^{-p}$ (with $p > 0$ set by the fixed-point exponents) [124,125]. Measuring the crossover (116) while toggling HCCB isolates Δc_- with enhanced contrast at $L \gg \ell_{\text{eq}}$.

- **Implementation.** Multi-component edges arise naturally at $\nu = 2/3$ and higher hierarchy states; they can also be created by *edge reconstruction* using finger gates. Controlled short-range disorder (narrow gate arrays producing smooth random potentials) accelerates equilibration into the universal regime of Ref. [124], suppressing device-to-device variability and ensuring $c_{-, \text{eff}} \rightarrow c_-$ in the absence of HCCB.

Why this enhances HCCB sensitivity. In the equilibrated regime, $c_{-, \text{eff}} = c_- + \Delta c_-$, so the *entire* slope of $\kappa_{xy}(T)$ reflects the HCCB drift. In the unequilibrated regime, parasitic contributions from counter-propagating channels raise the baseline (toward $N_d + N_u$) and dilute sensitivity. Hence, deliberately engineering fast equilibration is a metrological amplifier for Δc_- .

5.2.3. Driving with Controlled Disorder and with Non-Abelian Quasiparticles

Controlled disorder. At $\nu = 2/3$, the Kane–Fisher–Polchinski (KFP) theory shows that random inter-channel tunneling is RG-relevant and drives the edge to a fixed point with a downstream charged mode and an upstream neutral mode [124]. In practice, one can *gate-engineer* the random tunneling strength (via lithographically patterned serrated gates) and measure $\kappa_{xy}(L, T)$ over a set of path lengths L . Fitting to (116) extracts $\ell_{\text{eq}}(T)$ and projects out the universal, anomaly-controlled piece. Because HCCB modifies the mapping-class scalar (hence c_-) but *not* the microscopic disorder rates, the difference between equilibrated and unequilibrated data isolates Δc_- .

Non-Abelian quasiparticles as a coherence probe. Interferometric geometries (Fabry–Pérot or Mach–Zehnder) that *change* the topological charge enclosed by an edge loop manipulate the braiding/fusion data of conformal blocks [126,127]. While interference primarily probes F/R symbols, the *same* categorical ingredients enter the anomalous mapping-class phase (the scalar in the Dehn twist) that sets c_{eff} . A practical protocol is:

1. Use a quantum point contact (QPC) to form an island that traps a controlled number of localized non-Abelian quasiparticles.
2. Interleave interference measurements (to confirm the anyonic sector) with thermal Hall measurements (to read out $c_{-, \text{eff}}$).
3. Search for correlated changes: when a topological sector switch toggles the categorical defect that implements HCCB, κ_{xy} should jump by $(\pi^2 k_B^2 / 3h) \Delta c_- T$, while interference phases shift by the corresponding F/R -data update [126].

This cross-check couples a *phase-sensitive* probe (interferometry) to a *thermodynamic* probe (thermal Hall), tying both to the same coherence deformation.

5.2.4. Ancilla Coupling via Moiré Chern Bands and Artificial Gauge Potentials

A powerful way to modulate the mapping-class scalar—and hence c_{eff} —is to stack or *side-couple* an *ancilla* topological channel with tunable Chern index and edge content. Graphene/hexagonal-boron-nitride (hBN) moiré superlattices realize Hofstadter minibands and, at high quality, host both integer and *fractional* Chern insulators (FCIs) with gate-tunable Chern number $\mathcal{C}$ and gaps [128–131]. We propose two architectures:

(A) *Lateral ancilla edge.* An electrostatically defined strip of a moiré Chern phase is formed parallel to the QH edge at separation d . Tunnel couplers (short QPCs) hybridize the two chiral systems. In the strong-coupling regime, the modular data multiply, and the Dehn-phase shifts add; equivalently,

$$c_{-, \text{eff}} = c_{-}^{(\text{QH})} + c_{-}^{(\text{ancilla})} + \Delta c_{-}^{(\text{HCCB})}. \quad (117)$$

Sweeping the ancilla across a topological transition (e.g., $\mathcal{C} = 1 \rightarrow 2$ or an FCI plateau) produces *step changes* in κ_{xy} with height $\Delta \kappa_{xy} = (\pi^2 k_B^2 / 3h) \Delta c_{-}^{(\text{ancilla})} T$, on top of the HCCB drift. Because the ancilla can be tuned in situ, this implements a differential measurement that cancels many systematic errors.

(B) *Embedded ancilla junctions.* Alternatively, one can carve a short segment of the main edge into a moiré miniband regime (by local gates) so that chiral modes traverse a *topological junction*. The junction acts as a controlled *coherence defect*: the fusion/braiding matrices for blocks that cross this segment are conjugated by the ancilla’s categorical data, injecting a scalar into the torus Dehn twist. In transport, this yields a localized contribution to the energy magnetization detectable by comparing configurations with the segment switched on/off.

5.2.5. Putting It Together: A Measurement Protocol

1. **Device.** A Hall bar with one edge engineered to be multi-component ($\nu = 2/3$ or $3/5$), equipped with (i) serrated gates to tune randomness (KFP regime), (ii) one or two QPCs to form an interferometer/island, and (iii) a lateral moiré ancilla strip with gate-tunable $\mathcal{C}$.
2. **Calibration.** Verify channel counting at high field and low temperature in the decoupled regime: $\kappa_{xy}^{\text{dec}} \approx (N_d + N_u) \kappa_0 T$. Then increase disorder/interactions to reach the equilibrated regime and confirm $\kappa_{xy}^{\text{eq}} \approx c_- \kappa_0 T$ within error bars set by (115).
3. **HCCB toggle.** Activate the categorical deformation by (a) changing the trapped topological charge in the interferometer island (non-Abelian case) or (b) toggling the

moiré ancilla across a topological transition. Record the *difference* in κ_{xy} at fixed T and the *slope change* in $\kappa_{xy}(T)$; both return Δc_- .

4. **Cross-verification.** Simultaneously monitor interference fringes to correlate F/R -data changes with the κ_{xy} step. Consistency across these orthogonal channels is a stringent check on HCCB.

5.2.6. Remarks on Practicality

(i) Using multi-component edges is not optional but *beneficial*: it turns equilibration into a controllable amplifier. (ii) Controlled disorder (KFP) is compatible with maintaining well-quantized *electrical* Hall plateaus; it only affects neutral modes and their thermalization. (iii) Moiré ancillae offer large, gate-tunable Chern bands and demonstrated FCIs, making them ideal, integrable coherence “knobs” [128–131]. (iv) The Büttiker edge picture supplies a transparent baseline for interpreting the metrology [121], while microscopic theories of dephasing provide guidance on length scales and operating temperatures [125].

How to read the next three geometries (null tests, not Tier-1 observables). The three devices that follow realize, in gate-defined hardware, the elementary coherence comparisons of the categorical framework: re-bracketing (pentagon), braid–fuse ordering (hexagon), and bracketing of energy splits (associator in transport). We state their logical status precisely, in line with the route-blindness result of Section 5.1: in any inflow-consistent realization the compared routes are different representatives of the *same* topological operation, so the framework itself predicts *null* outcomes, $\varepsilon_F = \varepsilon_H = 0$ and $\Delta c_-^{\text{HCCB}} = 0$, in these geometries at fixed control parameters. Their value is twofold. First, as precision null tests: a reproducible nonzero result, surviving the chirality-reversal and junction-scaling diagnostics below, would falsify the modular-functor axioms for the edge data—a statement far stronger than a parameter measurement, and one that present hardware can bound at the 10^{-2} level in phase and at the percent level in κ_{xy}/T . Second, as calibrated platforms for the *cyclic* protocol of Section 5.2.10, where the same gates are driven around closed loops in control space and a quantized, route-blind pumping response is a Tier-1 prediction.

5.2.7. Pentagon Interferometer: $(ab)c$ Versus $a(bc)$

Design and intuition (Figure 3). A Fabry–Pérot cell is formed by two quantum point contacts; three gate-defined antidots (or trapped island anyons) a, b, c sit inside it. Electrostatic routing gates select the order in which the interfering partial wave associates the localized charges: route A implements the fusion bookkeeping $(ab)c$ —the amplitude is composed by first associating a with b and then the pair with c —while route B implements $a(bc)$. The two routes are deformable into one another without crossing any worldline: they are two book-keepings of the *same* topological process, related by exactly the re-association move whose consistency is the pentagon identity. The device therefore measures, *directly*, the relative phase between the two sides of the pentagon. By route blindness (Section 5.1) the consistent framework predicts strict equality, $\varepsilon_F = 0$: this is a precision *null test*, with sensitivity at the 10^{-2} rad level in present Fabry–Pérot technology. Three controls discriminate a genuine coherence phase from instrumentation: (i) the single-antidot configuration, in which no re-association exists and any residual offset is instrumental; (ii) magnetic-field reversal, under which a genuine coherence phase must be odd while charging artifacts are not; (iii) permuting the antidot visit order at fixed enclosed area and charge, which isolates order dependence from Aharonov–Bohm physics. The quantitative protocol, and what a reproducible nonzero outcome would mean physically, are given in Section 6.4.

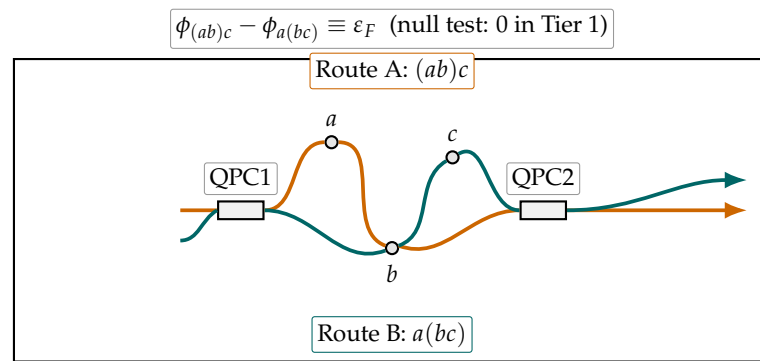


Figure 3. Pentagon interferometer. Two topologically equivalent but differently parenthesized paths are realized by gate routing. Within the inflow-consistent framework the two routings carry identical phases (route blindness, Section 5.1), so $\varepsilon_F = 0$ is the prediction; the device is a precision null test, and a reproducible nonzero offset would signal coherence failure beyond the invertible class. The two coloured paths (orange, route A; teal, route B) trace the parenthesizations $(ab)c$ and $a(bc)$ of the same fusion process.

5.2.8. Hexagon Interferometer: $R \circ F$ Versus $F \circ R$

Design and intuition (Figure 4). A third quantum point contact extends the cell so that the partial waves can either first *braid* around the island pair x, y and then traverse the fusion-defining constriction (the $R \circ F$ ordering), or first traverse it and then braid ($F \circ R$). The compatibility of these two orderings—exchange performed before or after re-association—is precisely the content of the hexagon identities, which tie the braiding data R to the associator F . In any consistent braided theory, and in any Tier-1 deformation of it, the two sequences yield the same phase, $\varepsilon_H = 0$; a stable order-dependent offset would mean that exchange and fusion fail to commute in the manner that the axioms require. The same discriminating controls as in the pentagon device apply, with the single-island configuration as the instrumental null and field reversal as the parity test.

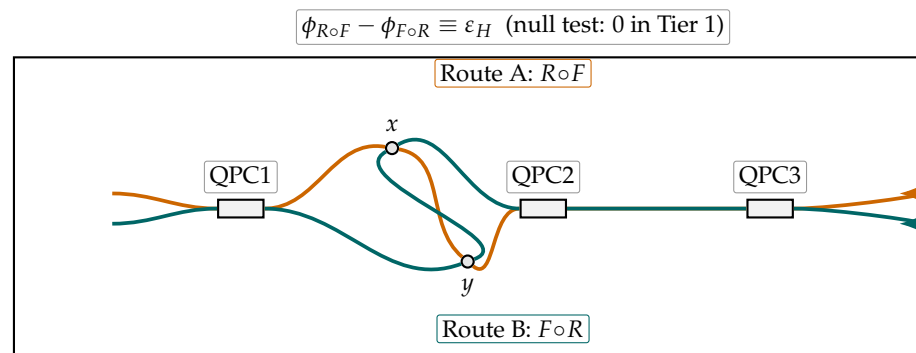


Figure 4. Hexagon interferometer. Braid-then-fuse vs. fuse-then-braid sequences agree in any exact MTC and in any inflow-consistent deformation thereof, so $\varepsilon_H = 0$ is the framework prediction; the geometry bounds order-dependent coherence failure beyond the invertible class. The two coloured paths (orange, $R \circ F$; teal, $F \circ R$) trace the two braid/fuse orderings.

5.2.9. Thermal Y-Network: Associator in Transport

A Y-junction is the elementary act of associativity in transport (Figure 5). A gate-defined junction splits one incoming chiral heat current into two descendant branches, or merges two branches into one; with three branches and two junctions in series, there are two physically distinct orders in which the splits and recombinations can be composed—first combine branches 1 and 2 and then bring in 3, or first combine 2 and 3 and then bring in 1. These are the transport realizations of the two bracketings $(12)3$ and $1(23)$: the Y-network is the calorimetric counterpart of the pentagon interferometer, with the abstract

re-association of fusion channels replaced by the concrete re-association of energy currents. Two properties make it uniquely valuable in the present program. First, it requires *no phase coherence*: the comparison is between net heat flows, governed only by chirality and energy conservation, so the device survives the dephasing that washes out interferometric fringes and can operate at path lengths and temperatures where no interferometer could. Second, it probes the *energy* channel directly—the gravitational-anomaly channel in which the chiral central charge itself lives—so that a bracketing dependence of heat flow, if it existed, would be a coherence failure in precisely the observable that this paper is about. Within the consistent framework the two routes are different decompositions of the same topological operation and the difference signal vanishes identically (route blindness, Section 5.1); the network is therefore a phase-insensitive null test of the sewing axioms. The same gate set, driven around a closed cycle in control space rather than compared between two static routings, becomes the hardware of the central-charge pump (Section 5.2.10).

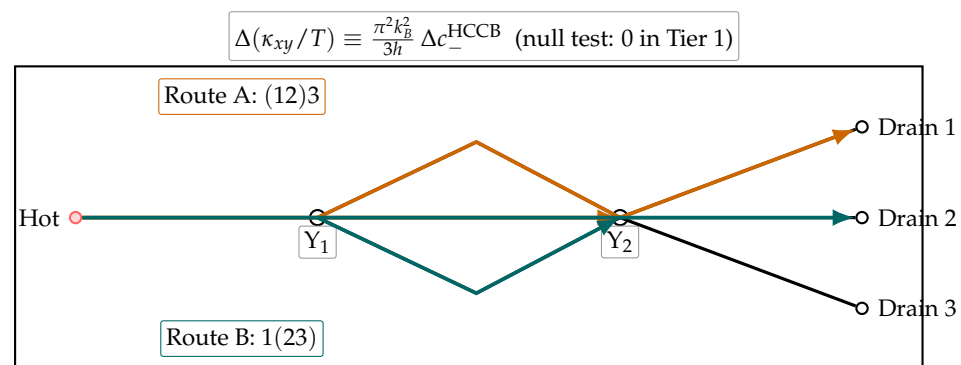


Figure 5. Thermal Y-network. Two bracketings of split/recombine operations are compared. Within the inflow-consistent framework, the bracketings are equivalent and the difference signal vanishes (route blindness, Section 5.1); therefore, the geometry bounds beyond-framework coherence failure and serves as the platform for the cyclic pumping protocol of Section 5.2.10, the quantized response of which is a Tier-1 prediction. The two coloured paths (orange, route A, (12)3; teal, route B, 1(23)) trace the two bracketings; where they share the central branch the colours coincide, which is schematic and does not affect the comparison.

5.2.10. From Route Comparisons to Cycles: A Central-Charge Pumping Protocol

The route-blindness theorem redirects rather than eliminates the search for higher-coherence physics in multi-junction devices: the consistent home for new invariants is not a comparison of two configurations at fixed couplings, but the *holonomy of a closed cycle* in the space of control parameters. Let λ collect the categorical controls of the device (wall positions and types, ancilla gate voltages, trapped topological charge) and let $\lambda(s)$, $s \in [0, 1]$, $\lambda(1) = \lambda(0)$ be a closed loop along which the bulk gap and the edge structure remain intact except at the discrete controlled events where the invertible sector is toggled. Families of gapped theories over a parameter manifold carry quantized invariants of exactly this kind—higher-dimensional generalizations of the Berry phase and, equivalently, anomalies in the space of coupling constants [132,133]—of which the Thouless charge pump is the classic degree-one ancestor [134]. In the present setting the relevant invariant is precisely the integer of Equation (172) (Section 7.2): the net Dehn-twist scalar accumulated around the cycle

$$\oint_{S^1} d\varphi(\lambda) = n \in \mathbb{Z}, \quad \Delta c_-^{\text{cycle}} = 24n \quad (\text{bosonic sector; } \frac{1}{2}\mathbb{Z}\text{-steps with spin structure}),$$

which is route-blind (it depends only on the homotopy class of the loop in control space) and yet strictly invisible to any static stacking description of a single configuration. Opera-

tionally: (1) calibrate the device in two static configurations λ, λ' and verify the quantized difference step of Section 5.3; (2) drive the controls around a closed loop passing through λ and λ' and back, counting the accumulated quantized steps of κ_{xy}/T per cycle, with the interferometric channel monitoring that each step coincides with a topological-sector switch; and (3) reverse the loop orientation, which must reverse the count, and contract the loop to a trivial one, which must return zero. A nonzero, orientation-odd, homotopy-invariant count is then a genuine higher-coherence observable—a “central-charge pump”—realized entirely within the inflow-consistent framework, with the extra cohomological degree supplied by the parameter space rather than by a fourth spacetime dimension.

In what sense route sensitivity survives. It is worth making the geometry of this statement explicit, because it resolves the apparent tension between route blindness and the pump. Route blindness (Section 5.1) forbids dependence on the bookkeeping route of a comparison performed *at fixed couplings*; it does not constrain open trajectories in control space. Along such a trajectory, the system is genuinely path-sensitive in a local sense: the instantaneous $c_{\text{eff}}(\lambda(s))$ traces a profile that depends on the path—two different routes from λ to λ' can visit different sectors, in different orders, with the thermometer registering different intermediate step sequences. Consistency constrains only the closed-loop total, and it constrains it to a topological quantity: the categorical data define, in effect, a *flat connection on the space of controls*, whose flatness is route blindness for contractible comparisons and whose *holonomy* around non-contractible loops is the pump count. Equivalently, n in Equation (172) counts the signed number of times the cycle winds around the codimension—two loci in control space where the invertible sector changes—which represent the control-space analogues of vortices. Therefore, local route sensitivity with quantized global holonomy is not an exception to route blindness; rather, it is its precise complement, and is the unique form in which the intuition of “route-dependent central charge” survives the consistency analysis—relocated from the routing of currents in real space to the routing of *controls* in parameter space.

A concrete cycle. In the lateral-ancilla architecture of Section 5.2.4, take $\lambda = (V_g, B_{\text{loc}})$ to be the ancilla gate voltage and a local field (or displacement field) that together tune the ancilla strip across its $C = 0 \leftrightarrow 1$ transition. The transition is a point (more generally, a codimension-two locus) in the (V_g, B_{loc}) plane. A closed loop that encircles this point once cannot be contracted without crossing it; per cycle it produces exactly one net quantum of pumped anomaly between the ancilla channel and the main edge—registered as one unit appearing and one unit disappearing in κ_{xy}/T at parameter values that differ between the two halves of the loop, with the interferometric channel confirming a sector switch at each step. A loop in the same plane that does *not* enclose the transition returns a net count of zero, whatever its shape (this is route blindness in control space); reversing the orientation of the enclosing loop reverses the sign of the count, and a loop enclosing the transition k times yields k quanta. These three checks—contractibility null, orientation oddness, and integer winding proportionality—are the experimental fingerprints that distinguish the pump from any adiabatic background, in direct analogy with the corresponding checks for the Thouless charge pump [134].

5.3. Expected Experimental Signatures

We now translate the higher categorical coherence breakdown (HCCB) mechanism into concrete observables for quantum Hall devices. The core prediction is a *drift* in the anomaly-controlled response coefficients that is *quantized* in ideal conditions and manifests most cleanly in the thermal Hall conductance. We develop a quantitative scaling theory that (i) isolates the HCCB contribution to the thermal response from edge-equilibration and parasitic channels, (ii) clarifies when nonlinear temperature dependences of κ_{xy} are ex-

pected, and (iii) prescribes robust difference measurements that return Δc_- with controlled error bars.

5.3.1. Thermal Hall Conductance with HCCB: Scaling Form

In the equilibrated low-temperature regime of a chiral edge, the thermal Hall conductance obeys

$$\kappa_{xy}(T; \lambda) = \kappa_0 [c_- + \Delta c_-(\lambda)] T + \delta \kappa_{xy}^{\text{corr}}(T; \lambda), \quad \kappa_0 \equiv \frac{\pi^2 k_B^2}{3h}, \quad (118)$$

where λ is a *categorical control* that turns the coherence deformation on/off (e.g., domain-wall configuration, ancilla phase, trapped topological charge) and $\Delta c_-(\lambda)$ is the resulting drift in the (chiral) central charge. The correction $\delta \kappa_{xy}^{\text{corr}}$ includes non-universal but *diagnosable* effects (finite equilibration length, phonon leakage, contact losses).

Thus, a first parameter-free prediction is the *difference* signal between two categorical configurations λ and λ' :

$$\Delta \kappa_{xy}(T; \lambda, \lambda') \equiv \kappa_{xy}(T; \lambda) - \kappa_{xy}(T; \lambda') = \kappa_0 \Delta c_- T + \Delta \delta \kappa_{xy}^{\text{corr}}(T), \quad (119)$$

with $\Delta c_- = \Delta c_-(\lambda) - \Delta c_-(\lambda')$. If the categorical control does not modify equilibration or leakage channels (a key design principle of Section 5.2), then $\Delta \delta \kappa_{xy}^{\text{corr}}(T) \approx 0$ and the *slope* of $\Delta \kappa_{xy}(T)$ directly returns Δc_- . We formalize these statements next.

5.3.2. Partial Equilibration and Exponential Approach to the Anomaly

Consider an edge with N co-propagating chiral modes labeled by i with chiralities $s_i = \pm 1$, velocities v_i , and “central charge weights” c_i (one per chiral CFT channel: 1 for a free chiral boson, 1/2 for a Majorana, etc.). In linear response with a small temperature bias ΔT between source ($x = 0$) and drain ($x = L$), let $T_i(x) = T_0 + \theta_i(x) \Delta T$ denote the local mode temperatures. Energy conservation and inter-mode exchange give the coupled first-order equations

$$s_i \frac{\pi}{6} \frac{k_B^2}{\hbar} c_i v_i \partial_x \theta_i(x) = - \sum_j \Gamma_{ij}(T_0) [\theta_i(x) - \theta_j(x)], \quad (120)$$

with boundary conditions $\theta_i(0) = +1/2$ for downstream modes entering from the hot source and $\theta_i(L) = -1/2$ for upstream modes entering from the cold drain. The matrix $\Gamma(T_0)$ parameterizes inter-mode energy exchange (from disorder and interactions). Diagonalizing the coupling in the co-moving basis reduces (120) to decoupled exponential relaxations with *equilibration lengths*

$$\ell_\alpha(T_0) = \frac{\hbar}{\pi k_B^2} \frac{\sum_i s_i c_i v_i u_i^{(\alpha)}}{\lambda_\alpha(T_0)}, \quad \lambda_\alpha(T_0) \text{ eigenvalues of } \Gamma(T_0), \quad (121)$$

where $u^{(\alpha)}$ are normalized right-eigenvectors. Integrating along the boundary yields the net heat current, hence

$$\kappa_{xy}(L, T_0) = \kappa_0 T_0 \left[\sum_i s_i c_i - \sum_\alpha A_\alpha e^{-L/\ell_\alpha(T_0)} \right], \quad (122)$$

with positive coefficients A_α fixed by boundary conditions and eigenvector overlaps. Equation (122) reproduces the known limiting cases observed experimentally: (i) the *decou-*

pled limit $L \ll \ell_\alpha$ measures the sum of absolute chiralities, while (ii) the *equilibrated* limit $L \gg \ell_\alpha$ measures the anomaly $\sum_i s_i c_i = c_-$ [135].

Crucially, the equilibration lengths are *temperature-dependent*. For random tunneling processes with scaling dimension Δ_{tun} , Fermi's golden rule and RG arguments give $\ell_\alpha(T) \propto T^{2\Delta_{\text{tun}}-1}$, hence

$$\ell_\alpha(T) \propto T^{1-2\Delta_{\text{tun}}}. \quad (123)$$

Thus, at fixed L , the deviations from the linear-in- T anomaly law inherit a *nonlinear* T -dependence through the exponent in (122). This is a benign diagnostic nonlinearity: it disappears as $L/\ell_\alpha(T) \rightarrow \infty$ and does not mimic an HCCB-induced *slope* change.

HCCB in the presence of partial equilibration. Switching a categorical control λ adds a constant *anomalous* piece Δc_- to the square bracket in (122) without changing ℓ_α (by design). Therefore, in the difference channel (119), the exponentials *cancel* to leading order and

$$\frac{\Delta\kappa_{xy}(T)}{T} \xrightarrow{L/\ell_\alpha \gtrsim 1} \kappa_0 \Delta c_- + O(e^{-L/\ell_{\min}(T)}), \quad (124)$$

so the *temperature derivative* $\partial_T[\Delta\kappa_{xy}(T)/T]$ provides a null test for residual nonequilibrium.

5.3.3. Nonlinear $\kappa_{xy}(T)$ and How to Interpret It

In addition to partial equilibration, two further mechanisms can bend the T -dependence:

1. *Topological-sector mixing.* If the categorical control involves trapping non-Abelian quasiparticles with a small energy splitting Δ_{topo} between sectors, thermal occupation leads to a T -dependent mixing weight $p(T)$ for the sector carrying the coherence defect. Then,

$$c_{-, \text{eff}}(T) = c_- + p(T) \Delta c_-, \quad p(T) = \frac{1}{1 + e^{\Delta_{\text{topo}}/T}} \quad (\text{two-sector toy model}), \quad (125)$$

and κ_{xy} acquires a smooth crossover instead of a sharp step. Independent interferometry of the island charge (switching between even/odd fusion channels) allows one to *calibrate* $p(T)$ and re-extract the asymptotic Δc_- from low- T data [136,137].

2. *Edge-phonon heat leaks.* Weak phonon coupling along the mesa edge adds a subleading channel for which the contribution typically scales faster than linear in T , leading to

$$\kappa_{xy}(T) = \kappa_0 (c_- + \Delta c_-) T + \sum_r b_r T^{\alpha_r}, \quad \alpha_r > 1. \quad (126)$$

A two-temperature slope fit of $\kappa_{xy}(T)/T$ removes such analytic backgrounds to first order, while on-chip noise thermometry can verify that the electronic temperature follows the heater calibration [138,139].

In all three cases (partial equilibration, topological-sector mixing, phonons), *difference* measurements between λ and λ' combined with an analysis of $\kappa_{xy}(T)/T$ isolate the HCCB drift as the residual *constant* offset at low T and large L .

5.3.4. Drift of the Effective Anomaly: Null Tests and Cross-Checks

Define the operational estimator

$$\widehat{\Delta c_-}(T) \equiv \frac{1}{\kappa_0} \frac{\Delta\kappa_{xy}(T)}{T}. \quad (127)$$

Three complementary signatures should coincide in a successful experiment:

1. **Slope test:** $\widehat{\Delta c_-}(T)$ becomes T -independent as $T \rightarrow 0$ and $L/\ell_\alpha(T) \rightarrow \infty$; the plateau value is the predicted rational (or integer) Δc_- .
2. **Length test:** At fixed low T , $\widehat{\Delta c_-}(L)$ approaches the same plateau as L increases; this tests the cancellation of partial-equilibration corrections in the difference channel (controlled by (122)).
3. **Interferometric correlation:** In devices with a non-Abelian island, changes in the interference phase (from F/R -symbol updates) correlate one-to-one with steps in κ_{xy} , as both derive from the *same* categorical deformation. This cross-check disambiguates HCCB from conventional changes in edge dispersion [140].

5.3.5. Auxiliary Observables

Two further anomaly-sensitive quantities provide independent confirmation:

- *Casimir amplitude on a ring.* On a mesoscopic annulus of circumference L , the free-energy density contains $-(\pi/6)(c_{\text{eff}}/v)T^2$; as in Section 4.2, an HCCB toggle changes the T^2 coefficient by $(\pi/6)(\Delta c_-/v)$. Measuring this small effect requires on-chip calorimetry but is a direct thermodynamic readout of the drift.
- *Chiral energy imaging.* Local thermometry along the edge (quantum-dot spectroscopy or quantum point-contact thermometry) maps $T_i(x)$ and tests Equation (120), extracting $\ell_\alpha(T)$ and verifying that HCCB changes the *offset* but not the *shape* of $T(x)$ profiles [135,138].

Synthesis. The *primary* HCCB signature is a *linear-in- T* step in κ_{xy} whose slope equals $\kappa_0 \Delta c_-$. Nonlinear T -dependences are expected from partial equilibration, sector mixing, and phonon leakage; they are not confounding if one uses difference measurements, length sweeps, and interferometric correlations. The resulting “triple coincidence” of slope, length, and phase tests provides a stringent identification of HCCB in quantum Hall devices. Table 1 summarizes how the HCCB signature is distinguished from every conventional mechanism known to shift the apparent c_- .

Table 1. Mechanisms affecting the measured thermal-Hall slope $\kappa_{xy}/(\kappa_0 T)$ and how the HCCB (Tier-1) signal is distinguished from each. “Quantized” refers to steps in units of $\frac{1}{2}$ (spin sectors) or 8 (purely bosonic stacking). The difference channel compares two categorical configurations λ, λ' at otherwise identical operating conditions (Section 5.3.1).

Mechanism	Quantized?	T -Dependence of κ_{xy}/T	L -Dependence	Odd $B \rightarrow -B$? Under $-B$?	Discriminating Test
HCCB/invertible-sector toggle (this work)	Yes ($\frac{1}{2}\mathbb{Z}$ or $8\mathbb{Z}$)	constant offset at low T	none at $L \gg \ell_{\text{eq}}$	Yes	step correlated one-to-one with interferometric sector switch; survives difference channel
Central-charge pump over control cycle (this work)	Yes (integer count per cycle)	constant per plateau	none	Yes (orientation of cycle)	homotopy invariance: contractible cycle returns zero
Partial equilibration (counter-propagating modes)	No	approaches anomaly value as $e^{-L/\ell_{\text{eq}}(T)}$, $\ell_{\text{eq}} \propto T^{1-2\Delta_{\text{tun}}}$	strong	No	cancels in difference channel; flat in L after cancellation
Edge reconstruction (extra non-topological modes)	No	device-specific, smooth	gate-voltage dependent	No	changes electrical edge structure (tunneling spectroscopy); not toggled by categorical control
Edge-velocity renormalization	—(does not shift κ_{xy}/T)	none in equilibrated limit	none	No	affects Casimir/specific-heat amplitudes but not the anomaly slope
Phonon and parasitic heat leaks	No	superlinear ($T^\alpha, \alpha > 1$)	geometry dependent	No	removed by two-temperature slope fits; insensitive to categorical toggle
Thermal sector mixing (quantized splitting Δ_{topo})	Quantized at low T	activated crossover, $p(T)$ of Section 5.3.3	none	Yes	activated form in T ; interferometric monitoring of sector occupancy
Anyon-condensation phase transition	Yes (changes $c_-^{(0)}$ itself)	step at transition	none	Yes	accompanied by bulk gap closing and change of fusion content (interference periods)
Tier-2 route dependence (bounded, not predicted)	—	—	scales with junction number	Yes	excluded by route-blindness; any residue falsifies modular-functor axioms

5.3.6. Nonlinearity in $\kappa_{xy}(T)$ Versus Genuine Δc_-

Finally, it is important to emphasize how to distinguish benign nonlinearities in $\kappa_{xy}(T)$ from a genuine categorical drift of the anomaly coefficient. As discussed above, partial equilibration, phonon heat leaks, or sector mixing can all produce departures from strict linearity in T , often sublinear. In particular, the presence of upstream neutral modes is well known to generate such trends even in otherwise well-calibrated devices.

Our proposal uses a difference channel between two categorical configurations λ and λ' . Provided that the toggle between λ and λ' does not itself alter equilibration lengths, these non-universal contributions cancel to leading order, leaving only the categorical offset Δc_- .

Empirical check. Experimentally, one should verify this cancellation by examining the L -dependence of $\Delta\kappa_{xy}(T)$. If the extracted ratio $\Delta\kappa_{xy}(T)/T$ is flat in L within error bars, then the residual signal can be reliably interpreted as the quantized HCCB-induced shift. If instead there remains a systematic L -dependence, this indicates incomplete cancellation and the need to refine equilibration control before attributing the effect to Δc_- .

6. Higher Categorical Coherence Breakdown and the Fractional Quantum Hall Effect

The fractional quantum Hall effect (FQHE) is one of the most striking manifestations of topological order and strong correlations in condensed matter physics. Its discovery revealed not only quantized electrical Hall conductance at fractional fillings but also emergent quasiparticles with fractional charge and statistics as well as candidate non-Abelian orders of direct relevance to topological quantum computation. Despite the broad success of the Chern–Simons and conformal field theory descriptions, several experimental puzzles remain unresolved: small but reproducible spreads in thermal Hall conductance, unexpected phase slips in interferometers, geometry-dependent tunneling exponents, and ambiguities between competing candidate orders at even-denominator fillings.

In this section, we propose that these anomalies are not merely experimental imperfections or uncontrolled edge reconstructions but rather signatures of a more fundamental effect, namely, *higher categorical coherence breakdown* (HCCB). At triangle level, the Virasoro central charge already encodes a well-known obstruction (Witt $\rightarrow$ Virasoro) and yields the chiral central charge c_- governing thermal Hall quantization. We argue that higher obstructions (3- and 4-cocycles) can appear at the pentagon and hexagon levels, producing tiny but measurable corrections to edge phases, visibilities, and effective central charges in complex edge networks.

We organize the rest of this section as follows: in Section 6.1, we review the integer and fractional QH effects and outline the open puzzles; in Section 6.2, we apply the HCCB framework to QH edge theories and define pentagon/hexagon anomalies; Section 6.3 reinterprets the puzzles through HCCB, while Section 6.4 proposes concrete experimental probes; finally, Section 6.5 synthesizes the implications and emphasizes the broader conceptual and experimental payoff.

In this section, we connect higher categorical coherence breakdown (HCCB) to concrete measurable anomalies in quantum Hall systems. At triangle level, the obstruction class (Virasoro central term) yields the well-known chiral central charge and its transport avatar, the thermal Hall conductance. At higher coherence (pentagon/hexagon), new weaker obstructions appear that can bias effective anomaly readouts in multi-edge networks. To set the stage, we first carefully and quantitatively review the integer/fractional quantum Hall frameworks and spell out the outstanding puzzles where HCCB can act.

6.1. Motivation: From Integer to Fractional QH and the Open Puzzles

6.1.1. Integer QHE: Landau Quantization, Chern Response, and Plateaus

Consider non-interacting electrons of charge $-e$ and effective mass m confined to two dimensions in a perpendicular magnetic field B . In Landau gauge $A = (0, Bx)$ the single-particle Hamiltonian

$$H = \frac{1}{2m} (p_x^2 + (p_y + eBx)^2)$$

has *Landau levels* (LLs) with energies

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right), \quad \omega_c := \frac{eB}{m}, \quad n = 0, 1, 2, \dots \quad (128)$$

Each LL exhibits a macroscopic degeneracy: on an area A , the number of orthogonal states is

$$N_\phi = \frac{BA}{\Phi_0}, \quad \Phi_0 := \frac{h}{e}, \quad (129)$$

so the degeneracy per unit area is eB/h . If the electron density is n_e , the *filling factor*

$$\nu := \frac{n_e}{eB/h} = \frac{n_e h}{eB} \quad (130)$$

counts how many LLs are filled (approximately, at $T=0$ and ignoring spin/Zee-man for brevity). When $\nu \in \mathbb{Z}$, all LLs below E_F are completely filled, the bulk spectrum has a mobility gap, and the Hall conductivity is quantized,

$$\sigma_{xy} = \nu \frac{e^2}{h}. \quad (131)$$

There are three complementary derivations of this quantization:

- (i) **Kubo–TKNN/Chern:** in a periodic band setting, σ_{xy} is the sum of first Chern numbers of filled bands. Each Landau level contributes Chern number $+1$, giving $\nu e^2/h$ in the continuum limit.
- (ii) **Topological effective action:** integrating out gapped electrons yields a Chern–Simons (CS) response for the background electromagnetic field A ,

$$S_{\text{eff}}[A] = \frac{\nu}{4\pi} \int_{M_{2+1}} A \wedge dA, \quad \Rightarrow \quad j^\mu = \frac{\delta S_{\text{eff}}}{\delta A_\mu} = \frac{\nu}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho, \quad (132)$$

which reproduces $\sigma_{xy} = \nu e^2/h$.

- (iii) **Edge–bulk correspondence:** a confining potential bends LLs at the boundary; each filled LL generates one chiral edge channel carrying charge current $I = (e^2/h)V$. The number of co-propagating channels equals ν .

Why plateaus? Real samples have disorder that broadens LLs into bands with localized states in the tails and extended states near band centers. As B or n_e vary, the Fermi energy E_F can reside in localized regions over finite parameter intervals; longitudinal transport vanishes ($\sigma_{xx} \approx 0$) while σ_{xy} stays pinned to the topological value of the nearest extended-state manifold. This yields the observed plateaus with robust quantization.

6.1.2. Fractional QHE: Interactions, Anyons, and Topological Order

At fractional fillings (e.g., $\nu = \frac{1}{3}, \frac{2}{5}, \dots$), non-interacting LL physics fails: Coulomb repulsion and projection to the lowest LL drive the system into a *topologically ordered*

incompressible quantum fluid. The prototypical Laughlin state at $\nu = 1/m$ (m odd for electrons) admits a compact universal bulk description by a $U(1)$ CS gauge field a :

$$S_{\text{bulk}}[a, A] = \frac{m}{4\pi} \int a \wedge da + \frac{1}{2\pi} \int A \wedge da, \quad (133)$$

where A is the physical electromagnetic field. Varying a yields $da = -(1/m) dA$; substituting into (133) gives

$$S_{\text{eff}}[A] = \frac{1}{4\pi m} \int A \wedge dA \Rightarrow \sigma_{xy} = \frac{1}{m} \frac{e^2}{h}. \quad (134)$$

Quasiparticles correspond to 2π flux insertions of a , and as such carry *fractional charge* $e^* = e/m$ (by the $A \wedge da$ coupling) and *fractional statistics* with exchange phase $\theta = \pi/m$ (from the $a \wedge da$ term).

More general Abelian FQH phases are captured by the K -matrix CS theory

$$S_K[a^I, A] = \frac{1}{4\pi} \int K_{IJ} a^I \wedge da^J + \frac{1}{2\pi} \int t_I A \wedge da^I, \quad (135)$$

with integer symmetric nondegenerate K , charge vector t , and emergent $U(1)^N$ gauge fields a^I . Universal data follow algebraically:

$$\sigma_{xy} = \frac{e^2}{h} t^\top K^{-1} t, \quad Q(\ell) = e \ell^\top K^{-1} t, \quad \theta(\ell) = \pi \ell^\top K^{-1} \ell, \quad (136)$$

for quasiparticles labeled by integer vectors $\ell \in \mathbb{Z}^N$. On a closed surface of genus g , the ground-state degeneracy is $|\det K|^g$, a hallmark of topological order.

Edge theory and central charge. Bulk CS theories induce gapless chiral edge modes. For Abelian phases, the edge is a multi-component chiral Luttinger liquid

$$\mathcal{L}_{\text{edge}} = \frac{1}{4\pi} \left(K_{IJ} \partial_t \phi^I \partial_x \phi^J - V_{IJ} \partial_x \phi^I \partial_x \phi^J \right) + \frac{e}{2\pi} t_I A_\mu \epsilon^{\mu\nu} \partial_\nu \phi^I, \quad (137)$$

with compact bosons $\phi^I \sim \phi^I + 2\pi$. The chiral central charge is the net number of right-moving minus left-moving gapless modes

$$c_- := c_R - c_L = \text{sgn}(K) = n_+ - n_-, \quad (138)$$

where $n_\pm$ are the numbers of positive/negative eigenvalues of K . (For purely downstream Abelian edges (positive-definite K), $c_- = N$. For non-Abelian phases, c_- receives contributions from both charged bosons and neutral sectors (e.g., Majorana/parafermion CFTs)). Physically, c_- controls the *gravitational anomaly* and the *thermal Hall conductance*,

$$\frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h} c_- \quad (T \rightarrow 0). \quad (139)$$

6.1.3. What Is Settled vs. What Remains Puzzling

The core paradigms above are well tested:

- Quantized *electrical* Hall conductance $\sigma_{xy} = \nu e^2/h$ across many integer and fractional plateaus.
- *Fractional charge* via shot-noise/Fano-factor measurements at QPCs (e.g., $e^* = e/3$ at $\nu = 1/3$).
- *Anyonic phases* through interferometry (with Coulomb-dominated caveats).

- *Thermal Hall* measurements that broadly align with predicted c_- in several Abelian states.

However, notable tensions persist, especially in regimes with neutral modes and suspected non-Abelian order (e.g., even-denominator states):

(A) *Thermal Hall at $\nu = \frac{5}{2}$ and related even denominators.* Several experiments report half-integer quantization $\kappa_{xy}/T \approx (n + \frac{1}{2})\kappa_0$ with $\kappa_0 := (\pi^2 k_B^2/3h)$, suggestive of neutral Majorana modes, yet device-to-device spreads and temperature/geometry dependences persist. Interpreting these consistently within a single edge theory is subtle due to edge reconstruction, upstream neutral modes, and equilibration.

(B) *Interferometer anomalies (phase slips and route dependence).* Fabry–Perot/Mach–Zehnder data sometimes display phase irregularities not fully captured by simple area/charge and Coulomb-dominated models, including apparent dependence on the order in which braids/fusions are implemented in multi-island geometries.

(C) *Non-universal tunneling exponents.* Point-contact I – V characteristics often deviate from clean Luttinger predictions $I \sim V^\alpha$ inferred from the putative edge K and t . Scatter correlates with device layout (number/ordering of junctions), not just with disorder strength.

(D) *Neutral mode equilibration puzzles.* Measured equilibration lengths of upstream neutral modes vary strongly with temperature and edge routing. Distinguishing intrinsic edge content (fixed by bulk topological order) from extrinsic reconstruction/equilibration remains challenging.

(E) *Particle–hole (PH) asymmetry at $\nu = \frac{5}{2}$.* Candidate orders (Pfaffian vs. anti-Pfaffian vs. PH-Pfaffian) predict the same σ_{xy} but different c_- and neutral content. Some trends appear to favor one orientation or show weak PH breaking in nominally PH-symmetric regimes.

6.1.4. Why Higher Coherence Effects Are Natural Suspects

All puzzles above share a common feature: they are most pronounced in *complex edge networks*—multiple QPCs, multi-island interferometers, serial Y-junctions—where one *composes* elementary processes (split, recombine, braid, fuse) in different sequences. The standard topological framework (triangle-level consistency) guarantees that topologically equivalent routes produce the same outcome. If, however, higher categorical coherence *very weakly* fails at the level of the associator (pentagon) or braiding hexagon, then:

1. Topologically equivalent but differently *parenthesized* compositions can acquire tiny relative phases/attenuations (pentagon angles and non-unitary leaks).
2. Braid-then-fuse and fuse-then-braid sequences can differ by small systematic angles (hexagon anomaly).
3. In transport, repeated split/recombine events can bias the *effective* anomaly coefficient read out by a thermal Hall protocol, shifting

$$c_-^{\text{eff}} = c_-^{(0)} + \Delta c_-^{\text{HCCB}}, \quad \frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h} c_-^{\text{eff}}. \quad (140)$$

Equation (140) does *not* alter the bulk topological order; it asserts that in multi-junction edge networks the measured anomaly can pick up a routing-dependent correction Δc_-^{HCCB} that vanishes for a single straight edge and flips sign under chirality reversal (magnetic-field inversion). This is precisely the regime where current discrepancies are largest and where targeted experiments can decisively test for higher coherence.

Roadmap. In the next subsections we formalize the HCCB couplings on edge networks (triangle vs. pentagon vs. hexagon), quantify their leading effects on phases, visibilities, and thermal transport, and propose concrete interferometric and thermal Y-network protocols that can bound or reveal ε_F , ε_H , and Δc_-^{HCCB} within present experimental precision.

6.2. The HCCB Framework Applied to Quantum Hall Edges

6.2.1. Triangle-Level Anomalies and the Virasoro Central Charge

At the most elementary level of coherence (the triangle diagram), the obstruction is a 2-cocycle that prevents the Witt algebra of circle diffeomorphisms from closing linearly on the Hilbert space of edge states. Instead, the algebra closes only after central extension to the Virasoro algebra,

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}. \quad (141)$$

Here, c is the Virasoro central charge, the “residue of regularization” that measures the quantum obstruction.

In the edge theory of a quantum Hall state, this c appears chirally: the system has a net chiral central charge

$$c_- \equiv c_R - c_L, \quad (142)$$

counting the imbalance of right- versus left-moving degrees of freedom. This imbalance is precisely the gravitational anomaly of the edge, canceled by inflow from the bulk gravitational Chern–Simons term.

Physically, c_- is directly observable through the *quantized thermal Hall conductance*,

$$\frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h} c_-. \quad (143)$$

This is the transport manifestation of the triangle-level coherence obstruction.

6.2.2. Pentagon-Level Breakdown: Associator Curvature

Higher categorical structure appears when we consider *composition of edge processes*. At the pentagon level, the relevant constraint is the associator; composing three processes f, g, h in different parenthesizations must give the same amplitude. Formally,

$$(fg)h = f(gh).$$

If coherence fails very weakly, this equality is modified by a small unitary or non-unitary factor

$$(fg)h = e^{i\varepsilon_F} \Lambda_{fg,h} f(gh), \quad (144)$$

with ε_F a small “pentagon angle” and $\Lambda_{fg,h}$ a possible attenuation factor (if the associator curvature carries a non-unitary component). This breakdown is governed by a 3-cocycle in the categorical cohomology.

Physical meaning. At a quantum Hall edge, split/merge events at Y-junctions or quantum point contacts (QPCs) implement such associative compositions of current channels. In the standard (topologically exact) theory, different routing orders give identical interference patterns. With pentagon breakdown, topologically equivalent routings differ by ε_F in phase and by $\Lambda_{fg,h}$ in visibility.

Thus, a putative associator curvature would produce two observable effects:

1. Small but systematic *phase shifts* between interferometer configurations related by re-parenthesization.
2. Possible *non-unitary leakage channels* leading to reduced visibility or effective renormalization of Luttinger exponents in tunneling.

Status within the consistent framework. We stress the logical status of ε_F and $\Lambda_{fg,h}$, in line with Sections 2.3.3 and 5.1: *no* inflow-consistent (Tier-1) deformation can produce them, since route blindness forces all re-parenthesization comparisons to close exactly. A nonzero

ε_F therefore requires Tier-2 physics: either a genuine H^4 obstruction, which demands a topologically non-trivial (3+1)-dimensional host that is absent in planar devices, or non-topological dynamics at the junctions (decoherence, hysteresis, internal junction states), in which case no quantization protects the effect and it must be modeled microscopically. Accordingly, in what follows ε_F , Λ , and ε_H are treated as *bounded hypothesis parameters* probed by null tests, not as predictions of the framework.

6.2.3. Hexagon-Level Breakdown: Braiding Anomalies

The next level of coherence is the hexagon diagram, expressing the compatibility of braiding with associativity. In categorical form, exchanging two anyons and then fusing should equal fusing and then exchanging. If the hexagon relation fails, we may write

$$R \circ F \neq F \circ R, \quad (145)$$

with a mismatch characterized by a “hexagon anomaly angle” ε_H (or more generally, a 4-cocycle class in the braiding cohomology).

Physical meaning. In interferometers with multiple antidots or islands, one can compare braid-then-fuse vs. fuse-then-braid sequences. In standard UMTC physics, these give the same phase. With hexagon breakdown, one expects

$$\Delta\phi_{\text{hex}} = \varepsilon_H \neq 0, \quad (146)$$

which would appear as stable order-dependent phase offsets in interference fringes.

6.2.4. Effective Central Charge Drift from Higher Coherence

Within the consistent (Tier-1) framework, the effective anomaly coefficient extracted from transport shifts as

$$c_{-}^{\text{eff}} = c_{-}^{(0)} + \Delta c_{-}^{\text{HCCB}}, \quad (147)$$

where $\Delta c_{-}^{\text{HCCB}}$ is set by the invertible sector realized along the measured edge—by the walls, condensate interfaces, and ancillary channels *present in the configuration*—and not by the routing or bracketing of operations: it is quantized ($8\mathbb{Z}$ bosonic, $\frac{1}{2}\mathbb{Z}$ with spin structure), it vanishes for a bare single edge with no engineered sectors, it flips sign under magnetic field reversal (since chirality reverses), and it does *not* accumulate with the number of junctions traversed at fixed topology. What does accumulate quantizedly is the pump count of closed control cycles (Section 5.2.10).

The corresponding thermal Hall conductance then reads

$$\frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h} (c_{-}^{(0)} + \Delta c_{-}^{\text{HCCB}}). \quad (148)$$

Because the shift is quantized, it cannot by itself produce *small continuous* spreads; the only Tier-1 route to intermediate apparent values is statistical mixing between topological sectors with a quantized splitting, which carries a characteristic activated temperature dependence (Section 5.3.3). This sharpens rather than weakens the experimental program: quantized steps and activated mixing are Tier-1 physics; anything else in the spreads must be conventional (equilibration, reconstruction, leakage) or Tier-2.

6.3. Explaining Experimental Anomalies with HCCB

In this subsection, we confront the higher categorical coherence breakdown (HCCB) framework with a series of long-standing anomalies in fractional quantum Hall (FQH) physics. We do so with the two-tier logic of Sections 2.3.3 and 5.1 made explicit in each

case: Tier-1 (inflow-consistent) HCCB can contribute only *quantized* sector effects and their thermal mixtures, while any genuinely route- or order-dependent residue would be Tier-2 physics to be bounded rather than asserted. Therefore, the purpose of the survey is not to claim that HCCB “explains” these anomalies—most have credible conventional explanations in equilibration, edge reconstruction, and electrostatics—but to identify for each the sharply defined component that HCCB *could* contribute, the signature that would isolate it, and the bound that a null result establishes.

6.3.1. Thermal Hall Conductance Spread at $\nu = 5/2$

Experiments at even-denominator filling $\nu = 5/2$ report thermal Hall conductance values close to half-integer multiples of the thermal conductance quantum $\kappa_0 = \pi^2 k_B^2 / 3h$, e.g.,

$$\frac{\kappa_{xy}}{T} \approx \left(n + \frac{1}{2}\right) \kappa_0, \quad (149)$$

as predicted by non-Abelian Pfaffian/anti-Pfaffian orders. However, device-to-device and geometry-to-geometry variations of order $\Delta(\kappa_{xy}/T)/\kappa_0 \sim 0.1$ are common, challenging a purely topological interpretation.

What can HCCB say here? A quantized Tier-1 shift

$$c_-^{\text{eff}} = c_-^{(0)} + \Delta c_-^{\text{HCCB}}, \quad \frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h} (c_-^{(0)} + \Delta c_-^{\text{HCCB}}) \quad (150)$$

cannot produce continuous $\sim 0.1 \kappa_0$ spreads, because its steps are no smaller than $\frac{1}{2} \kappa_0$ (spin sector). The honest statement is sharper and still useful: (i) if different devices stabilize different topological sectors along their measured edges (e.g., Pfaffian-type versus anti-Pfaffian-type domains, or Majorana sectors toggled by reconstruction), Tier-1 HCCB describes the resulting *half-integer* device-to-device differences, with the interferometric channel identifying the sector; (ii) intermediate, temperature-dependent values can arise within Tier 1 only through thermal mixing of quantizedly split sectors, with the activated form of Section 5.3.3 as its fingerprint; (iii) any residual continuous spread that survives equilibration and reconstruction diagnostics, is odd under $B \rightarrow -B$, and correlates with junction count would fall outside the consistent framework and is precisely what the null tests of Section 6.4 bound.

6.3.2. Interferometer Phase Slips and Route Dependence

Fabry–Perot and Mach–Zehnder interferometers probing anyon braiding often exhibit irregular phase slips not accounted for by simple area/charge relations or Coulomb domination. In standard modular tensor category (MTC) theory, topologically equivalent fusion/braiding routes yield identical phases.

A hypothetical pentagon- or hexagon-level breakdown would add stable phase angles ε_F (associator) and ε_H (braiding),

$$\phi_{(ab)c} - \phi_{a(bc)} = \varepsilon_F, \quad \phi_{R \circ F} - \phi_{F \circ R} = \varepsilon_H, \quad (151)$$

appearing as order-dependent offsets when switching between routings related by re-parenthesization or by braid–fuse order. By route blindness these vanish identically in Tier 1; the practical import is methodological: after excluding the conventional sources of slips (area breathing, charging events, quasiparticle traps), the *residual, orientation-odd, routing-correlated* component of the phase budget is a clean bound on Tier-2 coherence failure—currently the only proposed laboratory bound on the pentagon identity itself.

6.3.3. Non-Universal Tunneling Exponents

In ideal chiral Luttinger liquid edges, current–voltage characteristics at a quantum point contact scale as

$$I \sim V^\alpha, \quad \alpha = 2g^{-1} - 1, \quad (152)$$

with the Luttinger parameter g fixed by the K -matrix. Experimentally, however, α often drifts from these universal predictions, and scatter correlates with device geometry.

A Tier-2 associator curvature with a non-unitary component $\Lambda_{f,g,h}$ acting where channels recombine would renormalize the local g and make tunneling exponents depend on network parenthesization. We do not assert this mechanism—conventional explanations (edge reconstruction, finite-range interactions, dissipation at the QPC) are well documented and must be excluded first—but it defines a falsifiable residue: a component of the exponent scatter that correlates with bracketing order at fixed disorder and geometry, and reverses with chirality, is not produced by any of the conventional mechanisms nor by Tier-1 HCCB, and would constitute evidence of (non-unitary) coherence failure. Absent such a residue, these data bound $|\Lambda - 1|$.

6.3.4. Neutral Mode Equilibration Puzzles

Upstream neutral modes are observed in many FQH states, but their equilibration lengths vary strongly with temperature, edge layout, and device. Standard topological theory fixes the *existence* of neutral modes but not their equilibration dynamics.

Here, the two-tier logic cuts cleanly: tier-1 HCCB changes the *anomaly bookkeeping* of the edge (which sectors are present and their net chirality), but not the microscopic exchange rates; therefore, its imprint on equilibration data is indirect—a change of the asymptotic plateau that the profiles approach—and is removed in the difference channel of Section 5.3.2. Equilibration *lengths* themselves are non-universal physics (disorder, interactions, temperature) and their device dependence needs no categorical input. Only a routing-dependent, orientation-odd modulation of ℓ_{eq} at fixed microscopics would point to a Tier-2 inelastic channel; the local-thermometry protocol of Section 5.3.5 is the natural instrument to bound it.

6.3.5. Particle–Hole Asymmetry at $\nu = 5/2$

Pfaffian and anti-Pfaffian candidate states are related by particle–hole (PH) conjugation, predicting the same electrical Hall conductance but opposite edge orientation and different chiral central charges. Experiments sometimes favor one orientation, suggesting weak PH asymmetry.

The Tier-1 contribution of HCCB to this question is diagnostic rather than explanatory: the candidate orders differ precisely in c_- (and neutral content), so sector-resolved difference measurements—thermal Hall steps correlated with interferometric sector identification, as in Section 5.2—discriminate among Pfaffian, anti-Pfaffian, and PH-Pfaffian more sharply than time-averaged transport. A hypothetical Tier-2 hexagon bias ε_H , odd under $B \rightarrow -B$, could mimic weak PH breaking in finite devices; the same orientation-reversal protocol that identifies it would, in a null outcome, bound it.

Summary. All five anomalies arise in regimes where multiple edge processes are composed in series, which is what motivated examining them through the coherence lens. The outcome of the two-tier analysis is asymmetric by design. Tier-1 HCCB contributes *quantized* structure: half-integer sector differences between devices, activated sector-mixing crossovers, sector-resolved discrimination of candidate orders, and quantized pump counts over control cycles. The continuous routing-correlated components of the anomalies are *not* produced by the consistent framework; for those, the geometries of Section 6.4 provide null tests whose current sensitivities already imply bounds at the 10^{-2} level in phase and

the percent level in κ_{xy}/T . Both halves of this statement are experimentally valuable, and neither overreaches the mathematics.

6.4. Proposed Experimental Probes of HCCB

The two-tier structure makes the experimental logic of multi-edge networks sharp and falsifiable in both directions. For Tier-1 HCCB the framework *predicts* quantized, route-blind responses: half-integer steps under sector toggles and quantized pump counts over control cycles (Sections 5.2 and 5.3). For the route- and order-dependent parameters ($\varepsilon_F, \varepsilon_H, \Lambda$) the framework predicts exact *nulls*, so the geometries below are precision null tests whose nonzero outcome would falsify the modular-functor axioms themselves. In this subsection we specify the device geometries that realize both halves of this program through measurable phase, visibility, and thermal transport signatures.

What a nonzero outcome would mean physically. Since the null tests target axioms rather than parameters, it is worth stating concretely what a reproducible violation—one surviving the chirality-reversal, junction-scaling, and drive-dependence diagnostics below—would imply. The modular-functor axioms (Section 5.1) assert that anyonic amplitudes are well-defined linear maps determined by the topology of a process, composing locally under sewing. A confirmed route dependence would therefore admit only three readings, each of major significance. (a) *Hidden non-topological dynamics*: the junctions carry internal states (charge hysteresis, trapped quasiparticles, memory effects) so that the two “routes” are secretly different physical processes; this is the mundane reading, and the diagnostics are designed to exclude it—in particular, mundane junction dynamics has no reason to be odd under $B \rightarrow -B$ at fixed gating. (b) *Failure of the effective edge description*: the low-energy theory of the edge is not a consistent chiral conformal field theory with local sewing—locality or unitarity of the effective description breaks down at the junction scale. This would overturn the standard account of quantum Hall edges while leaving quantum mechanics intact. (c) *Genuine coherence breakdown*: the compositional structure of quantum amplitudes itself—the associativity of sequential process composition that the pentagon encodes—receives corrections, the direction explored at the level of general quantum dynamics in the companion work [22]. Reading (c) is the most radical, and is constrained by the dimensional analysis of Section 2.3.3: in a strictly planar device it has no inflow-consistent topological realization, so it would have to be non-topological in origin—which is exactly why the magnitude, scaling, and symmetry pattern of any observed violation would immediately discriminate among (a), (b), and (c). In all cases, the measured numbers are informative: null results convert routine interferometric and calorimetric precision into quantitative bounds on the pentagon and hexagon identities, which to our knowledge represents the first direct laboratory bounds on the coherence axioms of the topological description of matter.

6.4.1. Pentagon Interferometer

Design. A Hall bar with two QPCs forming a Fabry–Perot cell and three gate-defined antidots (or island anyons) inside the interferometer. Electrostatic gates allow one to route interfering paths in two different *parenthesizations*:

$$\text{Path A: } (ab)c, \quad \text{Path B: } a(bc).$$

Prediction. In standard modular tensor category (MTC) theory—and, by route blindness, in any Tier-1 HCCB deformation—A and B yield identical interference patterns. A Tier-2 coherence failure would instead give

$$\phi_A - \phi_B = \varepsilon_F, \quad \mathcal{V}_A - \mathcal{V}_B = \delta\lambda, \quad (153)$$

where ε_F is a small pentagon anomaly angle and $\delta\lambda$ an associator-induced visibility shift. Both vanish for a single antidot interferometer and flip sign under $B \rightarrow -B$ (chirality reversal).

6.4.2. Hexagon Interferometer

Design. Extend the Fabry–Perot with a third QPC, allowing sequences of the form *braid then fuse* ($R \circ F$) vs *fuse then braid* ($F \circ R$).

Prediction. In exact MTC theory, and in any Tier-1 deformation, the two sequences agree. A Tier-2 coherence failure would give

$$\phi_{R \circ F} - \phi_{F \circ R} = \varepsilon_H, \quad (154)$$

a stable order-dependent phase offset. A null result bounds hexagon-level coherence failure; a reproducible nonzero offset surviving the diagnostics below would falsify the braided modular-functor axioms for the edge data.

6.4.3. Thermal Y-Network

Design. Two serial Y-junctions split and recombine a hot edge channel into three branches, with two possible bracketing orders:

$$\text{Route A: } (1\,2)3, \quad \text{Route B: } 1(2\,3).$$

Prediction. In topologically exact theory and in any Tier-1 deformation, the net thermal current is independent of bracketing. A Tier-2 failure would give

$$\Delta\left(\frac{\kappa_{xy}}{T}\right) = \frac{\pi^2 k_B^2}{3h} \Delta c_-^{\text{HCCB}} \neq 0. \quad (155)$$

Such a correction would have to vanish in a single Y, grow with the number of splits/recombinations, and flip sign under chirality reversal—the discriminating pattern used in the falsifiability checklist below. Within Tier 1, the same hardware (driven around closed control cycles) measures the quantized pump count of Section 5.2.10.

6.4.4. Neutral-Mode Floquet Probes

Design. Apply a weak GHz drive to gates coupled primarily to *neutral* edge modes.

Prediction. If a Tier-2 associator curvature carries a non-unitary component, the bounded parameters ε_F and $\delta\lambda$ would acquire a controlled drive-amplitude dependence without affecting the quantized σ_{xy} ; a drive-independent null at all amplitudes strengthens the corresponding bound.

6.4.5. Scaling and Falsifiability

A genuine Tier-2 coherence failure would have to display a consistent pattern of signatures, which is what makes the null tests sharp:

- (i) *Vanishing in single straight edges.* A single chiral edge without recombination has $\varepsilon_F = \varepsilon_H = \Delta c_-^{\text{HCCB}} = 0$.
- (ii) *Chirality flip.* All anomaly parameters change sign when the magnetic field is reversed ($B \rightarrow -B$).
- (iii) *Network scaling.* Corrections grow with the number of junctions and recombinations, not with edge length or area.
- (iv) *Robustness vs disorder.* Signals persist after standard dephasing/phonon corrections are removed, distinguishing HCCB from mundane noise.

Summary. Pentagon interferometers, hexagon interferometers, and thermal Y-networks provide direct laboratory realizations of higher categorical diagrams. Within the consistent framework their route-comparison outcomes are exact nulls, so any reproducible order-of-operations dependence in phase, visibility, or thermal conductance—surviving the chirality-reversal, junction-scaling, and disorder diagnostics above—would falsify the modular-functor axioms for the edge data, a result of fundamental significance. Null results at current sensitivity ($\lesssim 1\%$ in κ_{xy}/T ; $\lesssim 10^{-2}$ rad in phase) place the first stringent laboratory bounds on ε_F , ε_H , and route-dependent Δc_- . The Tier-1 content of the same hardware—quantized sector steps and cyclic pump counts—is measured by the protocols of Sections 5.2.5 and 5.2.10.

6.5. Implications and Synthesis

The higher categorical coherence breakdown (HCCB) framework provides a unifying perspective on long-standing puzzles of the fractional quantum Hall effect. By elevating the analysis of edge anomalies beyond the triangle-level (Virasoro central extension), one gains access to a hierarchy of potential obstructions: pentagon-level (3-cocycles) associated with associator structure and hexagon-level (4-cocycles) associated with braiding compatibility. The practical consequence is two-sided. Within the consistent (Tier-1) framework, experimental readouts of the central charge remain route-blind topological data, but become *addressable*: they step quantizedly under sector toggles and pump quantizedly around control cycles. Route-dependent corrections, by contrast, are forbidden by the consistent framework, which converts multi-edge networks into precision null tests of the coherence axioms themselves.

6.5.1. Conceptual Payoff

- The Virasoro central charge c appears as the universal *residue of regularization*, measuring the obstruction that promotes Witt $\rightarrow$ Virasoro. Its chiral combination $c_- = c_R - c_L$ is directly tied to the thermal Hall conductance via gravitational anomaly inflow.
- HCCB extends this picture: consistent 3-cocycle deformations shift c_- in quantized steps (Tier 1), while genuine pentagon/hexagon failure ($\varepsilon_F, \varepsilon_H$) is an H^4 -level obstruction that no planar system can host (Tier 2), and is therefore a target for null tests rather than a tunable parameter.
- The anomalies are *hierarchical*: triangle-level c_- governs universal quantization; the higher coherence levels govern, on one side, quantized sector and pump responses, and on the other, the strict route-blindness whose violation the proposed geometries bound.

6.5.2. Methodological Payoff

- The framework decomposes the $\nu = 5/2$ thermal-Hall spreads into a quantized, sector-resolved component (half-integer steps and activated sector mixing, which Tier-1 HCCB describes and the interferometric channel identifies) and a continuous remainder that must be conventional (equilibration, reconstruction) or, if routing-correlated and orientation-odd, Tier-2—and it supplies the protocols that separate the two.
- Interferometer phase slips and geometry-dependent tunneling exponents acquire a disciplined phenomenology: after conventional sources are excluded, their residual routing-correlated components become quantitative bounds on associator and hexagon failure, replacing ad hoc reinterpretation by systematic cohomological bookkeeping.

- Neutral-mode equilibration data are cleanly split into non-universal rate physics (no categorical input) and anomaly bookkeeping (plateau values), with local thermometry bounding any putative Tier-2 inelastic channel.

6.5.3. Experimental Outlook

- Proposed devices—pentagon interferometers, hexagon interferometers, and thermal Y-networks—realize higher categorical diagrams directly in the laboratory.
- Reproducible order-of-operations dependence in phase, visibility, or thermal conductance—surviving chirality-reversal and junction-scaling diagnostics—would falsify the modular-functor axioms for the edge data: not a parameter measurement but a discovery-class result. The Tier-1 counterpart, equally concrete, is the quantized step and pump metrology of Sections 5.2.5 and 5.2.10.
- Conversely, null results at current experimental sensitivity will yield quantitative bounds:

$$|\varepsilon_F|, |\varepsilon_H| \lesssim 10^{-2} \text{ rad}, \quad \frac{|\Delta c_-^{\text{HCCB}}|}{c_-^{(0)}} \lesssim 1\%. \quad (156)$$

6.5.4. Broader Implications

The proposed interpretation links categorical physics to condensed matter experiment:

- It demonstrates that the “mysteries” of the FQHE—thermal Hall spreads, interferometer phase slips, non-universal exponents—can be interrogated within a single cohomological bookkeeping that says precisely which components may be topological (quantized, sector-resolved) and which cannot be, rather than leaving them as undifferentiated experimental imperfections.
- It provides a concrete arena where abstract categorical obstructions (cocycles, diagram coherence) meet measurable transport coefficients and interference phases.
- It points toward a more general program: to classify and probe higher-level anomalies in quantum systems, with quantum Hall edges serving as a natural laboratory.

Summary. In summary, HCCB offers a principled bookkeeping for experimental puzzles in the fractional quantum Hall effect. Triangle-level anomalies (Virasoro central charges) account for universal quantization; consistent higher-coherence deformations yield the quantized sector and pump responses that constitute our positive predictions; and the strict route-blindness of the consistent framework converts multi-junction anomalies into falsifiable null tests of the coherence axioms. Thus, the framework addresses open questions in FQHE without overreaching its mathematics, and exemplifies the power of categorical methods in physics: identifying universal obstructions and guiding new routes to experimental discovery on both sides of the theorem/hypothesis divide.

7. Theoretical Outlook

7.1. Linking to Other Physical Systems

The mechanism we have identified—higher categorical coherence breakdown (HCCB) shifting the anomaly coefficient read as an effective chiral central charge—has natural analogues and testbeds beyond semiconductor quantum Hall devices. We highlight three arenas: (i) quantum spin liquids with emergent gauge structure, where neutral edge modes and bulk topological order mirror the Chern–Simons/braided tensor category structure; (ii) ultracold atoms in synthetic gauge fields and topological bands, where clean, tunable platforms can realize fractional conformal edge theories; and (iii) string worldsheet theory, where controlled shifts of central charge occur through background charge and Liouville dressing. In each case we identify the mathematical mechanism, its physical imprint, and how a drift of c_{eff} arises or is constrained.

7.1.1. Spin Liquids with Emergent Gauge Structure

Chiral spin liquids and neutral edge CFTs. Chiral spin liquids (CSLs) provide a paradigmatic magnetic analogue of fractional quantum Hall (FQH) fluids. The Kalmeyer–Laughlin (KL) state is the $SU(2)$ -invariant spin version of the $\nu = 1/2$ bosonic Laughlin state [141]. Its low-energy topological field theory is a $SU(2)_1$ Chern–Simons theory, and the edge is a chiral $SU(2)_1$ WZW CFT with chiral central charge $c_- = 1$. Because the degrees of freedom are *neutral*, the electrical Hall response vanishes, but the edge energy current and bulk thermal Hall conductance retain their anomaly form,

$$J_{\text{edge}}^E(T) = \frac{\pi}{12} \frac{k_B^2}{\hbar} c_- T^2, \quad \kappa_{xy} = \frac{\pi^2 k_B^2}{3\hbar} c_- T. \quad (157)$$

From the categorical viewpoint, the fusion rules and F/R -symbols of $SU(2)_1$ control sewing of conformal blocks, and any coherence deformation consistent with an invertible $(2+1)d$ bulk would shift the Dehn twist phase by a scalar; hence, c_- by a quantized amount, exactly as in our QH analysis.

$U(1)$ and $\mathbb{Z}_2$ spin liquids; emergent gauge fields. Beyond CSLs, spin liquids exhibit emergent gauge fields ($U(1)$ or $\mathbb{Z}_2$) and fractionalized spinons/vasons [142–144]. In a $U(1)$ Dirac spin liquid in $(2+1)d$, the low-energy theory contains gapless Dirac spinons coupled to an emergent gauge field; absent explicit time-reversal breaking, c_- vanishes but a *chiral* mass term (e.g., induced by scalar spin chirality) gaps the spectrum and generates a Chern–Simons term with level k for the emergent gauge field, producing a neutral κ_{xy} proportional to k . In a $\mathbb{Z}_2$ spin liquid, chiral order can likewise arise through broken time-reversal, and a modular tensor category of anyons (spinons, vasons, their bound states) obtains. In both cases, the topological Gauss sum of the emergent category encodes c_- modulo 8, and consequently constrains any HCCB-induced drift.

Thermal Hall transport of neutral excitations. The thermal Hall effect of *neutral* bosonic excitations (magnons, Majorana fermions) is well established theoretically and observed experimentally in magnetic insulators. For magnons, a Berry curvature mechanism yields

$$\kappa_{xy} = -\frac{k_B^2 T}{\hbar} \sum_n \int_{\text{BZ}} \frac{d^2 k}{(2\pi)^2} c_2[n_B(\epsilon_{n\mathbf{k}})] \Omega_n(\mathbf{k}), \quad (158)$$

with n_B the Bose function, Ω_n the magnon Berry curvature, and $c_2(x) = (1+x)\left(\ln \frac{1+x}{x}\right)^2 - (\ln x)^2 - 2\text{Li}_2(-x)$ [145]. Experiments have observed nonzero κ_{xy} in ferromagnetic insulators (magnon Hall effect) [146]. In putative Kitaev spin liquids, reports of *half-quantized* thermal Hall plateaus suggest Majorana edge modes with $c_- = \frac{1}{2}$ [147]. In all such neutral systems, HCCB would manifest as *quantized* shifts of the low- T slope of $\kappa_{xy}(T)$ when the categorical control (e.g., domain walls of anyon condensation between spin-liquid regions) is toggled, directly paralleling our QH proposal.

Two-dimensional ferromagnets: how would one observe this concretely? It is useful to spell out the observational logic for two-dimensional magnetic insulators, distinguishing two regimes. (i) In a *topological magnon* regime (gapped magnon bands with Berry curvature, as in the magnon Hall systems above), κ_{xy} is the Bose-occupation-weighted curvature integral and is *not* quantized: it varies continuously with temperature and carries no protected anomaly coefficient. This regime is not a readout channel for c_- , and no HCCB signal should be sought there; it serves as the reference background. (ii) In a *fractionalized chiral* regime—a gapped spin liquid with chiral order, of which the Kitaev candidate materials are the prime examples—the edge carries a genuine chiral CFT and the low- T slope κ_{xy}/T is anomaly-quantized in units of $\frac{1}{2}\kappa_0$ per Majorana sector. There, the HCCB program translates directly: the categorical controls are the magnetic-field orientation and magnitude

(selecting the sign and presence of the chiral mass, i.e., the invertible sector), pressure or strain across phase boundaries, and engineered interfaces between regions stabilized in different sectors; the prediction is that κ_{xy}/T moves between half-integer plateaus in quantized steps, odd under field reversal, with no continuous drift within a sector. Practical caveats are essential in magnets: phonons and phonon–magnon coupling produce large non-quantized backgrounds, and the reported half-quantization in α -RuCl₃ remains debated, with oscillatory and sample-dependent contributions documented [147,148]. Consequently the difference-channel methodology of Section 5.2 (sector toggling at fixed phonon background, slope fits of κ_{xy}/T , orientation-odd extraction) is not optional but constitutive of any credible claim in this setting. Bilayer-graphene even-denominator platforms [149], where particle–hole-conjugate candidate orders differ precisely in c_- , offer the electronic counterpart in which the same sector-resolved methodology discriminates candidate orders.

Constraints and opportunities. Spin liquids often lack the lithographic control available in semiconductors, but they afford *bulk* control parameters (pressure, field, chemistry) that can tune between topological sectors. From a field-theoretic standpoint, emergent categorical data (fusion, braiding) are realized dynamically; if an HCCB-like deformation is implementable (e.g., by ordering a condensable boson to nucleate a domain wall), then c_- should drift in steps compatible with the Witt class of the category. The spin sector thereby provides a stringent “all-neutral” test of our mechanism.

7.1.2. Cold Atom Simulations of Fractional CFTs

Synthetic gauge fields and Chern bands. Ultracold atoms in optical lattices realize tight-binding Hamiltonians with *synthetic* magnetic fields via laser-assisted tunneling, generating Harper–Hofstadter or Haldane-type Chern bands [150,151]. The single-particle Hamiltonian reads

$$H = -J \sum_{\langle ij \rangle} e^{iA_{ij}} a_i^\dagger a_j + \text{h.c.} + \frac{U}{2} \sum_i n_i(n_i-1) + \dots, \quad (159)$$

with Peierls phases A_{ij} imprinting a magnetic flux per plaquette. At fractional fillings and sufficiently strong interactions, bosons (or fermions with interactions) are predicted to form *fractional* Chern insulators (FCIs) [152], which are lattice analogues of FQH states with the same topological order and edge CFTs. Optical *flux lattices* can reach flatter bands and stronger effective fields, favoring FCIs [153]. While direct observation of fractionalization remains challenging, chiral edge states have been visualized in synthetic Hall ribbons using synthetic dimensions [154,155].

Edge CFTs and energy transport in atomic channels. For a gapped bulk FCI, the edge hosts a chiral Luttinger/WZW CFT. The energy current in a 1D chiral channel remains $J^E = (\pi/12)(k_B^2/\hbar) cT^2$. Two experimental routes can access this with atoms: (i) *mesoscopic channel thermodynamics* (create a temperature imbalance between atomic reservoirs connected by a chiral channel and read out heat currents via time-of-flight calorimetry), and (ii) *noise thermometry* of density fluctuations along the edge. Because atoms are neutral, electrical parasitics are absent; the dominant systematics are atom losses and phonon coupling to the lattice. If a categorical control is engineered (e.g., by locally modifying interactions to nucleate an anyon condensate strip that acts as a domain wall), HCCB predicts a *step* of the low- T slope of the heat current proportional to Δc_- , providing a clean and tunable realization of our mechanism.

From microscopic couplings to categorical data. In optical lattices, microscopic knobs (laser phases, detunings, Feshbach-tuned interactions) map to effective K -matrices and fusion rules at low energy in Abelian FCIs. For non-Abelian FCIs (e.g., lattice Moore–Read analogues), engineered three-body interactions or Floquet drives can favor paired states

whose edge contains a Majorana sector. In either case, a controlled “twist” of the effective associator (realized by a patterned modulation that implements a categorical domain wall) inserts a scalar in the mapping-class action on edge blocks, hence shifts the Dehn twist and c_{eff} in the same way as in semiconductors.

7.1.3. Analogies in String Worldsheet Central Charge Shifts

Linear dilaton and Liouville: explicit central-charge shifts. Worldsheet CFTs provide a controlled laboratory in which central charge shifts are *computable* by deforming the stress tensor. For a free scalar with background charge Q (Feigin–Fuchs/linear-dilaton formulation), with the propagator normalization $\langle \phi(z)\phi(w) \rangle = -\log(z-w)$ used throughout Section 2, the holomorphic stress tensor

$$T(z) = -\frac{1}{2} :(\partial\phi)^2: + \frac{Q}{\sqrt{2}} \partial^2\phi \quad (160)$$

leads to the TT OPE

$$T(z)T(w) \sim \frac{1+6Q^2}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots, \quad (161)$$

so the central charge is $c = 1 + 6Q^2$ (the double contraction of the background-charge term contributes $6 \times (Q/\sqrt{2})^2 = 3Q^2$ to the coefficient of the quartic pole, i.e., $\Delta c = 12 \times Q^2/2$; in conventions where the background-charge term is $Q\partial^2\phi$ with the same propagator, one instead finds $c = 1 + 12Q^2$). In Liouville theory one commonly writes $Q = b + 1/b$ with $c_L = 1 + 6Q^2$, and the matter+ghost+Liouville anomaly cancellation in noncritical bosonic string theory enforces

$$c_{\text{matter}} + c_L + c_{\text{ghosts}} = 0, \quad c_{\text{ghosts}} = -26, \quad (162)$$

whence $c_L = 26 - c_{\text{matter}}$ [156–159]. Thus, shifting the background charge (or the matter central charge) changes the worldsheet c in a way fixed by anomaly cancellation, exactly mirroring how HCCB shifts the effective c by a scalar phase in the modular T -matrix.

KPZ gravitational dressing as a “categorical” renormalization. Coupling matter CFTs to 2D gravity modifies operator dimensions by the Knizhnik–Polyakov–Zamolodchikov (KPZ) relation: a matter primary of dimension Δ is “gravitationally dressed” to

$$\Delta_{\text{grav}} = \frac{\sqrt{1 - c_{\text{matter}} + 24\Delta} - \sqrt{1 - c_{\text{matter}}}}{\sqrt{25 - c_{\text{matter}}} - \sqrt{1 - c_{\text{matter}}}}, \quad (163)$$

expressing how the central-charge deficit controls scaling under random geometry [160,161]. Conceptually, HCCB plays a role akin to a *higher* anomaly: the categorical deformation changes modular phases by a scalar, which (by modular covariance) shifts c_{eff} in all anomaly-sensitive observables. In both settings, a single parameter that lives in an *obstruction* channel (Weyl anomaly for strings, Dehn twist scalar for edges) renormalizes scaling data in a universal manner.

Mapping-class phases: from worldsheet to edge. On a torus, worldsheet modular transformations act on characters by $T : \chi_a \mapsto e^{2\pi i(h_a - c/24)} \chi_a$. In noncritical strings, the Liouville sector contributes a universal shift of this phase through c_L ; in our condensed-matter setting, HCCB contributes a universal scalar to T via the bulk invertible TQFT. In both cases the *only* effect on modular data is an overall T -phase, hence a shift in c ; all other modular information (e.g., conformal weights modulo one, S -matrix) remains intact. This parallelism suggests that mathematical techniques developed for Liouville/KPZ (control

of background-charge deformations) may inform constructive models of HCCB in braided tensor categories.

Summary. Spin liquids, ultracold atoms, and worldsheet strings all realize the same pattern: a deformation in an *anomaly* or *coherence* channel shifts central-charge data while leaving local operator algebra intact. In magnets and atoms this becomes an experimentally accessible drift of κ_{xy} ; in strings, it is enforced by Weyl anomaly cancellation. The ubiquity and computability of these shifts strengthens the theoretical plausibility of HCCB and broadens the scope of experimental probes.

Remark on materials claims. It is worth noting that some reports in the thermal Hall literature, particularly half-quantized plateaus in magnetic insulators or phonon-dominated systems, remain experimentally and theoretically contested; in contrast, quantization in electronic quantum Hall platforms is on the firmest ground. For this reason, our proposal emphasizes electronic realizations as the clearest testbed, while neutral magnets and phononic systems are discussed as intriguing but more tentative extensions.

7.2. Implications for Physics

We close by distilling the broader message of our analysis. The central charge which appears in two-dimensional conformal field theory (CFT) and governs anomaly-sensitive responses (Casimir energy, modular phases, thermal Hall transport) can be understood as the low-energy avatar of a *coherence* invariant: when the categorical sewing data of conformal blocks are deformed in a way that is compatible with anomaly inflow, the mapping-class representation acquires a scalar phase, and the effective anomaly coefficient—hence, the *measured* chiral central charge—*shifts*. This viewpoint reframes anomalies as controlled breakdowns of higher coherence and elevates the “central charge” to a *tunable observable* that can be pumped, switched, and read out.

7.2.1. Rethinking Anomalies as Coherence Breakdowns

From Wess–Zumino consistency to categorical coherence. Let $\Gamma[A, g]$ be the Euclidean generating functional in background gauge field/geometry; an infinitesimal symmetry transformation with parameter α induces an anomalous variation

$$\delta_\alpha \Gamma = 2\pi \int_\Sigma \mathcal{I}_d^1(\alpha; A, g), \quad (164)$$

where $\mathcal{I}_d^1$ is obtained by descent from a closed $(d+2)$ -form $\mathcal{I}_{d+2}$ (the *anomaly polynomial*). The Wess–Zumino (WZ) consistency condition

$$\delta_{\alpha_1}(\delta_{\alpha_2} \Gamma) - \delta_{\alpha_2}(\delta_{\alpha_1} \Gamma) = \delta_{[\alpha_1, \alpha_2]} \Gamma \quad (165)$$

states that the anomaly is a Lie-algebra 2-cocycle (no obstruction beyond the Jacobi identity) [162]. Any two representatives related by addition of a local counterterm differ by a Bardeen–Zumino (BZ) polynomial; physically, this converts a *consistent* anomaly into a *covariant* one without changing its cohomology class [163,164].

In our categorical language, the triangle/pentagon/hexagon identities are the “WZ identities” for the sewing moves of conformal blocks; a higher-categorical coherence breakdown (HCCB) modifies these identities by *scalar* phases that obey the obvious consistency under composition. Algebraically, this is the passage from exact coherence to a projective (one-dimensional) anomaly in the mapping-class representation: the composition of elementary moves that implements a Dehn twist T acquires a scalar $e^{2\pi i \phi}$, and the theory is consistent iff these phases satisfy the analogues of (165). The *class* of these scalars is the obstruction; the BZ freedom is the categorical coboundary that rephases bases of blocks.

Descent, inflow, and the Dehn twist. For gravitational anomalies in $d = 2$, the anomaly polynomial is

$$\mathcal{I}_4 = \frac{c_-}{96\pi} \text{tr } R \wedge R, \quad (166)$$

for which the descent gives the gravitational Chern–Simons (gCS) action on a 3-manifold M^3 bounding Σ :

$$S_{\text{gCS}}[\omega] = \frac{\kappa_g}{4\pi} \int_{M^3} \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right), \quad c_- = 24\kappa_g, \quad (167)$$

with ω as the spin connection. Gluing along a mapping torus of the torus Dehn twist shows that a scalar phase $e^{2\pi i\phi}$ multiplying T corresponds to $\kappa_g = \phi$; therefore, it shifts the chiral central charge by

$$c_- \mapsto c_- + \Delta c_-, \quad \Delta c_- = 24\phi. \quad (168)$$

Proposition 1 (Scalar T -phase and central charge). *If the mapping-class group representation of the torus is modified by a scalar $e^{2\pi i\phi}$ on the Dehn twist T while S is left unchanged, then the effective central charge shifts as*

$$c \mapsto c + 24\phi.$$

Conversely, any shift $c \mapsto c + \Delta c$ with $\Delta c \in 24\mathbb{Q}$ arises from stacking with an invertible $(2+1)$ -dimensional bulk phase classified by $\phi = \Delta c/24$.

Proof. Characters transform as $\chi_a(\tau + 1) = e^{2\pi i(h_a - c/24)} \chi_a(\tau)$. Multiplying all characters by $e^{2\pi i\phi}$ under $\tau \mapsto \tau + 1$ shifts $c \mapsto c + 24\phi$, since the vacuum character ($h_a = 0$) acquires the extra phase $e^{-2\pi ic/24} e^{2\pi i\phi}$. Equivalently, the partition function on the mapping torus of the Dehn twist gains a bulk phase $e^{2\pi i\phi}$, which is exactly the contribution of an invertible gravitational Chern–Simons term with level $\kappa_g = \phi$. $\square$

Thus, coherence breakdown is the anomaly: the only allowed effect of HCCB on modular data is a one-dimensional T -phase, hence a shift of c_- .

Cobordism perspective and constraints. Modern classifications identify (bosonic) symmetry-protected topological (SPT) phases and anomalies with cobordism groups and, for on-site symmetries, with group-cohomology classes [165,166]. In this language, the HCCB scalar lives in the same “invertible” sector as the bulk inflow theory. Quantization of these groups restricts Δc_- (e.g., to integers modulo 8 in bosonic $(2+1)d$ systems), explaining why c_- can only *jump* when a topological obstruction is toggled, and must remain piecewise constant along smooth gapped paths.

7.2.2. “Dynamical Central Charge”: Definition, Pumping, and Hydrodynamics

Operational definition. Let λ be a *categorical control parameter* (domain-wall choice, ancilla phase, trapped topological charge). We define the *dynamical central charge*

$$c_{\text{eff}}(\lambda) := c_- + 24\phi(\lambda), \quad (169)$$

where $\phi(\lambda)$ is the Dehn-twist scalar generated by HCCB at control λ . In equilibrium transport this is the slope of the thermal Hall conductance,

$$\kappa_{xy}(T; \lambda) = \frac{\pi^2 k_B^2}{3h} c_{\text{eff}}(\lambda) T + o(T), \quad (170)$$

so that the *difference* measurement between two controls λ, λ' yields

$$\frac{\kappa_{xy}(T; \lambda) - \kappa_{xy}(T; \lambda')}{T} = \frac{\pi^2 k_B^2}{3h} [c_{\text{eff}}(\lambda) - c_{\text{eff}}(\lambda')] + o(1), \quad (171)$$

which is precisely the metrological content of our proposal.

Central-charge pump (adiabatic cycles). Consider an adiabatic cyclic variation $\lambda \in S^1$ of the control that returns the microscopic Hamiltonian to itself while the gap remains open. Let $\phi(\lambda)$ be smooth on the cycle except at isolated (allowed) points where the invertible bulk sector changes. Transgression of the gCS functional shows that the *net* change of c_{eff} over the cycle equals a topological index

$$\oint_{S^1} d\phi(\lambda) = n \in \mathbb{Z} \implies \Delta c_{\text{eff}} = 24n, \quad (172)$$

in exact analogy with Thouless' quantized pump for charge transport [134]. Equation (172) identifies a new kind of *central-charge pump*: a quantized transfer of gravitational anomaly between coupled chiral sectors driven by a cyclic HCCB control. Conceptually, (172) is an instance of the modern theory of quantized invariants of *families* of gapped systems, consisting of higher-dimensional generalizations of the Berry phase and anomalies in the space of coupling constants [132,133]: the extra cohomological degree that a single static configuration cannot carry (Section 7.3) is supplied consistently by the topology of the control manifold, which is also why the pump count is route-blind (it depends only on the homotopy class of the cycle), yet is not reducible to any single static stacking. The experimental protocol implementing (172) on Hall edges is given in Section 5.2.10.

Hydrodynamic imprint. Anomalies constrain hydrodynamics by fixing otherwise-undetermined transport coefficients. In particular, triangle and gravitational anomalies enter the constitutive relations as non-dissipative terms proportional to the anomaly coefficients, as required by the second law [167]. In 1+1 dimensions, gravitational anomalies contribute at zeroth order in gradients to the energy current in equilibrium [168]. Therefore, promoting the anomaly coefficient to the *control-dependent* $c_{\text{eff}}(\lambda)$ gives a hydrodynamic theory with a tunable non-dissipative coupling: switching λ instantaneously changes the hydrostatic transport by the quantized amount dictated by (169).

Piecewise constancy and jump conditions. Within a fixed topological sector (fixed invertible bulk), c_{eff} is rigid; any continuous change would contradict the quantization implied by cobordism. Hence, $c_{\text{eff}}(\lambda)$ is piecewise constant and can only jump when the invertible sector changes—precisely when the categorical control crosses a domain wall or when an ancilla passes through a topological transition. This sharp quantized “step” is the hallmark of the dynamical central charge.

7.2.3. Experimental Metrology and Cross-Platform Outlook

A metrological observable. Because the *quantum of thermal conductance* is universal:

$$\kappa_0 T := \frac{\pi^2 k_B^2}{3h} T, \quad (173)$$

a plateau-to-plateau change $\Delta c_{\text{eff}} \in \mathbb{Z}$ is a directly countable change $\Delta \kappa_{xy} = \Delta c_{\text{eff}} \kappa_0 T$ even outside electronic systems, e.g., in phononic channels [169]. In correlated insulators, thermal Hall responses from *neutral* excitations provide an additional arena; the recent discovery of large κ_{xy} in cuprates illustrates the diagnostic power of heat transport even when charge is localized [170]. Thus, the “dynamical central charge” supplies a unifying language and a practical calibration-ready observable across platforms.

From classification to control. Group-cohomology and cobordism approaches to SPT phases and anomalies [165,166] suggest concrete *knobs*: condensable bosons and domain walls implement categorical deformations, while proximitized ancillae realize invertible bulk stacks that additively shift c_{eff} . In each case, the HCCB scalar is the only degree of freedom that *must* appear in modular data and hydrodynamics, implying that *any* successful implementation across materials realizes the same low-energy phenomenology.

Summary and perspective. Anomalies can be reinterpreted as controlled cohomological failures of higher coherence; their coefficients become *device-level* control parameters. The central charge, read operationally through modular and transport probes, is therefore *dynamical* in the best sense: piecewise quantized, pumpable, and measurable with metrological precision.

7.3. Defining HCCB at the Minimal Structural Level

Up to this point, we have described higher categorical coherence breakdown (HCCB) in terms of “coherence defects” or “scalar phases” attached to associativity and braiding moves. To avoid ambiguity, we now state one precise minimal model that we adopt throughout this work.

The correct level of structure is a *modular functor* (or equivalently, the category of representations of a rational conformal field theory) equipped with associators and braidings that satisfy the pentagon and hexagon identities. Algebraically, this is a *ribbon fusion category*. Consistency requires the data (F, R) to solve the pentagon and hexagon equations exactly, so that the mapping class group of punctured surfaces acts linearly on spaces of conformal blocks.

Definition (consolidated, two tiers). Tier 1 (the definition adopted in this work). HCCB is the situation in which the associator is twisted by a normalized $U(1)$ -valued 3-cocycle Ω_3 ,

$$\alpha'_{a,b,c} = \Omega_3(a, b, c) \alpha_{a,b,c}, \quad \delta\Omega_3 = 1,$$

supplemented by the braiding/twist adjustment required to preserve the ribbon structure (classified in the pointed case by Eilenberg–Mac Lane Abelian cohomology, i.e., by shifts of the quadratic form; Section 2.3.4). The pentagon and hexagon close *exactly*; the deformation is equivalent to composing the boundary data with a $(2+1)$ -dimensional invertible phase. Its entire imprint on the modular functor is a one-dimensional scalar on the mapping-class action—a character $\{e^{2\pi i\phi(\gamma)}\}$ of the mapping class group determined by its value on the Dehn twist:

$$T \mapsto e^{2\pi i\phi} T, \quad c_- \mapsto c_- + 24\phi.$$

Because a character depends only on the group element, ϕ is *route-blind*, with words of elementary moves representing the same mapping class; in particular all rebracketing and braid/fuse-reordering loops, which represent the identity, carry no scalar (Section 5.1). The equivalence class of the deformation is captured by $[\Omega_3] \in H^3$ together with its braiding refinement, and its physically realizable values are quantized as in Section 7.5.

Tier 2 (excluded in planar devices; bounded by null tests). A deformation for which pentagon failure defines a non-trivial class in H^4 cannot be trivialized by any redefinition and cannot exist in a standalone $(2+1)$ -dimensional system; it requires a $(3+1)$ -dimensional invertible host, as in anomalous surface topological orders [49–51]. A globally defined 3-cochain with $\delta\Omega_3 \neq 1$ does *not* produce such a defect: its pentagon failure is an exact coboundary, and is removable by undoing the twist (Section 2.3.3). Therefore, tier-2 physics enters this paper only as the null hypothesis bounded by the route-comparison experiments.

Minimal stance. For the purposes of this paper, we adopt the Tier-1 ribbon-category/modular-functor definition of HCCB. It isolates the essential content that HCCB is

a cohomology class of consistent coherence twists that produces a one-dimensional anomaly factor in the modular representation—nothing more, and nothing route-dependent.

Note. Full higher-category formalisms (e.g., A_∞ - or E_n -categories and their higher coherence data) generalize this picture to homotopy levels beyond the pentagon/hexagon. Within the consistent framework, genuinely new invariants beyond a single Tier-1 twist arise for *families* of theories over a control manifold—the higher-Berry/coupling-space anomalies underlying the pumping protocol of Section 5.2.10 and the cycle quantization of Section 7.2 [132,133]. We defer the general homotopy-coherent treatment to future work, since it is not needed for the concrete transport predictions developed here.

7.4. HCCB Versus Stacking Invertible 3D Phases

A natural question raised by our mechanism is whether the observable effect of higher categorical coherence breakdown (HCCB) is genuinely distinct from simply stacking the boundary theory with an *invertible three-dimensional topological phase* whose bulk anomaly inflow produces the same scalar shift in the Dehn twist.

Our construction makes this equivalence explicit. Recall from Sections 5.1 and 7.2 that an HCCB deformation multiplies the torus T transformation by a scalar $e^{2\pi i\phi}$, yielding an effective central charge shift

$$c_- \mapsto c_- + 24\phi.$$

However, this is exactly the transformation produced by stacking with a (2+1)d invertible phase carrying gravitational Chern–Simons level $\kappa_g = \phi$. In particular, bosonic systems allow only shifts $\Delta c_- \equiv 0 \pmod{8}$, corresponding to multiples of the E_8 state, while spin systems allow half-integer steps (e.g., a single Majorana sector).

Therefore, from the viewpoint of experimental metrology, the two descriptions are *operationally equivalent*: both predict the same quantized drift in κ_{xy}/T and the same modular scalar in T . The advantage of phrasing the effect as HCCB is conceptual; the shift is seen not as the ad hoc addition of a bulk, but as the boundary imprint of a controlled categorical deformation of sewing relations. Practically, engineered domain walls, ancilla Chern bands, or trapped non-Abelian anyons act as *toggles* that implement such a deformation, and hence realize in situ what is mathematically equivalent to stacking.

In summary, HCCB should be understood as the categorical mechanism by which an effective stacking with an invertible 3D phase is generated and detected at the boundary. This perspective clarifies both the universality of the response and the operational handle by which experiments can probe it. One genuine extension beyond the static stacking picture survives this equivalence: the quantized pump count accumulated around a *closed cycle* of categorical controls (Sections 5.2.10 and 7.2) is an invariant of the *family* of boundary theories over the control manifold, not a property of any single configuration, and as such is not reducible to a statement about one static stack.

7.5. Quantization Domains and Periodicities

A further conceptual point concerns the *quantization* of the allowed shifts in the effective central charge. Because the scalar $e^{2\pi i\phi}$ multiplying the torus T is produced by an invertible (2+1)d bulk, the value of ϕ is not arbitrary but restricted by cobordism classes.

In bosonic topological phases without a spin structure, the only nontrivial invertible bulk is the E_8 state. This implies that the chiral central charge is only well defined modulo 8, and any HCCB-induced drift must satisfy

$$\Delta c_- \equiv 0 \pmod{8}.$$

In other words, boundary theories that differ by $\Delta c_- = 8$ are experimentally indistinguishable once bosonic invertible stacking is accounted for.

By contrast, in spin systems (fermionic platforms, including topological superconductors), the presence of a spin structure refines this periodicity. A single Majorana mode contributes $c_- = \frac{1}{2}$, and hence half-integer shifts are allowed. In this case, one has

$$\Delta c_- \equiv 0 \pmod{\frac{1}{2}}.$$

This explains why experiments in quantum Hall devices can in principle observe half-integer multiples of the thermal conductance quantum, while purely bosonic lattice models cannot.

Thus, it is important when interpreting data to specify the background structure: bosonic systems enforce an 8-fold periodicity, while spin systems enforce a 1/2-fold periodicity. Any measured drift Δc_- should be compared against these quantization domains in order to decide whether it represents a genuinely new categorical deformation or a redundancy from stacking with an invertible phase.

In summary, while HCCB shifts are always *quantized*, the step size depends on the underlying category of the system: 8 for bosonic phases and 1/2 for spin phases. This clarifies the periodicity structure that constrains dynamical central charge measurements.

7.6. Beyond a Single Central Charge: A Hierarchy of Anomaly Coefficients

This subsection makes explicit what is only implicit in ordinary presentations: once higher coherence is allowed to fail in a controlled and inflow-compatible way *beyond* the pentagon/hexagon level, the anomaly data no longer compress to a single number; instead, there is a *hierarchy* of scalar invariants, each attached to a different layer of coherence, that we organize as a vector of “central charges.”

Setup: Modular functor with scalar anomalies. Let $\mathcal{Z}$ be a modular functor assigning to each compact surface Σ (with boundary/punctures and labels suppressed) a finite-dimensional vector space $\mathcal{B}_\Sigma$, and to each mapping class $\gamma \in \text{MCG}(\Sigma)$ a linear map $\rho_\Sigma(\gamma) : \mathcal{B}_\Sigma \rightarrow \mathcal{B}_\Sigma$. In a *strictly* coherent theory, ρ_Σ is a genuine representation. Under higher categorical coherence breakdown (HCCB), composition is satisfied only up to a *scalar* phase,

$$\rho_\Sigma(\gamma_1)\rho_\Sigma(\gamma_2) = e^{2\pi i \Phi_\Sigma(\gamma_1, \gamma_2)} \rho_\Sigma(\gamma_1\gamma_2),$$

so that ρ_Σ is projective and Φ_Σ is a 2-cocycle valued in $U(1)$. Concretely, Φ_Σ is built from the scalar failures of coherence (associator, braiding, and their higher homotopies) accumulated along words in the elementary sewing moves corresponding to γ_1, γ_2 .

Decomposition by background structure and genus. The scalar anomaly Φ_Σ depends on the *structure* with which surfaces are equipped (orientation only, or spin/pin structure, Spin^c , etc.) and on the *topology* (genus g). It is useful to split the anomaly into a continuous (free) part and a torsion (discrete) part:

$$\Phi_\Sigma = \Phi_\Sigma^{\text{free}} + \Phi_\Sigma^{\text{tors}}.$$

The free part is generated by the gravitational (framing) anomaly, and is governed by a single real parameter κ_g (equivalently the chiral central charge $c_- = 24\kappa_g$). The torsion part lives in finite groups determined by the surface structure (e.g., spin-structure dependent $\mathbb{Z}_2$ or $\mathbb{Z}_{16}$ invariants in fermionic settings). Each independent torsion invariant yields an *additional*, quantized scalar.

Definition (hierarchy of central charges). We encode the anomaly parameters as a vector

$$\vec{c} \equiv (c^{(2)}; c_{\text{tors},1}^{(3)}, \dots, c_{\text{tors},r}^{(3)}; c_{\text{tors},1}^{(4)}, \dots),$$

with the following meaning:

- $c^{(2)} := c_- \in \mathbb{R}$ is the *primary* (chiral) central charge, determined by the scalar acquired by the Dehn twist on the torus T , i.e., by $\Phi_{\Sigma_1}(T, \cdot)$; it generates the continuous part and fixes the slope of the thermal Hall conductance.
- $c_{\text{tors},j}^{(3)}$ are *secondary* central charges: independent torsion coefficients that govern scalar phases on genus- $g \geq 2$ mapping class relations not reducible to T alone (e.g., chain/lantern relations, separating twists), and on surfaces carrying additional background structure (e.g., spin). They *do not* affect the torus thermal Hall slope but do control spin-structure dependence and genus-two partition-function phases.
- Higher entries $c_{\text{tors},\bullet}^{(n)}$ (if present) label scalar anomalies tied to higher coherence cells (homotopy associativity/commutativity beyond pentagon/hexagon). They appear only when the modular functor is promoted to include those higher cells (e.g., homotopy-coherent field theories) and, again, are torsion.

Relation to mapping-class cohomology and inflow. For fixed Σ , projective representations are classified by $H^2(\text{MCG}(\Sigma); U(1))$. In the purely bosonic, orientation-only case and for large genus, this group is (essentially) generated by a single free class (the Meyer class), hence only $c^{(2)}$ survives as an independent continuous parameter. When additional structures are present (spin/pin, G -bundles for global symmetries), H^2 acquires *torsion* summands. These torsion classes yield the entries $c_{\text{tors},j}^{(3)}$, etc., and correspond in the bulk to stacking with distinct *invertible* (2+1)d phases carrying purely *discrete* response (e.g., spin-structure dependent η -invariants). Thus

HCCB twist $\iff$ projective anomaly class $[\Phi_{\Sigma}] \iff$ invertible bulk inflow data.

The free component of $[\Phi_{\Sigma}]$ is exactly $c^{(2)}/24$, while the torsion components are the higher entries $c_{\text{tors},\bullet}^{(n)}$.

Measurement channels. Only $c^{(2)}$ couples to the *thermal Hall slope*:

$$\frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h} c^{(2)} \quad (\text{low } T, \text{ equilibrated edge}).$$

By contrast, the torsion entries manifest through:

1. *Spin-structure dependence* of partition functions and of edge spectra (e.g., periodic versus antiperiodic boundary conditions for fermions around non-contractible cycles). A change in spin structure multiplies the genus- g partition function by a torsion phase determined by $c_{\text{tors},\bullet}^{(3)}$.
2. *Genus-two modular transformations.* Certain relations in $\text{MCG}(\Sigma_2)$ (e.g., lantern/chain relations) admit scalar anomalies independent of T . Their phases probe combinations of the $c_{\text{tors},\bullet}^{(3)}$.
3. *Interferometric pumps sensitive to discrete inflow.* Domain-wall braiding or parity pumps that toggle background structures (e.g., insert a π flux of an emergent gauge field, or flip a spin structure along a cycle) read out the torsion entries via quantized phase steps, without affecting κ_{xy}/T .

HCCB as a multi-parameter deformation. With this bookkeeping, a general HCCB deformation acts linearly on the anomaly vector:

$$\vec{c} \longmapsto \vec{c} + \Delta \vec{c} = (c^{(2)} + 24\phi; c_{\text{tors},j}^{(3)} + \delta_j^{(3)}; c_{\text{tors},k}^{(4)} + \delta_k^{(4)}; \dots),$$

where $\phi \in \mathbb{R}$ (mod the appropriate periodicity) is the Dehn-twist scalar and $\delta_{\bullet}^{(n)}$ are discrete increments fixed by the allowed torsion classes for the chosen background structure. Our

proposed experiments directly access the *first* component ($c^{(2)}$) via the slope of $\kappa_{xy}(T)$; the remaining components call for spin- and genus-sensitive probes as outlined above.

Consistency and constraints. The entries of $\vec{c}$ are not arbitrary: (i) additivity under stacking/inflow implies linearity of $\vec{c}$; (ii) background-structure choice fixes which torsion components exist (bosonic, spin, $\text{pin}^\pm$, G -symmetry); (iii) cobordism periodicities quantize the allowed increments (e.g., bosonic 8-fold periodicity for $c^{(2)}$, spin half-integer granularity, discrete periodicities for torsion components).

Summary. From the perspective of low-temperature heat transport, HCCB behaves like a *single knob*— $c^{(2)}$ —because only the Dehn-twist scalar enters κ_{xy}/T . From the categorical and inflow viewpoints, the same HCCB deformation generally shifts a *vector* of anomaly coefficients $\vec{c}$: a continuous component (the chiral central charge) and a finite set of discrete components tied to higher coherence and background structure. This hierarchy reconciles the laboratory expectation of a *single* quantized slope with the deeper mathematical fact that coherence can fail in many, independently quantized ways.

8. Conclusions

Central charge as a unifying invariant. We began by systematizing the threefold meaning of the central charge in two-dimensional conformal field theory (CFT): (i) as an *effective count of degrees of freedom*, controlling the asymptotic density of states (Cardy growth) and universal entanglement amplitudes; (ii) as the *determinant of Casimir/thermal amplitudes*, fixing finite-size and finite-temperature coefficients via modular covariance; and (iii) as an *obstruction/anomaly*, the non-removable 2-cocycle that centrally extends the Witt algebra to Virasoro and generates the Schwarzian shift in stress-tensor transformations. These faces are mathematically equivalent and physically complementary: the same c that counts states also appears as the coefficient of the trace/gravitational anomaly, and this coincidence is *why* thermodynamic and transport measurements can “read out” a parameter that is defined algebraically.

Categorical reinterpretation and higher coherence. We then recast OPE associativity and braiding in the categorical language of monoidal/braided tensor categories. There, the pentagon/hexagon coherence relations encode crossing and monodromy of conformal blocks, while mapping-class actions on block spaces organize modular data. A controlled quantized deformation of coherence—which we call *higher categorical coherence breakdown* (HCCB)—multiplies the mapping-class representation by a one-dimensional *scalar* factor. On the torus, this shows up as a rephasing of the Dehn twist:

$$T \longmapsto e^{2\pi i \phi} T.$$

Since $T_{aa} = e^{2\pi i(h_a - c/24)}$, this scalar is *equivalent* to a drift of the effective central charge

$$c \longmapsto c_{\text{eff}} = c + 24\phi,$$

with the same statement for the chiral part c_- relevant to parity-odd transport. The upshot is concise: *coherence breakdown lives in the same anomaly pipeline as c* , so its only allowed modular imprint is a shift of c .

Quantum Hall edges as anomaly meters. Chiral edges of quantum Hall (QH) phases implement 1+1D CFTs coupled to a topological bulk. Their thermal Hall conductance is fixed at low temperature by the chiral central charge,

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_- T.$$

Because HCCB shifts the T-phase by ϕ , it necessarily shifts c_- by $\Delta c_- = 24\phi$, and hence the measurable slope of $\kappa_{xy}(T)$ by the universal amount $(\pi^2 k_B^2 / 3h) \Delta c_-$. Modular thermodynamics and Casimir amplitudes respond in lockstep: the free energy density acquires the T^2 coefficient $-(\pi/6)(c_{-,eff}/v)$ on a ring, and difference measurements between categorical configurations return the same Δc_- .

From principle to protocol. We translate this logic into an experimental program. The standard thermal Hall bar provides the baseline; its metrology is sharpened by three levers:

1. *Edge composition as a gain knob:* Engineering multi-component edges (e.g., $\nu = 2/3$) and driving them into the equilibrated regime suppresses parasitic nonequilibrium contributions so that the measured slope equals $c_{-,eff}$.
2. *Categorical toggles:* Non-Abelian islands (via QPC-defined interferometers) and moiré Chern-band ancillae both provide in situ switches of topological sector or invertible bulk stack, changing only the mapping-class scalar and thus $c_{-,eff}$.
3. *Difference metrology:* Comparing $\kappa_{xy}(T)$ across two categorical settings λ, λ' cancels device-specific backgrounds and isolates the constant offset Δc_- in the ratio $\Delta\kappa_{xy}(T)/T$ at low T and large path length L .

The same categorical switches also move *interference phases*, tying phase-sensitive (braiding/fusion) probes to thermodynamic (thermal Hall) probes and providing a stringent cross-check that the observed step is *anomaly-derived* rather than kinematic. Multi-junction route comparisons play a complementary role: within the consistent framework, they are exact nulls (route blindness, Section 5.1), so that they *bound* coherence failure beyond the invertible class rather than measuring a Tier-1 parameter; driven around closed control cycles, the same hardware realizes the quantized central-charge pump of Sections 5.2.10 and 7.2.

What would falsify the picture. The HCCB hypothesis is falsifiable. In devices where the categorical control does not affect equilibration rates or phonon leakage, the difference observable must show

$$\frac{\Delta\kappa_{xy}(T)}{T} \xrightarrow{T \rightarrow 0, L/\ell(T) \rightarrow \infty} \frac{\pi^2 k_B^2}{3h} \Delta c_-,$$

with (i) a *temperature-independent* plateau, (ii) convergence in L (exponential in L/ℓ), and (iii) one-to-one correlation with interferometric sector changes when present. Failure of any of these three signatures despite successful device calibration and channel counting would disfavor HCCB (or indicate that the control changes non-anomalous parameters such as mode velocities or leakages).

Actionable predictions.

- *Step height:* Switching a categorical control between two settings λ, λ' generates a linear-in- T step $\Delta\kappa_{xy} = (\pi^2 k_B^2 / 3h) \Delta c_- T$; for a single neutral Majorana sector toggled on/off, $\Delta c_- = 1/2$ and the step is $\frac{1}{2} \kappa_0 T$.
- *Length scaling:* The approach of $\kappa_{xy}(L, T)/T$ to its anomaly limit is exponential in $L/\ell(T)$; the *difference* channel inherits the same exponent but with vanishing prefactor to leading order, producing faster convergence.
- *Casimir echo:* The same Δc_- appears in the T^2 coefficient of the free energy on a ring; a small but consistent change in the specific heat per unit length, $\Delta(C/L) = (\pi/3)(\Delta c_- / v) T$, provides a thermodynamic echo of the transport step.

Beyond semiconductors: Condensed matter and high energy. The framework extends naturally to neutral platforms (spin liquids with chiral order, phononic/photon channels) where κ_{xy} is unencumbered by charge transport, to topological superconductors (treated

explicitly in Appendix D), and to ultracold atoms, where synthetic gauge fields and programmable interactions grant categorical control at the Hamiltonian level. In high-energy theory, the analogy is precise: just as linear-dilaton/Liouville backgrounds shift the worldsheet c by a background charge, HCCB shifts c_{eff} by a modular scalar coming from an invertible $(2+1)$ -dimensional bulk. The same cobordism constraints that quantize anomaly coefficients quantize the allowed Δc_- here, explaining the observed step structure and forbidding smooth drifts within a fixed topological sector.

Conceptual moral. Anomalies can be reinterpreted as *coherence invariants* of the sewing problem for conformal blocks. When coherence is deformed in a way that is globally consistent (via inflow), those invariants can *move* in a quantized fashion. The central charge, long regarded as a fixed label of a CFT, thus becomes a *dynamical*—yet still sharply quantized—device-level observable. This reframing does not dilute the rigidity of c ; rather, it clarifies the channel through which c can change: not by local perturbations within a phase but by stacking with (or inducing) an invertible topological sector that alters the mapping-class scalar.

Outlook. Three immediate directions stand out. First, *classification*: systematize the space of categorical deformations that produce nontrivial Dehn scalars, relating them to Witt classes of modular tensor categories and to invertible TQFTs. Second, *control*: realize tunable low-loss ancillae (moiré Chern bands; atomic flux lattices) to implement repeatable Δc_- steps and “central-charge pumps” over cycles of a control parameter. Third, *diagnostics*: integrate interference, calorimetry, and local thermometry to enforce triple coincidence (slope, length scaling, phase) across many platforms. If these programs succeed, then “dynamical central charge” will join quantized conductance as a standard metrological tool, bridging mathematical physics, mesoscopic engineering, and quantum simulation.

In sum, we have traced a direct quantitative path from a higher-categorical deformation of coherence data to a universal and experimentally accessible change in anomaly-controlled observables. Quantum Hall systems provide the cleanest near-term testbed, but the conceptual reach is broader: wherever chiral edges or worldsheet-like CFTs govern low-energy physics, HCCB offers a lever to move the central charge itself in a way that is predictable, quantizable, and measurable.

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Appendix A. Technical Derivation of the Thermal Hall Conductance Formula

This appendix collects complementary derivations of the chiral thermal Hall conductance

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} c_- T, \quad (\text{A1})$$

valid for a $(2+1)$ -dimensional gapped topological phase with a chiral edge conformal field theory (CFT). We keep the discussion self-contained and emphasize the role of energy magnetization terms that are essential in Kubo formulas for thermal transport.

Appendix A.1. Edge-CFT (Landauer) Route

Consider a single chiral sector with central charge c on a circle of length L and mode velocity v . In thermal equilibrium at temperature T , the (chiral) energy density is fixed by CFT thermodynamics:

$$\varepsilon(T) = \frac{\pi}{12} \frac{k_B^2}{\hbar} c T^2 \frac{v}{L} L = \frac{\pi}{12} \frac{k_B^2}{\hbar} c T^2, \quad (\text{A2})$$

and the energy current equals the momentum density for a chiral theory, yielding

$$J_Q^{\text{chiral}}(T) = J_E^{\text{chiral}}(T) = \frac{\pi}{12} \frac{k_B^2}{\hbar} c T^2. \quad (\text{A3})$$

Connecting two semi-infinite edges at temperatures T_L and T_R gives a net heat current

$$J_Q = J_E^{\text{chiral}}(T_L) - J_E^{\text{chiral}}(T_R) = \frac{\pi k_B^2}{12\hbar} c_- (T_L^2 - T_R^2). \quad (\text{A4})$$

Linearizing for a small temperature bias $\Delta T = T_L - T_R$ around $T = \frac{T_L + T_R}{2}$,

$$J_Q = \frac{\pi^2 k_B^2}{3h} c_- T \Delta T \equiv \kappa_{xy} \Delta T, \quad (\text{A5})$$

which is (A1). This derivation relies only on (i) chiral 1D CFT thermodynamics and (ii) local equilibration along the edge.

Appendix A.2. Kubo Formula with Energy Magnetization

A bulk Kubo formula for κ_{xy} must subtract circulating (magnetization) heat currents to isolate the *transport* response. Let Q^μ be the (microscopic) heat current density operator obtained from the Hamiltonian density and the continuity equation. (For interacting lattice models, care is required to construct Q^μ consistently with gauge invariance and to include improvement terms; see [171] for a systematic operator approach.) The naive Kubo conductivity

$$\tilde{\kappa}_{xy} = \lim_{\omega \rightarrow 0} \frac{1}{T} \frac{\text{Im } K_{xy}^R(\omega)}{\omega}, \quad K_{xy}^R(\omega) = -i \int_0^\infty dt e^{i\omega t} \langle [Q_x(t), Q_y(0)] \rangle \quad (\text{A6})$$

must be corrected by the *energy magnetization* M_z^E through

$$\kappa_{xy} = \tilde{\kappa}_{xy} + \frac{2}{T} \frac{\partial M_z^E}{\partial B}. \quad (\text{A7})$$

The thermodynamic quantity M_z^E is defined so that $\nabla \times \mathbf{M}^E$ reproduces circulating (non-transport) heat currents; microscopically, M^E can be obtained from the grand potential in a fictitious (Luttinger) gravitational potential or from the response to torsionful backgrounds.

In a fully gapped topological phase, the retarded correlator contribution $\tilde{\kappa}_{xy}$ vanishes in the $\omega \rightarrow 0$ limit, and the *entire* Hall response comes from the magnetization term (A7). Coupling to background geometry and using anomaly inflow for the parity-odd gravitational sector shows that

$$\frac{2}{T} \frac{\partial M_z^E}{\partial B} = \frac{\pi^2 k_B^2}{3h} c_- T, \quad (\text{A8})$$

reproducing (A1). This computation can be organized either by explicit operator constructions for Q^μ and M^E [171] or by a field-theoretic derivation using the gravitational Chern–Simons term [14].

Appendix A.3. Effective Action and Anomaly Inflow

A continuum low-energy description couples the gapped bulk to background geometry $(e^a_\mu, \omega_\mu^{ab})$. The parity-odd piece of the effective action includes the gravitational Chern–Simons term,

$$S_{\text{gCS}}[\omega] = \frac{\kappa_g}{4\pi} \int_{M^3} \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right), \quad c_- = 24\kappa_g. \quad (\text{A9})$$

Varying the action with respect to the background “gravitoelectromagnetic” fields (the analogues of A_0^g and A_i^g) yields the equilibrium energy current and, via the Sřreda-like formula for thermal transport, the Hall coefficient (A1). The metric-independence of the coefficient reflects its topological origin in anomaly inflow.

Each route—edge thermodynamics, bulk Kubo with energy magnetization, and geometric effective action—arrives at the same universal result (A1). In our main text, we showed how higher categorical coherence breakdown modifies the modular Dehn phase and hence shifts the anomaly coefficient c_- that controls all three derivations.

Appendix B. Sketch of Categorical Diagrams and Their Breakdown

This appendix provides a compact graphical (string-diagram) view of coherence in monoidal and braided categories and explains how higher categorical coherence breakdown (HCCB) appears as controlled scalar failures of these diagrams. Throughout, objects are wires and morphisms are boxes, tensor product is juxtaposition, and composition is vertical stacking; we follow standard string-diagram and monoidal-category conventions [172–174].

Appendix B.1. Triangle and Pentagon (Monoidal Coherence)

Triangle. Unit object $\mathbf{1}$ with unitors $\lambda_b : \mathbf{1} \otimes b \Rightarrow b$ and $\rho_a : a \otimes \mathbf{1} \Rightarrow a$:

$$\begin{array}{ccc} (a \otimes \mathbf{1}) \otimes b & \xrightarrow{\alpha_{a, \mathbf{1}, b}} & a \otimes (\mathbf{1} \otimes b) \\ & \searrow \rho_a \otimes \text{id}_b & \downarrow \text{id}_a \otimes \lambda_b \\ & & a \otimes b \end{array}$$

Graphically, the two ways of eliminating the trivial wire $\mathbf{1}$ agree.

Pentagon. Associator $\alpha_{a,b,c} : (a \otimes b) \otimes c \Rightarrow a \otimes (b \otimes c)$ must satisfy the pentagon: all five rebracketings of $a \otimes b \otimes c \otimes d$ connected by four α s commute.

$$\begin{array}{ccccc} ((a \otimes b) \otimes c) \otimes d & \xrightarrow{\alpha_{a,b,c} \otimes \text{id}_d} & (a \otimes (b \otimes c)) \otimes d & \xrightarrow{\alpha_{a,b \otimes c, d}} & a \otimes ((b \otimes c) \otimes d) \\ \downarrow \alpha_{a \otimes b, c, d} & & & & \downarrow \text{id}_a \otimes \alpha_{b, c, d} \\ (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\alpha_{a, b, c \otimes d}} & & & a \otimes (b \otimes (c \otimes d)) \end{array}$$

In words: starting from $((a \otimes b) \otimes c) \otimes d$, the “top path”

$$((a \otimes b) \otimes c) \otimes d \xrightarrow{\alpha_{a,b,c} \otimes \text{id}_d} (a \otimes (b \otimes c)) \otimes d \xrightarrow{\alpha_{a,b \otimes c, d}} a \otimes ((b \otimes c) \otimes d) \xrightarrow{\text{id}_a \otimes \alpha_{b, c, d}} a \otimes (b \otimes (c \otimes d))$$

equals the “bottom path”

$$((a \otimes b) \otimes c) \otimes d \xrightarrow{\alpha_{a \otimes b, c, d}} (a \otimes b) \otimes (c \otimes d) \xrightarrow{\alpha_{a, b, c \otimes d}} a \otimes (b \otimes (c \otimes d)).$$

This commutativity is the pentagon coherence law.

Appendix B.2. Braiding and the Hexagon

A braiding is a natural isomorphism $c_{a,b} : a \otimes b \Rightarrow b \otimes a$ compatible with the associator $\alpha_{a,b,c} : (a \otimes b) \otimes c \Rightarrow a \otimes (b \otimes c)$. This compatibility is encoded by two hexagon identities. They assert that “sliding a crossing past a fusion” is the same as “rebracket, cross, rebracket back.”

Hexagon (+) (moving a past $b \otimes c$).

$$\begin{array}{ccccc} & & A := a \otimes (b \otimes c) & & \\ & \nearrow^{\alpha_{a,b,c}} & & \searrow^{c_{a,b \otimes c}} & \\ S := (a \otimes b) \otimes c & & & & B := (b \otimes c) \otimes a \\ \downarrow^{c_{a,b} \otimes \text{id}_c} & & & & \uparrow^{\alpha_{b,c,a}^{-1}} \\ E := (b \otimes a) \otimes c & \xrightarrow{\alpha_{b,a,c}} & G := b \otimes (a \otimes c) & \xrightarrow{\text{id}_b \otimes c_{a,c}} & H := b \otimes (c \otimes a) \end{array}$$

Algebraically,

$$c_{a,b \otimes c} = \alpha_{b,c,a} \circ (\text{id}_b \otimes c_{a,c}) \circ \alpha_{b,a,c}^{-1} \circ (c_{a,b} \otimes \text{id}_c) \circ \alpha_{a,b,c}.$$

Hexagon (−) (moving $a \otimes b$ past c).

$$\begin{array}{ccccc} & & U := (a \otimes b) \otimes c & & \\ & \nearrow^{\alpha_{a,b,c}^{-1}} & & \searrow^{c_{a \otimes b, c}} & \\ T := a \otimes (b \otimes c) & & & & Y := c \otimes (a \otimes b) \\ \downarrow^{\text{id}_a \otimes c_{b,c}} & & & & \uparrow^{\alpha_{c,a,b}} \\ T' := a \otimes (c \otimes b) & \xrightarrow{\alpha_{a,c,b}^{-1}} & V := (a \otimes c) \otimes b & \xrightarrow{c_{a,c} \otimes \text{id}_b} & W := (c \otimes a) \otimes b \end{array}$$

Equivalently,

$$c_{a \otimes b, c} = \alpha_{a,b,c}^{-1} \circ (\text{id}_a \otimes c_{b,c}) \circ \alpha_{a,c,b} \circ (c_{a,c} \otimes \text{id}_b) \circ \alpha_{c,a,b}.$$

Appendix B.3. How HCCB Appears

A deformation controlled by a normalized $U(1)$ 3-cochain Ω_3 multiplies the associator by a phase:

$$\alpha'_{a,b,c} = \Omega_3(a, b, c) \alpha_{a,b,c}. \quad (\text{A10})$$

On four objects, the pentagon picks up the 4-coboundary $\delta\Omega_3(a, b, c, d)$. If $\delta\Omega_3 = 1$, coherence is preserved and the twist replaces the category by a generally inequivalent consistent one (Tier 1 of the main text); the modular imprint is the Dehn-twist scalar. If $\delta\Omega_3 \neq 1$ for a globally defined Ω_3 , the pentagon closes only up to the scalar $e^{2\pi i \delta\Omega_3(a,b,c,d)}$, but this failure is an exact coboundary that is removable by undoing the twist, and carries no anomaly; a

genuine non-removable failure is a non-trivial H^4 class and requires a (3+1)d invertible host (Section 2.3.3 of the main text). In braided settings, compatible deformations of $c_{a,b}$ are classified by Abelian (Eilenberg–Mac Lane) cohomology and shift the quadratic form.

Appendix B.4. Higher Coherence

Beyond the pentagon/hexagon, homotopy-coherent structures (A_∞, E_n) organize associativity/commutativity by higher cells whose indexing polytopes are associahedra (K_n) and related complexes [175]. Deforming these higher cells by phases is the “higher” in HCCB; closed loops of higher moves evaluate to scalars that depend only on the underlying mapping-class element. In particular, the Dehn twist on a torus obtains a scalar $e^{2\pi i\phi}$, shifting the effective central charge by 24ϕ in modular data.

Appendix C. Experimental Data Estimates: Noise Floors and Sensitivity

We collect back-of-the-envelope estimates useful for experiment design. The universal quantum of thermal conductance is

$$\kappa_0 = \frac{\pi^2 k_B^2}{3h} \approx 9.46 \times 10^{-13} \frac{\text{W}}{\text{K}^2}. \quad (\text{A11})$$

Hence, a single fully equilibrated downstream channel contributes $\kappa_0 T$; an HCCB-induced step with Δc_- channels contributes $\Delta \kappa_{xy} = \Delta c_- \kappa_0 T$.

Appendix C.1. Signal Sizes

At representative millikelvin temperatures,

$$T = 20 \text{ mK} \Rightarrow \kappa_0 T \simeq 1.89 \times 10^{-14} \text{ W/K}, \quad (\text{A12})$$

$$T = 30 \text{ mK} \Rightarrow \kappa_0 T \simeq 2.84 \times 10^{-14} \text{ W/K}, \quad (\text{A13})$$

$$T = 50 \text{ mK} \Rightarrow \kappa_0 T \simeq 4.73 \times 10^{-14} \text{ W/K}. \quad (\text{A14})$$

For a difference measurement toggling $\Delta c_- = \frac{1}{2}$ (e.g., a single Majorana sector on/off), the expected *step* of κ_{xy} is half the above numbers:

$$\Delta \kappa_{xy} \approx \begin{cases} 9.5 \times 10^{-15} \text{ W/K} & (20 \text{ mK}), \\ 1.4 \times 10^{-14} \text{ W/K} & (30 \text{ mK}), \\ 2.4 \times 10^{-14} \text{ W/K} & (50 \text{ mK}). \end{cases} \quad (\text{A15})$$

With a small temperature bias ΔT between source and drain, the corresponding heat-current step is $\Delta J_Q = \Delta \kappa_{xy} \Delta T$. Taking $\Delta T = 3 \text{ mK}$ at $T = 30 \text{ mK}$ gives $\Delta J_Q \approx 4.3 \times 10^{-17} \text{ W}$.

Appendix C.2. Thermal Fluctuation Noise and Bandwidth

For a channel with thermal conductance κ , the thermal-fluctuation (Johnson–Nyquist) noise of heat flow over bandwidth B obeys

$$\text{NEP} \equiv \sqrt{S_{J_Q}(0)} \simeq \sqrt{4k_B T^2 \kappa B}. \quad (\text{A16})$$

At $T = 30 \text{ mK}$ with $\kappa \sim 1.5 \kappa_0 T \approx 4.3 \times 10^{-14} \text{ W/K}$ and $B = 1 \text{ Hz}$,

$$\text{NEP} \approx 4.6 \times 10^{-20} \text{ W}/\sqrt{\text{Hz}}. \quad (\text{A17})$$

Comparing with the signal $\Delta J_Q \sim 4 \times 10^{-17}$ W shows an instantaneous signal-to-noise ratio $\gtrsim 10^3 / \sqrt{\text{Hz}}$ in the *ideal* limit set by thermal fluctuations; in practice, calibration errors and parasitic leaks dominate the error budget.

Appendix C.3. Temperature Calibration and Thermometer Noise

Let $\hat{c}_{-, \text{eff}} = (3h/\pi^2 k_B^2)(\kappa_{xy}/T)$ be the slope estimator. Propagating uncertainties,

$$\delta c_{-, \text{eff}} \approx \frac{3h}{\pi^2 k_B^2} \left(\frac{\delta \kappa_{xy}}{T} + \frac{\kappa_{xy}}{T} \frac{\delta T}{T} \right). \quad (\text{A18})$$

To resolve $\Delta c_- = \frac{1}{2}$ at 5σ with $T = 30$ mK, one needs

$$\frac{\delta \kappa_{xy}}{T} \lesssim \frac{1}{5} \kappa_0 \Delta c_- \approx 9.5 \times 10^{-14} \frac{\text{W}}{\text{K}^2}, \quad (\text{A19})$$

or equivalently, a fractional thermometer calibration error $\delta T/T \lesssim 10^{-2}$ if $\delta \kappa_{xy}$ is dominated by the temperature scale. On-chip noise thermometry and primary mesoscopic thermometers can reach this regime [176].

Appendix C.4. Systematic Effects and Diagnostics

- *Phonon leaks*: Add analytic T^α ($\alpha > 1$) backgrounds. The remedy consists of two-temperature slope fits of $\kappa_{xy}(T)/T$ and device geometries that minimize phonon conduction.
- *Partial equilibration*: Produces an exponential approach $\propto e^{-L/\ell(T)}$ to the anomaly limit; the *difference* observable cancels leading corrections if the categorical toggle does not change $\ell(T)$.
- *Contact losses*: Verified by channel counting in the decoupled regime and by reciprocity tests.

Appendix D. Generalizations to Topological Superconductors

Topological superconductors (TSCs) furnish a natural extension of our program. Charge conservation is broken at the mean-field level, but *energy* is conserved and chiral Majorana edge modes carry heat. The quantized thermal Hall effect becomes the primary anomaly probe; HCCB again enters by shifting the modular T-phase of the edge CFT.

Appendix D.1. Bulk–Edge Correspondence and Central Charge

In (2+1)d class-D superconductors—one entry of the tenfold-way classification of topological insulators and superconductors [177,178]—the bulk Bogoliubov–de Gennes (BdG) Hamiltonian has an integer Chern number $\nu \in \mathbb{Z}$. The boundary hosts $|\nu|$ chiral Majorana modes with chiral central charge

$$c_- = \frac{\nu}{2}. \quad (\text{A20})$$

Each Majorana contributes $c = \frac{1}{2}$; hence, the low- T thermal Hall response is

$$\kappa_{xy} = \frac{\pi^2 k_B^2}{3h} \frac{\nu}{2} T. \quad (\text{A21})$$

This matches the prediction from a gravitational Chern–Simons term with level $\kappa_g = \nu/48$ in the bulk effective action via $c_- = 24\kappa_g$.

Appendix D.2. Edge Theory and HCCB

The edge is a chiral Majorana CFT (Ising neutral sector). A categorical deformation that multiplies the Dehn twist by $e^{2\pi i\phi}$ shifts c_- by 24ϕ , just as in charged systems. Because the electrical Hall response vanishes in a superconductor, thermal transport and modular thermodynamics provide the cleanest windows into this shift. In particular, (A21) acquires $c_- \rightarrow c_- + \Delta c_-$ and the heat current changes by $\Delta J_Q = (\pi^2 k_B^2 / 3h) \Delta c_- T \Delta T$ under a small bias ΔT .

Appendix D.3. Microscopic Derivations and Anomaly Matching

Microscopic derivations of (A21) use either (i) edge CFT thermodynamics for Majorana fermions; (ii) Kubo formulas for BdG quasiparticles with correctly defined heat-current operators (including energy magnetization); or (iii) effective actions with gravitational Chern–Simons terms and anomaly inflow, in the spirit of the topological field theory of time-reversal-invariant insulators and superconductors and of cross-correlated topological responses [179,180]. All three routes agree and reproduce the half-quantized unit per Majorana channel. HCCB modifies only the *overall* scalar in the mapping-class representation, hence only c_- ; it leaves local operator algebra (and thus scattering) intact.

Appendix D.4. Platforms and Probes

Candidate platforms include chiral $p_x + ip_y$ superconductors (spinless or effectively spinless via spin–orbit locking) and engineered heterostructures (quantum anomalous Hall insulators proximitized by superconductors). In all cases, the hallmark signature is linear-in- T κ_{xy} with half-integer multiples of $\kappa_0 T$ [181]; an HCCB toggle should shift the slope by integer multiples of $\frac{1}{2} \kappa_0 T$.

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