

Article

An Architecture-Conditional Framework for Relative-Entropy Event Timing, Quantum Records, and Modular Recovery

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Abstract

The quantum measurement problem separates into operational questions: which observables are stable records, how outcome probabilities are represented, how conditional post-event states are updated, and how a detector event time is assigned. We give a compact architecture-conditional framework. In a finite-dimensional detector model, the detector-side relative-entropy flux is differentiated with the exact Fréchet derivative of the matrix logarithm. A noise-regularised timing distribution is defined and, under an explicitly assumed isolated non-degenerate maximum of the calibrated flux, Laplace asymptotics proves concentration at that maximum. Under stated fixed-point and detailed-balance hypotheses, the centre of the fixed-point algebra gives a canonical commutative record algebra. Outcome probabilities admit a POVM representation, and conditional updates use a completely positive (CP) instrument in its standard sense: CP maps whose traces give probabilities and whose normalised outputs give post-event states, summing to a trace-preserving map. Separately, an assumed modular-invariant inclusion of von Neumann algebras admits a state-preserving conditional expectation and CP retraction. A conditional quantum-error-correction lemma bounds accumulated record failure. These results do not derive unique outcomes from unitarity, construct a black-hole algebra inclusion, or resolve the black-hole information problem; they give a conditional framework, a worked illustration, and testable timing and record-stability criteria.

Keywords: quantum measurement problem; relative-entropy flux; Fréchet derivative; Laplace asymptotics; POVM; completely positive instrument; fixed-point algebra; modular conditional expectation; Petz recovery; quantum error correction

1. Introduction

1.1. What Problem Is Isolated

The quantum measurement problem asks how deterministic closed-system unitary evolution is reconciled with definite, probabilistically distributed outcomes recorded by laboratory detectors [1–3]. The present paper does not claim that a unique outcome is derived from unitarity alone. Instead, it isolates four operational tasks that any detector-level account must specify:

1. Stable record observables, related to environment-induced superselection and einselection [4–6];
2. Outcome probabilities, for which the positive-operator-valued-measure (POVM) formalism is the general affine representation [7–9];
3. Conditional state updates, represented by CP instruments [10–12];
4. Detector event timing, here defined through a calibrated relative-entropy flux.



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The black-hole information problem is mentioned only as an algebraic recovery analogy. By “emitted radiation”, we mean Hawking radiation: quantum fields on a black-hole background give outgoing quanta detected in the exterior radiation algebra, while tracing over inaccessible interior degrees of freedom gives a mixed exterior state in the semiclassical description [13,14]. The modern Page-curve and island or quantum-extremal-surface programme reformulates this question in terms of the recoverability of information from the radiation algebra [15,16]. The present manuscript uses that analogy only under an explicit modular-inclusion assumption; it does not construct that inclusion.

Two longer papers by the author give expanded background and applications of the same research programme in the *International Journal of Quantum Foundations*: one on the operational QMP architecture and one on a covariant BHIP recoverability framework [17,18]. The present article is deliberately narrower and self-contained: it states only the finite-dimensional timing, record, POVM/instrument, conditional modular retraction, and conditional QEC stability claims that are proven below.

1.2. Methods and Results in Narrative Form

The manuscript is organised as a method-to-result chain rather than as a bare list of formulae. Table 1 gives the logical role of each section. The following Figure 1 displays the dependency structure and shows that the modular recovery and QEC components use separate assumptions.

Table 1. Narrative map of the compact framework. Each result is conditional on the stated hypotheses; no row asserts an unconditional solution of measurement or black-hole information loss.

Operational Question	Method	Result	Location
How is the detector information rate computed?	Differentiate Umegaki relative entropy with the Fréchet derivative of the logarithm.	A finite-dimensional flux formula with no commutativity shortcut.	Section 2
When is the event registered?	Define a Gibbs–Laplace timing density on a pre-specified window.	Concentration at an assumed isolated maximum; calibration robustness.	Section 3
Which observables are records?	Use fixed points of a detailed-balance detector semigroup.	The centre of the fixed-point algebra is the canonical commutative record algebra.	Section 4
How are probabilities and updates represented?	Use normal affine maps and CP instruments.	POVM probabilities and Stinespring-realised conditional updates.	Section 5
How is algebraic recovery formulated?	Apply Tomita–Takesaki/Takesaki conditional-expectation theory under a modular-invariant inclusion.	A CP retraction and relative-entropy sufficiency statement.	Section 6
How are encoded records stabilised?	Assume a per-cycle logical-error bound below threshold.	Accumulated failure is bounded by a union bound and can be made small by code distance.	Section 7

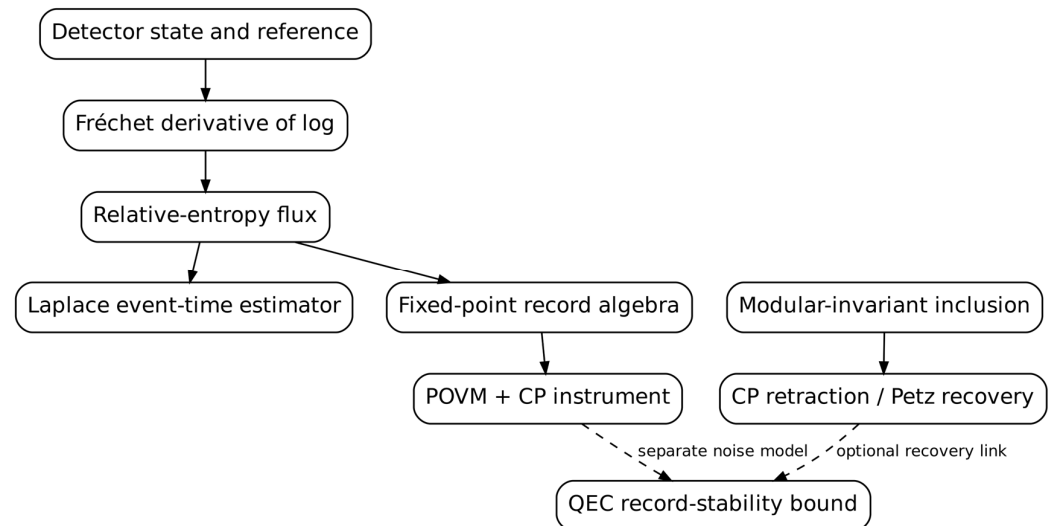


Figure 1. Dependency structure of the main results. Solid arrows denote mathematical dependence inside the finite-dimensional timing/record/instrument chain; dashed arrows denote optional recovery or record-stability links that require separate hypotheses.

1.3. Exact Claim Boundary

The results below are conditional implications. The timing distribution is an operational definition made within a calibrated detector model; the existence and uniqueness of its maximiser are explicit hypotheses. The record algebra is canonical only under the stated fixed-point assumptions. The POVM and instrument results are representation theorems and do not derive Born probabilities from unitary dynamics alone. The modular recovery theorem is conditional on the existence of a state-preserving modular-invariant inclusion. QEC stabilises encoded records under an assumed noise model but does not construct the modular inclusion. No unconditional resolution of either the quantum measurement problem or the black-hole information problem is claimed.

1.4. Reading Guide

Section 2 fixes the finite-dimensional detector model and the Fréchet flux. Section 3 defines the event-time estimator and proves Laplace concentration. Section 4 identifies the record algebra. Section 5 explains POVMs and CP instruments. Section 6 explains the modular-theoretic recovery statement. Section 7 gives the QEC record-stability bound. Section 8 works a numerical example and states a detector protocol. Section 9 summarises the scope and limitations.

2. Finite-Dimensional Detector Model and Relative-Entropy Flux

We work throughout the main theorem stack in a finite-dimensional detector Hilbert space $\mathcal{H}_R$, with $\dim \mathcal{H}_R = n < \infty$. Infinite-dimensional and type-III settings are discussed only through the conditional modular statement of Section 6; no theorem in Sections 2–5 is claimed beyond finite dimension.

2.1. Fréchet Derivative of the Matrix Logarithm

Lemma 1 (Fréchet derivative of the logarithm). *For a positive-definite operator σ on $\mathcal{H}_R$ and a self-adjoint perturbation X , the Fréchet derivative of $\sigma \mapsto \log \sigma$ is*

$$(D\log)_\sigma[X] = \int_0^\infty (\sigma + s\mathbf{1})^{-1} X (\sigma + s\mathbf{1})^{-1} ds. \quad (1)$$

It equals $\sigma^{-1}X$ if $[\sigma, X] = 0$, and in general, the non-commuting integral form must be used.

Proof. The integral representation $\log \sigma = \int_0^\infty \left[(1+s)^{-1} \mathbf{1} - (\sigma + s\mathbf{1})^{-1} \right] ds$ is valid for $\sigma > 0$. Differentiating the resolvent under the integral sign gives (2.1). In a common eigenbasis, the integral reduces to entrywise multiplication by $1/a_i$, which gives the commuting case. A finite dimension justifies dominated convergence [19,20]. $\square$

Lemma 2 (Trace identity). For positive-definite σ and self-adjoint X ,

$$\text{Tr}[\sigma(D\log)_\sigma[X]] = \text{Tr}X. \quad (2)$$

Proof. Using cyclicity, $\text{Tr}[\sigma(\sigma + s)^{-1}X(\sigma + s)^{-1}] = \text{Tr}[X(\sigma + s)^{-2}\sigma]$. In the eigenbasis $\sigma = \sum_i a_i |i\rangle\langle i|$, $\int_0^\infty a_i / (a_i + s)^2 ds = 1$. The integral therefore gives $\text{Tr}X$. $\square$

2.2. Detector-Side Relative-Entropy Flux

Let $\rho_R(t)$ and $\sigma_R(t)$ be C^1 trajectories of positive-definite density operators with unit trace. The Umegaki relative entropy is

$$D(\rho_R \| \sigma_R) = \text{Tr}[\rho_R(\log \rho_R - \log \sigma_R)]. \quad (3)$$

This convention fixes the sign used in the flux definition below [21]. Define the detector-side relative-entropy flux:

$$J_{\text{det}}(t) = -\frac{d}{dt} D(\rho_R(t) \| \sigma_R(t)). \quad (4)$$

Proposition 1 (Flux formula). If $\sigma_R(t) \equiv \sigma$ is a fixed faithful reference, then

$$J_{\text{det}}(t) = -\text{Tr}[\dot{\rho}_R(t)(\log \rho_R(t) - \log \sigma)]. \quad (5)$$

Proof. Differentiate $D = \text{Tr}[\rho_R \log \rho_R] - \text{Tr}[\rho_R \log \sigma]$. The first term contributes

$$\text{Tr}[\dot{\rho}_R \log \rho_R] + \text{Tr}[\rho_R (D\log)_{\rho_R} [\dot{\rho}_R]],$$

where $\text{Tr} \dot{\rho}_R = 0$ by Lemma 2. The second entropy term contributes $\text{Tr}[\dot{\rho}_R \log \sigma]$. Combining and applying the minus sign gives (5). $\square$

Operational reading. Equation (5) is not a new dynamical law. It is the detector-side information-rate signal obtained after the detector model, the state trajectory, and the faithful reference have been fixed.

Remark 1 (Moving reference term). The timing analysis below uses the fixed-reference case $\sigma_R(t) \equiv \sigma$. If a moving faithful reference $\sigma_R(t)$ is used, the exact finite-dimensional derivative is

$$J_{\text{det}}(t) = -\text{Tr}[\dot{\rho}_R(t)(\log \rho_R(t) - \log \sigma_R(t))] + \text{Tr}[\rho_R(t) (D\log)_{\sigma_R(t)} [\dot{\sigma}_R(t)]]. \quad (6)$$

The second term reduces to $\text{Tr}[\rho_R(t)\sigma_R(t)^{-1}\dot{\sigma}_R(t)]$ only in the commuting case. No moving-reference identity is used in Sections 3–5.

Remark 2 (No hidden infinite-dimensional claim). *Proposition 1 is the only differentiation identity used in the timing analysis. In trace-class and type-III settings, the relative-modular operator is generally unbounded, and an analogous derivative requires form-domain and integrability hypotheses not established here [22,23].*

3. Operational Event-Time Estimator and Laplace Concentration

The flux formula gives a calibrated scalar signal. This section defines, rather than derives, a detector-time estimator from that signal. The definition is operational: it tells an experimentalist which distribution to compute from a pre-specified observation window and noise scale.

Fix a calibration time-scale $\tau_0 > 0$, a flux-scale $J_0 > 0$, and set

$$x = t/\tau_0, \quad j(x) = J_{\text{det}}(\tau_0 x)/J_0, \quad \theta = \varepsilon/J_0, \quad (7)$$

where $\varepsilon > 0$ is the detector noise level. On a pre-specified dimensionless window $I = [a, b]$, define

$$f_\theta(x) = Z_\theta^{-1} \mathbf{1}_I(x) e^{j(x)/\theta}, \quad Z_\theta = \int_I e^{j(x)/\theta} dx. \quad (8)$$

Assumption 1 (Isolated non-degenerate maximum). *The calibrated flux j is C^4 on a neighbourhood of the observation window $I = [a, b]$. It has a unique interior maximiser $x^* \in (a, b)$. There are constants $r > 0$, $\kappa_0 > 0$, and $\gamma > 0$, with $[x^* - r, x^* + r] \subset (a, b)$, such that*

$$j''(x) \leq -\kappa_0 \quad \text{for all } |x - x^*| \leq r, \quad (9)$$

and

$$j(x^*) - \sup_{|x - x^*| \geq r} j(x) \geq \gamma. \quad (10)$$

Theorem 1 (Laplace concentration). *Under Assumption 1, if $X_\theta \sim f_\theta$, then as $\theta \downarrow 0$,*

$$\mathbb{E}[X_\theta] = x^* + O(\theta), \quad \text{Var}[X_\theta] = \frac{\theta}{|j''(x^*)|} (1 + O(\theta)), \quad (11)$$

and $X_\theta \rightarrow x^*$ in probability. In physical units, $\text{Var}[T_\varepsilon] = \tau_0^2 \varepsilon / (J_0 |j''(x^*)|) (1 + O(\theta))$.

Proof. This is Laplace's method for the family $e^{j/\theta}$ with a non-degenerate interior maximum [24–26]. With $x = x^* + \sqrt{\theta}u$, the C^4 regularity of Assumption 1 gives the Taylor expansion $j(x) = j(x^*) - |j''(x^*)|\theta u^2/2 + O(\theta^{3/2})$ with a remainder controlled by the local bound on $j^{(4)}$. The gap condition (10) controls the tails by $e^{-\gamma/\theta}$. Uniform integrability then gives the stated moment estimates with an $O(\theta)$ multiplicative correction. $\square$

Operational reading. The event-time distribution narrows when the calibrated flux has lower noise scale θ or larger local curvature $|j''(x^*)|$. Thus, the theorem gives a directly testable scaling law for detector-time jitter.

Proposition 2 (Calibration robustness). *Let $\tilde{j} = j + \delta j$, where $\delta j \in C^2(I)$ and*

$$\|\delta j\|_\infty \leq \eta, \quad \|\delta j'\|_\infty \leq \eta, \quad \|\delta j''\|_\infty \leq \eta.$$

If

$$\eta < \min\{\kappa_0/2, \kappa_0 r/2, \gamma/2\}, \quad (12)$$

then $\tilde{j}$ has a unique maximiser $\tilde{x}^*$ in $|x - x^*| \leq r$, no maximiser outside that ball, and

$$|\tilde{x}^* - x^*| \leq \frac{\|\delta j'\|_\infty}{\kappa_0 - \eta}. \quad (13)$$

Proof. On $|x - x^*| \leq r$, one has $\tilde{j}''(x) \leq -\kappa_0 + \eta < 0$, so $\tilde{j}$ is strictly concave on that ball. Moreover, $j'(x^* - r) \geq \kappa_0 r$ and $j'(x^* + r) \leq -\kappa_0 r$. Since $\|\delta j'\|_\infty \leq \eta < \kappa_0 r/2$, we have

$$\tilde{j}'(x^* - r) > 0, \quad \tilde{j}'(x^* + r) < 0.$$

The intermediate-value theorem gives a critical point $\tilde{x}^* \in (x^* - r, x^* + r)$, and strict concavity makes it the unique maximiser on that ball. At the perturbed critical point,

$$0 = \tilde{j}'(\tilde{x}^*) = j'(\tilde{x}^*) + \delta j'(\tilde{x}^*).$$

Since $j'(x^*) = 0$ and $j'' \leq -\kappa_0$ on the ball, the mean-value estimate gives

$$(\kappa_0 - \eta)|\tilde{x}^* - x^*| \leq \|\delta j'\|_\infty.$$

The bound $\eta < \kappa_0 r/2$ keeps $\tilde{x}^*$ inside the ball. For $|x - x^*| \geq r$,

$$\tilde{j}(x) \leq j(x^*) - \gamma + \eta < j(x^*) - \eta \leq \tilde{j}(x^*),$$

so, no outside point can be a global maximiser. $\square$

4. Canonical Record Algebra from Fixed-Point Hypotheses

A detector record should be stable under the detector dynamics and should have mutually compatible classical alternatives. The fixed-point algebra expresses stability; its centre expresses commutativity of recorded alternatives.

Let $\mathcal{D}^\dagger : \mathcal{B}(\mathcal{H}_R) \rightarrow \mathcal{B}(\mathcal{H}_R)$ be the Heisenberg generator of a finite-dimensional detector semigroup, with predual $\mathcal{D}$ acting on states.

Assumption 2 (Fixed points and detailed balance). *The semigroup $e^{t\mathcal{D}^\dagger}$ on $\mathcal{B}(\mathcal{H}_R)$ is unital and completely positive, admits a faithful invariant state σ , and satisfies σ -detailed balance, equivalently GNS self-adjointness of the generator [27,28]. We also work in the standard finite-dimensional detailed-balance setting in which the fixed-point space is the range of a σ -preserving conditional expectation E_{fix} and is therefore a unital $*$ -subalgebra of $\mathcal{B}(\mathcal{H}_R)$.*

Theorem 2 (Canonical commutative record algebra). *Under Assumption 2,*

$$\text{Fix}\mathcal{D}^\dagger = \{X : \mathcal{D}^\dagger X = 0\} \quad (14)$$

is a unital $$ -subalgebra of $\mathcal{B}(\mathcal{H}_R)$, and its centre*

$$Z_R := Z(\text{Fix}\mathcal{D}^\dagger) \quad (15)$$

is a canonical commutative subalgebra preserved by the dynamics.

Proof. By Assumption 2, the fixed-point space is the range of the σ -preserving conditional expectation E_{fix} . In the finite-dimensional detailed-balance setting, this range is a unital $*$ -subalgebra [29,30]. Its centre $Z_R = Z(\text{Fix}\mathcal{D}^+)$ is therefore a finite-dimensional commutative $*$ -subalgebra. Since E_{fix} commutes with the semigroup, the fixed-point algebra and its centre are preserved by the dynamics. $\square$

Operational reading. The fixed-point algebra identifies observables stable under the detector dynamics. Its centre gives the mutually compatible classical labels that can function as recorded outcomes.

Remark 3. The claim is not that Z_R is the only Abelian subalgebra of $\mathcal{B}(\mathcal{H}_R)$, nor that it must be maximal Abelian. The claim is that Z_R is the commutative algebra singled out by the invariant detector structure.

Proposition 3 (Relaxation to the fixed-point algebra). Assume Assumption 2. Suppose that $\mathcal{D}^+$ has spectral gap $\lambda > 0$ on the $L^2(\sigma)$ -orthogonal complement of $\text{Fix}\mathcal{D}^+$. Let E_{fix} be the σ -preserving conditional expectation onto $\text{Fix}\mathcal{D}^+$, and let $E_{\text{fix},*}$ denote its predual on states. Then, for every state ρ ,

$$\|e^{t\mathcal{D}}\rho - E_{\text{fix},*}(e^{t\mathcal{D}}\rho)\|_1 \leq C(\rho, \sigma)e^{-\lambda t}. \quad (16)$$

Here $C(\rho, \sigma)$ is a finite-dimensional norm-comparison constant.

Proof. Detailed balance makes $\mathcal{D}^+$ self-adjoint on $L^2(\sigma)$. The spectral gap gives exponential contraction on the orthogonal complement of $\text{Fix}\mathcal{D}^+$. Finite-dimensional equivalence of $L^2(\sigma)$ and trace norms convert this contraction into the stated trace-norm estimate for the predual dynamics [31]. $\square$

Remark 4 (Fixed points versus classical record labels). The dynamical relaxation statement is to the fixed-point algebra $\text{Fix}\mathcal{D}^+$. The commutative record algebra is the centre $Z_R = Z(\text{Fix}\mathcal{D}^+)$. Relaxation all the way to Z_R requires either $\text{Fix}\mathcal{D}^+ = Z_R$, as in a purely dephasing pointer model, or an additional centre-pinning/pinching mechanism acting inside the fixed-point algebra. The present manuscript uses Z_R only as the canonical commutative algebra of classical record labels, not as a consequence of the spectral decay of non-central fixed observables.

5. POVM Probabilities and Completely Positive Instruments

The record algebra identifies possible classical alternatives. A probability rule assigns numbers to those alternatives, and an instrument specifies the state update associated with each realised alternative.

5.1. POVM Representation

Theorem 3 (POVM representation). Let $O = \{1, \dots, m\}$ be a finite outcome set and let $p : S(\mathcal{H}_R) \rightarrow \Delta_m$ be a normal affine map from the state space to the probability simplex. Then there is a unique POVM $\{E_k\}_{k=1}^m$, with $E_k \geq 0$ and $\sum_k E_k = \mathbf{1}$, such that

$$p_k(\rho) = \text{Tr}[\rho E_k] \quad \text{for all } \rho \in S(\mathcal{H}_R). \quad (17)$$

Proof. Affinity gives a bounded linear functional on self-adjoint trace-class operators. Riesz representation gives a self-adjoint effect E_k . Positivity on states gives $E_k \geq 0$, and normalisation gives $\sum_k E_k = \mathbf{1}$ [8,9]. $\square$

Remark 5. Theorem 3 represents an outcome assignment already assumed to be a normal affine probability rule. It does not prove that the Born rule follows from unitary dynamics alone.

5.2. What a CP Instrument Means Operationally

A CP instrument is the standard mathematical object that joins probabilities and state updates. For each outcome k , the map $\mathcal{I}_k$ sends an input state ρ to an unnormalised positive state $\mathcal{I}_k(\rho)$. Its trace is the probability of that outcome,

$$p_k(\rho) = \text{Tr}[\mathcal{I}_k(\rho)], \quad (18)$$

and, when $p_k(\rho) > 0$, the conditional post-event state is

$$\rho_k' = \frac{\mathcal{I}_k(\rho)}{\text{Tr}[\mathcal{I}_k(\rho)]}. \quad (19)$$

Complete positivity is required because the detector may be entangled with an unobserved ancilla; positivity must be preserved after tensoring with arbitrary ancillary systems. Trace preservation of $\sum_k \mathcal{I}_k$ says that the non-selective measurement loses no total probability [10,11].

Theorem 4 (CP instrument representation). Let $R = \mathcal{B}(\mathcal{H}_R)$ and let $\{\mathcal{I}_k\}_{k=1}^m$ be completely positive maps on trace-class operators such that $\sum_k \mathcal{I}_k$ is trace-preserving. Define

$$E_k := \mathcal{I}_k^\dagger(\mathbf{1}). \quad (20)$$

Then $E_k \geq 0$, $\sum_k E_k = \mathbf{1}$, and

$$\text{Tr}[\mathcal{I}_k(\rho)] = \text{Tr}[\rho E_k]. \quad (21)$$

Moreover, there are an ancilla K , a unit vector $|\omega\rangle \in K$, a unitary U on $\mathcal{H}_R \otimes K$, and mutually orthogonal projections $Q_k \in \mathcal{B}(K)$ such that

$$\mathcal{I}_k(\rho) = \text{Tr}_K[(\mathbf{1} \otimes Q_k)U(\rho \otimes |\omega\rangle\langle\omega|)U^\dagger(\mathbf{1} \otimes Q_k)]. \quad (22)$$

Proof. Complete positivity of $\mathcal{I}_k$ makes the adjoint $\mathcal{I}_k^\dagger$ completely positive, so $E_k = \mathcal{I}_k^\dagger(\mathbf{1}) \geq 0$. For any state ρ , $\text{Tr}[\mathcal{I}_k(\rho)] = \text{Tr}[\rho \mathcal{I}_k^\dagger(\mathbf{1})] = \text{Tr}[\rho E_k]$, which is (5.5). Trace preservation of $\sum_k \mathcal{I}_k$ gives $\text{Tr}[\rho \sum_k E_k] = \text{Tr} \rho$ for all ρ , hence $\sum_k E_k = \mathbf{1}$. For the dilation, each CP map has a Kraus form $\mathcal{I}_k(\rho) = \sum_j A_{kj} \rho A_{kj}^\dagger$, and trace preservation of the total map gives $\sum_{k,j} A_{kj}^\dagger A_{kj} = \mathbf{1}$. Define an isometry by $V|\psi\rangle = \sum_{k,j} A_{kj} |\psi\rangle \otimes |k,j\rangle$, extend it to a unitary U on a larger space if necessary, and take $Q_k = \sum_j |k,j\rangle\langle k,j|$. This gives (22) [12,32]. $\square$

Operational reading. The POVM gives the probabilities of outcomes, while the CP instrument gives the corresponding conditional post-event states. The theorem states that this operational description has a standard unitary dilation with an ancilla.

Remark 6 (Lüders update). A Lüders projection is not forced by sharpness alone in degenerate sectors. It is obtained after adding the minimal-disturbance assumption; otherwise, block unitaries yield other repeatable instruments.

5.3. Petz Recovery

For a channel Φ and faithful reference state σ , the Petz map is defined on the support of $\Phi(\sigma)$ and recovers the reference state [33,34]. Exact recovery of another state ρ is equivalent to equality in the corresponding data-processing inequality. Without equality, one uses approximate recovery.

Proposition 4 (Approximate recovery bound). *For a CPTP map Φ and faithful σ , there is a rotated Petz recovery map $\mathcal{R}$ such that*

$$F(\rho, \mathcal{R}(\Phi(\rho)))^2 \geq \exp[-(D(\rho\|\sigma) - D(\Phi\rho\|\Phi\sigma))], \quad (23)$$

where F is fidelity [35,36]. Equality of the two relative entropies is the sufficiency case and gives exact recovery.

6. Conditional Modular Retraction

This section is the only place where von Neumann algebraic modular theory is used. The inclusion is assumed, not constructed.

Modular Theory in One Paragraph

Tomita–Takesaki theory assigns to a von Neumann algebra M and a faithful normal state φ : a one-parameter modular automorphism group σ_t^φ . In a finite matrix algebra, if h_φ is the positive density matrix representing φ , then $\sigma_t^\varphi(X) = h_\varphi^{it} X h_\varphi^{-it}$. In type-III algebras, there is no density-matrix trace calculus. The modular automorphism group therefore replaces the finite-dimensional logarithm and trace formulae as the natural operator-algebraic structure. Takesaki’s conditional-expectation theorem states that a φ -preserving normal conditional expectation from M onto a von Neumann subalgebra $N \subset M$ exists precisely when N is globally invariant under σ_t^φ [22,23,37,38].

Assumption 3 (Modular-invariant inclusion). *Let $\iota : N \rightarrow M$ be a normal unital $*$ -monomorphism between von Neumann algebras. Let φ be a faithful normal state on M , set $\psi = \varphi \circ \iota$, and assume $\iota(N)$ is globally invariant under σ_t^φ .*

Theorem 5 (Conditional modular retraction). *Under Assumption 3, there exists a unique φ -preserving normal faithful conditional expectation $E : M \rightarrow \iota(N)$. The map*

$$R = \iota^{-1} \circ E : M \rightarrow N \quad (24)$$

is normal, unital and completely positive, and satisfies $R \circ \iota = \text{id}_N$. Hence, N is a CP retract of M .

Proof. Takesaki’s theorem gives the conditional expectation exactly from the assumed modular invariance [22,37]. Composing with the $*$ -isomorphism ι^{-1} on the image yields a normal unital CP map. Since E fixes $\iota(N)$, $R \circ \iota = \text{id}_N$. $\square$

Remark 7. *Theorem 5 does not construct the inclusion and does not assert a unitary equivalence between M and N . It states only that, once the inclusion and modular invariance are assumed, the CP retraction follows.*

Operational reading. The modular result is conditional. If a state-preserving modular-invariant inclusion is supplied, Takesaki’s theorem gives the conditional expectation and hence the CP retraction. The theorem does not construct the black-hole inclusion.

Proposition 5 (State transport and Araki relative entropy). *For a normal state ρ on N , define its extension to M by $\tilde{\rho} = \rho \circ R$. Then, for normal states ρ_1, ρ_2 on N ,*

$$S_M(\tilde{\rho}_1 \| \tilde{\rho}_2) = S_N(\rho_1 \| \rho_2), \quad (25)$$

where S_M and S_N denote Araki relative entropy on the respective von Neumann algebras.

Proof. Because $R \circ \iota = \text{id}_N$, the extension followed by restriction along ι returns the original state. Monotonicity of Araki relative entropy under the unital normal completely positive maps ι and R gives both inequalities,

$$S_N(\rho_1 \| \rho_2) \leq S_M(\tilde{\rho}_1 \| \tilde{\rho}_2) \leq S_N(\rho_1 \| \rho_2),$$

and hence equality [33,34,39]. $\square$

7. Conditional QEC Record-Stability Bound

Quantum error correction is used here only as a record-stability estimate under a stated noise model. It does not construct the modular inclusion of Assumption 3.

Assumption 4 (Per-cycle logical-error hypothesis). *A code family of distance d has per-cycle logical-failure probability:*

$$p_L(d, p) \leq Ae^{-a(p)d}, \quad p < p_{\text{th}}, \quad d \geq d_0, \quad (26)$$

with constants $A > 0$, threshold p_{th} , and rate $a(p) > 0$ [40,41].

Proposition 6 (Accumulated failure). *Under Assumption 4, for $N = \lceil T/\tau_{\text{cyc}} \rceil$ correction cycles,*

$$P_{\text{fail}}(T) \leq NAe^{-a(p)d}. \quad (27)$$

A target $P_{\text{fail}}(T) \leq \delta$ is guaranteed by

$$d \geq d_\delta := \max\left\{d_0, \left\lceil a(p)^{-1} \log(NA/\delta) \right\rceil\right\}. \quad (28)$$

Proof. Failure over T occurs if at least one of the N cycles fails. The union bound gives (27). Solving $NAe^{-a(p)d} \leq \delta$ gives (28). $\square$

Remark 8. *The logical channel is a temporal composition of single-cycle channels. A diamond-norm statement requires an additional model relating logical-failure events to channel distance; here, the bound is stated only for failure probability.*

Operational reading. The QEC estimate is an accumulated failure-probability bound. It says that, under the assumed per-cycle threshold law, the probability of record failure over N cycles can be made smaller than a target δ by increasing the code distance.

8. Worked Finite-Dimensional Example and Detector Protocol

This section gives a reproducible finite-dimensional illustration of the timing distribution and the record algebra. The scalar flux profile below is used as a calibrated detector signal for the timing calculation; it is not an empirical validation claim and is not used to infer a new physical law.

Let $\mathcal{H}_R = \mathbb{C}^2$. For the record-algebra part, take a dephasing detector whose fixed observables are diagonal in the basis $\{|0\rangle, |1\rangle\}$. Then

$$Z_R = \text{span}\{|0\rangle\langle 0|, |1\rangle\langle 1|\} \quad (29)$$

is the commutative pointer algebra.

For the timing part, take the calibrated dimensionless flux signal

$$j(x) = 0.6 \operatorname{sech}(x - 2), \quad I = [0, 4], \quad \theta = 0.05. \quad (30)$$

Then $x^* = 2$ and $j''(x^*) = -0.6$. The leading Laplace prediction is

$$\operatorname{Var}(X_\theta) \sim \frac{\theta}{|j''(x^*)|} = \frac{0.05}{0.6} = 0.0833. \quad (31)$$

Direct deterministic quadrature of

$$f_\theta(x) = Z_\theta^{-1} \mathbf{1}_{[0,4]}(x) \exp(j(x)/\theta) \quad (32)$$

gives mean 2.000 and variance 0.109. The difference from the leading value in (32) is a finite-noise and finite-window effect and is consistent with the asymptotic correction term in Theorem 1.

For the dephasing generator, Theorem 2 identifies Z_R in (29) as the diagonal pointer algebra, and the two-outcome POVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ reproduces the diagonal Born statistics through (17).

In a laboratory implementation, the calibrated signal $j(x)$ would not be assumed. It would be obtained by reconstructing $\rho_R(t)$, choosing a faithful calibrated reference σ , computing $J_{\det}(t)$ from (5), nondimensionalising it by (7), and then applying (8). The example above tests the timing estimator after that calibrated scalar signal has been obtained. The model is rejected if the measured timing variance falls outside the pre-registered uncertainty interval obtained from the Laplace variance formula in Theorem 1, with calibration-induced shifts in the maximiser bounded by Proposition 2.

9. Discussion and Limitations

9.1. Relation to Existing Interpretations

The framework is compatible with, but does not subsume, established approaches. Decoherence and einselection explain suppression of off-diagonal coherence and emergence of a preferred basis [4,5]; Section 4 identifies a compatible pointer algebra under explicit fixed-point hypotheses. Quantum Darwinism concerns redundant environmental records [6]; the present paper treats a single detector record. Consistent histories provide a probability calculus for histories [42–44]; here, the timing estimator is a single-event detector object. Objective-collapse models modify dynamics [45–48]; this framework adds no collapse dynamics. Relational and QBist interpretations reinterpret the state [49,50]; the representation theorems here are interpretation-neutral.

9.2. Gravitational and Modular Scope

The modular theorem is a conditional recovery statement, not a construction of a black-hole inclusion. A physically meaningful BHIP application would require an independently justified state-preserving modular-invariant inclusion of an exterior radiation algebra into an appropriate detector or recovery algebra. The longer *International Journal of Quantum Foundations* BHIP article discusses broader recoverability architecture and observational motivation [18]; the present paper imports none of its stronger sectorial closure claims.

9.3. Quantum Speed Limits

The variance of the Gibbs–Laplace estimator is a property of the detector-time distribution, not a Mandelstam–Tamm or Margolus–Levitin evolution time [51–53]. Establishing a dynamical relation between timing curvature and energy variance would require an additional theorem and is not claimed.

10. Conclusions

We have presented a compact architecture-conditional framework for four operational tasks in the quantum measurement problem: computing a detector-side relative-entropy flux, assigning an event-time distribution, identifying stable record observables, and representing outcome probabilities and conditional updates. In finite dimensions, the flux formula follows from the exact Fréchet derivative of the matrix logarithm. The Gibbs–Laplace event-time density concentrates at an explicitly assumed isolated non-degenerate maximum. Under detailed-balance and fixed-point hypotheses, the detector dynamics selects a fixed-point algebra whose centre supplies canonical commutative record labels. Normal affine outcome assignments give POVMs, and CP instruments give the corresponding conditional post-event states.

The modular and QEC parts are separate conditional additions. If a state-preserving modular-invariant inclusion of von Neumann algebras is supplied, Takesaki’s theorem yields a normal completely positive retraction. If a code family satisfies the stated per-cycle logical-error hypothesis, the accumulated record-failure probability is bounded by a union bound and can be made smaller than a prescribed target by increasing the code distance. These statements do not derive unique outcomes from unitary dynamics alone, do not construct a black-hole algebra inclusion, and do not resolve the black-hole information problem. They provide a self-contained conditional framework and a reproducible timing illustration, with direct experimental validation left as an open task.

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