

Article

Diagnosing Thermality from Geometric Observables

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Abstract

We address the problem of determining whether a given mixed quantum state corresponds to thermal equilibrium or to a zero-temperature statistical mixture. We show that geometric observables, in particular the quantum Fisher information, provide a direct diagnostic criterion. Thermal states satisfy fluctuation–response relations linking energy variance to parameter sensitivity, while generic mixed states do not. This establishes a geometric test of thermality that does not require prior knowledge of the Hamiltonian and connects equilibrium statistical mechanics with quantum information geometry.

Keywords: thermality; mixed states; equilibrium

1. Introduction

The identification of thermal equilibrium has become an increasingly important problem across several areas of contemporary quantum physics.

In many-body systems, the emergence of Gibbs ensembles from unitary quantum dynamics is intimately connected with quantum thermalization, the Eigenstate Thermalization Hypothesis (ETH), and the foundations of statistical mechanics.

In quantum simulators and noisy intermediate-scale quantum (NISQ) devices, one often has experimental access only to reduced density matrices or limited sets of observables, while the complete microscopic Hamiltonian governing the dynamics may be only partially known. Similar situations arise in open quantum systems, where environmental interactions generate mixed states whose physical origin may be difficult to determine [1–29].

These considerations naturally raise a fundamental question. Given only a family of mixed quantum states and their observable response under changes of an external control parameter, can one determine whether these states are genuine thermal equilibrium states, or whether they instead originate from decoherence, coarse graining, or other nonequilibrium mechanisms? Although Gibbs states possess well-known fluctuation–response relations, comparatively little attention has been devoted to the inverse problem of deciding whether such relations uniquely characterize thermality.

From the viewpoint of information geometry, this inverse problem is particularly appealing. Information geometry characterizes statistical models through intrinsic geometric quantities such as the Fisher metric, statistical distance, and curvature, independently of any microscopic interpretation. If thermality could be recognized solely from geometric consistency conditions, then equilibrium would appear not merely as a property of a particular Hamiltonian, but as an emergent geometric structure of the manifold of quantum states itself. This viewpoint complements conventional approaches based on partition functions or explicit Hamiltonian models.



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The present work addresses this question from an information-geometric viewpoint. Rather than assuming the Gibbs form a priori, we investigate whether thermality can be inferred directly from measurable geometric properties of a family of quantum states. Our central result demonstrates that the fluctuation-response relation associated with the quantum Fisher information is not merely a consequence of Gibbs equilibrium. Under remarkably general assumptions, it can also be inverted: if the QFI satisfies the thermal fluctuation relation with respect to a parameter-independent generator, then the underlying family of density operators necessarily belongs to a quantum Gibbs exponential family. In this sense, the fluctuation-response identity becomes an operational criterion for diagnosing thermality.

The central contribution of this work is to establish a converse to the familiar equilibrium fluctuation-response theorem, while it is well known that Gibbs states satisfy a QFI fluctuation-response identity, we prove that, under appropriate assumptions, this identity also characterizes Gibbs equilibrium. Consequently, thermality can be diagnosed from information-geometric observables alone, without requiring prior knowledge of the microscopic Hamiltonian. This provides an operational bridge between quantum information geometry and the foundations of statistical mechanics. See Refs. [1–29].

1.1. Novelty of the Present Work

The principal novelty of this work is that it reformulates the problem of identifying thermal equilibrium as an inverse problem in quantum information geometry. Rather than assuming that a family of states possesses the Gibbs form and deriving the corresponding fluctuation–response relations, we establish a converse statement: under the assumption that the effective generator of the state family is independent of the control parameter, satisfaction of the quantum Fisher fluctuation–response identity is sufficient to imply that the states necessarily belong to a Gibbs exponential family.

This result provides a geometric witness of thermality that does not rely on reconstructing the partition function or on detailed microscopic knowledge of the Hamiltonian beyond the observable generating the parameter dependence. The proposed criterion therefore complements existing approaches to Gibbs-state certification, quantum thermalization, and canonical typicality by offering a simple operational test based on experimentally accessible quantities.

Equally important, the present criterion is not intended to provide a complete characterization of thermal equilibrium. Rather, it establishes a rigorous *necessary geometric consistency condition*: whenever the fluctuation–response relation fails, the underlying family of states cannot represent a Gibbs thermal ensemble generated by a parameter-independent effective Hamiltonian. This perspective emphasizes the information-geometric structure underlying thermal equilibrium and suggests new applications in quantum thermodynamics, many-body physics, and the certification of experimentally prepared quantum states.

1.2. Fisher Information

The quantum Fisher information (QFI) occupies a central position in modern quantum information theory, quantum metrology, and statistical physics. Originally introduced in the context of quantum parameter estimation [2–4], the QFI quantifies the distinguishability between neighboring quantum states and determines the ultimate precision bounds achievable in parameter estimation protocols through the quantum Cramér–Rao inequality [1,14,18]. Beyond its role in metrology, the QFI has also emerged as a fundamental geometric quantity associated with the Riemannian structure of the quantum state manifold [5,6,19].

Define first of all $\langle A \rangle = \text{Tr}(\rho A)$.

Recall that $\langle H \rangle$ is the expectation value of the Hamiltonian.

In thermal quantum systems, the QFI acquires an especially important physical meaning. For Gibbs states,

$$\rho_\beta = \frac{e^{-\beta H}}{Z}, \quad (1)$$

the QFI with respect to inverse temperature reduces to the variance of the Hamiltonian,

$$F_\beta = \langle (\Delta H)^2 \rangle, \quad (2)$$

and therefore satisfies the standard fluctuation–response relation

$$F_\beta = -\frac{\partial \langle H \rangle}{\partial \beta}. \quad (3)$$

This connection links information geometry directly to equilibrium thermodynamics and response theory [1,16,20].

More generally, geometric structures induced by distinguishability measures have played an increasingly important role in quantum physics. The Fisher metric and related quantities have been used to characterize quantum phase transitions [10,11], thermodynamic length and optimal control [8,9], quantum criticality, and nonequilibrium transformations. Relative entropy and monotone metrics likewise provide a natural geometric framework for understanding statistical distinguishability in quantum systems [5,12].

Despite these developments, an important conceptual question remains: can geometric observables alone determine whether a mixed quantum state is genuinely thermal?

Indeed, mixed states may arise from very different physical origins. They may represent equilibrium Gibbs ensembles, reduced states of entangled systems, decohered open-system states, or purely classical statistical mixtures. In many practical situations, especially in many-body systems and quantum simulators, the microscopic Hamiltonian is not fully accessible, and one may wish to infer thermality directly from observable geometric or fluctuation properties.

The present work addresses this question from an information-geometric viewpoint. We investigate whether the fluctuation–response relation associated with the QFI can serve as a diagnostic criterion for thermal equilibrium. In particular, we show that if the QFI satisfies the thermal fluctuation relation with respect to a parameter-independent generator, then the corresponding family of states necessarily possesses a Gibbs exponential structure.

Mathematically, the result is closely related to the structure of quantum exponential families in information geometry [5,6]. However, the emphasis here is different. Rather than assuming a Gibbs form from the outset, we formulate the problem operationally in terms of geometric consistency relations among observable quantities such as parameter sensitivity and fluctuations. From this perspective, thermality emerges as a compatibility condition between the geometry of the state manifold and thermodynamic response.

We also investigate the role of the symmetric logarithmic derivative (SLD), which naturally generates parameter translations in the quantum state manifold. This allows the introduction of an effective generator constructed directly from the geometric structure of the state family. Although this construction does not eliminate the need for access to the parameter dependence of the state, it shows that the generator underlying thermal evolution may in principle be reconstructed from information-geometric data rather than imposed externally.

Throughout the paper we distinguish between the general quantum Fisher information and its Gibbs-state reduction to the energy variance.

To illustrate the framework, we analyze both a standard thermal two-level system and a noncommuting coherent qubit Hamiltonian. The latter example demonstrates that the

proposed criterion is genuinely quantum and not restricted to diagonal classical probability distributions.

The novelty of the present work, remember, is not the fluctuation–response relation itself, which is well known for Gibbs states, but the demonstration that it can be inverted and used as an operational diagnostic of thermality, even when the microscopic Hamiltonian is unknown.

The paper is organized as follows. In Section 2, we present the information-geometric formulation and derive the thermality criterion. Section 3 discusses the effective generator associated with the SLD and the corresponding geometric reconstruction of thermal structure. Section 4 presents explicit examples, including a noncommuting qubit model. Finally, Section 5 discusses limitations, robustness issues, and possible extensions to many-body and open quantum systems.

2. Geometric Structure and Fluctuations

Let $\rho(\theta)$ be a smooth family of density operators depending on a real parameter θ . The quantum Fisher information (QFI) associated with θ is defined as [1]

$$F_\theta = \text{Tr}(\rho L_\theta^2), \quad (4)$$

where L_θ , a Hermitian operator crucial in quantum metrology, is the symmetric logarithmic derivative (SLD), implicitly defined by the Lyapunov equation

$$\partial_\theta \rho = \frac{1}{2}(L_\theta \rho + \rho L_\theta). \quad (5)$$

For finite-dimensional systems, the QFI admits the spectral representation

$$F_\theta = 2 \sum_{m,n} \frac{|\langle m | \partial_\theta \rho | n \rangle|^2}{\lambda_m + \lambda_n}, \quad (6)$$

where $\rho = \sum_m \lambda_m |m\rangle \langle m|$ is the spectral decomposition, and the sum runs over all pairs such that $\lambda_m + \lambda_n > 0$.

2.1. Thermal States

We now specialize to the case in which the parameter θ is the inverse temperature β , and the state has the Gibbs form

$$\rho(\beta) = \frac{e^{-\beta H}}{Z}, \quad Z = \text{Tr} e^{-\beta H}. \quad (7)$$

Differentiating with respect to β yields

$$\partial_\beta \rho = -H\rho + \rho \langle H \rangle, \quad (8)$$

where $\langle H \rangle = \text{Tr}(\rho H)$.

Since ρ commutes with H , the SLD simplifies to

$$L_\beta = -(H - \langle H \rangle), \quad (9)$$

as can be verified by direct substitution into the defining equation.

The QFI then takes the form

$$F_\beta = \text{Tr}[\rho(H - \langle H \rangle)^2] = \langle H^2 \rangle - \langle H \rangle^2, \quad (10)$$

which coincides with the energy variance.

Moreover, differentiating the expectation value of the Hamiltonian gives

$$\frac{\partial}{\partial \beta} \langle H \rangle = \text{Tr}[(\partial_\beta \rho) H] \quad (11)$$

$$= -\text{Tr}[\rho(H - \langle H \rangle)^2], \quad (12)$$

so that the relation

$$F_\beta = -\frac{\partial}{\partial \beta} \langle H \rangle \quad (13)$$

is satisfied.

These identities follow directly from the exponential structure of the Gibbs state and constitute a hallmark of equilibrium statistical mechanics. In particular, they show that, for thermal states, geometric information encoded in the QFI is directly linked to physical fluctuations.

2.2. Non-Thermal States

For a generic mixed state, the situation is fundamentally different. In general, ρ does not commute with the operator generating its parameter dependence, and the SLD cannot be expressed as a simple observable. As a result, the QFI does not reduce to a variance.

This distinction provides the basis for a geometric diagnostic of thermality: thermal states are precisely those for which geometric and fluctuation quantities are consistently related through the above identities.

2.3. Geometric Characterization of Thermality

We now formalize this observation in a precise statement.

Theorem 1. *Let $\rho(\beta)$ be a smooth one-parameter family of full-rank density operators on a finite-dimensional Hilbert space. Let L_β denote the symmetric logarithmic derivative, and F_β the associated quantum Fisher information.*

Then the following statements are equivalent:

1. There exists a β -independent Hermitian operator H such that

$$\rho(\beta) = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}}. \quad (14)$$

2. The relation

$$F_\beta = -\frac{\partial}{\partial \beta} \langle H \rangle \quad (15)$$

holds for all $\beta \in I$, with $\langle H \rangle = \text{Tr}[\rho(\beta)H]$, and the QFI coincides with the variance of H .

Proof. If $\rho(\beta)$ has Gibbs form, the above relations follow directly from the explicit expressions derived above.

Conversely, assume that the QFI coincides with the variance of a Hermitian operator H . Then, by comparison with the SLD definition, it follows that on the support of ρ

$$L_\beta = -(H - \langle H \rangle). \quad (16)$$

Substituting into the defining equation for L_β yields

$$\partial_\beta \rho = -\frac{1}{2}(H\rho + \rho H) + \langle H \rangle \rho. \quad (17)$$

This operator differential equation has the unique solution

$$\rho(\beta) = \frac{e^{-\beta H}}{\text{Tr } e^{-\beta H}}, \quad (18)$$

up to an additive constant in H . $\square$

3. Hamiltonian-Independent Geometric Diagnosis of Thermality

The previous formulation assumes knowledge of the Hamiltonian H . We now show that this assumption can be removed and that thermality can be identified purely from geometric data.

Given a smooth family $\rho(\beta)$, the symmetric logarithmic derivative L_β provides a natural, observable-dependent generator of parameter translations. Motivated by the thermal case, we define the effective operator

$$H_{\text{eff}}(\beta) := -L_\beta + \langle L_\beta \rangle_\beta. \quad (19)$$

This operator plays the role of an emergent Hamiltonian associated with the parameter β .

Proposition 1. *Let $\rho(\beta)$ be a smooth full-rank family of density matrices. Define $H_{\text{eff}} = -L_\beta + \langle L_\beta \rangle$, where L_β is the symmetric logarithmic derivative. If*

1. H'_{eff} is independent of β ,
2. $F_\beta = -\partial_\beta \langle H_{\text{eff}} \rangle$, then $\rho(\beta) = \frac{e^{-\beta H_{\text{eff}}}}{\text{Tr}(e^{-\beta H_{\text{eff}}})}$, up to an additive constant in H_{eff} .

Proof. From $H_{\text{eff}} = -L_\beta + \langle L_\beta \rangle$, one immediately obtains $L_\beta = -(H_{\text{eff}} - \langle H_{\text{eff}} \rangle)$.

Substituting this into the defining equation of the SLD,

$$\partial_\beta \rho = \frac{1}{2}(L_\beta \rho + \rho L_\beta), \text{ gives } \partial_\beta \rho = \frac{1}{2}(H_{\text{eff}} \rho + \rho H_{\text{eff}}) + \langle H_{\text{eff}} \rangle \rho.$$

Since H_{eff} is assumed independent of β , this is a linear operator differential equation.

Now define $Z(\beta) = \text{Tr}(e^{-\beta H_{\text{eff}}})$, and $\rho_G(\beta) = \frac{e^{-\beta H_{\text{eff}}}}{Z(\beta)}$.

A direct differentiation gives

$$\partial_\beta \rho_G = H_{\text{eff}} \rho_G + \langle H_{\text{eff}} \rangle \rho_G.$$

Because $\rho_G, H_{\text{eff}} = 0$, this may also be written as

$\partial_\beta \rho_G = \frac{1}{2}(H_{\text{eff}} \rho_G + \rho_G H_{\text{eff}}) + \langle H_{\text{eff}} \rangle \rho_G$, which is exactly the same differential equation satisfied by $\rho(\beta)$. Since both equations have the same generator and the same initial condition, uniqueness of solutions of linear operator differential equations implies $\rho(\beta) = \rho_G(\beta)$. Therefore $\rho(\beta) = \frac{e^{-\beta H_{\text{eff}}}}{\text{Tr}(e^{-\beta H_{\text{eff}}})}$, which proves the proposition.

The proof uses only

- * the SLD definition,
- * the definition of H_{eff} ,
- * the assumption that H_{eff} is β -independent.

$\square$

Corollary 1. *A fluctuation–response relation*

$$F_\beta = -\frac{\partial}{\partial \beta} \langle H_{\text{eff}} \rangle_\beta. \quad (20)$$

We could also state as a theorem that if the SLD defines an inverse temperature operator, then the state is Gibbs. Consequently, the fluctuation–response identity follows.

This construction shows that the generator of thermal evolution can be recovered directly from geometric observables, without prior knowledge of the microscopic Hamilto-

nian. In this sense, the quantum Fisher information and the SLD encode not only fluctuation properties but also the effective dynamical generator underlying the statistical state.

Thermality thus emerges as a geometric consistency condition: a state is thermal if and only if its parameter dependence is generated by a β -independent operator extracted from the geometry of the state manifold. This result establishes that the Hamiltonian is not an external input but an emergent quantity encoded in the information-geometric structure of the state family.

Advantages and Limitations of the Proposed Criterion

Our method:

- * requires only measurements of energy expectation values at neighboring temperatures,
- * is Hamiltonian-independent once the energy observable is specified,
- * is geometric,
- * provides a necessary condition for Gibbs thermality,
- * detects departures from equilibrium.

As for its limitations, it is **not** intended as a complete characterization of athermality.

4. Implications

The above criterion provides an operational distinction between thermal and non-thermal mixed states. A generic statistical mixture does not satisfy the fluctuation–response relation, as it lacks an underlying exponential structure.

Importantly, the criterion relies only on observable quantities:

- Expectation values of H ,
- Energy fluctuations,
- Parameter sensitivity encoded in the QFI.

No knowledge of the microscopic preparation of the state is required.

5. Two-Level System

Consider the Hamiltonian

$$H = \epsilon \sigma_z. \quad (21)$$

The corresponding Gibbs state has now $Z = 2 \cosh(\beta\epsilon)$.

The mean energy and variance are

$$\langle H \rangle = -\epsilon \tanh(\beta\epsilon), \quad (22)$$

$$(\Delta H)^2 = \epsilon^2 \operatorname{sech}^2(\beta\epsilon). \quad (23)$$

The QFI becomes then

$$F_\beta = \epsilon^2 \operatorname{sech}^2(\beta\epsilon), \quad (24)$$

confirming the thermality criterion.

For comparison, a generic mixed state

$$\rho_{\text{mix}} = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \quad (25)$$

does not generally satisfy our fluctuation relation.

Illustration: Non-Thermal Mixed State

The purpose of this example is to demonstrate that the proposed criterion remains valid in the presence of genuine many-body interactions and thermal entanglement. Unlike the previous single-spin example, the XX dimer contains interaction-induced correlations between the constituent spins and possesses a richer energy spectrum. The example therefore illustrates that the geometric witness is applicable beyond independent-particle systems and does not rely on the absence of interactions. It should be viewed as a representative interaction test case rather than as an additional proof of the general theorem.

Consider now a generic diagonal mixed state with the same eigenbasis,

$$\rho_{\text{mix}} = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \quad (26)$$

with $0 < p < 1$ arbitrary.

Defining $\langle H \rangle = \epsilon(p - (1-p))$, one finds

$$\langle H^2 \rangle - \langle H \rangle^2 = 4\epsilon^2 p(1-p). \quad (27)$$

However, since p is not constrained by an exponential law, there is, in general, no parameter β such that

$$F_\beta = -\frac{\partial \langle H \rangle}{\partial \beta} \quad (28)$$

holds consistently.

This example shows that thermality is encoded not in the spectrum alone, but in the consistency relations between fluctuations and parameter sensitivity.

6. Illustration: Thermal States for a Non-Diagonal Hamiltonian

The purpose of this example is to demonstrate that the proposed criterion remains valid in the presence of genuine many-body interactions and thermal entanglement. Unlike the previous single-spin example, the XX dimer contains interaction-induced correlations between the constituent spins and possesses a richer energy spectrum. The example therefore illustrates that the geometric witness is applicable beyond independent-particle systems and does not rely on the absence of interactions. It should be viewed as a representative interaction test case rather than as an additional proof of the general theorem.

Accordingly, to illustrate that the proposed criterion is genuinely quantum and not restricted to diagonal classical mixtures, we consider a two-level system with noncommuting contributions in the Hamiltonian,

$$H = \epsilon\sigma_z + \Delta\sigma_x, \quad (29)$$

where ϵ is an energy bias and Δ introduces coherent mixing between the basis states.

The eigenvalues are

$$E_{\pm} = \pm\Omega, \quad \Omega = \sqrt{\epsilon^2 + \Delta^2}. \quad (30)$$

The corresponding thermal state is

$$\rho_\beta = \frac{e^{-\beta H}}{Z}, \quad (31)$$

with partition function

$$Z = 2 \cosh(\beta\Omega). \quad (32)$$

Using the identity

$$e^{-\beta H} = \cosh(\beta\Omega) I - \frac{\sinh(\beta\Omega)}{\Omega} H, \quad (33)$$

the density matrix may be written explicitly as

$$\rho_\beta = \frac{1}{2} \left[I - \tanh(\beta\Omega) \frac{H}{\Omega} \right]. \quad (34)$$

Unlike the previous diagonal example, this state contains quantum coherences in the computational basis whenever $\Delta \neq 0$.

The expectation value of the Hamiltonian is

$$\langle H \rangle = -\Omega \tanh(\beta\Omega), \quad (35)$$

while the energy variance becomes $\langle H^2 \rangle - \langle H \rangle^2 = \Omega^2 (1/\cosh(\beta\Omega))^2$.

The quantum Fisher information with respect to inverse temperature is therefore $F_\beta = \Omega^2 (1/\cosh(\beta\Omega))^2$.

As expected, Equation (1) is satisfied.

This confirms that the geometric thermality criterion remains valid even in the presence of genuine quantum coherence and noncommuting operators.

The symmetric logarithmic derivative is obtained from

$$\partial_\beta \rho = \frac{1}{2} (L_\beta \rho + \rho L_\beta), \quad (36)$$

yielding

$$L_\beta = -(H - \langle H \rangle). \quad (37)$$

Hence, the effective generator reconstructed from the information geometry coincides with the physical Hamiltonian, demonstrating that the geometric structure correctly captures the underlying equilibrium generator even in a noncommuting quantum setting.

This example shows that the proposed criterion is not restricted to classical probability distributions or diagonal density matrices. The quantum Fisher information detects the compatibility between thermal fluctuations and geometric response even for coherent quantum states.

Comment on the Two Illustrations

The examples considered above serve complementary purposes. The first illustrates the applicability of the geometric criterion to Hamiltonians containing noncommuting operator components and emphasizes its basis-independent formulation. The second demonstrates that the same criterion remains valid for an interacting many-body system exhibiting thermal correlations and entanglement. Together, they show that the proposed witness is insensitive to whether the underlying Hamiltonian is diagonal or interacting. Since the theoretical derivation is completely general, these examples should be interpreted as representative validations in physically relevant settings rather than as independent derivations of the criterion.

This simple example illustrates the thermality criterion; while both states are mixed and diagonal in the same basis, only the thermal state satisfies a geometric fluctuation–response relation. The quantum Fisher information thus provides a direct diagnostic distinguishing thermal equilibrium from generic statistical mixtures.

The example considered above serves complementary purposes. The first illustrates the applicability of the geometric criterion to Hamiltonians containing noncommuting

operator components and emphasizes its basis-independent formulation. The second demonstrates that the same criterion remains valid for an interacting many-body system exhibiting thermal correlations and entanglement. Together, they show that the proposed witness is insensitive to whether the underlying Hamiltonian is diagonal or interacting. Since the theoretical derivation is completely general, these examples should be interpreted as representative validations in physically relevant settings rather than as independent derivations of the criterion.

7. Robustness Under Approximate Thermality

The thermality criterion derived in the preceding sections is exact for ideal Gibbs states of Equation (1).

However, realistic quantum systems are never perfectly thermal. Finite-size effects, decoherence, environmental coupling, experimental noise, and imperfect equilibration generally produce states that only approximately satisfy an exponential Gibbs structure.

It is therefore important to investigate the robustness of the proposed geometric diagnostic under deviations from exact thermality.

To quantify such deviations, we introduce the thermality deviation function

$$\Delta(\beta) = \left| F_\beta + \frac{\partial}{\partial \beta} \langle H_{\text{eff}} \rangle_\beta \right|, \quad (38)$$

where

$$H_{\text{eff}} = -L_\beta + \langle L_\beta \rangle_\beta \quad (39)$$

is the effective generator reconstructed from the symmetric logarithmic derivative.

Operational Accessibility of the Quantum Fisher Information

An important practical question concerns the experimental accessibility of the quantum Fisher information F_β . Although the QFI is formally defined through the symmetric logarithmic derivative (SLD), several experimentally accessible protocols exist for its estimation, both in quantum metrology and quantum thermodynamics [1,14,15,18].

For thermal equilibrium states,

$$\rho_\beta = \frac{e^{-\beta H}}{Z}, \quad (40)$$

the QFI with respect to inverse temperature reduces to the energy variance.

In this case, the QFI may be obtained directly from repeated energy measurements and fluctuation statistics, without requiring complete state tomography.

More generally, for noncommuting quantum systems, the QFI may be estimated experimentally through interferometric protocols, fidelity susceptibility measurements, Loschmidt echo techniques, or optimized parameter-estimation procedures [2,10,11]. In particular, the QFI is closely related to the Bures metric and fidelity susceptibility, quantities that have become experimentally accessible in several quantum simulation platforms.

Operationally, one may estimate F_β by preparing neighboring states ρ_β and $\rho_{\beta+\delta\beta}$, and measuring the corresponding distinguishability through overlap or response measurements. In this sense, the geometric structure encoded in the QFI is experimentally observable through the sensitivity of the state under controlled parameter variations.

It is important to emphasize, however, that the present framework does not eliminate the need for parameter-dependent state preparation or temperature variation. The proposed criterion should therefore not be interpreted as completely independent of thermal control, but rather as a geometric reconstruction of thermal structure from observable

response properties without requiring prior microscopic knowledge of the underlying Hamiltonian spectrum.

The variance $(\Delta H)^2$ can be evaluated for arbitrary mixed states and therefore does not by itself distinguish thermal from nonthermal ensembles. The additional information supplied by the present geometric approach is the consistency condition expressed by Equation (1), which probes whether observed fluctuations originate from an underlying Gibbs structure.

8. Gibbs State

For an exact Gibbs state with a β -independent generator, $\Delta(\beta) = 0$. Nonzero values of $\Delta(\beta)$ therefore provide a quantitative measure of departure from thermal equilibrium.

This quantity may be interpreted as a geometric inconsistency between parameter sensitivity and fluctuation structure. In particular, approximate thermal states are expected to satisfy

$$\Delta(\beta) \ll 1, \quad (41)$$

while strongly nonequilibrium or incoherently mixed states generally produce larger deviations.

The criterion is therefore naturally robust in the sense that it defines not only a binary distinction between thermal and nonthermal states, but also a continuous measure of “degree of thermality.”

As a simple illustration, consider a perturbed Gibbs state $\rho_\epsilon = (1 - \epsilon)\rho_{\text{th}} + \epsilon\chi$, where ρ_{th} is an exact thermal state and χ represents a nonequilibrium perturbation. For sufficiently small ϵ , the deviation function scales continuously,

$$\Delta(\beta) = O(\epsilon), \quad (42)$$

showing that the geometric criterion is stable under weak perturbations.

This robustness is particularly relevant in experimental platforms such as noisy intermediate-scale quantum devices, cold atoms, and open quantum systems, where exact Gibbs equilibrium is rarely achieved. In such settings, the proposed diagnostic may serve as a quantitative probe of proximity to thermal equilibrium rather than an exact binary test.

A more detailed analysis of robustness, including many-body systems and open-system dynamics, constitutes an interesting direction for future work.

9. Minimal Interacting Many-Body Example

To examine whether the proposed thermality criterion extends beyond a single two-level system, we consider an interacting two-spin XX model,

$$H = J(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)}) + h(\sigma_z^{(1)} + \sigma_z^{(2)}). \quad (43)$$

The eigenvalues are

$$E_1 = 2h, \quad (44)$$

$$E_2 = -2h, \quad (45)$$

$$E_{3,4} = \pm 2J. \quad (46)$$

The thermal partition function becomes

$$Z = 2 \cosh(2\beta h) + 2 \cosh(2\beta J). \quad (47)$$

The mean energy is

$$\langle H \rangle = -\frac{2h \sinh(2\beta h) + 2J \sinh(2\beta J)}{\cosh(2\beta h) + \cosh(2\beta J)}. \quad (48)$$

The Fisher information with respect to inverse temperature is

$$F_\beta = \frac{\partial^2}{\partial \beta^2} \ln Z, \quad (49)$$

yielding

$$F_\beta = 4 \left[\frac{h^2 \cosh(2\beta h) + J^2 \cosh(2\beta J)}{\cosh(2\beta h) + \cosh(2\beta J)} \right] - \langle H \rangle^2. \quad (50)$$

Direct differentiation gives the fluctuation relation above. Hence, the geometric criterion remains exactly satisfied for an interacting many-body system.

10. Explicit Nonthermal Family

To illustrate the diagnostic power of the proposed criterion, we consider a family of states that interpolates continuously between a genuine Gibbs state and a nonthermal coherent component,

$$\rho_\epsilon(\beta) = (1 - \epsilon)\rho_{\text{th}}(\beta) + \epsilon|+\rangle\langle+|, \quad 0 \leq \epsilon \leq 1, \quad (51)$$

where

$$\rho_{\text{th}}(\beta) = \frac{e^{-\beta H}}{Z} \quad (52)$$

is the thermal Gibbs state associated with a fixed Hamiltonian H , and

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad (53)$$

is a coherent superposition state independent of β .

For definiteness, consider the two-level Hamiltonian

$$H = \varepsilon \sigma_z. \quad (54)$$

Since the coherent contribution does not depend on β ,

$$\frac{\partial \rho_\epsilon}{\partial \beta} = (1 - \epsilon) \frac{\partial \rho_{\text{th}}}{\partial \beta}. \quad (55)$$

Consequently, the sensitivity of the state with respect to inverse temperature is reduced by the admixture of the nonthermal component.

For small values of ϵ , the quantum Fisher information satisfies, to first order in a heuristic perturbative treatment

$$F_\beta^{(\epsilon)} = (1 - \epsilon)^2 F_\beta + O(\epsilon^2), \quad (56)$$

where F_β denotes the Fisher information of the exact Gibbs state.

The mean energy becomes

$$\langle H \rangle_\epsilon = (1 - \epsilon) \langle H \rangle_{\text{th}} + \epsilon \langle + | H | + \rangle. \quad (57)$$

Because

$$\langle +|H|+ \rangle = 0, \quad (58)$$

one obtains

$$\langle H \rangle_\epsilon = (1 - \epsilon) \langle H \rangle_{\text{th}}, \quad (59)$$

and therefore

$$-\frac{\partial \langle H \rangle_\epsilon}{\partial \beta} = (1 - \epsilon) F_\beta. \quad (60)$$

Since $F_\beta^{(\epsilon)} \neq -\partial_\beta \langle H \rangle_\epsilon$ for $\epsilon \neq 0$, the thermality criterion is violated continuously as the state departs from the Gibbs manifold.

11. Emergent Thermality in Reduced States of an Interacting Fermionic System

One of the central questions in modern quantum statistical mechanics is how thermal behavior emerges in subsystems of globally isolated quantum systems. Although the full system may evolve unitarily and remain in a pure state, reduced density operators obtained by tracing over inaccessible degrees of freedom often display properties characteristic of thermal equilibrium. This phenomenon underlies canonical typicality, eigenstate thermalization, and the statistical description of many-body quantum systems.

The geometric criterion developed in the present work provides a natural framework for investigating such emergent thermality. Rather than assuming a Gibbs form a priori, one may ask whether a reduced density matrix satisfies a fluctuation–response consistency relation that characterizes equilibrium states.

Consider an interacting fermionic system partitioned into two subsystems A and B ,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B, \quad (61)$$

with total Hamiltonian

$$H = H_A + H_B + H_{\text{int}}. \quad (62)$$

Suppose that the full system is prepared in a pure state

$$|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B. \quad (63)$$

The reduced density operator associated with subsystem A is

$$\rho_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|). \quad (64)$$

Even though the global state is pure, ρ_A is generally mixed because of entanglement between the two subsystems.

The fundamental question is whether ρ_A admits an effective thermal description. Within the present information-geometric framework, this may be investigated through the quantum Fisher information associated with a control parameter β ,

$$F_\beta^{(A)} = \text{Tr}[\rho_A L_\beta^2], \quad (65)$$

where L_β denotes the symmetric logarithmic derivative of the reduced state.

Following the construction introduced in Section 3, we define the effective generator

$$H_{\text{eff}}^{(A)} = -L_\beta + \langle L_\beta \rangle_A. \quad (66)$$

The reduced state may then be tested for thermality through the deviation function

$$\Delta_A(\beta) = \left| F_\beta^{(A)} + \frac{\partial}{\partial \beta} \langle H_{\text{eff}}^{(A)} \rangle \right|. \quad (67)$$

If

$$\Delta_A(\beta) = 0, \quad (68)$$

the subsystem satisfies a geometric fluctuation–response relation and therefore possesses an exact Gibbs structure. More generally, small values of Δ_A indicate approximate thermality and the existence of an effective equilibrium description.

A particularly relevant example is provided by interacting two-level fermionic models based on collective SU(2) degrees of freedom. Consider the Hamiltonian

$$H = J_z - V \left(J^2 - J_z^2 - \frac{N}{2} \right), \quad (69)$$

where V denotes the interaction strength and N the particle number.

The thermal state of the full system is

$$\rho = \frac{e^{-\beta H}}{Z}. \quad (70)$$

Tracing over a subset of fermionic degrees of freedom generates a reduced density matrix ρ_A that is generally not Gibbsian with respect to the bare subsystem Hamiltonian. Nevertheless, strong entanglement between the retained and traced sectors may induce an approximately thermal reduced description.

Within the present framework, the quantity $\Delta_A(\beta)$ provides a direct information-geometric probe of this phenomenon. Three qualitatively distinct regimes may be identified:

1. Exact thermal regime:

$$\Delta_A = 0.$$

The reduced state possesses an exact Gibbs representation generated by a β -independent effective Hamiltonian.

2. Emergent thermal regime:

$$\Delta_A \ll 1.$$

The reduced state is not exactly Gibbsian but satisfies a fluctuation–response relation approximately. In this case, an effective thermodynamic description emerges from entanglement with the remainder of the system.

3. Nonthermal regime:

$$\Delta_A \not\approx 0.$$

The fluctuation structure and geometric response become incompatible, indicating the absence of an effective equilibrium description.

This viewpoint establishes a connection between information geometry and subsystem thermalization. Rather than characterizing thermality solely through entropy or local expectation values, the present criterion probes whether the reduced state possesses the internal fluctuation structure characteristic of a Gibbs ensemble.

Consequently, the quantum Fisher information may serve as a diagnostic of emergent equilibrium in interacting many-body systems, providing a potential link between information geometry, canonical typicality, and quantum thermalization.

Illustrative Toy Calculation for a Two-Fermion Example

To provide a concrete illustration of the proposed thermality diagnostic, consider the smallest nontrivial realization of the interacting fermionic model, namely $N = 2$ fermions. The present calculation is intended only as an illustrative toy example and not as a complete many-body analysis.

For the Hamiltonian

$$H = J_z - V \left(J^2 - J_z^2 - \frac{N}{2} \right), \quad (71)$$

the physically relevant sector corresponds to total spin $J = 1$, with magnetic quantum numbers $M = -1, 0, 1$.

For $N = 2$ and interaction strength $V = 0.5$, the energy spectrum becomes

$$E(M) = M - V [2 - M^2 - 1], \quad (72)$$

yielding

$$E_{-1} = -1, \quad E_0 = -0.5, \quad E_1 = 1. \quad (73)$$

The thermal state of the complete two-fermion system is

$$\rho(\beta) = \frac{e^{-\beta H}}{Z}, \quad Z = e^\beta + e^{0.5\beta} + e^{-\beta}. \quad (74)$$

Tracing over one fermion produces a one-particle reduced density matrix

$$\rho_A(\beta) = \begin{pmatrix} 1 - q(\beta) & 0 \\ 0 & q(\beta) \end{pmatrix}, \quad (75)$$

where

$$q(\beta) = p_1 + \frac{1}{2}p_0, \quad (76)$$

with

$$p_M = \frac{e^{-\beta E_M}}{Z}. \quad (77)$$

To quantify the degree of thermality of the reduced subsystem, we consider the effective one-particle Hamiltonian

$$H_A = \frac{\sigma_z}{2}, \quad (78)$$

and evaluate the deviation function

$$\Delta_A(\beta) = \left| F_\beta^{(A)} + \frac{\partial}{\partial \beta} \langle H_A \rangle \right|. \quad (79)$$

The resulting numerical values are displayed in Table 1.

Table 1. Illustrative values of the subsystem thermality deviation $\Delta_A(\beta)$ for the two-fermion model with $V = 0.5$.

β	$q(\beta)$	$F_\beta^{(A)}$	$\partial_\beta \langle H_A \rangle$	$\Delta_A(\beta)$
0.1	0.4673	0.4127	−0.3205	0.0922
0.5	0.3528	0.2706	−0.2485	0.0220
1.0	0.2518	0.1350	−0.1595	0.0245
2.0	0.1459	0.0388	−0.0696	0.0307
3.0	0.0931	0.0196	−0.0407	0.0211

Several features are noteworthy. First, the reduced state is not exactly Gibbsian with respect to the bare subsystem Hamiltonian H_A , and therefore $\Delta_A(\beta)$ does not vanish identically. Second, the deviation remains relatively small throughout the temperature range considered, indicating that the reduced subsystem admits an approximately thermal description despite the presence of interactions and entanglement. Finally, the smallest deviations occur near $\beta \approx 0.5$, where the subsystem is closest to satisfying a geometric fluctuation–response relation.

This simple example illustrates how the quantity $\Delta_A(\beta)$ can serve as a quantitative measure of emergent thermality in reduced states of interacting fermionic systems. Even when the reduced density matrix is not exactly Gibbsian, the geometric criterion provides a continuous indicator of proximity to thermal equilibrium.

Our examples are illustrative applications of the general theorem, not separate demonstrations of its validity.

12. Discussion and Conclusions

We have shown that geometric observables provide a direct and operational diagnostic of thermality, while the density matrix alone does not reveal whether a mixed state originates from thermal equilibrium or from classical statistical mixing; the quantum Fisher information exposes the underlying structure through fluctuation–response relations. In particular, thermal states are uniquely characterized by the consistency between geometric quantities and thermodynamic fluctuations, a property that generic mixed states do not possess.

This result has both conceptual and practical implications. From a practical perspective, it offers a simple criterion to identify equilibrium states using experimentally accessible quantities. The proposed criterion does not require detailed microscopic knowledge of the system Hamiltonian, although parameter variation and state preparation remain necessary in practice. This is especially relevant in current experimental settings, such as cold atoms and quantum simulators, where only partial information about the system is typically available.

At a conceptual level, our findings suggest that the quantum Fisher information carries a deeper physical meaning than is usually ascribed to it. Traditionally, the QFI is viewed as a statistical or information-theoretic quantity that quantifies parameter sensitivity and sets bounds on estimation precision. Here, however, it plays a more fundamental role: it acts as a diagnostic of the physical origin of a quantum state. In particular, the equality between the QFI and energy fluctuations, together with an associated fluctuation–response relation, provides a signature of thermal equilibrium.

In this sense, the quantum Fisher information acquires a form of operational “reality”: it is not merely a mathematical construct associated with distinguishability, but an observable quantity that encodes whether a state is consistent with an underlying equilibrium structure. More precisely, it captures the compatibility between the state’s geometric response and its fluctuation properties, thereby revealing whether an exponential (Gibbs) description is possible, while mathematically related to standard characterizations of quantum exponential families in information geometry [5,6], the present formulation emphasizes an operational diagnostic interpretation: thermality is identified through direct consistency relations among measurable geometric observables, rather than through prior specification of an exponential ansatz.

This perspective suggests a broader role for geometric quantities in quantum physics. Rather than serving only as tools for estimation or state discrimination, they may provide direct access to structural properties of physical systems, including their thermodynamic nature. This opens new directions for the study of inverse problems, where one seeks to infer not only the state of a system but also the mechanisms responsible for its preparation.

The present extensions strengthen the geometric framework in several directions. First, the criterion was shown to remain valid in a minimally interacting many-body system, indicating that the approach is not restricted to isolated two-level examples. Second, the comparison with conventional fluctuation observables clarifies that the geometric formulation probes the origin of fluctuations rather than merely their magnitude. Finally, the explicit depolarizing-noise analysis demonstrates that deviations from thermality evolve continuously under imperfections, providing evidence for robustness in experimentally realistic settings.

An important feature of the present approach is that it probes the origin of fluctuations rather than merely their magnitude, while the variance can be evaluated for arbitrary mixed states, the consistency thermality condition is characteristic of Gibbsian equilibrium and therefore provides an internal test of thermality.

Future work may explore extensions of the present framework to open quantum systems, non-equilibrium steady states, and many-body systems with emergent thermal behavior. It would also be of interest to investigate whether other geometric quantities exhibit similar diagnostic power and to what extent these ideas can be implemented in realistic experimental platforms.

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