A Rigorous Feature Extraction Algorithm for Spherical Target Identification in Terrestrial Laser Scanning

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Abstract: Precise and rapid extraction of spherical target features from laser point clouds is critical for achieving high-precision registration of multiple point clouds. Existing methods often use linear models to represent spherical target characteristics, which have several drawbacks. This paper proposes a rigorous estimation algorithm for spherical target features based on least squares configurations, in which the point-cloud data error is used as a random parameter, while the spherical center coordinates and radius are used as nonrandom parameters, emphasizing correlation between spherical parameters. The implementation details of this algorithm are illustrated by deriving calculation formulas for three variance–covariance matrices: variance–covariance matrices of the new observations, variance–covariance matrices of the new observation noise, and variance–covariance matrices of random parameters and the new observation noise. Experiments show that the estimation accuracy of sphere centers using our method is improved by at least 5.7% compared to classical algorithms, such as least squares, total least squares, and robust weighted total least squares.

Keywords: terrestrial laser scanning; spherical center fitting; linear parameter estimation; nonlinear parameter estimation; least squares configuration

1. Introduction

With a dramatic reduction in the cost of LiDAR technology, terrestrial laser scanning (TLS) technology has been widely applied in mobile mapping, 3D reconstruction, unmanned driving, forestry estimation, disaster monitoring, and many other fields [1–4]. These applications often involve point-cloud registration between different survey stations, and one of the most common methods is target-based point cloud registration. The accuracy of this method depends mainly on the extraction accuracy of the target. However, with the exception of spherical targets, the extraction accuracy of other shapes of targets will be affected by the scanning incidence angle [5–8].

The isotropic property of spherical targets ensures that the hemispherical point cloud (the unobscured part of the point cloud from a spherical target, which does not describe the shape of the whole point cloud) of the same target can be obtained from different measuring stations [9,10]; thus, spherical targets are the first choice for target point cloud registration, and it is of great practical value to study the feature extraction of spherical targets. For example, Yuriy [11] proposed a method to determine the optimal diameter of spherical objects based on a least squares (LS) adjustment model, and a value of 14 cm has been obtained as the optimal diameter. Liu [6] proposed a new method by integrating the adaptive dynamic random sample consensus algorithm (AD-RANSAC) and nonlinear least squares (NLS), and the resulting accuracy of sphere center coordinate estimation was...
less than 2 cm. The authors of [12] adopted the method of centering rod center to eliminate the influence of error caused by non-centering of the bubble, and the fitting accuracy within a short distance could reach about 4 mm.

Spherical target feature extraction algorithms can be mainly divided into linear and nonlinear categories [13]. Nonlinear algorithms are generally time-consuming and cannot use the law of error propagation to derive the extraction accuracy of geometric features [14,15]. Therefore, manufacturers and researchers mainly use linear extraction algorithms in real applications, where a fast speed of processing and extraction accuracy are demanded. Brazeal [16] studied the LS solutions for determining the parameters of a spherical TLS target, in which the coordinates of the spherical center and the radius were regarded as independent. Lu [17] proposed a total least squares (TLS) method for laser scanning target ball positioning, while Chen [9] proposed a weighted total least squares (WTLS) fitting method, which introduced a weight matrix according to the laser reflection intensity of the point cloud and partial correction of the vector in column A of the coefficient matrix. The authors of [10] determined the cofactor of each point according to the distance and the a priori incident angle, and then used a robust weighted total least squares (RWTLS) method to locate the target ball.

However, when the radius of spherical target is unknown, all algorithms assume that the sphere center coordinate parameter and sphere radius parameter are independent of each other. The negligence of their correlation would cause non-robust performance of the algorithms. Through considering the correlation between radius of the sphere and the coordinates of the sphere center, this paper proposes a high-precision estimation algorithm for spherical target sphere centers based on least squares configuration (LSC). Corresponding formulas and algorithm procedure steps for feature parameter estimation are derived in Section 2. Through simulation experiments and actual scanning experiments, Section 3 demonstrates the advantages of our proposed algorithm with respect to traditional algorithms, including LS, TLS, and RWTLS.

2. Materials and Methods

The LSC is a method to simultaneously determine random and nonrandom parameter estimations from observation data. It has been applied in conversion of the base level [18,19], determination of the geoid [20], and other fields. By treating the point-cloud data error as a random parameter and the spherical center coordinates and radius as nonrandom parameters, this section first constructs the LSC observation model of spherical target features and gives its parameter estimation model. Secondly, the rigorous calculation formulas of three variance–covariance matrices in the parameter estimation model are derived. Lastly, the realization of the algorithm steps is presented.

2.1. LSC Observation Model

Suppose the spherical point cloud contains \( n \) points. The true value, observation value, and observation error of the coordinates of the arbitrary point are \((\tilde{x}_i, \tilde{y}_i, \tilde{z}_i)\), \((x_i, y_i, z_i)\), and \((e_x, e_y, e_z)\), respectively, with \( i = 1, 2, \cdots, n \). The true value, approximate value, and approximate error of the sphere center coordinates are \((\tilde{a}, \tilde{b}, \tilde{c})\), \((a_0, b_0, c_0)\), and \((e_a, e_b, e_c)\), while those for the sphere radius are \( \tilde{r} \), \( r_0 \), and \( e_r \), respectively. Then, the observation model for the point cloud at point \( i \) of the sphere is

\[
\tilde{r}^2 = (\tilde{x}_i - \bar{a})^2 + (\tilde{y}_i - \bar{b})^2 + (\tilde{z}_i - \bar{c})^2, \quad i = 1, 2, \cdots, n,
\]

where \( \tilde{r} = r_0 + e_r \), \( \bar{a} = a_0 + e_a \), \( \bar{b} = b_0 + e_b \), \( \bar{c} = c_0 + e_c \), \( \tilde{x}_i = x_i + e_x \), \( \tilde{y}_i = y_i + e_y \), and \( \tilde{z}_i = z_i + e_z \).
According to Equation (1), the observation equation of the $i$-th observation value can be expressed as follows:

$$\frac{1}{2}[r_i^2 - (x_i - a_0)^2 - (y_i - b_0)^2 - (z_i - c_0)^2]$$

$$= [x_i - a_0, y_i - b_0, z_i - c_0] \begin{bmatrix} \varepsilon_{x_i} \\ \varepsilon_{y_i} \\ \varepsilon_{z_i} \end{bmatrix} - [x_i - a_0, y_i - b_0, z_i - c_0] \begin{bmatrix} \varepsilon_a \\ \varepsilon_b \\ \varepsilon_c \end{bmatrix} + \frac{1}{2}[(\varepsilon_{x_i} - \varepsilon_a)^2 + (\varepsilon_{y_i} - \varepsilon_b)^2 + (\varepsilon_{z_i} - \varepsilon_c)^2 - \varepsilon_r^2].$$ (2)

The observation model of LSC can then be obtained as follows:

$$L = BX + GY + \Delta,$$ (3)

where $L$ represents a new observation, $X$ denotes random variables, $Y$ denotes non-random variables, and $\Delta$ illustrates the noise of new observations.

$$L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}, l_i = \frac{1}{2}[r_i^2 - (x_i - a_0)^2 - (y_i - b_0)^2 - (z_i - c_0)^2], i = 1, 2, \ldots, n.$$ (4)

$$B = \begin{bmatrix} x_1 - a_0 & y_1 - b_0 & z_1 - c_0 \\ x_2 - a_0 & y_2 - b_0 & z_2 - c_0 \\ \vdots & \vdots & \vdots \\ x_n - a_0 & y_n - b_0 & z_n - c_0 \end{bmatrix},$$ (5)

$$G = \begin{bmatrix} %x_1 - a_0 & y_1 - b_0 & z_1 - c_0 & r_0 \\ x_2 - a_0 & y_2 - b_0 & z_2 - c_0 & r_0 \\ \vdots & \vdots & \vdots & \vdots \\ x_n - a_0 & y_n - b_0 & z_n - c_0 & r_0 \end{bmatrix}.$$ (6)

$$X = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \quad \varepsilon_i = \begin{bmatrix} \varepsilon_{x_i} \\ \varepsilon_{y_i} \\ \varepsilon_{z_i} \end{bmatrix}, \quad i = 1, 2, \ldots, n, \quad Y = \begin{bmatrix} \varepsilon_a \\ \varepsilon_b \\ \varepsilon_c \end{bmatrix}.$$ (7)

$$\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix}, \quad \Delta_i = \frac{1}{2}[(\varepsilon_{x_i} - \varepsilon_a)^2 + (\varepsilon_{y_i} - \varepsilon_b)^2 + (\varepsilon_{z_i} - \varepsilon_c)^2 - \varepsilon_r^2], \quad i = 1, 2, \ldots, n.$$ (8)

Assume that the error parameters $\varepsilon_a, \varepsilon_b, \varepsilon_c, \varepsilon_{x_i}, \varepsilon_{y_i}, \varepsilon_{z_i}, \varepsilon_{x_j}, \varepsilon_{y_j}, \varepsilon_{z_j} (i \neq j)$ are independent of each other, the observation error parameter satisfies $\varepsilon_{x_i} \sim N(0, \sigma^2)$, $\varepsilon_{y_i} \sim N(0, \sigma^2)$, and $\varepsilon_{z_i} \sim N(0, \sigma^2)$. From the least squares collocation theory of Equation (3), we can get
\[ \hat{Y} = (G^T H^{-1}G)^{-1}G^T H^{-1}L, \]  
\[ \hat{X} = (D_XB^T + D_{\Delta\Lambda})H^{-1}(L - G\hat{Y}), \]  
\[ D_{\bar{X}} = (G^T H^{-1}G)^{-1}G^T H^{-1}D_XH^{-1}G(G^T H^{-1}G)^{-1}, \]  
\[ D_X = D_X - (D_XB^T + D_{\Delta\Lambda})H^{-1}(I_n - GD_YG^T H^{-1})(BD_X + D_{\Delta\Lambda}), \]  
\[ H = BD_XB^T + D_X + BD_{\Delta\Lambda} + D_{\Delta\Lambda}B^T, \]

where \( D_X = I_{3n} \cdot \sigma^2 \), \( I_{3n} \) is the unit matrix of \( 3n \times 3n \), \( D_L \) denotes the variance–covariance matrix of new observations \( L \), \( D_{\Delta\Lambda} \) is the variance–covariance matrix of parameter \( X \) and the new observation noise \( \Delta \), and \( D_{\Delta} \) illustrates the variance–covariance matrix of new observation noise.

2.2. Variance–Covariance

2.2.1. Variance–Covariance of the New Observations

From Equations (A2) and (A3) and Equation (4), we can get

\[ l_i = \frac{1}{2}(r_{0i}^2 - a_i^2 - b_i^2 - c_i^2 - x_i^2 - y_i^2 - z_i^2 + 2a_0x_i + 2b_0y_i + 2c_0z_i), \]  
\[ u_i = \frac{1}{2}[(r_{0i}^2 - (\mu_y - a_i)^2 - (\mu_y - b_i)^2 - (\mu_z - c_i)^2 - 3\sigma^2], \]  
\[ l_i - u_i = -\frac{1}{2}[(x_i^2 + y_i^2 + z_i^2) - E(x_i^2 + y_i^2 + z_i^2) + 2a_0x_i + 2b_0y_i + 2c_0z_i], \]

where \( i = 1, 2, \ldots, n \).

Suppose the variance of the new observation \( l_i \) is \( \text{cov}(l_i, l_i) \), while the covariance of \( l_i \) and \( l_j \) is \( \text{cov}(l_i, l_j) \), then, from Equations (A2), (A3) and (16), we have

\[ E[(l_i - u_i)^2] = \frac{1}{4}D(x_i^2 + y_i^2 + z_i^2) + E(a_0x_i^2 + b_0y_i^2 + c_0z_i^2) + E(a_0x_i + b_0y_i + c_0z_i), \]  
\[ \text{cov}(l_i, l_j) = E[(l_i - u_i)(l_j - u_j)] = 0, \]

where \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, n \), and \( i \neq j \).

Substituting Equations (A3), (A5) and (A8) into Equation (17), we can get

\[ \text{cov}(l_i, l_i) = E[(l_i - u_i)^2] = \frac{3}{2}\sigma^4 + \sigma^2[(\mu_y - a_0)^2 + (\mu_y - b_0)^2 + (\mu_z - c_0)^2], \]

where \( i = 1, 2, \ldots, n \).

Assuming that \((\mu_y - a_0)^2 + (\mu_y - b_0)^2 + (\mu_z - c_0)^2 \approx r_0^2\), Equation (19) can be simplified as

\[ \text{cov}(l_i, l_i) = \frac{3}{2}\sigma^4 + \sigma^2r_0^2. \]

Then,

\[ D_L = \left[ \frac{3}{2}\sigma^4 + \sigma^2r_0^2 \right] I_n, \]
where \( i = 1, 2, \cdots, n \), and \( I_n \) is the unit matrix of \( n \times n \).

2.2.2. Variance–Covariance of the New Observation Noise

Suppose the expectation of new observation noise \( \Delta_i \) is \( \mu_{\Delta_i} \), the variance of the new observation noise \( \Delta_i \) is \( \text{cov}(\Delta_i, \Delta_i) \), and the covariance of \( \Delta_i \) and \( \Delta_j \) is \( \text{cov}(\Delta_i, \Delta_j) \). From Equations (6) and (A2)–(A5), we can get

\[
\begin{align*}
\begin{cases}
u_{\Delta_i} &= \frac{E(e_{x_i}^2 + e_{y_i}^2 + e_{z_i}^2)}{2} + \frac{1}{2}(e_{x_i}^2 + e_{y_i}^2 + e_{z_i}^2 - e_{e_i}^2) \\
u_{\Delta_j} &= \frac{E(e_{x_j}^2 + e_{y_j}^2 + e_{z_j}^2)}{2} + \frac{1}{2}(e_{x_j}^2 + e_{y_j}^2 + e_{z_j}^2 - e_{e_j}^2)
\end{cases}.
\end{align*}
\]

\[
\begin{align*}
\Delta_i - \nu_{\Delta_i} &= \frac{e_{x_i}^2 + e_{y_i}^2 + e_{z_i}^2 - E(e_{x_i}^2 + e_{y_i}^2 + e_{z_i}^2)}{2} - (e_{a_i}e_{x_i} + e_{b_i}e_{y_i} + e_{c_i}e_{z_i}), \\
\Delta_j - \nu_{\Delta_j} &= \frac{e_{x_j}^2 + e_{y_j}^2 + e_{z_j}^2 - E(e_{x_j}^2 + e_{y_j}^2 + e_{z_j}^2)}{2} - (e_{a_j}e_{x_j} + e_{b_j}e_{y_j} + e_{c_j}e_{z_j}),
\end{align*}
\]

\[
\text{cov}(\Delta_i, \Delta_i) = E((\Delta_i - \mu_{\Delta_i})^2) = \frac{D(e_{x_i}^2 + e_{y_i}^2 + e_{z_i}^2)}{4} + (e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2)\sigma^2,
\]

\[
\text{cov}(\Delta_i, \Delta_j) = E((\Delta_i - \mu_{\Delta_i})(\Delta_j - \mu_{\Delta_j})) = 0,
\]

where \( i = 1, 2, \cdots, n \), \( j = 1, 2, \cdots, n \), and \( i \neq j \).

Substituting Equation (A7) into Equation (24), we have

\[
\text{cov}(\Delta_i, \Delta_j) = \frac{3}{2}\sigma^2 + (e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2)\sigma^2,
\]

and

\[
D_{\Delta} = \left[ \frac{3}{2}\sigma^2 + (e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2)\sigma^2 \right]I_n,
\]

where \( i = 1, 2, \cdots, n \), and \( I_n \) is the unit matrix of \( n \times n \).

2.2.3. Variance–Covariance of the Random Parameter and New Observation Noise

From Equation (6), we know

\[
\begin{align*}
\Delta_i e_{x_i} &= \frac{1}{2}(e_{x_i}^2 + e_{y_i}^2 + e_{z_i}^2 + e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2 - e_{e_i}^2)e_{x_i} - (e_{x_i}e_{a_i} + e_{y_i}e_{b_i} + e_{z_i}e_{c_i})e_{x_i}, \\
\Delta_i e_{x_j} &= \frac{1}{2}(e_{x_j}^2 + e_{y_j}^2 + e_{z_j}^2 + e_{a_j}^2 + e_{b_j}^2 + e_{c_j}^2 - e_{e_j}^2)e_{x_j} - (e_{x_j}e_{a_j} + e_{y_j}e_{b_j} + e_{z_j}e_{c_j})e_{x_j},
\end{align*}
\]

where \( i = 1, 2, \cdots, n \), \( j = 1, 2, \cdots, n \), and \( i \neq j \).

Considering that \( e_{a_i}, e_{b_i}, e_{c_i} \) are nonrandom parameters, from Equations (A2)–(A4), (28) and (29), we can get

\[
E(\Delta_i e_{x_i}) = -e_a\sigma^2, E(\Delta_i e_{x_j}) = 0,
\]

and

\[
E(\Delta_i e_{x_j}) = -e_a\sigma^2, E(\Delta_i e_{x_j}) = 0,
\]

where \( i = 1, 2, \cdots, n \), \( j = 1, 2, \cdots, n \), and \( i \neq j \).

The same derivation of Equation (30) yields
\[ E(\Delta \varepsilon_{y_i}) = -\sigma^2 \varepsilon_y, E(\Delta \varepsilon_{z_j}) = 0, \]  
\[ E(\Delta \varepsilon_{z_i}) = -\sigma^2 \varepsilon_z, E(\Delta \varepsilon_{z_j}) = 0, \]  
where \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n, \) and \( i \neq j. \)

Assuming that covariances between new observation noise \( \Delta_i \) and the observation error \( \varepsilon_i = (\varepsilon_{x_i}, \varepsilon_{y_i}, \varepsilon_{z_i}) \) and \( \varepsilon_j = (\varepsilon_{x_j}, \varepsilon_{y_j}, \varepsilon_{z_j}) \) are \( \text{cov}(\Delta_i, \varepsilon_i) \) and \( \text{cov}(\Delta_i, \varepsilon_j) \), respectively, then, using Equations (30)–(32), we can get

\[ \text{cov}(\Delta_i, \varepsilon_i) = \begin{bmatrix} \sigma^2 & \varepsilon_y & \varepsilon_z \\ \varepsilon_y & \sigma^2 & \varepsilon_z \\ \varepsilon_z & \varepsilon_z & \sigma^2 \end{bmatrix}, \]

\[ \text{cov}(\Delta_i, \varepsilon_j) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \]  

Then,

\[ D_{XX} = -\begin{bmatrix} \varepsilon_a & \varepsilon_b & \varepsilon_c \\ \varepsilon_a & \varepsilon_b & \varepsilon_c \\ \varepsilon_a & \varepsilon_b & \varepsilon_c \end{bmatrix} \sigma^2, \]  

where \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n, \) and \( i \neq j. \)

### 2.3. LSC Calculation Steps

The above observation model and parameter estimation assume that the initial values are known. However, spherical center coordinates and the approximate value of the spherical radius are usually unknown; thus, an iterative algorithm is required to complete LSC estimation of the spherical target characteristic parameters, which specific calculation steps are described as follows:

1. Read observation values of point clouds on surface of the ball target \( Data = \{ p_i \}, \)
   \( p_i = (x_i, y_i, z_i), i = 1, 2, \ldots, n; \)
2. Use conventional LS algorithm to calculate spherical coordinates and approximate spherical radius \( a_0, b_0, c_0, r_0; \)
3. Assume the variance–covariance matrix of random parameters as \( D_x = I_{3n} \cdot \sigma^2, \) with the initial nonrandom parameter value \( Y_i = [0 \ 0 \ 0]^T; \)
4. Calculate the matrix of new observations \( L \) and coefficient matrices \( B \) and \( G \) using Equations (4)–(6), respectively;
5. Assume \( \varepsilon_a = Y_1(1,1), \ varepsilon_b = Y_1(2,1), \) and \( \varepsilon_c = Y_1(3,1). \) Calculate random parameter \( X \) and the variance–covariance matrix \( D_{XX} \) of new observed noise \( \Delta \) using Equation (35) and the covariance matrix \( D_{\Delta} \) of new observed noise \( \Delta \) using Equation (27);
6. Calculate matrix \( H \) using Equation (13);
7. Calculate \( \hat{Y} \) of the nonrandom parameters \( Y \) using Equation (9);
8. If \( (\hat{Y} - Y)^T(\hat{Y} - Y) \) is less than the iteration stop threshold (the value can be 0.000001), then continue; if not, update the initial value of nonrandom parameters \( Y_i = \hat{Y}, \) and go back to Step 5;
9. If \( \hat{Y}^T \hat{Y} \) is less than iteration stop threshold (the value can be 0.000001), then continue; otherwise, update the approximate values of sphere center coordinates and
sphere radius as \( a_0 = a_0 + \hat{Y}(1,1) \), \( b_0 = b_0 + \hat{Y}(2,1) \), \( c_0 = c_0 + \hat{Y}(3,1) \), \( r_0 = r_0 + \hat{Y}(4,1) \), and \( Y_i = [0 \ 0 \ 0]^T \), and go back to Step 4;

(10) Calculate the variance–covariance matrix \( D_L \) of new observations \( L \) using Equation (21);

(11) Calculate the variance–covariance matrix \( D_y \) of \( \hat{Y} \) using Equation (11);

(12) End.

3. Experimental Verification

In order to verify the effectiveness of the spherical target LSC algorithm, both simulations and actual scanning experiments are designed in this paper. In this section, we first introduce the evaluation indicators used in the experiments, then introduce experimental data and scenarios, and finally analyze experimental results through comparing the estimation accuracy of spherical parameters obtained from our method with LS, TLS, and RWTLS algorithms.

3.1. Evaluation Index

We used the root-mean-square error (RMSE) of sphere center coordinate parameter estimation (\( \text{RMSE}_{abc} \)) and sphere radius parameter estimation (\( \text{RMSE}_r \)) as the evaluation metrics to compare differences among LS, TLS, and the RWTLS algorithms. The two evaluation metrics are expressed as follows:

\[
\text{RMSE}_{abc} = \sqrt{\frac{1}{h} \sum_{j=1}^{h} (\hat{a}_j - \tilde{a}^2 + (\hat{b}_j - \tilde{b})^2 + (\hat{c}_j - \tilde{c})^2 ,
\]

\[
\text{RMSE}_r = \sqrt{\frac{1}{h} \sum_{j=1}^{h} (\hat{r}_j - \tilde{r})^2},
\]

where \( h \) represents the number of trials, and \( \hat{a}_j, \hat{b}_j, \hat{c}_j, \) and \( \hat{r}_j \) are estimations of the sphere center coordinate parameter and sphere radius parameter calculated for different algorithms.

3.2. Simulation Experiment

In order to compare the sphere center extraction accuracy of our methods, we imitate the laser scanning process and use random simulation to generate observations of spherical point clouds with different radii under different noise levels.

3.2.1. Simulation Experiment I

It is assumed that the center coordinate of the scanner is \((0m,0m,0m)\), true values of sphere center coordinates are \((\tilde{a}, \tilde{b}, \tilde{c}) = (10m,10m,10m)\), the true value of the sphere radius is \( \tilde{r} \in [2cm,3cm,\ldots,11cm] \), and the observed noise variance is \( \sigma \in [1mm,2mm,\ldots,10mm] \). We randomly simulated 1000 sets of spherical point cloud observations under different noises \( \{D_{\sigma(l,j)}, D_{\sigma(2l,j)}\ldots,D_{\sigma(1000,j)}\}_{l=2r,\ldots,11l,j=1,2,\ldots,1000} \). For the above simulated point cloud observation data, LS, TLS, RWTLS, and the LSC algorithm proposed in this paper were used to estimate the spherical center coordinates and spherical radius of the target. At the same time, Equation (36) was used to calculate the estimation accuracy of the spherical center coordinate parameters under different noise variances (see Figure 1), and Equation (37) was used to calculate the estimation accuracy of spherical radius parameters under different noise variances (see Figure 2 and Table 1).
Figure 1. Estimation accuracy of spherical parameters under different noise variances. The $y$-axes of each graph in the left and right columns represent the RMSE of spherical center coordinate parameter estimation and spherical radius parameter estimation, respectively. The $x$-axis of each graph represents the magnitude of the noise variance. The unit of the $x$-axis is mm, while the unit of the $y$-axis is mm.
Figure 2. Comparison of sphere parameter estimation accuracy of different algorithms under different noise variances. The y-axes of each graph in the left and right columns represent the RMSE of spherical center coordinate parameter estimation and spherical radius parameter estimation, respectively. The x-axis of each graph represents the magnitude of the noise variance. The unit of the x-axis is mm, while the unit of the y-axis is mm.

3.2.2. Simulation Experiment II

The sphere radius was set to $r \in [2\text{cm}, 3\text{cm}, \ldots, 11\text{cm}]$. The coordinates of the scanner center were still $(0\text{m}, 0\text{m}, 0\text{m})$, true values of sphere center coordinates $(\tilde{a}, \tilde{b}, \tilde{c})$ were $(10\text{m}, 10\text{m}, 10\text{m})$, and the observed noise variance was $\sigma \in [1 \text{ mm}, 2 \text{ mm}, \ldots, 10 \text{ mm}]$. We randomly simulated 1000 sets of spherical point cloud observations under different radii $\{D_{1,1,j}, D_{1,2,j}, \ldots, D_{10,10,j}\}_{i=1,2,\ldots,10; j=1,2,\ldots,1000}$. For the above simulated point cloud observation data, LS, TLS, RWTLS, and the LSC algorithm proposed in this paper were used to estimate spherical center coordinates and spherical radius of the target. At the same time, Equation (36) was used to calculate the estimation accuracy of spherical center coordinate parameters under different radii (see Figure 3), and Equation (37) was used to calculate the estimated accuracy of spherical radius parameters under different radii (see Figure 4 and Table 2).
Figure 3. The estimation accuracy of spherical parameters under different radii. The $y$-axes in the left and right columns represent the RMSE of spherical center coordinate parameter estimation and spherical radius parameter estimation, respectively. The $x$-axis of each figure represents the initial value of the sphere radius. The unit of the $x$-axis is cm, while the unit of the $y$-axis is mm.
Figure 4. Comparison of sphere parameter estimation accuracy of different algorithms under different radii. The y-axes in the left and right columns represent the RMSE of spherical center coordinate parameter estimation and spherical radius parameter estimation, respectively. The x-axis of each figure represents the initial value of the sphere radius. The unit of the x-axis is cm, while the unit of the y-axis is mm.

3.3. Actual Experiment

To further verify the practicability of fitting the spherical center of the spherical target based on our LSC model, this study also designed a scan experiment (see Figure 5) taking into account the effects of observation noise and target radius. We used a MS60 three-dimensional laser scanner to scan spherical targets with radii of 2.02 cm, 2.14 cm, 2.76 cm, 3.44 cm, and 4.19 cm for 10 (h = 10) repetitions to obtain spherical point cloud observation data. The LS, TLS, and RW TLS algorithms and the calculation steps in Section 2.3 were applied to process the point cloud data separately. Equations (36) and (37) were used to calculate the root-mean-square error of parameters under the three traditional algorithms and the algorithm of this paper. The processing results are shown in Figure 6. In addition, we also calculated the computation time of each algorithm, and the results are shown in Table 3.
Figure 5. Experimental scanning setup.
Spherical center coordinate error (left) and sphere radius error (right). Different colors represent different parameter estimation algorithms; the x-axis value represents the experiment number; the y-axes in the left and right columns represent the RMSE of spherical center coordinate parameter estimation and spherical radius parameter estimation, respectively. (a) Ball target with a radius of 2.02 cm; (b) ball target with a radius of 2.16 cm; (c) ball target with a radius of 2.76 cm; (d) ball target with a radius of 3.44 cm; (e) ball target with a radius of 4.19 cm.

3.4. Result Analysis

It can be seen from Figure 1 that, when the observation error was small, the estimation results of spherical center coordinate parameters from the four algorithms had little difference. The RMSE of spherical center coordinate parameters from RWTLs could reach a maximum of 12 mm, indicating that the observation error had a great influence on the accuracy of RWTLs algorithm. Table 1 shows that LSC spherical center coordinates had the highest fitting accuracy as noise increased, while the accuracy of ball parameter extraction with different radii gradually decreased. In terms of ball parameter extraction accuracy, the LSC algorithm extraction accuracy decreased more slowly than that of LS, TLS, and WRTL. Most of the differences between maximum and minimum RMSEs for LSC were smaller than for the other three methods, indicating that the LSC algorithm was more robust than other methods.

The comparison results of Figure 3 and Table 2 show that, as the radius of sphere increased, the extraction accuracy of sphere parameters under the influence of different noises gradually increased, among which the LSC algorithm’s extraction accuracy still changed more smoothly than that of LS, TLS, and RWTLs. In other words, the differences between the maximum and values of the RMSE were mostly the smallest in LSC, indicating that the LSC algorithm was less affected by the radius of the ball than the other three methods.

Figures 2 and 4 show that the ball parameter extraction accuracy of LS-LSC, TLS-LS, WRTLS-TLS was greater than zero in all cases, and the results imply that the ball parameter extraction accuracy of LSC was the highest, followed by LS, TLS, and WRTLS. At the same time, it can be seen from Figure 2 that the differences in extraction accuracy of the spherical center coordinate parameters for different algorithms were mostly about 1 mm, with a maximum of 4.5 mm, while those for the sphere radius parameter were mostly below 0.5 mm, with a maximum of 2.6 mm. This shows that the extraction accuracy of the
sphere radius parameter was more affected by the algorithm than the extraction of the sphere center coordinate parameter.

In addition, as can be seen from Tables 1 and 2, compared with other algorithms, the LSC algorithm could improve the extraction accuracy of parameters in all cases, and the ball parameters could be increased by a minimum of 0.4% and maximum of 78.4%.

Figure 6 proves that, when our proposed algorithm was adapted to the point-cloud scanning data of a real scene, its accuracy remained the highest. With increasing radius of the ball target, the calculation accuracy of the four algorithms was improved. The spherical center fitting errors of the four radii from the LSC algorithm were 4–6 mm, 2–4 mm, and 2–4 mm lower than those of LS, TLS, and RWTLS, respectively, while the radius parameter estimation errors were 5–7 mm, 4–7 mm, and 2–5 mm lower than those of LS, TLS, and RWTLS, respectively. It can also be seen from Table 3 that, with increasing radius, the computation time of all algorithms increased, among which LS exhibited the lowest computation time, while the computation time of TLS, RWTLS, and the LSC algorithm proposed in this paper was about the same.

Table 1. Comparison of different algorithms under different radii (unit: %).

<table>
<thead>
<tr>
<th></th>
<th>(LS − LSC)/LS</th>
<th>(TLS − LSC)/TLS</th>
<th>(RWTLS − LSC)/RWTLS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>min</td>
<td>max</td>
</tr>
<tr>
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<td>9.8</td>
</tr>
<tr>
<td></td>
<td>r2</td>
<td>6.6</td>
<td>21.9</td>
</tr>
<tr>
<td></td>
<td>r3</td>
<td>1.6</td>
<td>28.6</td>
</tr>
<tr>
<td></td>
<td>r4</td>
<td>3.5</td>
<td>16.8</td>
</tr>
<tr>
<td></td>
<td>r5</td>
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<td>31.6</td>
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<td>31.2</td>
</tr>
<tr>
<td></td>
<td>r7</td>
<td>2.4</td>
<td>36.4</td>
</tr>
<tr>
<td></td>
<td>r8</td>
<td>10.3</td>
<td>54.5</td>
</tr>
<tr>
<td></td>
<td>r9</td>
<td>13</td>
<td>60.7</td>
</tr>
<tr>
<td></td>
<td>r10</td>
<td>13.5</td>
<td>50.5</td>
</tr>
<tr>
<td>RMSEr</td>
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<td>1.6</td>
<td>29.4</td>
</tr>
<tr>
<td></td>
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<td>29.6</td>
</tr>
<tr>
<td></td>
<td>r3</td>
<td>4.3</td>
<td>69.6</td>
</tr>
<tr>
<td></td>
<td>r4</td>
<td>2.9</td>
<td>32.1</td>
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<td></td>
<td>r5</td>
<td>3.8</td>
<td>68.3</td>
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<td>52.2</td>
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<tr>
<td></td>
<td>r7</td>
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<td>40.2</td>
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<td></td>
<td>r8</td>
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<td>42.5</td>
</tr>
<tr>
<td></td>
<td>r9</td>
<td>3.1</td>
<td>10.6</td>
</tr>
<tr>
<td></td>
<td>r10</td>
<td>0.4</td>
<td>30.2</td>
</tr>
</tbody>
</table>
Table 2. Comparison of different algorithms under different noises (unit: %).

<table>
<thead>
<tr>
<th></th>
<th>(LS−LSC)/LS</th>
<th>(TLS−LSC)/TLS</th>
<th>(RWTLS−LSC)/RWTLS</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\sigma$ 1</td>
<td>min</td>
<td>max</td>
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<tr>
<td>$\sigma$ 1</td>
<td>4.9</td>
<td>54.5</td>
<td>21.4</td>
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<tr>
<td>$\sigma$ 2</td>
<td>3.1</td>
<td>41.3</td>
<td>19.6</td>
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<tr>
<td>$\sigma$ 3</td>
<td>4</td>
<td>42.9</td>
<td>19.9</td>
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<tr>
<td>$\sigma$ 4</td>
<td>1.6</td>
<td>36.4</td>
<td>17.8</td>
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<tr>
<td>$\sigma$ 5</td>
<td>3.5</td>
<td>28.3</td>
<td>16.4</td>
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<tr>
<td>$\sigma$ 6</td>
<td>3.5</td>
<td>35.5</td>
<td>20.1</td>
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<tr>
<td>$\sigma$ 7</td>
<td>9.8</td>
<td>49.2</td>
<td>22.1</td>
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<td>$\sigma$ 9</td>
<td>3.3</td>
<td>56.5</td>
<td>19.4</td>
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<tr>
<td>$\sigma$ 10</td>
<td>2.4</td>
<td>60.7</td>
<td>19.5</td>
</tr>
<tr>
<td>$\sigma$ 1</td>
<td>1.7</td>
<td>78.4</td>
<td>31.7</td>
</tr>
<tr>
<td>$\sigma$ 2</td>
<td>0.4</td>
<td>49.6</td>
<td>33</td>
</tr>
<tr>
<td>$\sigma$ 3</td>
<td>1.4</td>
<td>30.9</td>
<td>9.9</td>
</tr>
<tr>
<td>$\sigma$ 4</td>
<td>1.3</td>
<td>32.1</td>
<td>60.2</td>
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<tr>
<td>$\sigma$ 5</td>
<td>3.1</td>
<td>27.1</td>
<td>11.6</td>
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<tr>
<td>$\sigma$ 6</td>
<td>0.7</td>
<td>47.7</td>
<td>22.1</td>
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<tr>
<td>$\sigma$ 7</td>
<td>1.7</td>
<td>38.8</td>
<td>18.5</td>
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<tr>
<td>$\sigma$ 8</td>
<td>3.3</td>
<td>34.2</td>
<td>17.3</td>
</tr>
<tr>
<td>$\sigma$ 9</td>
<td>5.3</td>
<td>26.9</td>
<td>14.8</td>
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<tr>
<td>$\sigma$ 10</td>
<td>5.8</td>
<td>15.2</td>
<td>9.7</td>
</tr>
</tbody>
</table>

Table 3. Computation time of different algorithms.

<table>
<thead>
<tr>
<th>Radius (cm)</th>
<th>Time (s)</th>
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<tbody>
<tr>
<td></td>
<td>LS</td>
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<tr>
<td>2.02</td>
<td>0.078</td>
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<tr>
<td>2.16</td>
<td>0.093</td>
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<tr>
<td>2.76</td>
<td>0.140</td>
</tr>
<tr>
<td>3.44</td>
<td>0.156</td>
</tr>
<tr>
<td>4.19</td>
<td>0.187</td>
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</table>

4. Conclusions

When the sphere radius is unknown, the correlation between the sphere center coordinate and the sphere radius is not considered in all existing estimation algorithms, which leads to inaccurate estimation of the sphere center coordinate. This paper constructed the LSC model of spherical point clouds and derived rigorous variance–covariance matrix estimation formulas for the new observations, new observation noise, and random parameters and the new observation noise. Then, a high-precision spherical target sphere estimation algorithm together with implementation steps based on LSC was proposed.

Through comparison with the traditional LS, TLS and WRTLS algorithms, it was found that the proposed method was less affected by noise and sphere radius. The ball parameter extraction accuracy from the proposed LSC method was the highest, followed by LS, WRTLS, and TLS. Under the influence of different noise variances, the LSC algorithm sphere center coordinate parameter extraction accuracy was improved by more than 5.7%, 12.1%, and 8.7% compared with the LS, RWTLs, and TLS algorithms, respectively. In terms of extraction accuracy of the sphere radius parameter, the LSC algorithm
was also superior to LS, WRTL, and TLS, with improvements of 13.7%, 13.6%, and 10.1%, respectively, on average.

With respect to the influence of different sphere radii, the sphere center coordinate parameter extraction accuracy obtained by our proposed rigorous LSC algorithm was improved by 16.4%, 24.1%, and 9.7%, respectively, on average as compared with the LS, WRTL, and TLS algorithms. The LSC algorithm could also improve the extraction accuracy of the sphere radius parameter by an average of more than 9.7%, 10.6%, and 9.6%, respectively, as compared with the other three methods. In particular, in the absence of gross errors, the accuracy of the RW TLS algorithm was not optimal. In addition, although the TLS algorithm takes into account the influence of the coefficient matrix error, there was no substantial improvement in the accuracy of ball parameter estimation.

**Author Contributions:** Conceptualization, R.Y., X.M. and Y.Y.; methodology, R.Y., J.L. and Y.Y.; software, J.L.; validation, R.Y. and J.L.; formal analysis, R.Y. and X.M.; investigation, J.L.; resources, R.Y. and X.M.; data curation, J.L.; writing—original draft preparation, J.L.; writing—review and editing, R.Y. and X.M.; visualization, J.L.; supervision, R.Y., X.M. and Y.Y.; project administration, R.Y. and X.M.; funding acquisition, R.Y. All authors have read and agreed to the published version of the manuscript.

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**Data Availability Statement:** The datasets and source codes used in this study is available from the corresponding author on a reasonable request.

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**Conflicts of Interest:** The authors declare no conflict of interest.

**Appendix A**

Suppose the expected values of coordinate observations \((x_i, y_i, z_i)\) and new observations \(\hat{\ell}\) are \(\mu_x, \mu_y, \mu_z, \mu_{\hat{\ell}}\), respectively, and the variances of coordinate observation \((x_i, y_i, z_i)\) and observation error \((\varepsilon_x, \varepsilon_y, \varepsilon_z)\) are \(D_x, D_y, D_z, D_{\varepsilon_x}, D_{\varepsilon_y}, D_{\varepsilon_z}\), respectively. Suppose \(\varepsilon_x, \varepsilon_y, \varepsilon_z\) are independent of each other, and \(\varepsilon_x \sim N(0, \sigma^2)\), \(\varepsilon_y \sim N(0, \sigma^2)\), \(\varepsilon_z \sim N(0, \sigma^2)\). Then, we can get

\[
\begin{align*}
    x_i & \sim N(\mu_x, \sigma^2) \\
    y_i & \sim N(\mu_y, \sigma^2) \\
    z_i & \sim N(\mu_z, \sigma^2) \\
    \frac{x_i}{\sigma} & \sim N\left(\frac{\mu_x}{\sigma}, 1\right) \\
    \frac{y_i}{\sigma} & \sim N\left(\frac{\mu_y}{\sigma}, 1\right) \\
    \frac{z_i}{\sigma} & \sim N\left(\frac{\mu_z}{\sigma}, 1\right)
\end{align*}
\]

(A1)
\[
\begin{align*}
\mu_x &= \bar{x}_i = E(x_i) \\
\mu_y &= \bar{y}_i = E(y_i) \\
\mu_z &= \bar{z}_i = E(z_i),
\end{align*}
\] (A2)

With the properties of the expectation and variance [19], we can get

\[
\begin{align*}
E(x_i^2) &= D(x_i) + [E(x_i)]^2 = \sigma^2 + \mu_x^2 \\
E(y_i^2) &= \sigma^2 + \mu_y^2, \quad E(e_i^2) = D(e_i) + [E(e_i)]^2 = \sigma^2, \\
E(z_i^2) &= \sigma^2 + \mu_z^2
\end{align*}
\] (A3)

\[
\begin{align*}
E(x_i^2) &= E[(\mu_x - e_i)^2] = \mu_x^2 + 3\mu_x\sigma^2 \\
E(y_i^2) &= \mu_y^2 + 3\mu_y\sigma^2, \quad E(e_i^2) = E(\mu_x^3) = E(e_i^3) = 0, \\
E(z_i^2) &= \mu_z^2 + 3\mu_z\sigma^2
\end{align*}
\] (A4)

\[
\begin{align*}
E(x_i^2e_i) &= E[(\mu_x - e_i)^2e_i] = -2\mu_x\sigma^2 \\
E(y_i^2e_i) &= -2\mu_y\sigma^2, \\
E(z_i^2e_i) &= -2\mu_z\sigma^2
\end{align*}
\] (A5)

where \(E(\cdot)\) represents the calculation of expectation, and \(i = 1, 2, \cdots, n\).

With Equation (A1) and the relationship between the chi-square distribution and the normal distribution, we can get

\[
\begin{align*}
\frac{x_i^2}{\sigma^2} &\sim \chi^2_{(1, \lambda_1)} \quad \text{and} \quad \lambda_1 = \frac{\mu_x^2}{\sigma^2} \\
\frac{y_i^2}{\sigma^2} &\sim \chi^2_{(1, \lambda_2)} \quad \text{and} \quad \lambda_2 = \frac{\mu_y^2}{\sigma^2} \\
\frac{z_i^2}{\sigma^2} &\sim \chi^2_{(1, \lambda_3)} \quad \text{and} \quad \lambda_3 = \frac{\mu_z^2}{\sigma^2}
\end{align*}
\] (A6)

\[
\frac{e_i^2}{\sigma^2} \sim \chi^2_{(1,0)} \quad \frac{e_i^2}{\sigma^2} \sim \chi^2_{(1,0)} \quad \frac{e_i^2}{\sigma^2} \sim \chi^2_{(1,0)}
\]

where \(i = 1, 2, \cdots, n\).

As the variance of the noncentral distribution \(\chi^2(m, \lambda)\) is \(D(\chi^2_{(m, \lambda)}) = 2m + 4\lambda\), with the above equation, we can get

\[
\begin{align*}
D(e_i^2) &= D(\mu_i^2) = 2\sigma^4, \quad D(\mu_i^2 + \mu_i^2) = 6\sigma^4, \\
D(x_i^2) &= 2\sigma^4 + 4\sigma^2\mu_i^2 \\
D(y_i^2) &= 2\sigma^4 + 4\sigma^2\mu_i^2, \quad D(x_i^2 + y_i^2 + z_i^2) = 6\sigma^4 + 4\sigma^2(\mu_i^2 + \mu_i^2 + \mu_i^2), \\
D(z_i^2) &= 2\sigma^4 + 4\sigma^2\mu_i^2
\end{align*}
\] (A7)

where \(i = 1, 2, \cdots, n\).
References


