Multi-Channel SAR Imaging on Cruising Ships with Sub-Orbital Spaceplane

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Abstract: A multi-channel synthetic aperture radar (SAR) on board a spaceplane orbiting near the top of the atmosphere is proposed to acquire images of cruising ships. Low pulse repetition frequency (PRF) is required for high-resolution wide-swath (HRWS) imaging, leading to inevitable problems of azimuth spectrum aliasing (ASA) and azimuth Doppler ambiguity (ADA). In this work, we propose a phase matching technique to solve the ASA problem in restoring the azimuth spectrum. A multi-stage compressive-sensing (CS) technique is also proposed to solve both ADA and ASA problems. Five similar types of cruising ship are simulated to verify the efficacy of the proposed approach, at different levels of signal-to-noise ratio. Indices of geometry match, intensity match, and structural similarity are used to identify different ships from the acquired SAR images.

Keywords: synthetic aperture radar (SAR); cruising ship; azimuth Doppler ambiguity (ADA); azimuth spectrum aliasing (ASA); compressive sensing (CS); phase matching

1. Introduction

High-resolution wide-swath (HRWS) synthetic aperture radar (SAR) imaging on ground moving target (GMTIm) has become increasingly important for homeland security and conflict control in real time over a vast area. The recent literature on SAR imaging indicates the feasibility of HRWS SAR imaging is in general constrained by the azimuth spectrum aliasing (ASA) and azimuth Doppler ambiguity (ADA) problems, which are inevitable as low pulse repetition frequency (PRF) is required for wide-swath SAR imaging. In this work, we explore the feasibility of using a suborbital spaceplane as platform to carry out SAR missions, and a rigorous SAR imaging approach is developed and verified by simulations.

A spaceplane orbiting near the top of atmosphere takes a vantage point for surveillance of cruising ships over a vast ocean area. To acquire images at fair spatial resolution over a wide swath [1], low pulse repetition frequency is required to avoid range ambiguity. However, the effects of azimuth Doppler ambiguity (ADA) and azimuth spectrum aliasing (ASA) will emerge if the pulse repetition frequency is lower than the Doppler bandwidth of the backscattered signals from the targets [2], resulting in multiple ghost targets and range error.

Wide Doppler spectrum implies a high azimuth resolution, but the spectrum will be wrapped into multiple segments in the azimuth baseband of \([-F_a/2, F_a/2]\), with an offset of \(N_s F_a\) for each segment, where \(F_a\) is the pulse repetition frequency, and \(N_s\) is the aliasing ambiguity number. This phenomenon is called azimuth spectrum aliasing (ASA). The radial velocity of a moving target induces a Doppler frequency shift \(N_d F_a\), with \(N_d\) the Doppler ambiguity number, which will arouse ambiguity between different targets at different along-track locations. This phenomenon is called azimuth Doppler ambiguity (ADA). Thus, the total Doppler frequency shift in each spectrum segment adds up to \((N_s + N_d) F_a\). If the ASA and ADA issues are not solved, the subsequent range-cell migration correction
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(RCMC) and phase compensation will not be properly implemented, resulting in artifacts such as ghost targets and range error.

Figure 1 shows the schematic of original azimuth spectrum before wrapping due to ASA and ADA. The baseband of Doppler frequency, \( f_d \in [-F_a/2, F_a/2] \), is labeled with \( N_s = 0 \). The spectrum of received signals is wrapped into multiple segments of bandwidth \( F_a \) each labeled with a different aliasing ambiguity number \( N_s \). The \( N_s \) associated with any spectrum segment will be determined first, then the true Doppler frequency \( f_d \) will be determined from its baseband counterpart \( f_d \) after estimating the Doppler ambiguity number \( N_a \). Many methods have been proposed to solve the ADA issue for SAR imaging of fast moving targets, but few of them considered the ASA issue which is crucial for HRWS applications. The ASA issue is more complicated than the ADA issue since a moving target is associated with only one \( N_a \), but multiple spectrum segments are associated with multiple \( N_s \)’s.

![Figure 1. Schematic of original azimuth spectrum before wrapping due to ASA and ADA.](image)

The ASA issue has been tackled in [1,2]. In [1], a deramp space-time adaptive process (STAP) was applied on a multi-channel SAR system, composed of one transmitter and an array of receivers, to tackle the ASA issue and coarsely focus the ground moving targets (GMTs) in a Doppler frequency-angle domain. The ambiguity number was then estimated by maximizing the signal-to-clutter-plus-noise ratio (SCNR) of GMTs in the image. However, prior knowledge of GMT along-track velocity was needed for the deramp function, otherwise the acquired image would be blurred. Performance between our proposed approach and the method in [1] will be compared by future simulations. It will be shown that the SAR image generated with the method in [1] maybe blurred if practical parameters for HRWS applications are imposed.

In [2], a bistatic SAR system operated at PRF of 75 Hz for GMT imaging was composed of a GEO satellite-borne transmitter and an airborne multi-channel receiver. The signals from the multi-channel receiver were Fourier transformed to obtain a spatial spectrum, which was composed of multiple sinc functions separated at intervals of \( F_a \), and a specific spectrum segment was extracted with a specific bandpass filter. However, \( F_a \) may not be commensurate with the baseband width of the spatial spectrum, causing additional wrapping of the spatial spectrum and inducing other artifacts. Although this method can focus SAR image, artifacts or ghost targets cannot be suppressed to an acceptable level, as will be shown later in the simulations. In this work, we propose a phase matching technique to tackle the ASA issue for acquiring images that meet the requirements of HRWS SAR on GMTim. By using a uniform array of receivers and exploiting the phase relation among the received signals, with proper receiver number \( M \) and receiver spacing \( d \), the spectrum segment associated with specific \( N_s \) can be completely extracted.

Many techniques have been developed to acquire HRWS images. In [3], multiple high-power beams with different azimuth phase centers were radiated from a satellite-borne HRWS SAR, around which three transponder-like MirrorSAR satellites orbited to configure multiple interferometric baselines in a single pass. The simulated resolution was 1.5 m and the swath width was 20 km. Significant trajectory curvature of spaceborne SAR is difficult to compensate for completely via signal processing, and the nested helix orbit of
MirrorSAR in [3] makes this issue even worse, leading to degraded resolution. Compared with practical HRWS applications, resolution of 1.5 m and swath width of 20 km seem to be insufficient. Take the simulation parameters in our work for example, where the resolution is 0.5 m and the swath width is 150 km.

At low PRF, the received signal over a single channel is not sufficient to estimate the Doppler centroid with conventional methods. In [4], a multi-channel SAR approach was proposed to estimate the Doppler ambiguity number embedded in the backscattered signal from static objects by exploiting the range-variant feature of Doppler centroid, which was perturbed by ground moving targets.

Two types of method were mentioned in [5] for removal of Doppler ambiguity. The first type utilizes a multiple-input multiple-output (MIMO) array to form several independent subapertures along track. The second type unwraps the Doppler spectrum by using multi-channel digital beamforming. Several identical linear frequency modulation (LFM) waveforms were emitted at tunable delays, making beams at different constituent frequencies point in different directions. The backscattered signals from a specific subswath could thus be extracted by using a band-pass filter. The range resolution was 0.75 m, compared with 0.25 m if the whole spectrum was radiated upon a fixed subswath. However, the benefit of solving range ambiguity other than Doppler ambiguity of HRWS SAR was not mentioned in [5]. Based on the simulation results, a swath width of 100 km is not very wide considering the platform height of 600 km, and the range resolution of 0.75 m is not impressive with a signal bandwidth of 1200 MHz. As a comparison, our simulations can achieve a resolution of 0.5 m with signal bandwidth of 350 MHz.

In [6], a ScanSAR was proposed by varying the elevation angle of a beam with slow time to cover a wide swath. The resolution became poorer as the illumination time on each subswath was curtailed. Since the strengths of backscattered signals from different subswaths were different, streak pattern was induced on the acquired image. Multiple subarrays were aligned along track, with tunable time delays between adjacent subarrays. The beams from different subarrays could be steered towards different directions, thus different frequency components of the LFM signals were emitted at different squint angles, achieving frequency-dependent along-track beam-steering to mitigate streak pattern and Doppler ambiguity. The swath width was about 400 km and the spatial resolution was about 5 m. Based on the simulation parameters in [6], the resolution was improved from 20 m to 5 m, but is still not fine enough for practical HRWS applications, even for imaging a large target such as a ship.

Next, we will review some methods on imaging of ground moving targets. Taking steady images of ground moving targets (GMTs) is useful to many civic and military applications [2,7–9]. Some focusing techniques are required, especially when the aperture time is long or the target moves fast [10–12]. Typical approaches to focus the image of a GMT require the estimation of its motion parameters from features embedded in its backscattered signal, such as Doppler centroid, modulation rates, and other higher-order parameters.

In [13], an airborne SAR system, composed of one transceiver and multiple receivers, was proposed for GMT imaging, with the Doppler frequency and chirp rate estimated from an Lv distribution. Clearer images were acquired than those with Radon–Wigner transform (RWT) and fractional Fourier transform (FrFT), but were not clear enough for target identification. The method in [13] may be suitable for moving target detection, but the image quality is not fine enough for identification purpose.

In [14], an airborne SAR system, composed of one transceiver and multiple receivers, was used for GMT imaging. A third-order polynomial phase signal (PPS) model was used to focus the GMT of interest, with coefficients estimated using a minimum entropy method. Minimum entropy method may work on a point target, but may not work as well on more complex or large targets, which were not presented in [14]. In addition, the method of [14] could focus the image of one moving target at a time, but the other moving targets would appear blurred.
In [15], a SAR system composed of an airborne transmitter and a companion airborne receiver was applied to acquire image of a ship at squint angle of 60°, with resolution of 1 m, which maybe sufficient for imaging large targets such as ships, as compared with our resolution of 0.5 m. Moreover, no real data processing was verified in [15].

In [16], a three-channel circular SAR was proposed to survey a specific area, at resolution of 2 m. The motion parameters of GMTs were estimated, but the image quality was not sufficient for target identification. Although circular SAR is a feasible method for imaging fixed point target, it cannot be extended for large area surveillance since the illuminated area is fixed.

In [17], SAR imaging on ships over a vast ocean was studied. A squint minimization (SM) method was implemented for the compensation of third range compression, which was caused by ship translation. The squint angle was 70°, low PRF of 400 Hz was used to avoid range ambiguity, and the spatial resolution was 1 m, which may be sufficient for ship identification purpose. Moreover, no real data processing was verified in [17] and the image quality may degrade in real applications.

In [18], an azimuth multi-channel HRWS SAR was proposed to acquire images of moving ships, with swath width of 30 km and resolution of 3 m. With proper processing, the image is coarsely focused, then the Doppler parameters can be estimated and used in subsequent focusing process. However, resolution of 3 m maybe too large for identification purpose, only suitable for detecting large targets.

In [7], the Doppler frequency was estimated by curve fitting the signal trace after range compression. A modified Keystone transform was then applied to compensate the linear range cell migration (RCM). However, signals from a real target are contributed by a bunch of scatterers, their signal traces are mingled and difficult to separate.

In [19], a focusing method on maneuvering target was proposed by improving the Keystone transform with a coherently integrated cubic phase function (CICPF), but the acquired images were not well focused; maybe the CICPF was not accurate enough for compensating phase shift, or some higher-order phase term in the simulation data was too large.

In [20], an airborne bistatic forward-looking SAR (BFL-SAR) was proposed. The received signals were processed with Keystone transform to compensate the linear RCM, followed by a mismatched compression in range to estimate the Doppler parameters of GMT. However, this technique might become laborious to focus image of large moving targets.

Compressive-sensing technique can be applied on the received signal matrix to minimize the rank of clutter signal matrix and the zero-norm of target signal matrix. In [8], a robust principal component analysis (RPCA) was proposed for SAR imaging with a multi-channel airborne platform. The target signal matrix was further decomposed into three matrices, embedding the radial velocity, nearest range, number and echo strength, respectively, of the GMT. Images could thus be acquired under strong clutters, but the ignored along-track velocity could compromise the image quality.

An airborne SAR system for GMT imaging proposed in [21] required no prior knowledge of motion parameters. The received signals after range compression, Keystone transform and RCMC were reversely processed in slow time, taken conjugate, and multiplied with its original version to eliminate the quadratic phase term in slow time. The peak in the resulting signals in fast-time and Doppler-frequency domain was used to estimate the closest range and radial velocity. However, the along-track velocity of GMT was ignored, which could compromise the image quality.

In an airborne SAR imaging method on GMTs [22], RCMC was conducted by using a sequence of filters and Stolt interpolation after clutter removal. The best-fit of range and along-track velocity component were derived from the focused image. This method is basically a trial-and-error process, and the computational cost could be high.

In this work, we propose a multi-channel SAR on board a spaceplane orbiting near the top of atmosphere to acquire images of cruising ships. For surveillance over a vast
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ocean area, low pulse repetition frequency is required, inducing ADA and ASA issues. A modified azimuth spectrum reconstruction method on multi-channel signals is proposed. A compressive-sensing technique is applied to improve the estimation accuracy of the baseband Doppler frequency, with proper number and spacing of receivers, and a phase matching technique is proposed to determine the aliasing ambiguity number of each baseband spectrum segment. After all the spectrum segments are unwrapped, the Doppler ambiguity number is estimated from their slope.

The rest of this paper is organized as follows. The range model and signal model are presented in Section 2. The estimation methods of motion parameters and ambiguities, along with simulations on a moving point target, are presented in Section 3. The simulation results on five similar types of cruising ship are presented and discussed in Section 4. Conclusions are drawn in Section 5.

2. Range Model and Signal Model

Figure 2 shows the schematic of airborne multi-channel SAR imaging on a moving point target. The platform flies at a constant speed of $V$ in $y$ direction, at height $H$ right above the $y$ axis laid on the mean sea surface level. The platform carries a linear array of $M$ antennas aligned in $y$ direction, at uniform spacing $d$. The first antenna is connected to a transceiver and the other antennas are connected to passive receivers. Without loss of generality, set $\eta = 0$ when the first antenna passes above the origin.

The target area is centered at a reference point $(x_0, 0, 0)$. The $q$th point target is located at $(x_{q0}, y_{q0}, 0)$ at $\eta = 0$ and moves at speed $[v_{qx}, v_{qy}, 0]^T$, with $q = 1, \cdots, Q$, where $Q$ is the number of point targets. The slant range between the $m$th receiver and the $q$th point is given by

$$R_{mq}^q(\eta) = \left\{ (x_{q0} + v_{qx}\eta)^2 + (y_{q0} + (v_{qy} - V)\eta + (m - 1)d)^2 + H^2 \right\}^{1/2} \quad (1)$$

The total path length from the transmitting antenna to the $q$th point and back to receiving antenna $m$ is given by

$$R_m^q(\eta) = R_{mq}^q(\eta) + R_{mq}^q(\eta) \quad (2)$$

2.1. Simulation Scenario

Table 1 lists the default parameters used in the simulations. A spaceplane is assumed as the SAR platform in the simulations, orbiting at velocity of 3 mach and altitude of 100 km. A commonly used look angle of $\theta_l = 60^\circ$ is chosen. The radar beam covers the off-nadir angles of 45–70$^\circ$, rendering a swath width of 150 km with a furthest range of $H \sec 70^\circ \simeq 300$ km. To avoid range ambiguity, the PRF is set to 250 Hz, at which the round-trip time of signal to the furthest point on the swath is about one half of pulse repetition...
interval. The carrier frequency is \( f_0 = 10 \text{ GHz} \) [1,15,17]. The pulse width is \( T_c = 10 \mu s \) and the chirp rate is \( K_r = 35 \text{ THz/s} \), rendering range bandwidth of \( B_r = 350 \text{ MHz} \) and range resolution of 0.5 m. The aperture time is \( T_a = 11 \text{ s} \), leading to a cross-range resolution of 0.53 m. The array element spacing is \( d = 16.32 \text{ cm} \), and the element number \( M \) is chosen to facilitate the phase matching technique.

### Table 1. Default parameters in simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of antennas</td>
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<td>50</td>
</tr>
<tr>
<td>platform speed</td>
<td>( V )</td>
<td>1020 m/s</td>
</tr>
<tr>
<td>height of platform</td>
<td>( H )</td>
<td>100 km</td>
</tr>
<tr>
<td>carrier frequency</td>
<td>( f_0 )</td>
<td>10 GHz</td>
</tr>
<tr>
<td>antenna spacing</td>
<td>( d )</td>
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<tr>
<td>PRF</td>
<td>( F_a )</td>
<td>250 Hz</td>
</tr>
<tr>
<td>aperture time</td>
<td>( T_a )</td>
<td>11 s</td>
</tr>
<tr>
<td>chirp rate</td>
<td>( K_r )</td>
<td>35 THz/s</td>
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<tr>
<td>bandwidth</td>
<td>( B_r )</td>
<td>350 MHz</td>
</tr>
<tr>
<td>pulse width</td>
<td>( T_c )</td>
<td>10 (\mu)s</td>
</tr>
<tr>
<td>look angle</td>
<td>( \theta_\ell )</td>
<td>60°</td>
</tr>
<tr>
<td>reference point</td>
<td>((x_0, y_0))</td>
<td>(173.205, 0) km</td>
</tr>
</tbody>
</table>

In this section, the backscattered signals from a moving point target are processed to verify different stages of the proposed approach. Without loss of generality, the moving point target is located at \((x_{10}, y_{10}) = (173.205, 0) \) km at \( \eta = 0 \) and moves with velocity \((v_{x1}, v_{y1}) = (10, -10) \) m/s. More simulation results on different cruising ship models will be presented in Section 4.

### 2.2. Preliminary Processing

Linear frequency modulation (LFM) pulses are periodically emitted at rate of \( F_a \), with each pulse given by

\[
s_t(\tau) = \text{rect}\left(\frac{\tau}{T_c}\right)e^{j2\pi f_0 \tau}e^{j\pi K_r \tau^2}
\]

where \( \tau \) is the fast time and

\[
\text{rect}(t) = \begin{cases} 
  1, & |t| \leq 1/2 \\
  0, & \text{otherwise}
\end{cases}
\]

is a window function.

The backscattered signal from the \( q \)th point target to the \( m \)th receiving antenna, after demodulation, is given by

\[
s_{mb}^q(\tau, \eta) = \sigma_q \text{rect}\left(\frac{\tau - R_{mq}(\eta)/c}{T_c}\right)e^{j\pi K_r (\tau - R_{mq}(\eta)/c)^2} \text{rect}\left(\frac{\eta}{T_a}\right)e^{-j2\pi f_0 R_{mq}(\eta)/c}
\]

where \( \sigma_q \) is the backscattering coefficient of the \( q \)th point target. Figure 3a shows the magnitude of received signal \( s_{1b}^q(\tau, \eta) \).
Figure 3. Magnitude of (a) $s_{1b}^q(\tau, \eta)$, received signal after demodulation, (b) $s_{21}^q(\tau, \eta)$, received signal after range compression, and (c) $S_{41}^q(\tau, f_\eta)$, wrapped Doppler spectrum, with $\tau_0 = 2R_0^0/c$.

Figure 4 shows the flow-chart of the proposed multi-channel SAR imaging on sea-surface moving targets (SSMTs). Conventional processes of range compression, phase compensation, and range-cell migration correction (RCMC) are applied in sequence to the received signals at each of the $M$ receivers. Then, a compressive-sensing (CS) technique is applied to estimate the (wrapped) Doppler frequency $\tilde{f}_{qd}$ in the major band (baseband), followed by a phase-matching technique to resolve the azimuth spectrum aliasing (ASA) issue. The Doppler ambiguity number $N_{qd}$ is estimated from the slope of signal traces in the $\tau$-$f_\eta$ plane. Finally, a focused image of SSMTs (moving point target in this section) is acquired.

To begin with, take the range Fourier transform of $s_{mb}^q(\tau, \eta)$ to have

$$S_{mb}^q(f_\tau, \eta) = \mathcal{F}_\tau \{s_{mb}^q(\tau, \eta)\} \simeq c_1 \sigma q \text{rect} \left( \frac{f_\tau}{B_r} \right) e^{-j2\pi(f_0+f_\tau)R_0^q(\eta)/c} e^{-j\pi f_\tau^2/K_r} \text{rect} \left( \frac{\eta}{T_a} \right)$$

where $c_1$ is a constant of integration. Then, design a matched filter for range compression as

$$H_{mf}(f_\tau) = \mathcal{F}_\tau \{s_{b}^q(-\tau)\} \simeq c_2 \text{rect} \left( \frac{f_\tau}{B_r} \right) e^{j\pi f_\tau^2/K_r}$$

where $c_2$ is a constant of integration and $s_{b}(\tau)$ is the baseband signal of $s_{i}(\tau)$ in (3). Multiply (6) and (7) to obtain a range compressed signal

$$S_{1m}^q(f_\tau, \eta) = S_{mb}^q(f_\tau, \eta) H_{mf}(f_\tau) = c_2 c_1 \sigma q \text{rect} \left( \frac{f_\tau}{B_r} \right) e^{-j2\pi(f_0+f_\tau)R_0^q(\eta)/c} \text{rect} \left( \frac{\eta}{T_a} \right)$$
which is inverse Fourier transformed in range to have

\[
\tilde{s}_{2m}^{q}(\tau, \eta) = \mathcal{F}_{r}^{-1}\left\{ \tilde{s}_{1m}^{q}(f_{r}, \eta) \right\} = c_{1}c_{2}B_{r}e^{q}\text{sinc}\left\{ B_{r}[\tau - R_{m}^{q}(\eta)/c] / c \right\} \text{rect}\left( \eta / T_{a} \right) e^{-j2\pi f_{0}R_{m}^{q}(\eta)/c}
\]

(9)

Figure 3b shows the magnitude of the range-compressed signal \( \tilde{s}_{21}^{q}(\tau, \eta) \), where the signal trace is bent due to the target motion.

Determine a reference slow time \( \eta_{q} \) when the transceiver and the \( q \)th point target reach the same \( y \) coordinate, namely, \( V_{q} \eta = y_{q0} + v_{qy} \eta_{q} \), which leads to \( \eta_{q} = y_{q0} / (V - v_{qy}) \). The total path length \( R_{m}^{q}(\eta) \) can be approximated by the first three terms of its Taylor’s series in \( \eta \), about \( \eta_{q} \), as

\[
R_{m}^{q}(\eta) \approx R_{0}^{q} + R_{mr}^{q}(\eta_{q}) - \lambda_{0} f_{dem}^{q}(\eta - \eta_{q}) - \frac{\lambda_{0}}{2} K_{r}^{q}(\eta - \eta_{q})^{2}
\]

(10)

where

\[
R_{mr}^{q}(\eta_{q}) \approx R_{0}^{q} + \frac{(m - 1)^{2}d^{2}}{2R_{0}^{q}}
\]

(11)

\[
R_{0}^{q} = \sqrt{(x_{q0} + v_{qy} \eta_{q})^{2} + H^{2}}
\]

(12)

The Doppler frequency is given by

\[
f_{dem}^{q} = -\frac{1}{\lambda_{0} d\eta} R_{mr}^{q}(\eta_{q}) = f_{dm}^{q} + (1 - m) f_{em}^{q}
\]

(13)
where

\[ f_{dm}^q = - \frac{(x_0 + v_x \eta_q)v_x}{\lambda_0} \left( \frac{1}{R_0^q} + \frac{1}{R_{m}^{qR}(\eta_q)} \right) \]  \hspace{1cm} (14)

\[ f_{em}^q = \frac{(v_y - V)d}{\lambda_0 R_{m}^{qR}(\eta_q)} \]  \hspace{1cm} (15)

are the Doppler frequency base and Doppler frequency bias, respectively. The Doppler chirp rate is given by

\[
K_{rm}^q = -\frac{1}{\lambda_0} \frac{d^2}{d\eta^2} R_m^q(\eta_q) = -\frac{v_{qx}^2 + (v_y - V)^2}{\lambda_0 R_m^q} - \frac{[(x_0 + v_x \eta_q)v_y]^2}{\lambda_0 |R_{m}^{qR}(\eta_q)|^3} - \frac{v_{qx}^2 + (v_y - V)^2}{\lambda_0 R_{m}^{qR}(\eta_q)} - \frac{[(x_0 + v_x \eta_q)v_x + (m-1)d(v_y - V)]^2}{\lambda_0 |R_{m}^{qR}(\eta_q)|^3}
\]  \hspace{1cm} (16)

For convenience, the reference point \((x_0, 0, 0)\) is labeled as the zeroth point target \((q = 0)\). The total path length from the transmitting antenna to the reference point back to the \(m\)th receiving antenna is approximated by the first three terms of its Taylor’s series in \(\eta\), about \(\eta = 0\), as

\[
R_0^m(\eta) \simeq 2R_0^0 + \frac{(m-1)^2d^2}{2R_0^0} + \frac{(1-m)dV}{R_{m}^{qR}(0)} \eta + \left\{ \frac{V^2}{2R_0^0} + \frac{V^2}{2R_{m}^{qR}(0)} + \frac{(1-m)dV^2}{2|R_{m}^{qR}(0)|^3} \right\} \eta^2
\]  \hspace{1cm} (17)

where

\[
R_0^0(0) \simeq R_0^0 + \frac{(m-1)^2d^2}{2R_0^0}
\]  \hspace{1cm} (18)

\[
R_0^0 = \sqrt{x_0^2 + H^2}
\]  \hspace{1cm} (19)

2.3. ADA and ASA Issues

To take a quick glance at how ADA and ASA issues emerge, first take the azimuth Fourier transform of \(s_{2m}^q(\tau, \eta)\) as

\[
s_{3m}^q(\tau, f_\eta) = \mathcal{F}_\eta \left\{ s_{2m}^q(\tau, \eta) \right\} \simeq c_1c_2c_3B \sigma_\nu \sigma_\eta e^{-j4\pi f_\eta R_0^0 / \nu} \exp \left\{ -j\pi \frac{(m-1)^2d^2}{\lambda_0 R_0^0} \right\} \begin{aligned}
\text{sinc} \left\{ B_\tau \left[ \tau - \frac{1}{c} \left( 2R_0^0 + \frac{(m-1)^2d^2}{2R_0^0} - \frac{\lambda_0}{2R_{m}^{qR}(0)}(f_{\eta}^r - (f_{\eta}^{dem}))^2 \right) \right] \right\} \\
\text{rect} \left\{ \frac{f_\eta - f_{\eta}^{dem}}{K_{rm}^q T_a} + \frac{\eta_q}{T_a} \right\} e^{-j\pi(f_\eta - f_{\eta}^{dem})^2/2(K_{m}^{qR})^2} e^{-j2\pi f_\eta \eta_0} \end{aligned}
\]  \hspace{1cm} (20)

where \(c_3\) is another constant of integration. This Fourier transform is implemented on samples of \(s_{2m}^q(\tau, \eta)\) at uniform spacing of \(\Delta \eta\), resulting in samples of \(S_{3m}^q(\tau, f_\eta)\) at uniform spacing of \(\Delta f_\eta\), which are wrapped to the major band of \(f_\eta \in [-F_a/2, F_a/2]\) if the Doppler bandwidth of \(S_{2m}^q(\tau, \eta)\) is wider than \(F_a\). As a consequence, the Doppler frequency base \(f_{dm}^q\) is wrapped to its major-band counterpart, \(f_{dm}^q \in [-F_a/2, F_a/2]\), as

\[
f_{dm}^q = f_{dm}^q + N_0^q F_a
\]  \hspace{1cm} (21)
where \( N_d^q \) is the Doppler ambiguity number. Similarly, the Doppler frequency is wrapped to \( f_{\text{dem}}^q \) as

\[
f_{\text{dem}}^q = f_{\text{dem}}^q + N_d^q F_a
\]

with \( f_{\text{dem}}^q = (1 - m) f_{\text{em}}^q + f_{\text{dem}}^q \). The azimuth frequency \( f_q \) is wrapped to its major-band alias, \( f_q \in [-F_a/2, F_a/2] \), as

\[
f_q = f_q + (N_d^q + N_s) F_a
\]

where \( N_s \) is the aliasing ambiguity number.

In other words, the original signal \( S_{4m}^q(\tau, f_q) \) is cleaved to multiple spectrum segments of \( f_q \in [N_s F_a - F_a/2, N_s F_a + F_a/2] \), and all these segments are wrapped into the major band, \( f_q \in [-F_a/2, F_a/2] \), forming

\[
S_{4m}^q(\tau, f_q) = \sum_{N_s} c_1 c_2 c_3 B_q \eta_q e^{-j4\pi f_0 R_0^q/c} \exp \left\{ -j\pi \frac{(m - 1)^2 d^2}{\lambda_0 R_0^q} \right\}
\]

\[
sinc \left\{ B_q \left[ \tau - \frac{1}{c} \left( 2R_0^q - \tau^q_m(\eta_q) \right) \right] \right\} \left\{ f_q - (1 - m)f_{\text{em}}^q - \frac{\eta_q}{F_a} \right\}
\]

\[
\exp \left\{ -j\pi \left[ f_q - (1 - m)f_{\text{em}}^q - \frac{\eta_q}{F_a} \right] \right\} e^{-j2\pi(f_q + (N_d^q + N_s) F_a) \eta_q}
\]

where the range migration takes the explicit form of

\[
r_m^q(f_q) = \frac{\lambda_0}{2K_m^q} \left\{ f_q^2 - (1 - m)^2 f_{\text{em}}^q \right\} - 2(1 - m)f_{\text{em}}^q f_{\text{dem}}^q - (f_{\text{dem}}^q)^2 + 2(N_d^q + N_s) F_a f_q
\]

\[
-2N_d^q F_a \left[ (1 - m)f_{\text{em}}^q + f_{\text{dem}}^q \right] + (N_d^q + N_s)^2 F_a^2 - (N_d^q)^2 F_a^2 = \frac{(m - 1)^2 d^2}{2R_0^q}
\]

Figure 3c shows the range-compressed signal, \( S_{4m}^q(\tau, f_q) \), which contains multiple segments wrapped from the higher-order bands in \( f_q \). Before further processing, \( S_{4m}^q(\tau, f_q) \) must be properly unwrapped by resolving the ADA and ASA issues.

3. Estimation of Motion Parameters and Ambiguities

3.1. Range Cell Migration Correction

To reduce the complexity of the functional form in (24), we first try to remove the squared term \((f_{\text{em}}^q)^2\) in the phase of \( S_{4m}^q(\tau, f_q) \) by designing a phase compensation filter

\[
H_{\text{pem}}(f_q, m) = e^{j\pi(m-1)^2 f_{\text{em}}^q / R_m^q} \exp \left\{ j\pi \frac{(m - 1)^2 d^2}{\lambda_0 R_0^q} \right\} e^{-j\pi(1-m)f_{\text{em}}^q / R_m^q}
\]

which is a function of baseband Doppler frequency \( f_q \) and is applied to all the spectrum segments of different \( N_s \)’s. Spectrum segment of a specific \( N_s \) will be extracted by using a phase matching technique later. In (26), the parameters \( R_m^q \) and \( f_{\text{em}}^q \) are estimated as

\[
R_m^q \approx K_m^q = -\frac{V^2}{\lambda_0 R_0^q} - \frac{V^2}{\lambda_0 R_{\text{mir}}^q(0)} - \frac{|(1 - m) V d^2|}{\lambda_0 |R_{\text{mir}}^q(0)|^3}
\]

\[
f_{\text{em}}^q \approx -\frac{V d}{\lambda_0 R_{\text{mir}}^q(\eta_q)}
\]
where we assume $|v_{xy}| \ll V_r (m-1)d \ll R_0^0$ and $R_0^q \simeq R_0^0$. Note that $R_{mr}^q(\eta_q)$ can be further approximated as

$$R_{mr}^q(\eta_q) \simeq R_0^0 + \frac{(m-1)^2d^2}{2R_0^0} \tag{29}$$

From (14), $f_{dm}^q$ is further approximated as

$$f_{dm}^q \simeq -\frac{(x_{q0} + v_{qx\eta_q})v_{qx}^2}{\lambda_0 R_0^q} = f_d^q \tag{30}$$

which is substituted into (21) to have

$$f_{dm}^q \simeq f_d^q - N_d^q F_a = f_d^q \tag{31}$$

which is independent of $m$. By multiplying the phase compensation filter $H_{pc1}(f_q, m)$ with $S_{4m}^q(\tau, f_q)$, we obtain

$$S_{6m}^q(\tau, f_q) = H_{pc1}(f_q, m)S_{4m}^q(\tau, f_q) = \sum_{N_a} c_1 c_2 c_3 B_r c_r e^{-j4\pi f_0 R_0^0/c}$$

\[ \times \text{sinc} \left\{ B_r \left[ \tau - \frac{1}{c} (2R_0^q - r_m^q(f_q)) \right] \right\} \text{rect} \left( \frac{f_q - (1-m)f_{em}^q - f_a^q + N_a F_a}{K_{rm}^q T_a} + \frac{\eta_q}{T_a} \right) \]

\[ \exp \left\{ -j2\pi c(1-m)(f_d^q - N_a F_a)^2 / K_{rm}^q \right\} \]

\[ e^{-j2\pi(1-m) (f_d^q - N_a F_a)^2 / K_{rm}^q} e^{-j2\pi N_a F_a/\eta_q} \]

which is Fourier transformed in range to have

$$S_{6m}^q(f_r, f_q) = F(f_r, S_{6m}^q(\tau, f_q)) = \sum_{N_a} c_1 c_2 c_3 B_r c_r e^{-j4\pi f_0 R_0^0/c} \frac{1}{B_r} \text{rect} \left( \frac{f_r}{B_r} \right)$$

\[ \times \exp \left\{ -j2\pi c (2R_0^q - r_m^q(f_q)) / B_r \right\} \text{rect} \left( \frac{f_q - (1-m)f_{em}^q - f_a^q + N_a F_a}{K_{rm}^q T_a} + \frac{\eta_q}{T_a} \right) \]

\[ \exp \left\{ -j2\pi c(1-m)(f_d^q - N_a F_a)^2 / K_{rm}^q \right\} \]

\[ e^{-j2\pi(1-m) (f_d^q - N_a F_a)^2 / K_{rm}^q} e^{-j2\pi N_a F_a/\eta_q} \]

Next, we try to reduce the complexity of functional form in the range migration $r_{dm}^q(f_q)$ by designing an RCMC filter

$$H_{rcmc1}(f_r, m) = \exp \left\{ -j2\pi f_r \left[ -\frac{\lambda_0}{2c K_{rm}^q} (1-m)^2 (f_{em}^q)^2 - \frac{(m-1)^2d^2}{2c R_0^q} \right] \right\} \tag{34}$$
where \( \bar{K}_{rm}^q \) and \( f_{em}^q \) are given in (27) and (28), respectively. By multiplying (33) with (34), we obtain

\[
S_{7m}^q(f_r, f_\eta) = H_{rcme}(f_r, m)S_{6m}^q(f_r, f_\eta) = \sum_{N_v} c_1 c_2 c_3 B_{r} e^{-j4\pi f_0 R^0_{rm}/c} \frac{1}{B_r} \text{rect} \left( \frac{f_r}{B_r} \right) \\
\exp \left\{ -j2\pi f_r \left[ 2R^0 - \Delta r^q_{rm}(f_\eta) \right] \right\} \text{rect} \left( \frac{f_\eta - (1 - m)f_{em}^q - \frac{f_{em}^q N_v F_d}{K_{rm}^q T_a} + \frac{\eta_q}{T_a}}{K_{rm}^q} \right) \\
\exp \left\{ -j\pi \left[ f_{em}^q - 2f_\eta f_{em}^q - N_v F_d \right] + (f_{em}^q - N_v F_d)^2 \right\} \\
e^{-2\pi j(1 - m)(f_{em}^q N_v F_d) / 2} e^{-j2\pi (f_\eta + N_v F_d) / \eta_q} (35)
\]

where the range migration \( r^q_{rm}(f_\eta) \) has been reduced to a residual range migration,

\[
\Delta r^q_{rm}(f_\eta) \approx \frac{\lambda_0}{2K_{rm}^q} \left[ (f_{em}^q - f_{em}^q)^2 + 2(N_v^2 + N_r) F_d f_\eta - 2N_v^2 F_d f_{em}^q \right. \\
\left. + (N_v^2 + N_r^2) F_d^2 - (N_v^2 F_d)^2 \right] = \Delta r^q(f_\eta)
\]

which is independent of \( m \). Note that \( \frac{\lambda_0}{2K_{rm}^q} - 2(1 - m)f_{em}^q f_r \approx \lambda_0 \frac{2N_v^2 F_d (1 - m)f_{em}^q}{K_{rm}^q} \) and \( \frac{\lambda_0}{K_{rm}^q} (N_v^2 + N_r^2) F_d f_\eta \) are neglected under the parameters considered in this work.

In addition, \( K_{rm}^q \) in (27) can be further approximated as

\[
K_{rm}^q \approx -\frac{2V^2}{\lambda_0 R^0_{rm}} = K_r^q (37)
\]

where \( R^0_{rm}(0) \approx R^0 \) and \( (m - 1)d \ll R^0_{rm}(0) \). Similarly, \( f_{em}^q \) in (28) can be further approximated as

\[
f_{em}^q \approx -\frac{Vd}{\lambda_0 R^0_{rm}} (38)
\]

where \( R^0_{rm}(\eta_q) \approx R^0_{rm}(0) \approx R^0 \). Now, both \( K_{rm}^q \) and \( f_{em}^q \) are independent of \( m \), and their ratio becomes

\[
\alpha = \frac{f_{em}^q}{K_{rm}^q} \approx -\frac{(v_{yy} - V) d}{\lambda_0 R^0_{rm}} \approx \frac{d}{2(v_{yy} - V)} \approx \frac{d}{2V} \approx \frac{d}{2V} (39)
\]

where (15) and (16) are used, \( R^0_{rm}(\eta_q) \approx R^0_{tr} \) and \( |v_{xy}|, |v_{yy}| \ll V \). Thus, the shift in the argument of the rect function in (35), \( \frac{f_{em}^q}{K_{rm}^q T_a} \approx \frac{d}{2VT_a} \ll 1 \), can be neglected.

With all these efforts, the inverse Fourier transform of \( S_{7m}^q(f_r, f_\eta) \) in range becomes

\[
S_{8m}(\tau, f_\eta) = F^{-1} \{ S_{7m}^q(f_r, f_\eta) \} = \sum_{N_v} A_{N_v}(\tau, f_\eta) e^{j2\pi (m - 1) \eta_q (f_{em}^q - N_v F_d)} (40)
\]
where \( m \) appears only in the phase term, and

\[
A_{N_f}(\tau, f_\eta) \simeq c_1 c_2 c_3 B_\tau c_\eta e^{-j2\pi f_\eta R_3^3/\tau} e^{-j2\pi \left[ (N_f^0 + N_f) f_\eta \right]} \frac{1}{\tau} \text{sinc} \left( \frac{2R_3^3 - \Delta r^3(f_\eta)}{\tau} \right) \text{rect} \left( \frac{f_\eta - f_\eta^0 + N_f F_a}{K_d T_a} + \eta/2 \right) \exp \left\{ -j\pi D_f \left[ 2f_\eta(f_\eta^0 - N_f F_a) + (f_\eta^0 - N_f F_a)^2 \right] \right\}
\]

(41)
is the \( N_f \)th spectral segment of the unwrapped signals.

3.2. Estimation of Doppler Shift in Major Band

Conventional compressive-sensing (CS) technique is applied to estimate the Doppler shift in the major band. Define a sparse vector with complex elements,

\[
\hat{A}_f = [A_{f1}, \ldots, A_{f_{u}}, \ldots, A_{f_{N_f}}]^t
\]

(42)
on a grid

\[
\hat{f}_d = [f_{d1}, \ldots, f_{d_{u}}, \ldots, f_{d_{N_f}}]^t
\]

(43)
with \( f_{dn} = -F_a/2 + (n - 1) F_a/N_f \). The sparse vector is determined by applying the CVX to solve an optimization problem [23]

\[
\hat{A}_f = \arg \min_{\hat{A}_f} \| \hat{A}_f \|_1, \quad \text{under the constraint of}
\]

\[
\| \tilde{D}_f \cdot \hat{A}_f - S^q_S(\tau, f_\eta) \|_2 < \varepsilon
\]

(44)
where \( \varepsilon \) is a tolerance, \( \tilde{D}_f \) and \( S^q_S(\tau, f_\eta) \) are given by

\[
\tilde{D}_f = \begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\varepsilon^{j2\pi(m-1)\alpha(N_f F_a - f_d)} & \cdots & \varepsilon^{j2\pi(m-1)\alpha(N_f F_a - f_{d_{N_f}})} \\
\vdots & \ddots & \vdots \\
\varepsilon^{j2\pi(M-1)\alpha(N_f F_a - f_d)} & \cdots & \varepsilon^{j2\pi(M-1)\alpha(N_f F_a - f_{d_{N_f}})}
\end{bmatrix}
\]

(45)

\[
S^q_S(\tau, f_\eta) = [S^q_S(\tau, f_\eta), \ldots, S^q_S(\tau, f_\eta), \ldots, S^q_S(\tau, f_\eta)]^t
\]

(46)
Based on (40), \( S^q_S(\tau, f_\eta) \) is a product of complex amplitude \( A_{N_f}^q(\tau, f_\eta) \) and phase term \( e^{j2\pi(m-1)\alpha(f_\eta^0 - N_f F_a)} \). The idea of CS in this case is to take the least number of Doppler-shift grid points, by minimizing the first norm of \( \hat{A}_f \), under the constraint that the reconstructed signal, \( \tilde{D}_f \cdot \hat{A}_f \), closely resembles \( S^q_S(\tau, f_\eta) \). Figure 5 shows the sparse vector \( \hat{A} \) for estimating \( f_\eta^0 \). The estimated Doppler frequency is \( f_\eta^0 = -77.5 \text{ Hz} \), compared with the true value of \( f_\eta^0 = -77.3502 \text{ Hz} \).
3.3. Estimation of y-Speed and Retrieval of Spectral Segments

Equation (39) implies that the y component of target speed can be estimated as

$$\tilde{v}_{qy} \simeq V - \frac{d}{2\tilde{\alpha}}$$

(47)

which relies on an estimated value of $\tilde{\alpha}$. Similar CS technique is applied to estimate the value of $\alpha$ by first defining a sparse vector

$$\tilde{A}_\alpha = [A_{\alpha 1}, \cdots, A_{\alpha n}, \cdots, A_{\alpha Na}]^T$$

(48)

on a grid

$$\tilde{\alpha} = [\alpha_1, \cdots, \alpha_n, \cdots, \alpha_{Na}]^T$$

(49)

The sparse vector is determined by applying the CVX to solve an optimization problem [23]

$$\tilde{A}_\alpha = \arg\min_{A} \|\tilde{A}_\alpha\|_1 \text{ under the constraint of }$$

$$\left\|\tilde{D}_\alpha \cdot \tilde{A}_\alpha - \tilde{S}_q^8(\tau, f_\eta)\right\|_2 < \varepsilon$$

(50)

where

$$\tilde{D}_\alpha = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ e^{i2\pi (M-1)\alpha_1 (N_s F_a - f_\eta d)} & \cdots & e^{i2\pi (M-1)\alpha_n (N_s F_a - f_\eta d)} \\ \vdots & \ddots & \vdots \\ e^{i2\pi (M-1)\alpha_1 (N_s F_a - f_\eta d)} & \cdots & e^{i2\pi (M-1)\alpha_n (N_s F_a - f_\eta d)} \end{bmatrix}$$

(51)

The estimated $\alpha$ is substituted into (47) to obtain an estimated y-speed of $	ilde{v}_{qy} = 9.8 \text{ m/s}$, compared with the true value of $v_{qy} = 10 \text{ m/s}$.

Next, the $N_s$th spectral segment is retrieved by multiplying $e^{i2\pi (m-1)\tilde{\alpha} (N_s F_a - \tilde{f}_d)}$ to (40) and summing over $1 \leq m \leq M$ to have

$$S_q^q(\tau, f_\eta) = \sum_{m=1}^{M} e^{i2\pi (m-1)\tilde{\alpha} (N_s F_a - \tilde{f}_d)} S_{8m}^q(\tau, f_\eta)$$

$$\simeq \sum_{m=1}^{M} \sum_{N'_s} A_{N'_s}^q(\tau, f_\eta) e^{i2\pi (m-1)\tilde{\alpha} (\tilde{f}_d - \tilde{f}_q d)} + (N_s - N'_s) F_a$$

(52)
Now, the phase matching technique is applied to the multichannel signals. If \( \tilde{f}_d \) is an accurate estimation of \( f_d \), the summation will reduce to

\[
\sum_{m=1}^{M} e^{j2\pi(m-1)\alpha(N_s-N'_s)F_a} = \begin{cases} 
M, & N_s = N'_s \\
0, & N_s \neq N'_s
\end{cases}
\]

where \( \alpha F_a = \frac{d}{2V}F_a = 0.02 \) and \( M = 1/(\alpha F_a) = 50 \). Figure 6 shows the schematic of adding phasors in (53) under two different conditions between \( N'_s \) and \( N_s \), given that \( M\alpha F_a = 1 \).

The spectrum segment with index \( N'_s = N_s \) in \( S_{q_m}^{(8)}(\tau, f_\eta) \) of (52), which matches the phase of \( e^{j2\pi(m-1)\alpha(N_s-N'_s)F_a} \), will add up with its counterparts from \( M \) receivers to reach \( M \), while the spectrum segments with index \( N'_s \neq N_s \) from \( M \) receivers will add out of phase to zero. Thus, the \( N_s \)th spectrum segment is extracted from (52) as

\[
S_{q_s}^{(9)}(\tau, f_\eta) \simeq MA_{N_s}q_\eta(\tau, f_\eta)
\]

**3.4. Estimation of Doppler Ambiguity Number**

The unwrapped Doppler frequency in the \( N_s \)th spectral segment corresponding to \( f_\eta \) is given by

\[
f'_\eta = f_\eta + N_s F_a = f_\eta - N'_s F_a
\]

where (23) is used to relate \( f'_\eta \) to the Doppler ambiguity number \( N'_d \). The signal in (54) is then rephrased in terms of \( f'_\eta \) as

\[
S_{q_s}^{(9)}(\tau, f'_\eta) = c_1c_2c_3B_r\sigma_q Me^{-j4\pi f_\eta R_0^d/c}\text{sinc}\left\{ B_r \left[ \tau - \frac{1}{c} \left( 2R_0^d - \Delta r^d(f'_\eta) \right) \right] \right\} \text{rect}\left( \frac{f'_\eta - f_d^d}{K_r^d T_a} + \frac{\eta_q}{T_a} \right) e^{-j2\pi(f_d^d_Nf_\eta)} \exp\left\{ -j\pi \frac{f_{\eta}^2 - 2f'_\eta f_d^d + (f_d^d)^2}{K_r^d} \right\}
\]

where the residual range cell migration in (36) is rephrased as

\[
\Delta r^d(f'_\eta) = \frac{\lambda_0}{2K_r^d}\left[ (f'_\eta)^2 - (f_d^d)^2 + 2N'_d F_a f'_\eta - 2N_d F_a f_d^d \right]
\]

Figure 7a shows the magnitude of \( S_{q_s}^{(9)}(\tau, f'_\eta) \) in (56). The spectral segments in Figure 3c have been stitched into one continuous trace.
Next, take the range Fourier transform of (56) to have

\[
S_{10}^{q}(f_{r}, f_{\eta}') = \mathcal{F}_{r}\{S_{0}^{q}(\tau, f_{\eta}')\} = c_{1}c_{2}c_{3}\sigma_{q}M e^{-j4\pi f_{\eta}' R_{0}^{q}/c} \text{rect}\left(\frac{f_{r}}{B_{r}}\right) \\
\exp\left\{-j2\pi f_{r} [2R_{0}^{q} - \Delta \tau^{q}(f_{\eta}')]\right\} \text{rect}\left(\frac{f_{\eta}' - f_{d}^{q}}{K_{d}^{q} T_{d}} + \frac{\eta_{q}}{T_{a}}\right) e^{-2\pi(\eta_{q}^{2} + N_{d}^{q} f_{d})}\eta_{t} \\
\exp\left\{-j\pi \frac{f_{\eta}'^{2} - 2f_{\eta}' f_{d}^{q} + (f_{d}^{q})^{2}}{K_{d}^{q}}\right\}
\]  

(58)

To compensate for the \(f_{\eta}'^{2}\) term in \(\Delta \tau^{q}(f_{\eta}')\), design a second RCMC filter

\[
H_{\text{rcmc2}}(f_{r}, f_{\eta}') = \exp\left\{-j2\pi f_{r} \frac{\lambda_{0}}{2cK_{d}^{q} f_{\eta}^{2}}\right\}
\]

and multiply it with (58) to obtain

\[
S_{11}^{q}(f_{r}, f_{\eta}') = H_{\text{rcmc2}}(f_{r}, f_{\eta}') S_{10}^{q}(f_{r}, f_{\eta}') = c_{1}c_{2}c_{3}\sigma_{q}M e^{-j4\pi f_{\eta}' R_{0}^{q}/c} \text{rect}\left(\frac{f_{r}}{B_{r}}\right) \exp\left\{-j2\pi f_{r} [2R_{0}^{q} - \Delta \tau^{q}(f_{\eta}')]\right\} \\
\text{rect}\left(\frac{f_{\eta}' - f_{d}^{q}}{K_{d}^{q} T_{d}} + \frac{\eta_{q}}{T_{a}}\right) e^{-2\pi(\eta_{q}^{2} + N_{d}^{q} f_{d})}\eta_{t} \exp\left\{-j\pi \frac{f_{\eta}'^{2} - 2f_{\eta}' f_{d}^{q} + (f_{d}^{q})^{2}}{K_{d}^{q}}\right\}
\]

(60)

where the residual cell migration is reduced to

\[
\Delta \tau^{q}(f_{\eta}') = \gamma_{q}(N_{d}^{q}) f_{\eta}' + c_{q}(N_{d}^{q})
\]

with

\[
\gamma_{q}(N_{d}^{q}) = \frac{\lambda_{0} N_{d}^{q} f_{d}}{K_{d}^{q}}
\]

(62)

\[
c_{q}(N_{d}^{q}) = -\frac{\lambda_{0}}{2K_{d}^{q}} (f_{d}^{q})^{2} + 2N_{d}^{q} f_{d} f_{d}^{q}
\]

(63)

Then, take the inverse Fourier transform of (60) in range to have

\[
S_{12}^{q}(\tau, f_{\eta}') = \mathcal{F}_{r}^{-1}\{S_{11}^{q}(f_{r}, f_{\eta}')\} = c_{1}c_{2}c_{3}B_{r} \sigma_{q} M e^{-j4\pi f_{\eta}' R_{0}^{q}/c} \text{sinc}\left\{B_{r} \left(\tau - \frac{1}{2}(2R_{0}^{q} - \Delta \tau^{q}(f_{\eta}'))\right)\right\} \\
\text{rect}\left(\frac{f_{\eta}' - f_{d}^{q}}{K_{d}^{q} T_{d}} + \frac{\eta_{q}}{T_{a}}\right) e^{-2\pi(\eta_{q}^{2} + N_{d}^{q} f_{d})}\eta_{t} \exp\left\{-j\pi \frac{f_{\eta}'^{2} - 2f_{\eta}' f_{d}^{q} + (f_{d}^{q})^{2}}{K_{d}^{q}}\right\}
\]

(64)

Figure 7b shows the magnitude of \(S_{12}^{q}(\tau, f_{\eta}')\). The curved trace in Figure 7a has been straightened.

The slope parameter \(\gamma_{q}(N_{d}^{q})\) in (62) is estimated by applying Hough transform to \(|S_{12}^{q}(\tau, f_{\eta}')|\), and \(N_{d}^{q}\) is estimated as

\[
N_{d}^{q} = \text{round}\left\{\frac{K_{d}^{q} \gamma_{q}(N_{d}^{q})}{\lambda_{0} f_{d}}\right\}
\]

(65)
An unambiguous Doppler frequency \( f_\eta \) is derived as \( f_\eta = f'_\eta + N_d^q F_a \), then \( S_{12}^q(\tau, f_\eta') \) is rephrased in terms of \( f_\eta \) as

\[
S_{12}^q(\tau, f_\eta) = c_1 c_2 c_3 B_r \sigma_q M e^{-j4\pi f_0 R^q_0/c} e^{-j2\pi f_\eta \eta_0} \quad \text{sinc} \left\{ B_r \left[ \tau - \frac{1}{c} \left( 2R^q_0 - \Delta r^q_f(f_\eta) \right) \right] \right\} \text{rect} \left\{ \frac{f_\eta - f_d^q}{K_d^q T_a} + \frac{\eta_q}{T_a} \right\} \\
\exp \left\{ -j\pi \left( f_\eta - N_d^q F_a \right)^2 - 2 \left( f_\eta - N_d^q F_a \right) f_d^q + (f_d^q)^2 \right\}
\] (66)

Figure 7c shows the magnitude of \( S_{12}^q(\tau, f_\eta) \), where the Doppler ambiguity number is estimated as \( N_d^q = -2 \), implying an azimuth frequency shift of \(-2F_a = -500 \text{ Hz}\) from that in Figure 7b.

3.5. Estimation of x-Speed and Image Focusing

By using (30), \( v_{qx} \) is estimated as

\[
\tilde{v}_{qx} \simeq -\frac{\lambda_0 R^q_0}{2x_q0} f_d^q = -\frac{\lambda_0 R^q_0}{2x_q0} (F_d^q + N_d^q F_a)
\] (67)
where $|v_{qz}| \ll x_{q0}$ is assumed. With $\tilde{K}_d^q = -77.5$ Hz, we have $\delta_{qz} \simeq -10.0026$ m/s, compared with the true value of $v_{qz} = -10$ m/s. From (16), $K_{rm}^q$ is estimated as

$$K_{rm}^q \simeq -2\frac{v^2 + \sigma_{qz}^2}{\lambda_0 R_0^q} = K_r^q$$  \hfill (68)

based on which a second phase compensation filter is designed as

$$H_{pc2}(f_q) = e^{j4\pi f_0 R_0^q/c} \exp \left\{ j\pi \frac{(f_q - K_r^q F_a)^2 - 2(f_q - N_d^q F_a)K_r^q + (K_r^q)^2}{K_r^q} \right\}$$  \hfill (69)

which is multiplied with $S_{12}^q(\tau, f_q)$ to have

$$S_{13}^q(\tau, f_q) = H_{pc2}(f_q)S_{12}^q(\tau, f_q) = c_1 c_2 c_3 B_r \sigma_q M \text{rect} \left( \frac{f_q - f_r^q}{K_r^q T_a} + \frac{\eta_q}{T_a} \right) e^{-j2\pi f_q \eta_q}$$  \hfill (70)

The argument in sinc function indicates an azimuth-dependent residual range cell migration $\Delta r_1^q(f_q)$, which will be compensated next.

Take the range Fourier transform of $S_{13}^q(\tau, f_q)$ to have

$$S_{14}^q(f_r, f_q) = \mathcal{F}_r \left\{ S_{13}^q(\tau, f_q) \right\} = c_1 c_2 c_3 B_r \sigma_q M \text{rect} \left( \frac{f_r}{B_r} \right) e^{-j2\pi f_r \Delta r_1^q(f_q)} \text{rect} \left( \frac{f_q - f_r^q}{K_r^q T_a} + \frac{\eta_q}{T_a} \right) e^{-j2\pi f_q \eta_q}$$  \hfill (71)

Design a third RCMC filter as

$$H_{rcmc3}(f_r, f_q) = e^{-j2\pi f_r \Delta r_1^q(f_q)}/c = e^{-j2\pi f_r [\gamma_q(N_d^q)(f_q - N_d^q F_a) + c_9(N_d^q)]}/c$$  \hfill (72)

where $\gamma_q(N_d^q)$ and $c_9(N_d^q)$ are computed by using (62) and (63), respectively. Apply the third RCMC filter to $S_{14}^q(f_r, f_q)$ to obtain

$$S_{15}^q(f_r, f_q) = H_{rcmc3}(f_r, f_q)S_{14}^q(f_r, f_q) = c_1 c_2 c_3 B_r \sigma_q M$$

$$\text{rect} \left( \frac{f_r}{B_r} \right) e^{-j4\pi f_r K_r^q/c} \text{rect} \left( \frac{f_q - f_r^q}{K_r^q T_a} + \frac{\eta_q}{T_a} \right) e^{-j2\pi f_q \eta_q}$$  \hfill (73)

which is inverse Fourier transformed in range to have

$$S_{16}^q(\tau, f_q) = \mathcal{F}_r^{-1} \left\{ S_{15}^q(f_r, f_q) \right\} = c_1 c_2 c_3 B_r \sigma_q Me^{-j2\pi f_q \eta_q}$$

$$\text{sinc} \left\{ B_r \left( \tau - \frac{2R_0^q}{c} \right) \right\} \text{rect} \left( \frac{f_q - f_r^q}{K_r^q T_a} + \frac{\eta_q}{T_a} \right) e^{-j2\pi f_q \eta_q}$$  \hfill (74)

where the argument in sinc function indicates a focus in range at $\tau = 2R_0^q/c$. Figure 8a shows the magnitude of $S_{16}^q(\tau, f_q)$. The signal trace becomes a vertical line after correcting
the range-cell migration. Finally, take the inverse Fourier transform of (74) in azimuth to have

\[
s_{17}^{\theta}(\tau, \eta) = F^{-1}_\eta \left\{ S_{16}^{q}(\tau, f_{\eta}) \right\} = c_1c_2c_3B_r |T_a\sigma_q M\text{sinc} \left\{ B_r \left( \tau - \frac{2K_q}{c} \right) \right\} \\
s\text{sinc} \{ K_q T_a(\eta - \eta_q) \} e^{-j2(\eta - \eta_q)/f_q} e^{jK_q r T_a(\eta - \eta_q)}
\]

(75)

Figure 8b shows the final image, \(s_{17}^{\theta}(\tau, \eta)\), mapped to the xy plane. The moving point target has been focused to a steady point target.

4. Simulations and Discussions

In this section, the proposed approach is applied to acquire the images of five similar types of cruising ship, using the default parameters listed in Table 1. The first one is an Arleigh Burke-class destroyer, which moves southeast at speed of 27.5 knots (\(\simeq 14.14 \text{ m/s}\)) [24]. The SAR image of sea surface as shown in Figure 9 [25] is used to simulate the backscattered signals from the sea surface. The sea waves move at 3 m/s in the direction perpendicular to the wave crests, under wind speed of 5 m/s [26]. The wakes move along with the ship stern. In the simulations, the background sea-wave pattern moves at 3 m/s in the direction perpendicular to the wave crests, and the ship wake is attached to the ship stern. The relative magnitudes of backscattered signals from sea surface, wakes, and the ship models are adjusted to be compatible to the scenario shown in Figure 9.

Figure 9. SAR image of sea surface [25].

4.1. Imaging on Various Destroyers

Figure 10a shows the magnitude of received signals after demodulation. The backscattered signals from the ship model are much stronger than that from the sea surface. Figure 10b shows the signal in the range-Doppler frequency domain after phase compensation with (26) and RCMC with (34). Compared with Figure 3c of a point target, with
signal segments clearly separated, Figure 10b displays complicated signal streaks attributed to multiple scatterers on the ship.

![Figure 10](image-url)

**Figure 10.** Magnitude of signals for imaging an Arleigh Burke-class destroyer, (a) $s_{q1}^1(\tau, \eta)$, received signal after demodulation, (b) $s_{q41}^1(\tau, f_\eta)$, wrapped Doppler spectrum, and (c) $s_{q9}^1(\tau, f_\eta')$, unwrapped Doppler spectrum after solving ASA issue.

A bunch of samples with significant amplitude are selected from Figure 10b and an $\hat{f}_d$ is estimated on each sample. The average of these estimated values is $\bar{f}_d = -77.3$ Hz, compared with the true value of $f_d = -77.174$ Hz.

Next, $\alpha$ is estimated as $\hat{\alpha} \approx 7.92618 \times 10^{-5}$ s, compared with the true value of $\alpha \approx 7.92233 \times 10^{-5}$ s. Then, $v_{qy}$ is estimated with (47) as $v_{qy} \approx -9.5$ m/s, compared with the true value of $v_{qy} = -10$ m/s. Figure 10c shows the signal $s_{q9}^1(\tau, f_\eta')$ after unwrapping the signal streaks in Figure 10b.

Figure 11a shows signal $s_{q12}^1(\tau, f_\eta')$ after applying the second RCMC filter in (59). The residual range cell migration $\Delta r_{q12}^1(f_\eta')$ in (64) indicates slant lines, of which the slope can be estimated with Hough transform. Then, the Doppler ambiguity number is estimated by using (65) as $N_{q12}^1 = -2$, which is also the true value. Figure 11b shows the magnitude of $S_{q12}^1(\tau, f_\eta)$, with the Doppler frequency of signals shifted by $-2F_a = -500$ Hz from that in Figure 11a. With the estimated $N_{q12}^1$, the second phase compensation filter in (69) and the third RCMC filter in (72) are applied to obtain the signal $s_{q16}^1(\tau, f_\eta)$, as shown in Figure 11c.
Figure 12a shows the acquired SAR image $s_{12}(\tau, \eta)$, mapped to the $xy$ plane, of the Arleigh Burke-class destroyer and its wakes on sea surface, with all the scatterers frozen during the aperture time.

Typical sea-surface height in a short time frame can be approximated as a superposition of standing-wave patterns at different wavenumbers, moving slowly in specific direction. In this work, time-varying profile of sea-surface height is adopted to demonstrate its relative scattering intensity in the acquired image. An improvised moving sea surface is simulated by shifting a radar image of sea surface at 3 m/s along an assumed wind direction during the aperture time of 11 s. The backscattering signal from a wake is also taken from a radar image, which is assumed to move along with the target ship. The relative intensities of the backscattering signals from both sea surface and wake are calibrated with respect to that of a ship in the same radar image.

Figure 12b shows the acquired SAR image of the ship dragging its wakes on the sea surface, with the waves moving at given speeds just mentioned. The image is well focused and the Arleigh Burke-class destroyer is well recognizable. The image intensity of the sea surface in Figure 12b is much weaker than that in Figure 12a. As the wavy sea surface moves during the aperture time, the backscattered signals from a given spatial cell vary in amplitude and phase with respect to slow time, smoothing out their contributions during signal processing. In addition, the Doppler frequency embedded in the sea scattered signals is different from that of the ship, the algorithm designed to focus the image of ship tends to blur the sea surface.
To verify these thoughts, consider a cruising ship on a static sea surface, with the acquired image shown in Figure 12c. The image of ship remains the same due to its dominant scattering signals. The features of sea surface is similar to those in Figure 12a, but their intensity becomes weaker since the Doppler frequency embedded in the sea scattered signals is different from that in the ship scattered signals. However, the features on sea surface are more obvious than those in Figure 12b because a static sea surface scatters more steady signals than a moving one does. Figure 12d shows the scan image of an Arleigh Burke-class destroyer from its 3D model. These images consistently indicate strong scattering parts in radar aerials, mast, bridge, and superstructure that have ridged edges.

Figure 13 shows the images of a Ticonderoga-class cruiser, cruising in the same direction and speed as the Arleigh Burke-class destroyer. Figure 13a shows the acquired SAR image on moving sea surface. It has similar size and shape but heftier superstructure than an Arleigh Burke-class destroyer.

The estimated Doppler frequency is $\tilde{f}_d = -76.78$ Hz, compared with the true value of $f_d = -77.174$ Hz. The estimated Doppler ambiguity number is $\tilde{N}_d = -2$, which is the true value. The velocity in $y$ direction is estimated as $\tilde{v}_y \approx -9.7$ m/s, compared with the true value of $v_y = -10$ m/s. The velocity in $x$ direction is estimated as $\tilde{v}_x \approx 9.99$ m/s, compared with the true value of $v_x = 10$ m/s. Figure 13b shows the scan image from
its 3D model. The strong scattering parts include radars, mast, and some edges on the superstructure.

![Figure 13](image)

Figure 13. (a) Acquired image of Ticonderoga-class cruiser, in scale of (40, 90) dB and (b) scan image of 3D model.

Figure 14 shows the images of a Slava-class cruiser, cruising at the same speed as the Arleigh Burke-class destroyer, but in southwest direction. The Slava-class cruiser has quite distinctive appearance from Arleigh Burke-class destroyer or Ticonderoga-class cruiser, and is larger in size than the latter two.

![Figure 14](image)

Figure 14. (a) Acquired image of Slava-class cruiser, in scale of (40, 90) dB and (b) scan image of 3D model.

The Doppler frequency is estimated as $\tilde{f}_q = 77.5 \text{ Hz}$, compared with the true value of $f_q = 77.174 \text{ Hz}$. The Doppler ambiguity number is estimated as $\tilde{N}_q = 2$, same as the true value. The velocity in $y$ direction is estimated as $\tilde{v}_{qy} \simeq -10.4 \text{ m/s}$, compared with the true value of $v_{qy} = -10 \text{ m/s}$. The velocity in $x$ direction is estimated as $\tilde{v}_{qx} \simeq -10.003 \text{ m/s}$, compared with the true value of $v_{qx} = -10 \text{ m/s}$. Figure 14b shows the scan image. Aside from the antennas, mast, and some edges on the superstructure, the 16 missile launchers on both sides of the bridge are strong scatterers, as shown in Figure 14a.

Figure 15 shows the images of a Type 055 destroyer, cruising at the same speed and direction as the Arleigh Burke-class destroyer. Figure 15a shows the acquired SAR image, in which both deck gun and missile silo hood are recognizable. Due to its stealth design,
only the antenna scatters significant signals. Figure 15b shows the scan image from its 3D model. It has a similar size to Slava-class cruiser, but carries stealth features of flat and smooth hull to reduce radar cross section.

![Figure 15](image1.png)

(a) Acquired image of Type 055 destroyer, in scale of (40, 90) dB and (b) scan image of 3D model.

The Doppler frequency is estimated as $\tilde{f}_q^d = -77.19$ Hz, compared with the true value of $f_q^d = -77.174$ Hz. The Doppler ambiguity number is estimated as $\tilde{N}_q^d = -2$, same as the true number. The velocity in $y$ direction is estimated as $\tilde{v}_{qy} \simeq -10.8$ m/s, compared with the true value of $v_{qy} = -10$ m/s. The velocity in $x$ direction is estimated as $\tilde{v}_{qx} \simeq 9.997$ m/s, compared with the true value of $v_{qx} = 10$ m/s.

Figure 16 shows the images of a Type 45 destroyer, cruising at the same speed and direction as the Slava-class cruiser. Type 45 destroyer has similar length to Arleigh Burke-class destroyer, but is wider than the latter. Figure 16a shows the acquired SAR image, in which the deck gun and the missile silo are well focused, and the tall main mast scatters strong signal. Figure 16b shows the scan image from its 3D model. A salient hallmark is its tall main mast for extending the radar surveillance range, which can be recognized in both images. Similar stealth features, as in Type 055 destroyer, are put on the hull design to reduce radar cross section.

![Figure 16](image2.png)

(a) Acquired image of Type 45 destroyer, in scale of (40, 90) dB and (b) scan image of 3D model.
The Doppler frequency is estimated as \( \tilde{f}_d = 77.18 \, \text{Hz} \), compared with the true value of \( f_d = 77.174 \, \text{Hz} \). The Doppler ambiguity number is estimated as \( \tilde{N}_d = 2 \), same as the true number. The velocity in \( y \) direction is estimated as \( \tilde{v}_{qy} \simeq -9.8 \, \text{m/s} \), compared with the true value of \( v_{qy} = -10 \, \text{m/s} \). The velocity in \( x \) direction is estimated as \( \tilde{v}_{qx} \simeq -9.997 \, \text{m/s} \), compared with the true value of \( v_{qx} = -10 \, \text{m/s} \).

The proposed method can also be applied to acquire image of multiple ships. Figure 17 shows an acquired image of an Arleigh Burke-class destroyer sailing alongside a Ticonderoga-class cruiser. Both ships are clearly focused.

![Figure 17.](image)

4.2. Recognition of Various Destroyers

To demonstrate the robustness of the proposed approach in acquiring SAR images of good quality that can be used for identification, a few basic recognition methods are used for verifying the usability of the acquired images. We first try to identify these five types of destroyer from the geometrical and radar features in their SAR images. Figure 18 shows the schematic of comparing the ship footprint on two SAR images. The contour of ship in a SAR image is first extracted and mapped to orient in the same bow-to-stern direction. Then dissect each ship contour into slices in \( x \) (horizontal) direction, and record the length of each slice in order of its \( y \) coordinate. As exemplified in Figure 18, \( \ell_1 \) and \( \ell_2 \) are the slice lengths of two ships at the same \( y \) coordinate. The difference of \( \ell_1 \) and \( \ell_2 \) is compared with a threshold \( \Delta_\ell \). If \( |\ell_1 - \ell_2| \leq \Delta_\ell \), then these two slices are claimed a match. After comparing all the corresponding slices, the percentage of match is called the index of geometry match.
Table 2 lists the index of geometry match among these five ships, with $\Delta \ell = 1$ m to tolerate perturbations in the simulations. In field measurements, this threshold may be used to tolerate jittering motion caused by ocean waves. It is shown that Arleigh Burke-class destroyer and Ticonderoga-class cruiser have a good chance to be identified as each other, and are less likely to be confused with the other three types. Similarly, Sl and T055 have a good chance to be identified as each other, and are less likely to be confused with the other three types.

Table 2. Index of geometry match among five ships.

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<td>0.0853</td>
<td>0.0352</td>
<td>0.1408</td>
<td>1</td>
</tr>
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</table>

Next, we try to use the backscattered signal strength or radar cross section to recognize different ships. Figure 19 shows the cumulative distribution function (CDF) of intensity (in dB) over all pixels (resolution cells) within the contour of ship in the acquired SAR image. The Slava-class cruiser has comparatively strong scattered signals due to its complicated superstructure, missile launchers, main mast, and other weaponry on the deck. Arleigh Burke-class destroyer and Ticonderoga-class cruiser have similar superstructure and layout, hence their CDFs look similar. Type 055 destroyer and Type 45 destroyer are designed with a stealth concept; their hull is composed of large flat surfaces to reduce radar cross section. The CDF curves show that their RCS is about 10 dB lower than the other three types. Type 45 destroyer has slightly higher RCS than Type 055, possibly due to its tall mast.
Figure 19. Cumulative distribution function of pixel intensity on SAR image. ———: Arleigh Burke-class destroyer, −−−−: Ticonderoga-class cruiser, ————: Slava-class cruiser, −−−−−: Type 055 destroyer, ·····: Type 45 destroyer.

The statistics of pixel intensity can also be used to identify different ships. Define an index of intensity match between ships $\alpha$ and $\beta$, with the centers of both ships overlapped and their orientations aligned, as

$$D_{\alpha\beta} = \frac{1}{N_p} \sum_n |S_{\alpha n} - S_{\beta n}| \text{ (dB)}$$

(76)

where $S_{\alpha n}$ is the intensity (in dB) of the $n$th pixel in the SAR image of ship $\alpha$, $N_p$ is the number of pixels within the contours of both ships. The intensity of a pixel outside of ship contour is set to a default value of 30 dB, which is the lowest value recorded in Figure 19.

Table 3 lists the index of intensity match among five ships. By comparing Arleigh Burke-class destroyer with the other ships, the index versus Ticonderoga-class cruiser is the smallest because both ships have similar superstructure. The index of Arleigh Burke-class destroyer versus Slava-class cruiser is attributed to their size difference, and the index versus Type 055 destroyer or Type 45 destroyer is attributed to the difference of pixel intensity profile on ship. Similar observations apply in comparing Ticonderoga-class cruiser with the other ships.

Table 3. Index of intensity match among five ships.

<table>
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<tr>
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The Slava-class cruiser has larger hull and stronger image intensity than the other four ships. Both Arleigh Burke-class destroyer and Ticonderoga-class cruiser have strong scatterers located at different spots than at Slava-class cruiser. Thus, the index of intensity match between Slava-class cruiser versus Arleigh Burke-class destroyer or Ticonderoga-class cruiser is large.

It is interesting to observe that the index of intensity match between Slava-class cruiser versus Type 055 destroyer or Type 45 destroyer is smaller than that versus Arleigh Burke-class destroyer or Ticonderoga-class cruiser. It is possibly because the former three types
have similar superstructure layout, although stealth technology is applied to both Type 055 destroyer and Type 45 destroyer, making their image intensity about 10 dB lower than the other three ships, as shown in Figure 19. The index of intensity match between Type 055 destroyer versus Type 45 destroyer is the smallest among all the off-diagonal entries in Table 3.

4.3. Effects of Noise

To study the effects of noise on the proposed method, Figure 20 shows the SAR images under different SNR levels. The estimation error of Doppler frequency is 0.2 % on Figure 20a,b, 0.3 % on Figure 20c and 0.5 % on Figure 20d. As SNR decreases, the estimation error increases and the image becomes blurrier. Figure 20d shows that the ship is almost engulfed in the noisy background, but is still discernible at SNR = −15 dB.

4.4. Similarity Index and Entropy

Structural similarity (SSIM) index [27] has been used to evaluate the similarity between two images. The fidelity of the SAR image acquired with the proposed approach can be evaluated with the SSIM index between the SAR image and the corresponding scan image. The SSIM index between images $a$ and $b$ is defined as

$$
\text{SSIM}(a, b) = \left( \frac{2\mu_a\mu_b + C_1}{\mu_a^2 + \mu_b^2 + C_1} \right)^{\alpha} \left( \frac{2\sigma_a\sigma_b + C_2}{\sigma_a^2 + \sigma_b^2 + C_2} \right)^{\beta} \left( \frac{2\sigma_{ab} + C_3}{\sigma_a\sigma_b + C_3} \right)^{\gamma}
$$

(77)
where the three factors are based on brightness, contrast and structure, respectively; \( \mu_p \) and \( \sigma_p \) are the mean and standard deviation, respectively, of image \( p \), with \( p = a, b \); \( \sigma_{ab} \) is the covariance between images \( a \) and \( b \); \( C_1, C_2 \) and \( C_3 \) are stability constants; \( \alpha, \beta \) and \( \gamma \) are weighting coefficients. The SSIM index lies in \([0, 1]\), with higher score on more similar images.

Table 4a lists the SSIM indices of SAR versus scan images of five ships, in the absence of noise. The coefficients in (77) are set to \( C_1 = 0.01\mu_a\mu_b, C_2 = 0.01\sigma_a\sigma_b, C_3 = 0.01\sigma_{ab} \) \( \alpha = 1, \beta = 1 \) and \( \gamma = 1 \). Since the table is symmetric with respect to the diagonal, redundant indices are not listed. These indices confirm the similarity between Arleigh Burke-class destroyer and Ticonderoga-class cruiser, as well as among Slava-class cruiser, Type 055 destroyer, and Type 45 destroyer. Table 4b lists the mean values of SSIM indices at \( \text{SNR} = -10 \text{ dB} \), which are averaged over 30 simulation samples, and Table 4c lists the associated standard deviations. It is observed that the mean indices of all pairs become smaller compared with their counterparts in Table 4a, but the SSIM index between SAR image and scan image of the same ship is still much higher than that between different ships. The standard deviations are very small, implying the mean values listed in Table 4 are creditable for identification purpose.

**Table 4.** SSIM indices of SAR versus scan images of five ships, (a) noise free, (b) mean value at \( \text{SNR} = -10 \text{ dB} \), and (c) standard deviation at \( \text{SNR} = -10 \text{ dB} \).

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<td></td>
</tr>
<tr>
<td>Sl</td>
<td></td>
<td>0.0032</td>
<td>0.0023</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>T055</td>
<td></td>
<td>0.0024</td>
<td></td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>T45</td>
<td></td>
<td>0.0026</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 21 shows the SSIM indices of all possible combinations of ship pair under different levels of SNR, where A/a, T/t, S/s, C/c, and B/b represent Arleigh Burke-class, Ticonderoga-class, Slava-class, Type 055, and Type 45, respectively. It is observed that at \( \text{SNR} \geq -15 \text{ dB} \), the acquired SAR images are clear enough to be distinguished with the SSIM indices. At \( \text{SNR} = -20 \text{ dB} \), the acquired SAR images are too fuzzy to recognize with the SSIM indices.
Entropy has been used to judge whether an image is well focused or not [28]. If a SAR image is well focused and free of artifacts, the contrast between ship and background will be significant, leading to lower entropy than an unfocused SAR image. The entropy of an image is defined as [28]

\[ E = -\sum_{\ell=0}^{L} P_\ell \ln P_\ell \]  

where \( P_\ell \) is the likelihood of a pixel having grey level \( \ell \), with \( 0 \leq \ell \leq L \). The entropies of images in Figure 20a–d are 3.8354, 4.2318, 5.1468, and 6.7385, respectively. As a reference, if the gray level is uniformly distributed over \( 0 \leq \ell \leq L \), the entropy is \( E = 6.9066 \). The entropy of the image in Figure 20d, with SNR = -15 dB, is close to that of random noise, but the ship image can still be acquired with the proposed method.

Machine learning methods, such as convolutional neural network (CNN), have been applied to identify ships in SAR images. In [29], a region of interest (ROI) was first selected by using a support vector machine (SVM), then a regional-based CNN was applied to accurately identify ships in the ROI. The classification of small ships was further improved with a customized CNN, called VGG16-based Faster-RCNN. Semi-supervised classification of SAR images was addressed in [30] in case only a small number of labeled SAR images were available. Such restriction could be relieved by supplementing with relevant electro-optical images. In [31], an image identifying CNN, YOLOv3, was improved by adding a dilated attention module for extracting discriminative features of ships, to achieve more accurate classification among ships with different sizes. According to [32], nearly 47% of the literature about deep learning methods for ship detection on SAR images were trained with the SAR ship detection dataset (SSDD), which contained SAR images of ships, including labels and percentage of inshore and offshore scenes. Conventional CNN-based methods were trained by using rectangular subimages with ships aligned in a specific direction, while real ships on SAR images are oriented in arbitrary directions. In [33], a multiscale rotation region detection method was proposed to retrieve ship images aligned in a given direction, improving the training and testing results.

### 4.5. Experiment on Real Data

In this subsection, the proposed method is applied to the real dataset collected by TerraSAR-X [34]. Table 5 lists the parameters associated with this dataset. The backscattered signals were received with two antennas, separated by 2.4 m. The Doppler bandwidth is \( K_{br} T_a \approx -2V_e^2 T_a/(\lambda_0 R_0) \approx 23,333 \) Hz, where \( V_e = \sqrt{VV_g} \) is the effective platform velocity. Since the ratio between the Doppler bandwidth and the pulse repetition frequency is \( 23,333/3807.03 \approx 6.14 \), the Doppler spectrum is wrapped into at least seven segments. The phase matching technique proposed in this work demands that \( MAF_2 \) be close to an
integer. With the parameters listed in Table 5, we have $MaF_a = MdF_a/(2V_e) \approx 1.24$, which is not close to an integer and its consequence will be manifested later.

**Table 5. Parameters in real data experiment [34].**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of antennas</td>
<td>$M$</td>
<td>2</td>
</tr>
<tr>
<td>platform speed</td>
<td>$V$</td>
<td>7682.782 m/s</td>
</tr>
<tr>
<td>ground speed</td>
<td>$V_g$</td>
<td>7065.511 m/s</td>
</tr>
<tr>
<td>height of platform</td>
<td>$H$</td>
<td>500 km</td>
</tr>
<tr>
<td>carrier frequency</td>
<td>$f_0$</td>
<td>9.65 GHz</td>
</tr>
<tr>
<td>antenna spacing</td>
<td>$d$</td>
<td>2.4 m</td>
</tr>
<tr>
<td>PRF</td>
<td>$F_a$</td>
<td>3807.03 Hz</td>
</tr>
<tr>
<td>aperture time</td>
<td>$T_a$</td>
<td>2.5 s</td>
</tr>
<tr>
<td>look angle</td>
<td>$\theta$</td>
<td>35°</td>
</tr>
<tr>
<td>reference point</td>
<td>$(x_0, y_0)$</td>
<td>(350, 0) km</td>
</tr>
</tbody>
</table>

To begin with, the received signals are range compressed with the matched filter in (7), followed by phase compensation with the filter in (26). Then, RCMC is conducted for the first time with the filter in (34) so that the signal in each channel is aligned in the fast time-Doppler domain. Next, apply the most important processes of baseband Doppler-shift estimation with compressive-sensing technique and phase matching in (52) and (53). The phase matching technique can fetch separate Doppler spectrum segments and stitch them into a continuous Doppler spectrum, solving the ASA issue. RCMC is conducted for the second time with the filter in (59) to compensate for curved range cell migration, then the ADA is solved by referring to the slope rate of Doppler spectrum on the fast time-Doppler domain, as in (65). Finally, phase compensation is conducted for the second time and RCMC is conducted for the third time to compensate for linear range cell migration.

Figure 22 shows the SAR image acquired with the proposed method. The bright ship at the center appears the same as in the original SAR image. Several ghost ships manifest because the criterion of $MaF_a$ being close to an integer is not satisfied by the experiment dataset. The estimated velocity of the ship is $(\tilde{v}_{qx}, \tilde{v}_{qy}) = (5.9, 11.8)$ m/s. Although the true velocity is unknown, the well-focused SAR image suggests the estimated velocity is creditable.

![Figure 22. SAR image of ship acquired from the backscattered signals of ships on the Strait of Dover, acquired with TerraSAR-X [34].](image-url)
4.6. Comparison with Other Methods

In this subsection, the proposed method in solving ADA and ASA issues is compared with the HRWS SAR imaging methods on GMTs in [1,2]. In [1], the multi-channel signals are first processed with deramp function, coarsely focusing the signal spectrum in the Doppler frequency-spatial frequency domain and tackling the ASA issue. Conventional techniques such as the Keystone transform and RCMC are applied, aided with a guessed Doppler ambiguity number. The SCNR of the acquired SAR image is expected to reach the highest value if the Doppler ambiguity number is correctly guessed.

In [2], a space-airborne bistatic multichannel SAR was considered. The key step for tackling ASA issue and estimating Doppler frequency is to derive a spatial spectrum by Fourier transforming the signals with respect to the channel index. The spectrum segments originally wrapped in the Doppler frequency domain are unwrapped into multiple sinc functions in the spatial frequency domain, separated by \( N_s F_a \). Then a spectrum segment of specific \( N_s \) can be retrieved with a bandpass filter, claiming to resolve the ASA issue. In addition, the peak location in the spatial spectrum segment with \( N_s = 0 \) indicates the baseband Doppler frequency.

Table 6 lists the simulation parameters for comparing the proposed method with those in [1,2]. The antenna spacing \( d \) and the platform speed \( V \) are the same as in [2]. The number of antennas in [2] is 24 for bistatic SAR. To achieve the same spatial frequency resolution as in bistatic SAR, the number \( M \) of monostatic SAR should be doubled. In addition, the proposed method requires \( dF_a M / (2V) \) to be equal or close to an integer, thus we choose \( M = 50 \).

Table 6. Parameters for method comparison.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of antennas</td>
<td>( M )</td>
<td>50</td>
</tr>
<tr>
<td>platform speed</td>
<td>( V )</td>
<td>300 m/s</td>
</tr>
<tr>
<td>height of platform</td>
<td>( H )</td>
<td>5 km</td>
</tr>
<tr>
<td>carrier frequency</td>
<td>( f_0 )</td>
<td>1.25 GHz</td>
</tr>
<tr>
<td>antenna spacing</td>
<td>( d )</td>
<td>1.6 m</td>
</tr>
<tr>
<td>PRF</td>
<td>( F_a )</td>
<td>250 Hz</td>
</tr>
<tr>
<td>aperture time</td>
<td>( T_a )</td>
<td>2 s</td>
</tr>
<tr>
<td>chirp rate</td>
<td>( K_r )</td>
<td>35 THz/s</td>
</tr>
<tr>
<td>bandwidth</td>
<td>( B_r )</td>
<td>700 MHz</td>
</tr>
<tr>
<td>pulse width</td>
<td>( T_c )</td>
<td>20 μs</td>
</tr>
<tr>
<td>look angle</td>
<td>( \theta_f )</td>
<td>45°</td>
</tr>
<tr>
<td>reference point</td>
<td>((x_0, y_0))</td>
<td>(5, 0) km</td>
</tr>
<tr>
<td>target initial position</td>
<td>((x_{q0}, y_{q0}))</td>
<td>(5, 0) km</td>
</tr>
<tr>
<td>target velocity</td>
<td>((v_x, v_y))</td>
<td>( -22, -22 ) m/s</td>
</tr>
</tbody>
</table>

Figure 23 shows the acquired SAR images with the proposed method and those in [1] and [2], respectively. The image in Figure 23b with method of [1] is not focused. Figure 24a shows the signal spectrum after deramping drags a long trace in the Doppler frequency-spatial frequency domain, explaining why the final image cannot be well focused. The image in Figure 23c with method of [2] reveals some artifacts. Figure 24b shows the signal after tackling ASA issue forms a straight line in the fast time-Doppler frequency domain. However, separating signal segments with bandpass filter in the spatial frequency domain can only scoop the main-lobe of a sinc-like spatial spectrum, the remaining spatial spectrum scooped by other bandpass filters induces artifacts in the acquired image. For
comparison, Figure 24c shows the signal of a point target after solving the ASA issue with
the proposed method.

Figure 23. Acquired images of a point target with parameters listed in Table 6, (a) proposed method,
(b) [1], and (c) [2].

Figure 24. Cont.
4.7. Computational Load

In the proposed method, the CS technique takes the most computational load, on the order of $O(N^3)$, where $N$ is the dimension of sparse vector. The computational load in [1,2] is comparable to conventional fast Fourier transform, on the order of $O(N \log N)$. A personal computer with CPU i9-10900k and memory of 128 GB is used to obtain the results in Figure 23. The elapse time is 60.548 s with the proposed method, 5.8313 s with the method in [1], and 8.8731 s with the method in [2]. Excluding the CPU time for running CS, these three methods take similar amount of computational load.

4.8. Highlights of This Work

Highlights and novelties of this work are briefly summarized as follows:

1. A phase matching technique is proposed to solve the azimuth spectrum aliasing problem.
2. A compressive-sensing technique is proposed to estimate the baseband Doppler shift.
3. A multi-stage compressive-sensing technique is proposed to estimate the target velocity in both directions.
4. The simulation results verify that the acquired image with the proposed approach outperforms conventional imaging methods on moving ships.
5. The proposed approach can acquire image without ghost targets in high-resolution wide-swath SAR imaging at low pulse repetition rate.

5. Conclusions

A multi-channel synthetic aperture radar (SAR) on board a spaceplane orbiting near the top of atmosphere is proposed to acquire images of cruising ships. Low pulse repetition frequency (PRF) is required for high-resolution wide-swath imaging, leading to inevitable problems of azimuth spectrum aliasing (ASA) and azimuth Doppler ambiguity (ADA). A rigorous approach has been developed by proposing a phase matching technique to solve the ASA problem and a multi-stage compressive-sensing technique to solve both the ADA and ASA problems. The imaging of a moving point target has been demonstrated to verify the efficacy of the proposed approach. Indices of geometry match, intensity match, and structural similarity on pairs of acquired SAR images are used to accurately recognize ships from among the five types of ship.

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References


