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Spatial–Temporal Joint Design and Optimization of Phase-Coded Waveform for MIMO Radar

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Abstract: By simultaneously transmitting multiple different waveform signals, a multiple-input multiple-output (MIMO) radar possesses higher degrees of freedom and potential in many aspects compared to a traditional phased-array radar. The spatial–temporal characteristics of waveforms are the key to determining their performance. In this paper, a transmitting waveform design method based on spatial–temporal joint (STJ) optimization for a MIMO radar is proposed, where waveforms are designed not only for beam-pattern matching (BPM) but also for minimizing the autocorrelation sidelobes (ACSLs) of the spatial synthesis signals (SSSs) in the directions of interest. Firstly, the STJ model is established, where the two-step strategy and least squares method are utilized for BPM, and the L2p-Norm of the ACSL is constructed as the criterion for temporal characteristics optimization. Secondly, by transforming it into an unconstrained optimization problem about the waveform phase and using the gradient descent (GD) algorithm, the hard, non-convex, high-dimensional, nonlinear optimization problem is solved efficiently. Finally, the method's effectiveness is verified through numerical simulation. The results show that our method is suitable for both orthogonal and partial-correlation MIMO waveform designs and efficiently achieves better spatial–temporal characteristic performances simultaneously in comparison with existing methods.

Keywords: MIMO radar; waveform design; spatial–temporal joint optimization; beam-pattern matching; L2p-Norm; spatial synthesis signals; gradient descent



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1. Introduction

Unlike a traditional phased-array radar, which typically emits phase-shifted copies of an identical waveform in each channel to form a focused narrow beam in the desired direction, a multiple-input multiple-output (MIMO) radar allows each element to emit waveforms with a certain degree of independence. This waveform diversity makes the MIMO radar superior in many aspects, such as having higher degrees of freedom and flexibility and better multi-target detection performance [1–3].

As the most important part of the MIMO radar, in recent years, MIMO radar waveform design has received widespread attention and has been researched by scholars both domestically and internationally. Generally, centralized MIMO radar waveform design can be divided into two categories: The first category is the transmitting waveform design problem in the presence of signal-correlation clutter and color noise, which aims to suppress the signal-correlation clutter and improve the detection probability. It includes the waveform design based on the maximum signal-to-interference-and-noise ratio (SINR) criterion [4–6] and the waveform design based on information theories such as mutual information or relative quotient [7–12]. The second type is the waveform design optimization problem based on beam-pattern control, which aims to enhance the emission energy in the

direction of the targets or the direction of interest while reducing the radiation energy in the noninterest or interference directions by designing the correlation between waveforms, and thus improve the resource utilization efficiency and target detection performance.

Generally speaking, two types of methods are commonly used in waveform design for beam-pattern matching (BPM): One approach is to directly design and optimize the transmitting waveform to achieve the desired transmitting beam pattern [13–15]. A transmitting waveform is synthesized by minimizing the error between the designed beam and the expected beam in [16], the waveform design problem of the minimum beam peak sidelobe for a centralized MIMO radar is studied in [17], and the design of transmitting waveforms with good beam performance and compatibility with the communication spectrum is investigated in [18], which are all solved using the ADMM. Subsequently, the waveform design for jointly suppressing the interference of spatial and frequency domains in multi-objective scenarios is discussed in [19]. The MIMO radar waveform design problem, which minimizes the spatial–temporal integral sidelobe level (ISL) while considering different practical constraints, such as the total transmitting power constraint, peak-to-average ratio (PAR) constraint, constant modulus (CM) constraint, and discrete phase table constraint, is elaborated on, and a coordinate descent (CD)-based algorithm is proposed for solving it in [20]. Although this type of method is more conducive to obtaining accurate transmitting beam patterns, the model is extremely complex and difficult to solve.

Another type of waveform design method for BPM is the two-step method, namely, first, designing a covariance matrix based on the expected transmitting beam pattern, and then, synthesizing the transmitting waveform according to the obtained covariance matrix under some practical constraints. For the first step, many methods have been proposed [3,21–24], wherein, under semi-positive definite constraints and equal power constraints of array elements, Refs. [3,21], respectively, proposed an efficient sequence quadratic program (SQP) algorithm and a semi-definite programming (SDP) algorithm for problem solving. To reduce target-characteristic distortion and beam sidelobe (SL) interference, Ref. [22] proposed a covariance design method for controlling the main lobe fluctuations and peak sidelobe level (PSL) of the transmitting beam. However, the computational complexity of these methods is still somewhat high. Afterward, for the covariance design problem of uniform linear arrays (ULAs), a closed-form solution that only involves FFT operation and Toeplitz matrix generation operation is proposed in [23,24], which further greatly improves the computational efficiency. For the second step, Refs. [25,26] proposed an efficient cyclic algorithm (CA) to synthesize a partially correlated waveform based on the given covariance matrix and discussed the situation of containing some constraints, such as constant modulus constraints and PAR constraints. Refs. [27–29] proposed a finite phase number constant-envelope waveform design method that approximates a given covariance matrix. Ref. [30] proposed a coordinate descent method framework for generating transmitting waveforms from covariance matrices in both the finite phase code case and continuous phase code case.

It should be noted that although the methods mentioned above possess good BPM performance, they hardly consider the temporal domain performance of the waveform. In [31], under the constant modulus constraint and waveform similarity constraint, a waveform design method for jointly optimizing the beam-pattern performance and ambiguity-function characteristics is studied by using the cyclic algorithm. However, it does not actually have much practical significance, as the waveform similarity constraint requires an existing waveform as a reference. In [15], the joint optimization of BPM and the autocorrelation sidelobe levels (ACSLs) and cross-correlation levels (CCLs) of the direction of interest is conducted, wherein an unconstrained optimization model is established and solved using the iterative quasi-Newton method. However, some high peak sidelobes appear due to the integral sidelobe criterion, which easily causes false alarms and missed alarms.

Under no prior information on target orientation, an orthogonal waveform is better, whose beam pattern is omnidirectional. To easily separate the signals of different channels through matched filtering at the receiving end and considering the limitations of radar amplifiers, many studies design orthogonal waveform sequences by minimizing their ACSLs

and CCLs under a unimodular constraint [32–36], wherein, based on the ISL criteria, [32] introduces the new cyclic algorithm (CAN) to design waveform sequences, and later, [33] proposes the Multi-CAN method by expanding it to the MIMO scenarios. In [34], an algorithm based on the general majorization–minimization (MM) method is developed to tackle the non-convex problem. The alternating direction method of multipliers (ADMM) is adopted in [35] to design MIMO waveforms considering spectral assignment constraints. In [36], the weighted p -norm of auto- and cross-correlation sidelobes is adopted as the objective function, and block successive upper-bound minimization (BSUM) is used to solve it. It is worth noting that the omnidirectionality of these methods is usually unsatisfactory, as it does not constrain the beam-pattern constraints directly. Moreover, good autocorrelation and cross-correlation performances cannot guarantee that the sidelobe performance is still good after equivalent transmit beamforming at the receiving end, making it difficult to ensure low missing alarm and false alarm rates.

To this end, we propose a spatial–temporal joint (STJ) optimization method for MIMO waveform design. Firstly, we consider the MIMO orthogonal waveform as a special case of a MIMO partially related waveform and establish a unified STJ optimization model for designing it. On the one hand, to reduce complexity, we adopt the two-step method for beam-pattern design, namely, first, designing a covariance matrix based on the expected transmitting beam pattern, and then, synthesizing the transmitting waveform according to the obtained covariance matrix under some practical constraints. On the other hand, for the optimization of the waveform temporal performance, considering that the spatial synthetic signals ultimately determine the detection performance of the MIMO radar, this article constructs the L2p-Norm sidelobe (L2pSL) function of spatial synthetic signals in the directions of interest as the criterion for temporal characteristics optimization. It should be noted that using the ISL criterion often leads to some high sidelobes, while the peak sidelobe criterion is complex and difficult to solve. Different from the iterative weighted integral sidelobe (IWISL) method [37], where each sidelobe is weighted by the amplitude of itself iteratively to achieve the goal of suppressing peak sidelobes, our method adopts the L2p-Norm criterion. Our L2p-Norm criterion is more flexible and efficient. It can avoid the shortcomings of both ISL and PSL criteria, as it can be compromised between the ISL and PSL by setting different p -values. Secondly, we transform the constant-modulus, high-dimensional, non-convex problem of STJ optimization into an unconstrained optimization problem, derive the gradient of the objective function in detail, and then, solve it efficiently by using a gradient descent (GD) algorithm. Finally, we evaluate the performance of the proposed method via numerical simulation. The results of BPM and the autocorrelation sidelobe levels of spatial synthetic signals in the direction of interest verify the effectiveness of our method. Compared to existing methods, our method possesses a better spatial–temporal performance and faster convergence speed.

The structure of this article is as follows: Section 2 elaborates on the construction of our spatial–temporal joint optimization model for a MIMO radar. In Section 3, we derive the gradient of the high-dimensional, nonlinear, complex objective function of the model in detail and provide the solution steps for the waveform optimization problem. Section 4 describes the numerical simulation experiments and discusses the experimental results. Section 5 provides a summary of this article.

Notations: In this paper, we use italic letters for scalars and adopt bold italic letters for vectors or matrices. j represents imaginary units. I_{M_t} represents the identity matrices of M_t rows and M_t columns. The superscripts $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ represent transposes, conjugates, and conjugate transposes, respectively. The symbols $\bullet$ and $\otimes$ denote Hadamard product and convolution operations, separately. $|\cdot|$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ denote absolute values, 2-norm, and infinite norm operations, respectively. $\text{xcorr}(\cdot)$ represents correlation operations. $\text{tr}(\cdot)$ denotes the traces of a matrix. $\text{vec}(\cdot)$ denotes stacking the matrix by column, and $\text{Im}(\cdot)$ denotes imaginary parts of a complex variable. $d(\cdot)$ and $\nabla(\cdot)$ denote differential and gradient operations, respectively.

2. Problem Formulation and Proposed Models

Consider a uniform, linear-array MIMO radar system composed of M_t transmitting antennas. Let $\{x_m(n), n = 1, 2, \dots, N\}$ represent the base-band signal emitted by the m th transmitting antenna, where N represents the number of discrete signal points. Assume the array signal matrix is represented by $\mathbf{X} = [x_1, x_2, \dots, x_{M_t}]^T$ and $\mathbf{a}(\phi)$ indicates the transmitting steering vector in the direction ϕ . Then, the spatial synthesis signal in the direction ϕ can be expressed as

$$\mathbf{y}(\phi, n) = \mathbf{a}^T(\phi) \mathbf{X}(n) \quad (1)$$

The average power of the MIMO radar spatial synthesis signal within the transmitting pulse, namely the transmitting beam pattern, can be formulated as

$$\begin{aligned} P(\phi) &= \frac{1}{N} \sum_{n=1}^N |\mathbf{y}(\phi, n)|^2 = \frac{1}{N} \sum_{n=1}^N [\mathbf{y}^*(\phi, n) \mathbf{y}^T(\phi, n)] \\ &= \frac{1}{N} \sum_{n=1}^N [\mathbf{a}^H(\phi) \mathbf{X}^*(n) \mathbf{X}^T(n) \mathbf{a}(\phi)] = \mathbf{a}^H(\phi) \mathbf{R} \mathbf{a}(\phi) \end{aligned} \quad (2)$$

It reflects the distribution of radar emission waveform energy in various directions, where $\mathbf{R} = \frac{1}{N} \mathbf{X}^H \mathbf{X}$ denotes the covariance matrix of the emission waveform. Similarly, to characterize the mutual interference intensity between two directional signals, the average emission energy between two different directions, ϕ_{k1} and ϕ_{k2} , can be defined as $P(\phi_{k1}, \phi_{k2}) = \mathbf{a}^H(\phi_{k1}) \mathbf{R} \mathbf{a}(\phi_{k2})$. It can be seen that the transmitting beam pattern is closely related to the waveform covariance matrix, and beam-pattern control can be achieved by designing the covariance matrix.

Let the receiving array also be a uniform linear array composed of M_r receiving elements, $\mathbf{b}(\phi)$ represent the receiving steering vector in the direction ϕ , and the target be located in the direction ϕ_{tg} ; then, the target echo signal received by the receiving array can be expressed as

$$\mathbf{y}_r(n, \phi_{tg}) = \mathbf{b}(\phi_{tg}) [\mathbf{a}^T(\phi_{tg}) \mathbf{X}(n)] \quad (3)$$

Let $\mathbf{z}$ represent the processing results of matched filtering and beamforming, including equivalent transmit beamforming and receive beamforming. Then, we have

$$\begin{aligned} \mathbf{z}(:, \phi, \phi_{tg}) &= \mathbf{b}^H(\phi) \mathbf{y}_r(n, \phi_{tg}) \otimes \mathbf{X}^H(-n) \mathbf{a}^*(\phi) \\ &= \mathbf{b}^H(\phi) [\mathbf{b}(\phi_{tg}) \mathbf{a}^T(\phi_{tg}) \mathbf{X}(n)] \otimes \mathbf{X}^H(-n) \mathbf{a}^*(\phi) \end{aligned} \quad (4)$$

where the convolution operation is performed on the fast time variable n , and the beamforming operations are performed on the angle variable ϕ . Thus, the convolution and matrix multiplication are independent of each other. Then, the convolution and matrix multiplication of Formula (4) can be flexibly combined according to the associative law. So, Formula (4) can be derived as

$$\begin{aligned} \mathbf{z}(:, \phi, \phi_{tg}) &= \mathbf{b}^H(\phi) \mathbf{b}(\phi_{tg}) [\mathbf{a}^T(\phi_{tg}) \mathbf{X}(n)] \otimes [\mathbf{a}^T(\phi) \mathbf{X}(-n)]^H \\ &= \mathbf{b}^H(\phi) \mathbf{b}(\phi_{tg}) \text{xcorr}(\mathbf{a}^T(\phi_{tg}) \mathbf{X}(n), \mathbf{a}^T(\phi) \mathbf{X}(n)) \end{aligned} \quad (5)$$

By writing the correlation operation in (5) in the form of matrix multiplication, we have

$$\mathbf{z}(k, \phi, \phi_{tg}) = \mathbf{b}^H(\phi) \mathbf{b}(\phi_{tg}) [\mathbf{a}^T(\phi_{tg}) \mathbf{X}] \mathbf{J}_k [\mathbf{a}^T(\phi) \mathbf{X}]^H \quad (6)$$

wherein $k \in [-N+1, N-1]$ indicates the number of delay points. $\mathbf{J}_k$ represents the shift matrix, and the following is true:

$$\mathbf{J}_k = \begin{bmatrix} \mathbf{0}_{(N-k) \times k} & \mathbf{I}_{N-k} \\ \mathbf{0}_{k \times k} & \mathbf{0}_{k \times (N-k)} \end{bmatrix}, \mathbf{J}_{-k} = \mathbf{J}_k^T, 0 \leq k \leq N-1 \quad (7)$$

From Formulas (4) to (6), two interesting conclusions can be drawn: Firstly, the distance–dimension sidelobes are only related to matched filtering and transmit beamforming but are independent of receive beamforming [38]. Secondly, the autocorrelation processing of the spatial synthesis signal, namely the pulse synthesis processing, which combines matched filtering and transmit beamforming together, is completely equivalent to sequentially performing matched filtering and equivalent transmit beamforming [38,39]. So, the autocorrelation of the spatial synthesis signal directly determines the detection performance of the radar, and the best strategy is minimizing the ACSL of the spatial synthesis signal of the transmitting waveform to ensure that the output response of the MIMO radar possesses the lowest distance–dimension sidelobes. It should be noted that unlike the narrow beam of the phased-array mode, the transmitting beams of partially correlated modes and orthogonal modes are generally relatively wide. Therefore, to accurately determine the range and azimuth of the target, when processing the echo signal at the receiving end according to Equation (5) or (6), multiple receiving beams and pulse synthesizers should be formed at certain angle intervals within the transmitting main beam.

2.1. Covariance Matrix Design

Many methods can be used to obtain the covariance matrix based on the expected beam pattern, such as the beam-pattern matching method or minimizing beam-pattern sidelobe method. For simplicity, here, we only consider the beam-pattern matching method. Let $\mathbf{P}_e$ represent the desired beam pattern; then, the beam-pattern matching model, considering the interference between different beams, can be represented as

$$\begin{aligned} \min & \left\{ \frac{\omega_m}{K_e} \sum_{k=1}^{K_e} [P(\phi_k) - P_e(\phi_k)]^2 + \frac{2\omega_c}{K_c^2 - K_c} \sum_{k_1=1}^{K_c-1} \sum_{k_2=k_1+1}^{K_c} |P(\phi_{k_1}, \phi_{k_2})|^2 \right\} \\ \text{s.t.} & \\ & \mathbf{R}_{mm} = c, m = 1, \dots, M_t \\ & \mathbf{R} \succeq 0 \end{aligned} \quad (8)$$

Among them, K_e represents the number of directional grid points in the desired beam pattern, ω_m is the beam-pattern matching weight, K_c is the number of beams, and ω_c is the interference weight between different beams. Obviously, the objective function of the above model is a quadratic function of the optimization variable $\mathbf{R}$, and its constraints are linear functions about the optimization variable $\mathbf{R}$. So, it belongs to convex optimization (CO) problems. Actually, due to the semi-positive definite property of the covariance matrix, it is a semi-definite programming problem, a subset of convex optimization [21]. So, it can be solved by using the CVX toolbox [40], which is a MATLAB software package for solving CO problems. Specifically, for the omnidirectional beam-pattern mode, $\mathbf{R} = \mathbf{I}_{M_t}$.

2.2. STJ Optimization of Transmitting Waveform

Although the transmitting waveform can be directly obtained using the CA method based on the covariance matrix [25,26], the range sidelobes of its obtained waveform are high, as it only considers the beam-pattern characteristics but does not take temporal performance into account. So, it is difficult to meet the detection requirements. In this chapter, a spatial–temporal joint optimization model is established. Firstly, in terms of beam-pattern matching performance, we use the mean square error (MSE) between the covariance matrix of the waveform to be solved and the expected covariance matrix as the objective function. Moreover, for temporal characteristics optimization, the L2pSL function of the spatial synthesis signal in the direction of interest is constructed. Last but not least, by taking the weighted sum of the two objective functions as the final optimization objective, a transmitting waveform design model for good spatial–temporal performance is simultaneously obtained.

To obtain a greater optimization space, this paper considers a continuous, multiphase-encoded waveform. In practical applications, in order to allow each power amplifier to operate at maximum efficiency, a constant modulus constraint is required. Let $\mathbf{X}_{mn} = \sqrt{c}e^{j\theta_{mn}}$,

$m = 1, 2 \dots M_t, n = 1, 2 \dots N$ denote the constant-modulus, multiphase-encoded waveform, where θ denotes the waveform phase matrix, and $\sqrt{c}$ is the signal amplitude, which is a constant. To obtain the waveform sequences with a specific beam pattern, the following covariance matrix-matching error function is defined:

$$f_1(\mathbf{X}) = \frac{1}{M_t^2 N^2} \|\mathbf{X}\mathbf{X}^H - \mathbf{R}\|_2^2 \quad (9)$$

where $\frac{1}{M_t^2 N^2}$ is the normalization coefficient of the covariance matrix-matching error used to facilitate the subsequent weighted joint optimization and unified evaluation under different parameters.

To optimize the range sidelobes of the spatial synthesis signals (SSSs) of the transmitting waveform in the directions of interest, the following function is defined as the objective function, which, essentially, is the L2p-Norm of the normalized ACSLs of SSSs:

$$f_2(\mathbf{X}) = \left[\frac{1}{B(2N-2)} \sum_{i=1}^B \sum_{n=1, n \neq N}^{2N-1} \omega_{ik} g_{ik}^p(\mathbf{X}) \right]^{\frac{1}{2p}} = \left[\frac{2}{B(2N-2)} \sum_{i=1}^B \sum_{n=1}^{N-1} \omega_{ik} g_{ik}^p(\mathbf{X}) \right]^{\frac{1}{2p}} \quad (10)$$

$$g_{ik}(\mathbf{X}) = \beta_i \left| \mathbf{a}^T(\phi_i) \mathbf{X} \mathbf{J}_k \mathbf{X}^H \mathbf{a}^*(\phi_i) \right|^2, i = 1, 2 \dots B; k = 1, 2 \dots N-1 \quad (11)$$

$$\beta_i = \frac{1}{|\mathbf{a}^T(\phi_i) \mathbf{R} \mathbf{a}^*(\phi_i)|^2} \quad (12)$$

where $g_{ik}(\mathbf{X})$ is the normalized modulus value of the k th autocorrelation sidelobe of the spatial synthesis signal in the direction ϕ_i ; β_i is the normalization coefficient relative to its main lobe to prevent numerical accuracy issues from excessive gradient values of the objective function. ω_{ik} is the weight coefficient for the sidelobes in each direction and distance. $\frac{1}{B(2N-2)}$ is the average coefficient for the sidelobe sums in various directions and distances. p is half of the norm order, $p \in \mathbb{Z}^+$. The closer the value of p approaches 1, the more the optimization strategy tends to minimize the ISL. The larger the value of p , the more the optimization strategy tends to minimize the PSL. Specifically, when p equals 1, the objective function is the 2-norm of the ACSL of the spatial synthesis signal, namely the ISL of spatial synthesis signal in the direction of interest. When p approaches infinity, it is the PSLs of spatial synthesis signals in the directions of interest. So, different strategies for minimizing the sidelobe levels can be achieved using different p values.

To design the transmitting waveform for both beam-pattern matching and temporal performance simultaneously, we take the weighted sum of the above two objective functions as the final optimization objective, as shown below.

$$f_{obj}(\mathbf{X}) = \mu_1 \cdot f_1(\mathbf{X}) + \mu_2 \cdot f_2(\mathbf{X}) \quad (13)$$

3. Method for Solving STJ Model

The above spatial-temporal joint optimization problem can be summarized as the following optimization problem:

$$\begin{aligned} & \min \{ f_{obj} \} \\ & \text{s.t.} \\ & |\mathbf{X}_{mn}| = \sqrt{c}, m = 1, 2 \dots M_t, n = 1, 2 \dots N \end{aligned} \quad (14)$$

Due to the non-convexity of constant modulus constraints and the high-dimensional nonlinearity of the joint objective function, it is difficult to directly solve this problem and obtain the $\mathbf{X}$ solution. Fortunately, the joint optimization model established above can also be regarded as a nonlinear optimization problem about the phase matrix of the transmitting waveform, where its variables are continuous. Then, the SQP method can be used to solve

it. However, with SQP, it is difficult to achieve satisfactory effect, as its complexity is high, and the iteration time is very long. This chapter first derives the gradient of the objective function and then provides the algorithm flow of the fastest descent algorithm for this STJ optimization model.

3.1. Gradient Derivation

In the steepest descent algorithm, the key is to obtain the gradient of the objective function. Below, the gradient of the objective function in our model is derived.

3.1.1. Deriving the Gradient of $f_1(\mathbf{X})$

In Equation (9), $f_1(\mathbf{X})$ is a polynomial complex function with the highest order to the power of 2 for $\mathbf{X}$ and $\mathbf{X}^*$, respectively. According to the theory of complex functions, it is easy to represent $f_1(\mathbf{X})$ in the form of a holomorphic function $f_1(\mathbf{X}, \mathbf{X}^*)$. Thus, it is convenient to obtain the gradients $\nabla_{\mathbf{X}} f_1(\mathbf{X}, \mathbf{X}^*)$ and conjugate gradients $\nabla_{\mathbf{X}^*} f_1(\mathbf{X}, \mathbf{X}^*)$ about the complex vector $\mathbf{X}$. Let $\nabla_{\theta} f_1(\mathbf{X})$ be the gradient of the objective function $f_1(\mathbf{X})$ concerning the phase matrix θ . From the gradient properties of composite functions, it can be concluded that

$$\nabla_{\theta} f_1(\mathbf{X}) = \nabla_{\mathbf{X}} f_1(\mathbf{X}, \mathbf{X}^*) \bullet \nabla_{\theta} \mathbf{X} + \nabla_{\mathbf{X}^*} f_1(\mathbf{X}, \mathbf{X}^*) \bullet \nabla_{\theta} \mathbf{X}^* \quad (15)$$

Substitute Equations (A6) of Appendix A and (A14) of Appendix D into Equation (15) above; then, $\nabla_{\theta} f_1(\mathbf{X})$ can be represented as shown below

$$\begin{aligned} \nabla_{\theta} f_1(\mathbf{X}) &= \frac{2}{M_i^2 N^2} \{ [(X^* X^T - R^T) X^*] \bullet (jX) + [(XX^H - R) X] \bullet (-jX^*) \} \\ &= -\frac{4}{M_i^2 N^2} \text{Im} \{ [(X^* X^T - R^T) X^*] \bullet (X) \} \end{aligned} \quad (16)$$

3.1.2. Deriving the Gradient of $f_2(\mathbf{X})$

According to the gradient properties of composite functions, the gradient of $f_2(\mathbf{X})$ concerning the phase matrix θ can be expressed as

$$\nabla_{\theta} f_2(\mathbf{X}) = \nabla_{g_{ik}(\mathbf{X})} f_2(\mathbf{X}) \nabla_{\theta} g_{ik}(\mathbf{X}) = 2p\eta \sum_{i=1}^B \sum_{n=1}^{N-1} \omega_{ik} g_{ik}^{p-1}(\mathbf{X}) \nabla_{\theta} g_{ik}(\mathbf{X}) \quad (17)$$

where $\eta = \left[\frac{1}{B(2N-2)} \right]^{\frac{1}{2p}} \frac{1}{2p} \left[2 \sum_{i=1}^B \sum_{n=1}^{N-1} \omega_{ik} g_{ik}^p(\mathbf{X}) \right]^{\frac{1}{2p}-1}$. From Appendix B, the following can be known:

$$\nabla_{\theta} g_{i,k}(\mathbf{X}) = 2\beta_i \text{Im} \left\{ g_{1,i,k}(\mathbf{X}) [(a^*(\phi_i) a^T(\phi_i) \mathbf{X} J_k^T) \bullet \mathbf{X}^* - (a(\phi_i) a^H(\phi_i) \mathbf{X}^* J_k) \bullet \mathbf{X}] \right\} \quad (18)$$

So, the unambiguous formula of $\nabla_{\theta} f_2(\mathbf{X})$ can be obtained by substituting the above formula with Equation (17).

3.1.3. Deriving the Gradient of $f_{obj}(\mathbf{X})$

Based on the derivations of Sections 3.1.1 and 3.1.2, we can easily obtain the gradients of $f_{obj}(\mathbf{X})$, as shown below.

$$\nabla_{\theta} f_{obj}(\mathbf{X}) = \mu_1 \cdot \nabla_{\theta} f_1(\mathbf{X}) + \mu_2 \cdot \nabla_{\theta} f_2(\mathbf{X}) \quad (19)$$

$$\nabla_{\theta} f_1(\mathbf{X}) = -\frac{4}{M_i^2 N^2} \text{Im} \{ [(X^* X^T - R^T) X^*] \bullet (X) \} \quad (20)$$

$$\nabla_{\theta} f_2(\mathbf{X}) = 2p\eta \sum_{i=1}^B \sum_{n=1}^{N-1} \omega_{ik} g_{ik}^{p-1}(\mathbf{X}) \nabla_{\theta} g_{ik}(\mathbf{X}) \quad (21)$$

$$\nabla_{\theta} g_{i,k}(\mathbf{X}) = 2\beta_i \text{Im} \left\{ g_{1,i,k}(\mathbf{X}) [(a^*(\phi_i) a^T(\phi_i) \mathbf{X} J_k^T) \bullet \mathbf{X}^* - (a(\phi_i) a^H(\phi_i) \mathbf{X}^* J_k) \bullet \mathbf{X}] \right\} \quad (22)$$

3.2. Flows of STJ Optimization

The algorithm flow for the spatial–temporal joint optimization of the MIMO radar multiphase-encoded waveform based on gradient descent is shown in Figure 1. Firstly, the system parameters are set, including the number of transmitting array elements M_t , the number of waveforms encoding N , the desired beam pattern P_e , the weight u_1 of the beam-pattern objective function, the weight u_2 of the ACSL of the spatial synthesis signal, the sidelobe strategy parameter p , the maximum number of iterations $IterNums_{max}$, the termination threshold ε , and the maximum step size λ_{max} for gradient descent. The phase matrix of the transmitting waveform θ is initialized. Secondly, if it is in omnidirectional emission mode, $R = I_{M_t}$ is set; otherwise, the covariance matrix R is calculated based on the beam-pattern template P_e . Thirdly, the gradient of the objective function and the value of the objective function under the current phase matrix are calculated according to (20,21,19) and (9,10,13), respectively. Fourthly, the GD method is used to update the phase matrix of the transmitting waveform; namely, θ is updated using $\theta_{i+1} = \theta_i - \lambda \frac{\nabla_{\theta} f_{obj}}{\|\nabla_{\theta} f_{obj}\|_2}$, where the step size λ is set as an empirical value or determined using a one-dimensional linear search method. Finally, the objective function value is judged to determine if it meets the termination threshold condition or if the maximum number of iterations has been reached; if it is not true, Step 3 and Step 4 are repeated; otherwise, the loop is exited, the solved waveform phase matrix θ is output, and the transmitting waveform matrix X is obtained according to $X = \sqrt{c}e^{j\theta}$. It should be noted that in this paper, the Armijo criterion [41] is adopted for the one-dimensional linear search. Its algorithm steps are shown as follows:

- (1) One-dimensional linear search variable initialization is performed; namely, the initial values of β and σ are set under $\beta \in (0, 1)$ and $\sigma \in (0, 0.5)$, and $\lambda_{max} = 0.2\pi$, with $i = 0$.
- (2) λ is updated according to $\lambda = \lambda_{max}\beta^i$; if $f_{obj}(\theta_i - \lambda \frac{\nabla_{\theta} f_{obj}}{\|\nabla_{\theta} f_{obj}\|_2}) < f_{obj}(\theta_i) - \sigma\beta^i \|\nabla_{\theta} f_{obj}\|_2^2$, the one-dimensional linear search is stopped, and the step size λ is output; otherwise, with $i = i + 1$, λ is continually updated, and the inequality conditions are judged.

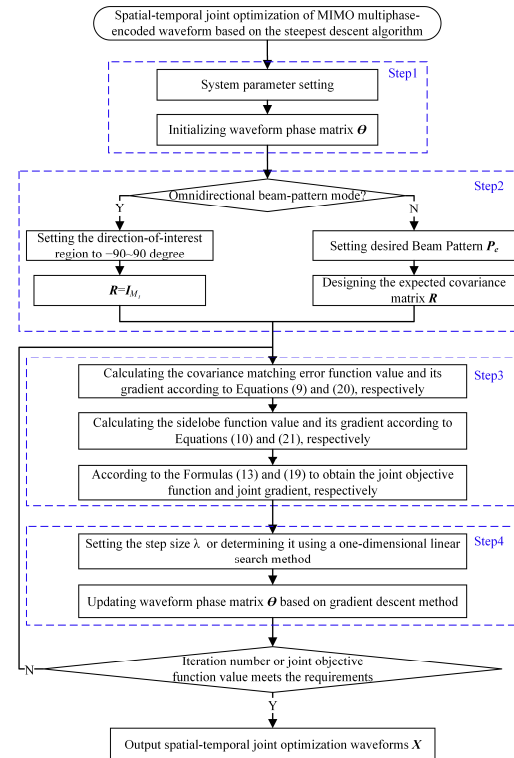


Figure 1. The algorithm flow for the spatial–temporal joint optimization of the MIMO radar multiphase-encoded waveform based on gradient descent.

4. Results and Discussion

This chapter describes the numerical simulation experiments used to verify the feasibility of our STJ optimization method for the multiphase coding waveform of the MIMO radar. The optimization results under different radar operating modes are given, including the beam pattern, cross-correlation between different beams, and autocorrelation characteristics of spatial synthesis signals in the directions of interest, and the algorithm convergence is evaluated comparatively. Unless otherwise specified, the following default parameters are used in this article: the spacing between transmitting elements is equal to half the wavelength, the carrier frequency $f_c = 3$ GHz, the amplitude of the transmitting signal $\sqrt{c} = 1$, the number of transmitting elements $M_t = 16$, the number of waveform codes $N = 128$, the sidelobe norm strategy $p = 16$, the maximum step size of gradient descent $\lambda_{\max} = 0.2\pi$, the weight coefficient $u_1 = 1$ and $u_2 = 1$, the maximum number of iterations $IterNums = 100$, and the initial waveform is generated randomly.

4.1. Orthogonal Waveform STJ Optimization

The orthogonal waveform mode is commonly used for holographic gaze detection of targets in all directions. Ideally, its covariance matrix is the identity matrix, and its beam pattern is uniformly distributed in all directions. To verify the feasibility and superiority of our method in designing an orthogonal waveform, this experiment is conducted. In this experiment, let the number of transmitting elements $M_t = 4$ and the covariance matrix $\mathbf{R} = \mathbf{I}_{M_t}$. To ensure good detection performance in all spatial directions, the interest direction domains are set to $-90^\circ \sim 90^\circ$.

Figure 2a,b show the autocorrelation and cross-correlation of the transmitting waveform, where the correlation result between every two waveforms is represented by a different color. It can be seen that the cross-correlation peak level of the orthogonal waveform designed using our STJ-L2pSL-GD method is close to that of the Multi-CAN method, but the autocorrelation peak sidelobe level is higher than that of the Multi-CAN method. However, the spatial synthesis signal (SSS) is the actual echo of the target, and the autocorrelation sidelobes (ACSLs) of the spatial synthesis signal ultimately affect the detection performance. Figure 2c,d show the autocorrelation performances of SSSs in the directions $-90^\circ \sim 90^\circ$ through the Multi-CAN method and our STJ-L2pSL-GD method, respectively. Among them, each color corresponds to the autocorrelation of a direction. Figure 2e shows the projection of autocorrelation of spatial synthetic signals in the $-90^\circ \sim 90^\circ$ directions of the waveform optimized using several methods to the distance dimension. It can be seen that the SSS of the waveform obtained using the Multi-CAN method or BSUM method has high autocorrelation peak sidelobes, indicating that good autocorrelation and cross-correlation performances between waveform sequences cannot guarantee that its sidelobe performance is still good after equivalent transmit beamforming at the receiving end. However, the peak sidelobes of the spatial synthesis signal from the STJ methods, including STJ-QSP, STJ-IWISL-GD, and STJ-L2pSL-GD, are flatter and better suppressed. Figure 2f indicates that the beam pattern of waveform designed using those methods, such as random, Multi-CAN, and BSUM, fluctuates relatively greatly, as it lacks constraints on the beam pattern. However, the beam pattern of the waveform optimized using the STJ methods fluctuates less. It should be noted that compared to other methods, the ACSL of the SSS of the waveform designed using the STJ-L2pSL-GD method is lower, and simultaneously, the beam pattern obtained using the STJ-L2pSL-GD methods is closer to the ideal omnidirectional pattern, which making it more conducive to holographic gaze detection.

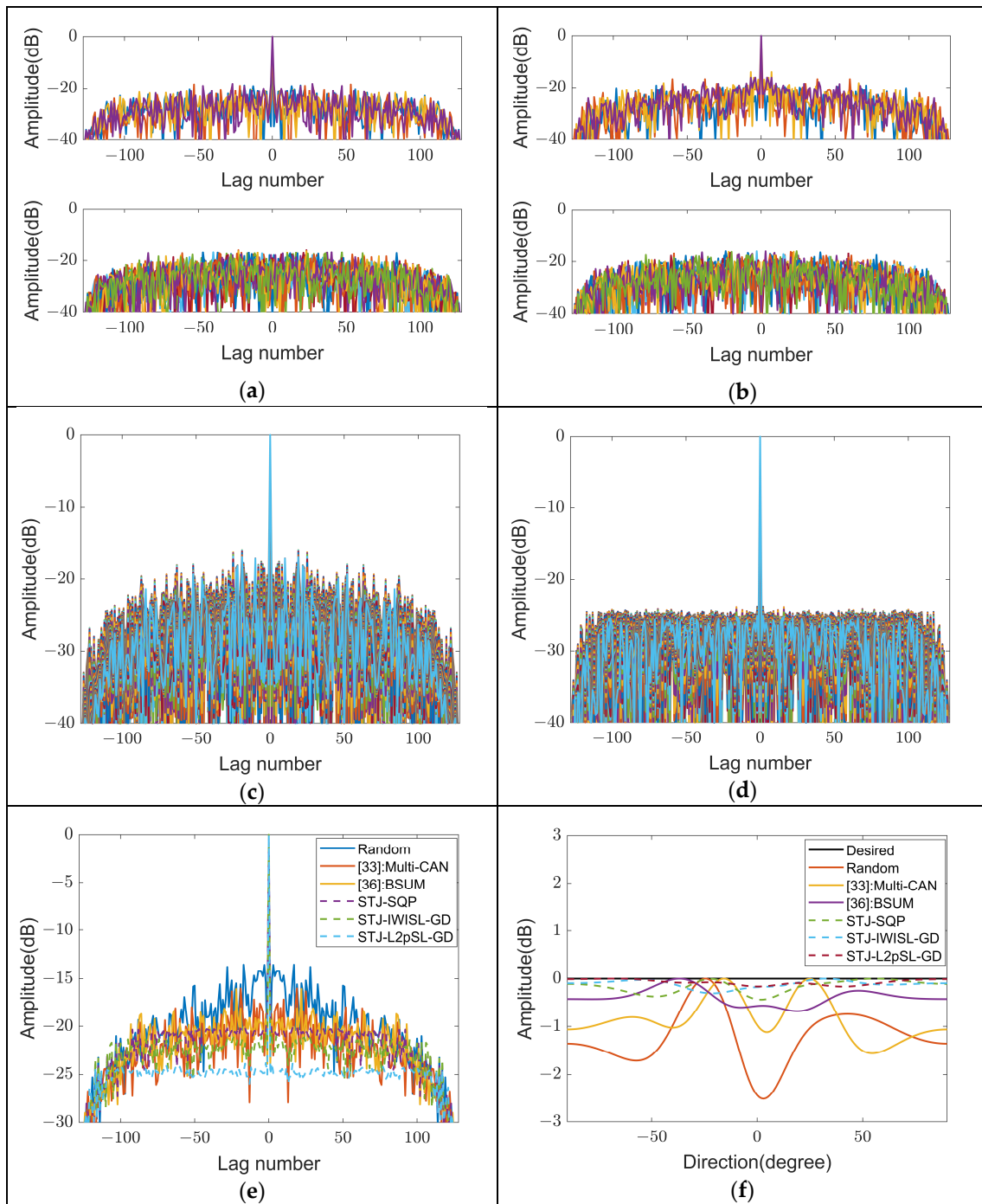


Figure 2. The spatial–temporal characteristics of the orthogonal waveform designed using several methods. (a) The autocorrelation and cross-correlation of the waveform obtained using the Multi-CAN method. (b) The autocorrelation and cross-correlation of waveform designed using the STJ-L2pSL-GD method. (c) The autocorrelation of spatial synthesis signals in the directions $-90^\circ \sim 90^\circ$ using the Multi-CAN method. (d) The autocorrelation of spatial synthesis signals in the directions $-90^\circ \sim 90^\circ$ using the STJ-L2pSL-GD method. (e) The projection of autocorrelation of spatial synthetic signals in the $-90^\circ \sim 90^\circ$ directions to the distance dimension. (f) The beam pattern of the waveform designed using several methods.

4.2. Partial-Related Waveform STJ Optimization

In this chapter, firstly, a covariance matrix that satisfies a desired beam pattern is designed, and then, the transmitting waveform is solved using our STJ optimization

method. In practical applications, depending on the requirements of different scenarios, the beam may need to be wide or narrow, and it may need to form one beam or multiple beams. Here, we take two beams of a certain width as examples to illustrate this. When evaluating the STJ optimization algorithm for designing waveforms, it is initialized with the same random number, and all algorithms are iterated the same number of times, such as 100 times.

4.2.1. Designing Covariance Matrix

Here, we set the number of transmitting elements $M_t = 16$, and the directions of interest are $[-10^\circ, 10^\circ]$ and $[40^\circ, 60^\circ]$. The covariance matrix optimization results obtained using the beam-pattern matching method described in Section 2.1 are shown in Figure 3. Here, Figure 3a is the modulus value graph of the designed covariance matrix, which shows that all values on the diagonal are equal to 1, and the matrix is symmetric about the diagonal, as expected. Figure 3b is the beam-pattern results of the designed covariance matrix, which is consistent with the desired beam pattern. As the beam pattern is designed with two main lobe directions of 0 degrees and 50 degrees, we analyze the cross-correlation characteristics of the beam in the 0° and 50° directions with the beams in other directions to illustrate that the mutual interference between different beams can be suppressed using the interference constraints of Formula (8). The results are shown in Figure 3c,d. As expected, the mutual interference intensity is greatly reduced when $w_c = 1$ compared to the one up to -10 dB when $w_c = 0$.

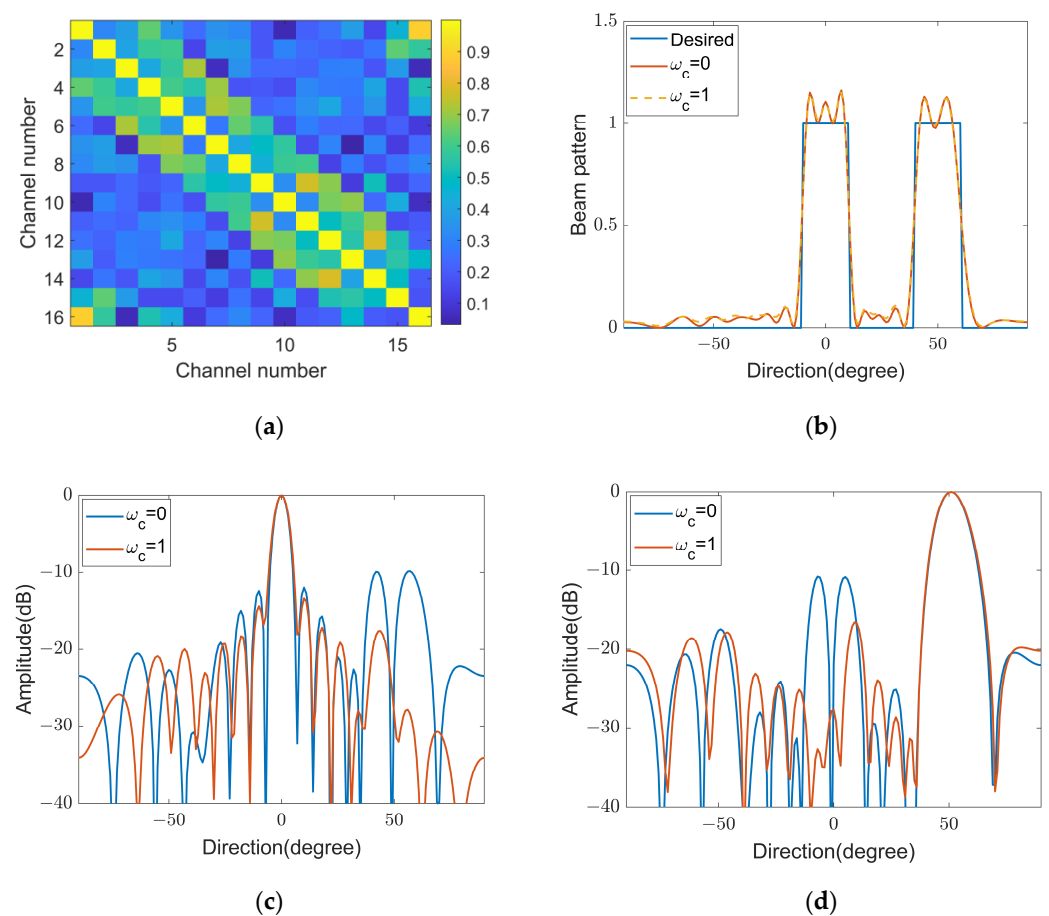


Figure 3. The covariance matrix results through the beam-pattern matching method. (a) The modulus value graph of the designed covariance matrix when $w_c = 1$. (b) The beam-pattern results of the designed covariance matrix. (c) Correlation characteristics between the beam in the 0° direction and the beams in other directions. (d) Correlation characteristics between the beam in the 50° direction and the beams in other directions.

4.2.2. Designing Waveform Based on Covariance Matrix

Based on the covariance matrix designed in Section 4.2.1, the transmitting waveform is obtained using several methods, respectively, and the results are shown in Figure 4. Specifically, Figure 4a is the beam pattern of the designed waveform. It indicates that compared to the beam pattern of the expected covariance matrix, even without considering the temporal-domain sidelobes, the beam-pattern performance of the obtained waveform is reduced due to the inevitable errors in the procedure of synthesizing the waveform from the expected covariance matrix. Fortunately, nevertheless, after considering the autocorrelation sidelobe of the spatial synthesis signal, further deterioration of the beam-pattern performance is seldom, and the PSL of the beam pattern still reaches nearly -18 dB, which is sufficient to meet the application requirements. Figure 4b,c show the autocorrelation characteristics of the spatial synthesis signals of the waveforms designed using the STJ-L2pSL-GD and STJ-SQP methods, separately, after 100 iterations, where each color corresponds to the autocorrelation of a direction. Figure 4d is the projection of autocorrelation of spatial synthetic signals of the directions of interest, $[-10^\circ, 10^\circ]$ and $[40^\circ, 60^\circ]$, to the distance dimension. It indicates that only when considering the BPM optimization in the STJ method, namely $\mu_2 = 0$, or adopting the CA method, the autocorrelation peak sidelobes of the synthesis waveform are relatively high, about -13 dB and -15 dB, respectively. When including the temporal-domain sidelobes constraint, namely $\mu_2 = 1$, a significant improvement in the time-domain sidelobe performance is achieved at the cost of a small decrease in the beam-pattern performance, wherein the STJ-L2pSL-GD method, respectively, achieves a sidelobe of about -25 dB, which is more significant than the STJ-SQP algorithm's -20 dB and the STJ-IWISL-GD method's -22 dB.

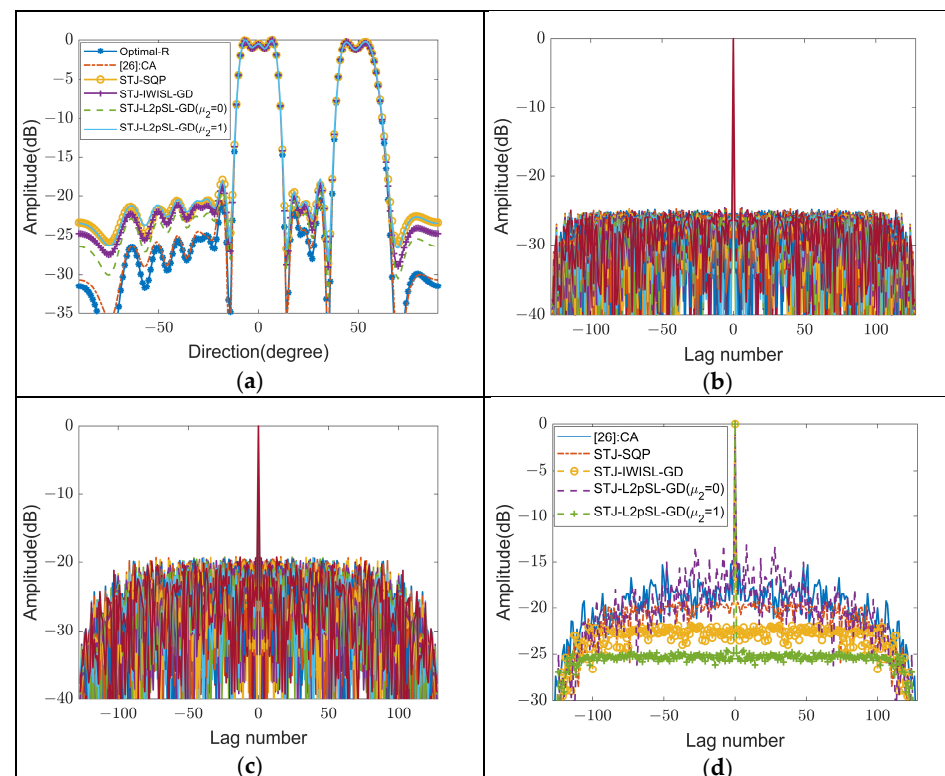


Figure 4. Spatial-temporal joint optimization results for partial-related waveform. (a) The comparison of beam-pattern characteristics from different methods. (b) The autocorrelation of the spatial synthesis signal of the waveform designed using the STJ-L2pSL-GD method, where $\mu_2 = 1$ in the directions of interest. (c) The autocorrelation of spatial synthesis signals of the waveform designed using the STJ-SQP method, where $\mu_2 = 1$ in the directions of interest. (d) The projection of autocorrelation of the spatial synthetic signals in the directions of interest, $[-10^\circ, 10^\circ]$ and $[40^\circ, 60^\circ]$, to the distance dimension.

4.3. Waveform Processing Results

To further validate the effectiveness of the method proposed in this paper, the following experiment is conducted. Pulse synthesis and receive beamforming processing are performed in sequence on the designed orthogonal waveform and partial-related waveform echo of zeros distance target from radar, respectively. And the results are shown in Figure 5, where the red line indicates the target. Overall, the results show that the target is well detected, which confirms the effectiveness of the proposed method. In addition, comparing Figure 5a,b, it can be seen that after the number of receiving elements increases from 1 to 16, the amplitudes of both the main lobes and sidelobes is increased by $20 \log_{10}(M_r) = 24$ dB simultaneously, evidencing that receive beamforming does not affect the range sidelobe ratio, although it contributes to focusing on specific directions by synthesizing multiple receiving channels and, thus, effectively improves the SNR. From Figure 5b,c, it can be seen that the processing output energy of the partial-related waveform is higher than that of the orthogonal waveform. This is because partial-related waveforms have a certain emission energy gain in the target direction compared to orthogonal waveforms. Taking Figure 5d compared with Figure 5c, it can be seen that the distance sidelobes of the multi-target echo are higher than those of the single-target echo. This is due to the inevitable cross-correlation between target echoes of different orientations, although it has been alleviated by adding the cross-correlation constraints of beams in different directions when designing the covariance matrix in Section 4.2.1.

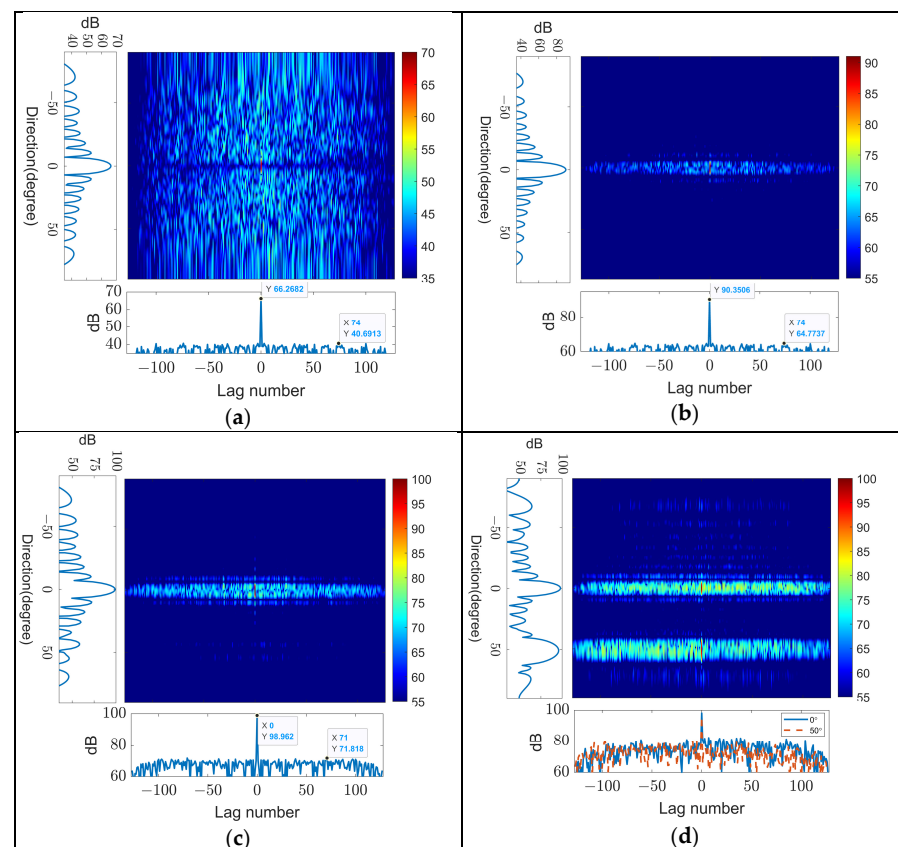


Figure 5. The echo processing results of the waveform designed using the STJ-L2pSL-GD method, where $M_t = 16$ and $L = 128$. (a) Echo processing results of a single target in a 0-degree azimuth under orthogonal waveform mode, without receive beamforming ($M_r = 1$). (b) Echo processing results of a single target in a 0-degree azimuth under orthogonal waveform mode, with receive beamforming ($M_r = 16$). (c) Echo processing results of a single target in a 0-degree azimuth under multi-beam of Figure 4a, with receive beamforming ($M_r = 16$). (d) Echo processing results of bi-target in a 0-degree azimuth and 50-degree azimuth, respectively, under multi-beam of Figure 4a, with receive beamforming ($M_r = 16$).

4.4. Analyzing the Influence of p on Temporal Sidelobe

This section analyzes the influence of p on the ACSLs of spatial synthesis signals in the directions of interest under the L2p-Norm criterion. The results are shown in Figure 6. Figure 6a shows the convergence curve of L2pSL of spatial synthesis signals with different p values under the STJ-L2pSL-GD method. It indicates that the convergence curve of L2pSL completely overlaps with the ISL curve when p is equal to 1, and as p increases, the convergence curve of L2pSL quickly approaches the PSL curve, which confirms the analysis in Section 2. So, by setting p to different values, various criterion effects, including the ISL criterion, approximate PSL criterion, and their compromise, can be achieved. Figure 6b shows the ISL convergence curves of our STJ-L2pSL-GD method under different p values. It indicates that the smaller the p , the faster the convergence speed of the ISL. This is because the smaller the p , the more inclined the L2p-Norm criterion is towards the ISL criterion. Figure 6c,d show the convergence curves of the PSL and the autocorrelation response of the spatial synthesis signal for 100 iterations under different p values, respectively. It can be seen that since the PSL is not the objective function, its monotonic convergence cannot be guaranteed. Additionally, the convergence speed of the PSL increases first and then decreases as p increases. When p is about 4, the fastest PSL convergence speed and the lowest PSL are achieved. This indicates that taking an appropriate value of p will result in a faster PSL convergence speed and a better PSL suppression effect than adopting the PSL criterion.

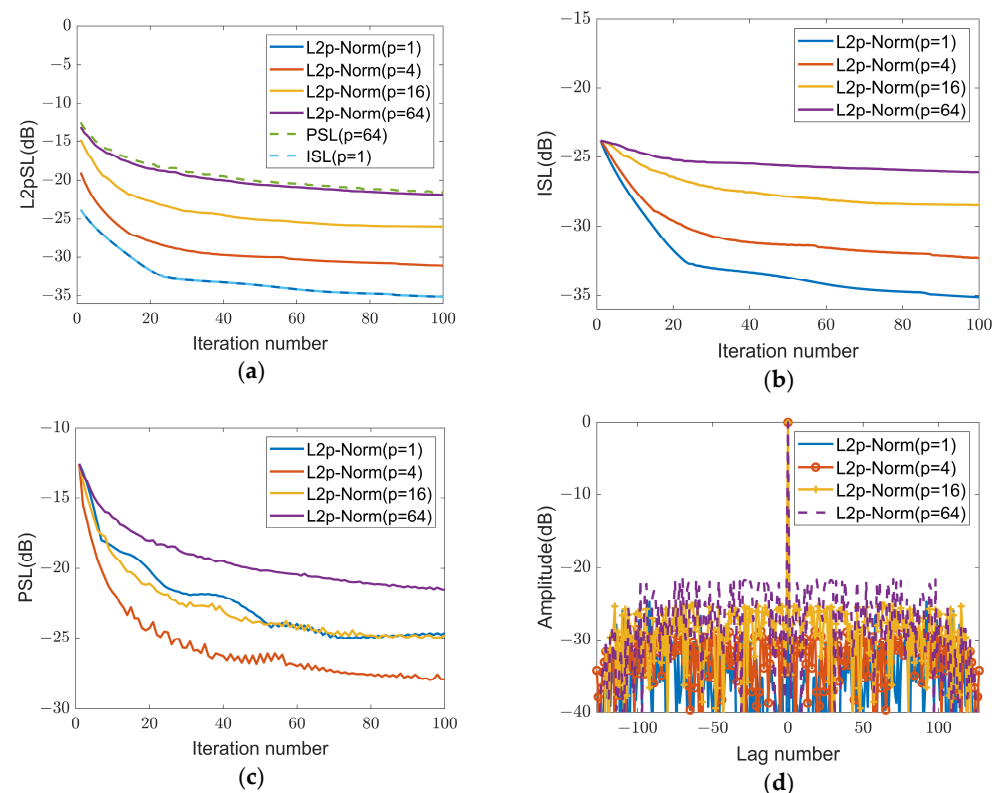


Figure 6. The sidelobe situations of spatial synthesis signals of our STJ-L2pSL-GD method in the directions of interest under different p -values using the linear search strategy and under a multi-beam mode. (a) L2pSL convergence curve under different p -values. (b) PSL convergence curve under different p -values. (c) ISL convergence curve under different p -values. (d) Autocorrelation of spatial synthesis signals in 0° direction under different p -values.

4.5. Evaluation of Algorithm Convergence and Complexity

To evaluate the convergence and efficiency of our algorithm, this experiment is conducted by using a regular laptop computer with a CPU configuration of Intel Core i7-11800H, a main frequency of 2.3 GHz, and a memory configuration of 40 GB. Figure 7a is

the convergence curve of the MSE for beam-pattern matching in covariance designing. It indicates that the convergence is achieved with only one iteration, the converging MSE value is only 0.103, and the time taken is only 5.20 s. Figure 7b,c show the convergence curves of the MSE for covariance matching and the convergence curve of L2pSL of spatial synthesis signals in the directions of interest for our STJ-L2pSL-GD method, respectively. It indicates that both they gradually decrease with the number of iterations. Additionally, when the step size λ of gradient descent is determined through the one-dimensional linear search method, the L2pSL value of the spatial synthesis signals monotonically decreases with the number of iterations, indicating good convergence of the algorithm. Figure 7d shows the convergence curves for the sum of PSLs in the temporal domain and MSEs from the covariance matrix matching of the STJ-SQP algorithm or STJ-L2pSL-GD method under different modes and parameters. It indicates that the objective function values of both methods decrease with the number of iterations, reaching about -15 dB and -25 dB after 100 iterations, respectively. Obviously, under the same number of iterations, the effects of our STJ-L2pSL-GD method are far superior to the STJ-SQP method. Moreover, as the essence of the SQP method is transforming nonlinear problems into a series of quadratic programming problems using Taylor expansion, it requires calculating the Hessian matrix and gradient of its objective function at each iteration, while our STJ-L2pSL-GD method only needs to calculate the gradient of the objective function. Therefore, the time required for each iteration using our STJ-L2pSL-GD method is significantly reduced compared to the STJ-SQP method. Specifically, the STJ-SQP algorithm takes 4319.44 s to iterate 100 times, while our STJ-L2pSL-GD method takes 69.08 s and 235.03 s to iterate 100 times under a fixed step size and linear search for step size, respectively.

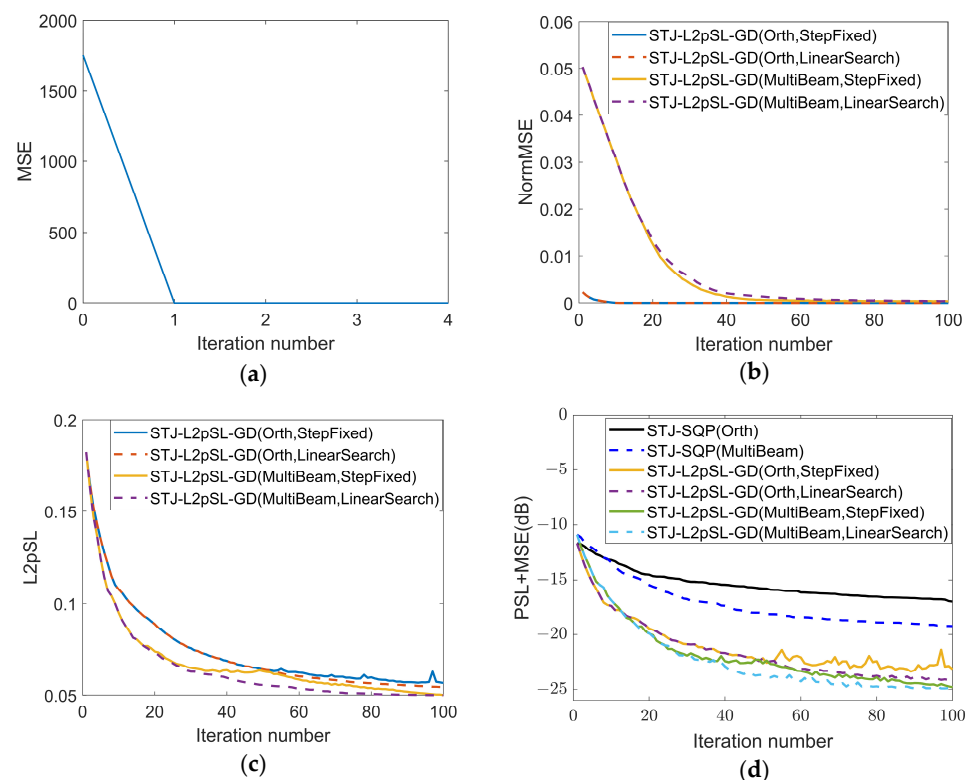


Figure 7. Convergence evaluation of waveform STJ optimization algorithm, where $M_t = 16$, $N = 128$, $\mu_1 = 1$, $\mu_2 = 1$, and $step_{max} = 0.2\pi$. (a) Convergence curve of MSE for BPM in covariance designing. (b) Convergence curve of MSE for covariance matrix matching in waveform designing using STJ-L2pSL-GD method. (c) Convergence curve of L2pSL of spatial synthesis signals in directions of interest for STJ-L2pSL-GD method. (d) Convergence curves for the sum of PSLs in the temporal domain and MSEs in covariance matrix matching under different modes and parameters of STJ-SQP and STJ-L2pSL-GD algorithms.

5. Conclusions

This paper elaborates on the waveform design problem of a centralized MIMO radar and proposes an STJ optimization method for multiphase coding sequences, where, in the optimization of beam-pattern matching, to reduce complexity, a two-step strategy is adopted, namely, first, designing a waveform covariance matrix, and then, synthesizing the waveform. In terms of optimizing the temporal-domain performance, considering that the autocorrelation of spatial synthesized signals is the final response of the MIMO waveform, our paper aims to minimize the L2pSL of the spatial synthesis signal. By setting different p -values, a compromise is achieved between ISL and PSL suppression. To solve the non-convex, high-dimensional, nonlinear, hard optimization problem efficiently, we transform it into an unconstrained optimization problem about the waveform phase and then use the gradient descent algorithm. The effectiveness of our method is verified through numerical simulation experiments. Compared to existing methods, our method possesses many advantages. Firstly, our method is suitable for designing both partial-correlation and orthogonal waveforms. Moreover, our method achieves both good beam-pattern matching and good autocorrelation characteristics of the spatial synthesis signal. Last but not least, our method is more flexible and efficient.

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Appendix A. The Derivations of $\nabla_X f_1(X, X^*)$ and $\nabla_{X^*} f_1(X, X^*)$

According to the proposition 3.5.1 of [42], given a complex scalar function f with a complex matrix X as its variable, its gradient matrix and conjugate gradient matrix concerning the complex matrix variable have the following unique identification correspondence with its differential [42]:

$$df(X, X^*) = \text{tr}(AdX + BdX^*) \Leftrightarrow \begin{cases} \nabla_X f(X, X^*) = A^T \\ \nabla_{X^*} f(X, X^*) = B^T \end{cases} \quad (\text{A1})$$

So, to obtain the gradient of a function, one can first find its differential. Considering that the objective Function (9) is the inner product of a matrix, the following holds true:

$$f_1(X, X^*) = \frac{1}{M_t^2 N^2} \|XX^H - R\|_2^2 = \frac{1}{M_t^2 N^2} [\text{vec}^H(XX^H - R)] \text{vec}(XX^H - R) \quad (\text{A2})$$

Additionally, according to the relationship between the vectorization operator and the trace function, $\text{tr}(A^T B) = \text{vec}^T(A) \text{vec}(B)$; then, the above formula can be further expressed as

$$f_1(X, X^*) = \frac{1}{M_t^2 N^2} \text{tr}[(XX^H - R)^H (XX^H - R)] \quad (\text{A3})$$

It should be noted that by swapping the order of trace operations and differential operations and utilizing the properties of differentiation and trace, the differential of $\|XX^H - R\|_2^2$ can be derived, as shown below:

$$\begin{aligned} d[\|XX^H - R\|_2^2] &= d\left\{\text{tr}[(XX^H - R)^H(XX^H - R)]\right\} \\ &= \text{tr}\left\{[d[(XX^H - R)^H](XX^H - R)]^T\right\} + \text{tr}[(XX^H - R)^H d(XX^H - R)] \\ &= \text{tr}\left\{(X^*X^T - R^T)[d(X^*)X^T + X^*d(X^T)]\right\} + \text{tr}\left\{(XX^H - R^H)[d(X)X^H + Xd(X^H)]\right\} \\ &= \text{tr}\left\{X^T(X^*X^T - R^T)d(X^*)\right\} + \text{tr}\left\{X^H(XX^H - R)d(X)\right\} + \\ &\quad \text{tr}\left\{X^H(XX^H - R^H)d(X)\right\} + \text{tr}\left\{X^T(X^*X^T - R^*)d(X^*)\right\} \end{aligned} \quad (A4)$$

As the waveform covariance matrix R is the Hermite matrix, the above equation can be further written as

$$d[\|XX^H - R\|_2^2] = 2\text{tr}\left\{X^H[XX^H - R]d(X)\right\} + 2\text{tr}\left\{X^T[X^*X^T - R^T]d(X^*)\right\} \quad (A5)$$

So, according to (A1), the expressions $\nabla_X f_1(X, X^*)$ and $\nabla_{X^*} f_1(X, X^*)$ can be obtained, as shown below:

$$\begin{aligned} \nabla_X f_1(X, X^*) &= \frac{2}{M^2 N^2} \{X^H[XX^H - R]\}^T = \frac{2}{M^2 N^2} [X^*X^T - R^T]X^* \\ \nabla_{X^*} f_1(X, X^*) &= \frac{2}{M^2 N^2} \{X^T[X^*X^T - R^T]\}^T = \frac{2}{M^2 N^2} [XX^H - R]X \end{aligned} \quad (A6)$$

Appendix B. The Derivation of $\nabla_{\theta} g_{ik}(X)$

For the convenience of elaboration, let $g_{1,ik}(X) = a^T(\phi_i)XJ_kX^H a^*(\phi_i)$ and $g_{2,ik}(X) = a^T(\phi_i)XJ_k^T X^H a^*(\phi_i)$; then, $g_{i,k}(X)$ can be represented as

$$\begin{aligned} g_{i,k}(X) &= \beta_i [a^T(\phi_i)XJ_kX^H a^*(\phi_i)][a^T(\phi_i)XJ_kX^H a^*(\phi_i)]^H \\ &= \beta_i [g_{1,ik}(X)g_{2,ik}(X)] \end{aligned} \quad (A7)$$

Then, according to the gradient properties of composite functions, $\nabla_{\theta} g_{i,k}(X)$ can be expressed as

$$\nabla_{\theta} g_{i,k}(X) = \beta_i \{g_{2,ik}(X)[\nabla_X g_{1,ik}(X) \bullet (\nabla_{\theta} X) + \nabla_{X^*} g_{1,ik}(X) \bullet (\nabla_{\theta} X^*)] + g_{1,ik}(X)[\nabla_X g_{2,ik}(X) \bullet (\nabla_{\theta} X) + \nabla_{X^*} g_{2,ik}(X) \bullet (\nabla_{\theta} X^*)]\} \quad (A8)$$

Equations (A12) and (A13) of Appendix C and (A14) of Appendix D can be substituted into the above equation; then, we have

$$\begin{aligned} \nabla_{\theta} g_{i,k}(X) &= \beta_i \{g_{2,ik}(X)[(a(\phi_i)a^H(\phi_i)X^*J_k^T) \bullet (jX) + (a^*(\phi_i)a^T(\phi_i)XJ_k) \bullet (-jX^*)] + \\ &\quad g_{1,ik}(X)[(a(\phi_i)a^H(\phi_i)X^*J_k) \bullet (jX) + (a^*(\phi_i)a^T(\phi_i)XJ_k^T) \bullet (-jX^*)]\} \end{aligned} \quad (A9)$$

Additionally, as $g_{2,ik}(X)$ is a matrix with one row and one column, then, $g_{2,ik}(X) = [g_{2,ik}(X)]^T = [g_{1,ik}(X)]^*$. So, the above formula can be further derived as

$$\nabla_{\theta} g_{i,k}(X) = 2\beta_i \text{Im}\{g_{1,ik}(X)[(a^*(\phi_i)a^T(\phi_i)XJ_k^T) \bullet X^* - (a(\phi_i)a^H(\phi_i)X^*J_k) \bullet X]\} \quad (A10)$$

Appendix C. The Gradient Derivations of $g_{1,i,k}(X)$ and $g_{2,i,k}(X)$ with Respect to X and X^* , Respectively

As $g_{1,i,k}(X)$, namely $\mathbf{a}^T(\phi_i)\mathbf{X}\mathbf{J}_k\mathbf{X}^H\mathbf{a}^*(\phi_i)$, is a matrix of one row and one column, the following holds true:

$$\begin{aligned} d[g_{1,i,k}(X)] &= d[\mathbf{a}^T(\phi_i)\mathbf{X}\mathbf{J}_k\mathbf{X}^H\mathbf{a}^*(\phi_i)] = d[\text{tr}\{\mathbf{a}^T(\phi_i)\mathbf{X}\mathbf{J}_k\mathbf{X}^H\mathbf{a}^*(\phi_i)\}] \\ &= \text{tr}\{\mathbf{a}^T(\phi_i)(d\mathbf{X})\mathbf{J}_k\mathbf{X}^H\mathbf{a}^*(\phi_i)\} + \text{tr}\left\{\left[\mathbf{a}^T(\phi_i)\mathbf{X}\mathbf{J}_k d(\mathbf{X}^H)\mathbf{a}^*(\phi_i)\right]^T\right\} \\ &= \text{tr}\{\mathbf{J}_k\mathbf{X}^H\mathbf{a}^*(\phi_i)\mathbf{a}^T(\phi_i)d\mathbf{X}\} + \text{tr}\{[\mathbf{J}_k^T\mathbf{X}^T\mathbf{a}(\phi_i)\mathbf{a}^H(\phi_i)d(\mathbf{X}^*)]\} \end{aligned} \quad (\text{A11})$$

Then, according to (A1) of Appendix A, we can know

$$\begin{aligned} \nabla_X g_{1,i,k} &= [\mathbf{J}_k\mathbf{X}^H\mathbf{a}^*(\phi_i)\mathbf{a}^T(\phi_i)]^T = [\mathbf{a}(\phi_i)\mathbf{a}^H(\phi_i)\mathbf{X}^*\mathbf{J}_k^T] \\ \nabla_{X^*} g_{1,i,k} &= [\mathbf{J}_k^T\mathbf{X}^T\mathbf{a}(\phi_i)\mathbf{a}^H(\phi_i)]^T = [\mathbf{a}^*(\phi_i)\mathbf{a}^T(\phi_i)\mathbf{X}\mathbf{J}_k] \end{aligned} \quad (\text{A12})$$

Similarly, the gradient of $g_{2,i,k}(X)$ can be obtained, as shown below:

$$\begin{aligned} \nabla_X g_{2,i,k} &= [\mathbf{a}(\phi_i)\mathbf{a}^H(\phi_i)\mathbf{X}^*\mathbf{J}_k] \\ \nabla_{X^*} g_{2,i,k} &= [\mathbf{a}^*(\phi_i)\mathbf{a}^T(\phi_i)\mathbf{X}\mathbf{J}_k^T] \end{aligned} \quad (\text{A13})$$

Appendix D. The Gradients of X and X^* with Respect to θ

Considering $X = \sqrt{c}e^{j\theta}$ and $X^* = \sqrt{c}e^{-j\theta}$, it can be easy to obtain the gradients of X and X^* with respect to θ , as shown below:

$$\begin{aligned} \nabla_\theta X &= j\sqrt{c}e^{j\theta} = jX \\ \nabla_\theta X^* &= -j\sqrt{c}e^{-j\theta} = -jX^* \end{aligned} \quad (\text{A14})$$

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