

Article

A Weighted Image-Point-Measurement Method of Laser Altimetry Points for Improving Laser-Altimetry-Data-Assisted Positioning Accuracy of Small-Satellite Images

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Highlights

What are the main findings?

- This study develops a laser altimetry point-measurement method specifically for small-satellite imagery and constructs a combined geometric positioning model that integrates virtual control points, laser altimetry points, and image-matching tie points. Experimental results confirm that this approach consistently and substantially improves vertical positioning accuracy across all evaluated image-combination scenarios.

What is the implication of the main finding?

- This method can facilitate the application of laser altimetry data for geometric positioning across multi-satellite, multi-sensor, and multi-resolution small-satellite imagery.

Abstract

For small-satellite imagery, traditional image points derived from the direct back-projection of laser altimetry points onto the imagery often exhibit deviations of tens of pixels or more, owing to the relatively limited attitude and orbit determination accuracy of small-satellite platforms. Moreover, the diversity of imaging sensors, variations in image resolution, and inherently weak image geometric configurations further complicate the accurate acquisition of image-space coordinates for laser altimetry points. To facilitate the application of laser altimetry data for geometric positioning across multi-satellite, multi-sensor, and multi-resolution small-satellite imagery, this study proposes a measurement method for laser altimetry points tailored to small-satellite images and establishes a combined geometric positioning model that integrates virtual control points, laser altimetry points, and image-matching tie points. The framework comprises four key procedural components: (1) an image-point-measurement strategy for laser altimetry points; (2) the construction of a laser altimetry data-assisted geometric positioning model for small-satellite imagery; (3) the solution of the geometric positioning model using a total least

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squares approach based on the partial-EIV (errors-in-variables) models; and (4) a comprehensive accuracy assessment conducted under multiple image-combination scenarios, including single-satellite single-stereo, single-satellite multi-stereo, dual-satellite single-stereo, and multi-satellite multi-stereo imagery configurations. Experimental validation is carried out using Jilin-1 small-satellite panchromatic images (KF01A, GF02A, and GF02B) acquired over the Henan region of China. The experimental results demonstrate that, with the laser altimetry point-measurement method and the combined geometric positioning model, the vertical positioning accuracy is substantially improved across all tested image-combination scenarios. These findings further confirm the capability in enhancing the vertical geometric positioning performance of stereoscopic small-satellite imagery characterized by multi-satellite platforms, multi-sensors, and multi-resolutions over terrain conditions similar to those tested.

Keywords: geometric positioning; small-satellite imagery; laser altimetry data; vertical positioning accuracy

1. Introduction

Geometric positioning accuracy denotes the degree of precision attained in aligning image points with their corresponding ground coordinates during the geometric correction of satellite imagery, a prerequisite for the accurate representation of the geographical information [1–3]. High-precision geometric positioning is of critical importance to the utility and quality of satellite imagery, as it underpins the reliability and validity of the imagery across a broad spectrum of applications, ranging from scientific research to commercial services, while furnishing users with accurate spatial references [4,5]. In defense contexts, for instance, superior geometric accuracy significantly facilitates mission planning and execution, thereby enhancing the likelihood of operational success.

Over the past decades, improvements in spatial resolution have enabled the extraction of increasingly fine ground details from satellite imagery, enhanced side-swinging capabilities have shortened the fixed-point revisit intervals of satellites, and the growing number of operational sensors has rendered remote sensing data ever more accessible. Presently, high-resolution satellites are equipped with diverse observation capabilities, encompassing panchromatic, multi-spectral, infrared, synthetic aperture radar (SAR), video, night-lighting, etc. This constitutes an efficient, diversified, and rapid integrated space-ground data acquisition system, characterized by multi-source (multi-platform, multi-sensors, and multi-view) and high-resolution (spatial, temporal, spectral, and radiation) capabilities [6–9]. Simultaneously, there has been a marked increase in the demand for high-precision geometric positioning of satellite images in numerous application areas, particularly in national defense-related projects. Consequently, the systematic integration of multi-source geospatial datasets for the purpose of achieving high-precision geometric positioning of satellite images persists as a significant research issue within the domain of high-resolution Earth satellite observation systems.

In general, achieving high-precision geometric positioning accuracy becomes more challenging when attitude and orbit parameters are subject to uncertainty, or when ground control points (GCPs) are insufficient. To mitigate this issue, recent studies have increasingly turned to the utilization of multi-source auxiliary data for geometric positioning. For instance, numerous studies [10–15] have leveraged distinct digital elevation models (DEMs) as elevation constraints in bundle adjustment models to enhance geometric positioning accuracy. Likewise, digital terrain models (DTMs) have also been integrated into bundle adjustment models in various research efforts (e.g., Guerin [16];

Alisisto et al. [17]) to improve the geometric positioning accuracy. As a crucial and autonomous mapping resource, laser altimetry data have been widely used in the realm of aerospace photogrammetry due to their proficiency in highly precise elevation measurements, as referenced in [18–20].

However, most existing studies have focused on the surveying and mapping of large platform satellites utilizing either a three-line array or a two-line array configuration. There exists a paucity of research for the laser altimetry data-assisted positioning of small platform satellite imagery. In comparison to the larger platform satellites employing three-line or two-line array surveying and mapping techniques, small satellites exhibit a significantly reduced volume, swifter response, and cost-effective nature. Nonetheless, owing to their relatively lower attitude-orbit accuracy, their geometric performance tends to be unstable and generally inferior. Consequently, when it comes to non-mapping applications of small-satellite images, due to the small size and light weight of the satellite, the accuracy of its initial position and attitude is relatively poor. Image points obtained by directly back-projecting laser altimetry data into small-satellite images can have tens or more pixel deviations, and the diversity and resolution of the sensors and the weak geometric relationships between the images add to the complexity of obtaining the coordinates of the image points. Therefore, with more and more small platform satellites with short operational cycles, compact size, rapid response capabilities, cost-effectiveness, and the ability to form multi-satellite networks, it is necessary to design a method to determine the image point coordinates of laser altimetry points and construct combined geometric positioning models for the laser altimetry data-assisted geometric positioning of multi-satellite, multi-sensor, and multi-resolution scenarios of small-satellite images.

Therefore, aiming to exploit laser altimetry data for assisted geometric positioning of multi-satellite, multi-sensor, and multi-resolution small-satellite images, this study proposes a measurement method for laser altimetry points tailored to small-satellite images and establishes a combined geometric positioning model that integrates virtual control points, laser altimetry points, and image-matching tie points. The framework comprises four key procedural components: (1) an image-point-measurement strategy for laser altimetry points; (2) the construction of a laser altimetry data-assisted geometric positioning model for small-satellite imagery; (3) the solution of the geometric positioning model using a total least squares approach based on the partial-EIV (errors-in-variables) models to circumvent the inherent limitations of traditional Markov models; and (4) a comprehensive accuracy assessment conducted under multiple image-combination scenarios, including single-satellite single-stereo, single-satellite multi-stereo, dual-satellite single-stereo, and multi-satellite multi-stereo imagery configurations. Experimental validation is carried out using Jilin-1 small-satellite panchromatic images (KF01A, GF02A, and GF02B) acquired over the Henan region of China. The experimental results demonstrate that, with the laser altimetry point-measurement method and the combined geometric positioning model, the vertical positioning accuracy is substantially improved across all tested image-combination scenarios. These findings further confirm the strong capability in enhancing the vertical geometric positioning performance of stereoscopic small-satellite imagery characterized by multi-satellite platforms, multi-sensors, and multi-resolutions over terrain conditions similar to those tested.

The subsequent sections are organized as follows: related works on laser altimetry data-assisted geometric positioning are introduced in Section 2. Section 3 provides a detailed explanation of the methodology, including an image-point-measurement method of laser altimetry points, a laser altimetry data-assisted geometric positioning model construction, calculation of the geometric positioning model, and accuracy analysis. In Section 4, the study data are described, the experimental design is given, and the experi-

mental results are presented. Section 5 provides a detailed discussion, including comprehensive comparisons of direct positioning accuracy, laser altimetry data-assisted positioning accuracy, and positioning accuracy constrained by actual ground points, which are demonstrated. Finally, the conclusions and outlook of the paper are summarized in Section 6.

2. Related Works

Laser altimetry data, as a crucial and autonomous mapping resource, has undergone substantial advancement over the past decades and has been widely adopted in aerospace photogrammetry, owing to its capability for high-precision elevation measurements.

Laser altimetry data-assisted geometric positioning has initially been utilized in deep-space exploration contexts, particularly for the Moon and Mars, where direct ground-truth measurements are inherently unattainable. Extensive studies have since investigated the synergistic integration of laser altimetry data and extraterrestrial optical imagery pertaining to the Moon and other planetary bodies. For instance, Spiegel et al. [21] harnessed laser altimetry data as constraints in bundle adjustment models, aiming to enhance the geometric positioning accuracy of Mars stereo images via the orienteering slice method. Their analysis delved into the effect of the number of tie points and the interval of orienteering slices on the geometric positioning of these stereo images. Ultimately, they achieved a planar accuracy of approximately 200 m and an elevation accuracy of up to 10 m for the digital surface model (DSM) of Mars. Almost simultaneously, Heipke et al. [22] generated a DSM from the data acquired by the Mars Orbital Altimeter by utilizing it as a constraint for block bundle adjustment, thereby improving the positioning accuracy of Mars Express images in the elevation direction. Yoon and Shan [23] leveraged the range data from the Mars Orbital Laser Altimeter (MOLA) as a constraint for performing the combined bundle adjustment with the image data obtained by the Mars Orbiter Camera (MOC), thereby eliminating the inconsistencies between the MOC images and the MOA range data. Through the combined bundle adjustment, the attitude error was compensated, so as to achieve the effective optimization of the attitude-orbit parameters. Lin and Muller et al. [24] proposed a method for registering laser altimetry data with Mars optical imagery. Initially, a DSM was created utilizing High-Resolution Stereo Camera (HRSC) imagery, subsequently registering it with the DSM derived from laser altimetry data. Afterward, the DSM generated from HRSC stereo imagery served as a reference for achieving registration with the DSM generated from the High Resolution Imaging Science Experiment (HiRISE) imagery. In 2010, Wang and Shu et al. [25] furnished a comprehensive depiction of the Chang'e-1 laser altimeter for lunar exploration purposes. Based on this, Wu et al. [26] used the laser altimetry data acquired by the Chang'e-1 to assist the combined bundle adjustment of optical images and achieved a better geometric positioning accuracy. Di et al. [27] achieved the accurate registration of the laser altimetry points and the Chang'e-1 optical images by using the cross-adjustment method along with sensor model optimization. Di et al. [28] generated laser control points with higher accuracy through the adjustment of the intersection points of ascending and descending orbits, and matched the Chang'e-1 stereo images and laser altimetry data based on the closest-point iteration algorithm to eliminate the inconsistency between the two types of data through combined bundle adjustment, so that the image registration errors were reduced from 1.5 pixels to the sub-pixel level, the spatial registration accuracy in the planar direction was improved from 300 m to 20 m, and the elevation accuracy was improved to 8 m. Wu et al. [29] harnessed the Chang'e-2 imagery and laser altimetry data from NASA's Lunar Reconnaissance Orbiter (LRO) to conduct a comprehensive block bundle adjustment, leveraging both orbital altitude and local surface constraints. This approach aimed to enhance the accuracy of the orientation parameters for both the Chang'e-

2 imagery and laser altimetry data. The adjusted Chang'e-2 images subsequently enabled the generation of a DEM with a resolution of 20 m, demonstrating significant consistency with the DEM derived from laser altimetry data.

In contrast to the combined processing of optical images and laser altimetry data for extraterrestrial bodies such as the Moon and Mars, the utilization of laser altimetry data in the integrated geometric processing of Earth-observation satellite imagery has emerged relatively later. Prior to the launches of various satellites, including the Gaofen-7, ZY-3 02, and ICESat-2, ICESat-1 held the distinction of being the sole Earth observation satellite that had been successfully launched into orbit, equipped with a laser altimeter, albeit devoid of an optical camera for imagery capture. The ICESat/GLAS data exhibit remarkably high elevation accuracy, achieving centimeter-level precision in flat regions [30–33]. This precision underscores the significant value of ICESat/GLAS data in serving as elevation constraints during block bundle adjustment procedures. Despite the ICESat/GLAS data boasting a high level of elevation accuracy, its footprint diameter spans approximately 70 m. Factors such as significant ground roughness, cloud reflection, and waveform saturation can adversely impact this accuracy. As a result, it is imperative to automate the extraction of ICESat/GLAS data that can be used as elevation control points, based on appropriate criteria [19,34,35]. Over the last decade, the field of Earth observation has witnessed noteworthy advancements in the integrated processing of laser altimetry data and optical satellite imagery. E et al. [36] applied the laser altimetry data to the optical stereo images of some areas in the Antarctic and generated a topographic map of the South Pole with a scale of 1:50,000 for some areas. Gonzalez et al. [35] conducted a preliminary analysis on how to extract effective elevation control points from ICESat/GLAS data. Yu et al. [37] conducted a comparative analysis to examine the influence of three distinct data combination schemes on the geometric positioning accuracy in both planar and elevation directions. These schemes encompassed the integration of single-line array optical imagery with LIDAR data, double-line array optical imagery coupled with laser altimetry data, and a combination of three-line array optical imagery and imagery from small-sized array cameras. Takaku et al. [38] successfully derived a DEM with good accuracy by the utilization of ALOS-1 satellite imagery, encompassing a global scale. Wang et al. [39] theoretically demonstrated the viability of incorporating laser altimetry data into the bundle adjustment model of two-line array optical satellite imagery based on simulated image data. Following the triumphant launch of the ZY-3 02 satellite and GF-7 satellite, Li and Tang et al. [40] conducted pivotal experiments to validate the viability of utilizing ICESat/GLAS data as generalized elevation control points to enhance the geometric positioning accuracy of ZY-3 satellite images in the elevation directions. The preliminary results indicated that employing ICESat/GLAS data as generalized elevation control points can significantly improve the uncontrolled elevation accuracy of ZY-3 satellite images, achieving an accuracy superior to 3 m. Furthermore, Li and Tang et al. [41] used the laser altimetry data obtained from the ZY-3 02 satellite to assist the stereoscopic geometric positioning of the ZY-3 02 satellite imagery, achieving a significant improvement in the uncontrolled geometric positioning accuracy in elevation. Cao et al. [42] proposed a refined general theory of satellite stereo-image geometric model assisted by laser altimeter data. The experiments in Hubei and Qinghai have shown that the elevation accuracy of the satellite geometric model refined with altimeter data can reach 1.97 m and 3.23 m, respectively. Tang and Liu et al. [43] proposed a method for combined adjustment of GF-7 stereo images and laser altimetry data, which employs footprint images and stereo images for registration to extract elevation control points. Experimental results indicate that after using laser altimetry data, the root mean square error (RMSE) of elevation significantly improved from the original 2.15 m to 0.75 m, while the maximum elevation error is less than 1.6 m. Considering the high planar accuracy of SAR images and the high elevation accuracy of laser altimetry

data, Zhang et al. [44] proposed an adjustment method of combining the two control sources, which firstly realizes the acquisition of planar control points based on SAR digital orthophoto maps by multi-modal matching, then filters the laser altimetry data to obtain laser altimetry points, and finally carries out the combined bundle adjustment; the experimental results of ZY-3 satellite images demonstrate that the method can achieve an accuracy of 7 m in plane and 3 m in elevation without GCPs. Xie et al. have obtained a series of datasets based on laser altimetry data, e.g., references [45,46].

However, most existing studies have focused on the surveying and mapping of large platform satellites utilizing either a three-line array or a two-line array configuration. There exists a paucity of research for the laser altimetry data-assisted positioning of small platform satellite imagery. Therefore, it is necessary to exploit laser altimetry data for assisted geometric positioning of multi-satellite, multi-sensor, and multi-resolution small-satellite images.

3. Methodology

Section 3 can be broken down into 4 main important components: an image-point-measurement method of laser altimetry points, a laser altimetry data-assisted geometric positioning model construction of multi-satellite, multi-sensor, multi-resolution small platform satellite images, calculation of the geometric positioning model, and accuracy analysis.

3.1. An Image-Point-Measurement Method of Laser Altimetry Points

At present, the laser altimetry data acquired by onboard laser altimeters of satellites such as ZY-3 02, GF-7, and other satellites have not yet been made public or open-source. ICESat-1/GLAS data and ICESat-2/ATLAS data are two kinds of open-source data. ICESat-1 has been withdrawn from active service, but the data it collects have high research value, and unlike ICESat-1/GLAS, ICESat-2/ATLAS data have a smaller spot size and higher density, and acquire more refined surface information. Taking the ICESat-1/GLAS data as an example, during the acquisition process, the ICESat-1/GLAS laser altimetry data are prone to a significant number of gross errors, stemming primarily from factors like atmospheric conditions. However, under favorable circumstances, such as excellent atmospheric transmittance, flat terrain, and the absence of vegetation, the satellite laser altimetry achieves remarkably high accuracy in measuring the elevation of ground footprint points [47]. The criteria for excluding significant error values in laser altimetry data are outlined below: (1) the threshold for eliminating data rows based on attitude quality label is set at 50 and 100 rows; (2) data rows with reflectance label values below 10% should be discarded; (3) all data bearing a gain value label exceeding 200 must be excluded [48]. Following the aforementioned screening procedure, we will acquire high-precision laser altimetry data, wherein the elevation reference shall be established as the TOPEX/Poseidon reference ellipsoid. Additionally, the geodetic height within the TOPEX/Poseidon ellipsoid shall be transformed to the WGS84 coordinate system. According to literature [41,49,50], the semi-major axis of the TOPEX/Poseidon ellipsoid differs from that of the WGS84 ellipsoid by approximately 0.7 m. As reported in reference [50], when converting coordinates for the same ground point between these two ellipsoidal reference frames, the corrections in longitude and latitude are negligible; only the elevation adjustment needs to be considered, and this adjustment can be controlled to within millimeter accuracy using the empirical formula provided below.

$$dh = -\cos^2 B da - \sin^2 B db \quad (1)$$

where B represents the latitude, da and db are the differences in the semi-major and semi-minor axes between the WGS84 ellipsoid and the TOPEX/Poseidon ellipsoid, respectively,

and dh is the elevation correction required when converting from the TOPEX/Poseidon reference frame to the WGS84 frame. After rigorous screening and coordinate system transformation, the laser altimetry points are primarily situated in regions with flat terrain and less vegetation coverage, and the elevation values within the footprint are considered to be consistent.

For large platform satellite images, such as the ZY-3 satellite images, in the absence of laser footprint imagery, the corresponding image points for the laser altimetry points proceed as follows: Initially, the three-dimensional coordinates of the central point within the laser footprint, where the laser elevation point is situated, are back-projected into the three-line array nadir-looking imagery, thereby determining the corresponding image point coordinates. Subsequently, these image point coordinates in the nadir-looking imagery are matched with the forward-looking and backward-looking images to ascertain the respective corresponding image points within those images. The primary reason lies in the fact that the planar nominal accuracy of the GLAS laser footprint approximates 10 m, whereas the ZY-3 satellite image boasts a planar positioning accuracy of roughly 15 m. Consequently, when laser altimetry data is directly back-projected into the nadir-looking imagery of the ZY-3 satellite, there are a few pixel deviations of approximately 3 to 4 pixels, representing a minor discrepancy in planar position that falls within the permissible accuracy threshold. However, when it comes to non-surveying and mapping applications of small-satellite imagery, owing to the satellite's small size and light weight, the accuracy of its initial position and attitude is relatively poor. There will be dozens or more pixel deviations for the image points obtained by back-projecting laser altimetry data directly into small-satellite imagery.

Therefore, as shown in Figure 1, we design a method for measuring laser altimetry point image coordinates tailored to multi-satellite, multi-sensor, and multi-resolution small-satellite imagery. As follows, (1) the small-satellite imagery are aligned by image matching and gross errors are eliminated to derive the image point coordinates; (2) for the obtained image point coordinates as shown by the purple cross markers in Figure 1, a meticulous manual visual inspection is conducted to fine-tune them, thereby achieving an image matching precision that extends to the sub-pixel domain, approximately 0.5 pixels; (3) based on the rational function model parameters, the satellite imagery is forward-intersected to obtain the calculated ground point coordinate, and the ground point coordinates (see the red dot marker in Figure 1) are back-projected to each small-satellite image to determine the calculated corresponding image point coordinates, as shown by the red cross markers in Figure 1; (4) utilizing these calculated image point coordinates along with those image points from step (2), we compute the residuals of the image point coordinates in row and column directions; (5) subsequently, taking each image as a basic statistical unit, we calculate the mean residuals of all image point coordinates in row and column directions; (6) the three-dimensional coordinates of the center of laser footprint (See the blue dot marker in Figure 1), which correspond to the screened laser altimetry points, are back-projected to each small-satellite image to derive the coordinates of the back-projection image points in row and column directions; (7) the mean residuals of the back projection in row and column directions, calculated in step (5) based on each image as a basic statistical unit, is used to compensate for the back projection image point coordinates of the laser altimetry data in row and column directions in step (6), so as to obtain the image point coordinates of the laser altimetry points in each image.

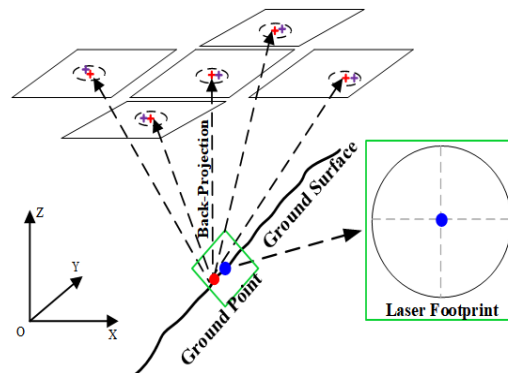


Figure 1. Image-point-measurement method of laser altimetry points.

3.2. Laser Altimetry Data Assisted Geometric Positioning Model Construction of Multi-Satellite Multi-Sensor Multi-Resolution Small Platform Satellite Images

In the laser altimetry data-assisted geometric positioning model, there are three types of points included: image tie points obtained through image matching, virtual control points (VCPs) and their corresponding image points, and laser altimetry points and their corresponding image points.

3.2.1. Error Equation Construction of Image Tie Points Obtained Through Image Matching

For the image tie points obtained through image matching, the parameters to be determined are the coefficients of the image distortion correction model and the three-dimensional coordinates of the ground points. The corresponding error equation can be formulated as follows:

$$\begin{bmatrix} v_{x_{ip}} \\ v_{y_{ip}} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{ip}}{\partial e_0} & \frac{\partial x_{ip}}{\partial e_1} & \frac{\partial x_{ip}}{\partial e_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial y_{ip}}{\partial f_0} & \frac{\partial y_{ip}}{\partial f_1} & \frac{\partial y_{ip}}{\partial f_2} \end{bmatrix} \begin{bmatrix} \Delta e_0 \\ \Delta e_1 \\ \Delta e_2 \\ \Delta f_0 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial x_{ip}}{\partial X} & \frac{\partial x_{ip}}{\partial Y} & \frac{\partial x_{ip}}{\partial Z} \\ \frac{\partial y_{ip}}{\partial X} & \frac{\partial y_{ip}}{\partial Y} & \frac{\partial y_{ip}}{\partial Z} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} - \begin{bmatrix} \Delta x_{ip} \\ \Delta y_{ip} \end{bmatrix}, \quad P_{ip} \quad (2)$$

where (x_{ip}, y_{ip}) represent the image coordinates, (X, Y, Z) are the corresponding coordinates in object space, $(\Delta x_{ip}, \Delta y_{ip})$ represent the constant vectors, $(v_{x_{ip}}, v_{y_{ip}})$ refer to the coordinate point residual vectors, $(e_0, e_1, e_2, f_0, f_1, f_2)$ are the affine transformation coefficients in image space, $(\Delta e_0, \Delta e_1, \Delta e_2, \Delta f_0, \Delta f_1, \Delta f_2)$ refer to the corrected vectors of the affine transformation coefficients in image space, and P_{ip} represents the weight matrix. In the absence of any a priori information, one can first use the original rational function model parameters to perform forward intersection for the tie points obtained from image matching, to obtain the three-dimensional coordinates of ground points. Then, project the obtained three-dimensional coordinates of the ground coordinates back to the image space, calculate the pixel residuals RUV of the tie point coordinates in the image space, and use $1/(RUV \times \text{image resolution})$ as the weight of this image tie point during the adjustment process.

3.2.2. Error Equation Construction of Virtual Control Points

In the complete absence of measured ground control points, the filtered laser altimetry points are directly introduced into the adjustment model as elevation constraints for bundle adjustment to solve the pending adjustment parameter, which will not be able to converge due to the lack of a planar constraint, resulting in the pathology of the normal equation. The traditional method introduces parameters to be adjusted into the bundle

adjustment model by processing them into weighted observations based on a priori information to correct the state of the adjustment model. However, due to the differences and correlations in the physical meaning of various parameters, this method has certain limitations for different image data sources [51]. This paper introduces virtual control points as weighted observations into the adjustment model. The steps for obtaining virtual control points from a single image are detailed in reference [52]. The parameters to be solved are the deviation correction model coefficients in image space, and the corresponding error equations can be constructed as follows:

$$\begin{bmatrix} v_{x_vcp} \\ v_{y_vcp} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{vcp}}{\partial e_0} & \frac{\partial x_{vcp}}{\partial e_1} & \frac{\partial x_{vcp}}{\partial e_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial y_{vcp}}{\partial f_0} & \frac{\partial y_{vcp}}{\partial f_1} & \frac{\partial y_{vcp}}{\partial f_2} \end{bmatrix} \begin{bmatrix} \Delta e_0 \\ \Delta e_1 \\ \Delta e_2 \\ \Delta f_0 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} - \begin{bmatrix} \Delta x_{vcp} \\ \Delta y_{vcp} \end{bmatrix} P_{vcp} \quad (3)$$

where (x_{vcp}, y_{vcp}) represent the corresponding image point coordinates of virtual control points, (X, Y, Z) are the corresponding coordinates in object space, $(\Delta x_{vcp}, \Delta y_{vcp})$ represent the constant vectors, (V_{x_vcp}, V_{y_vcp}) refer to the image point residual vectors, $(e_0, e_1, e_2, f_0, f_1, f_2)$ are the affine transformation coefficients in image space, $(\Delta e_0, \Delta e_1, \Delta e_2, \Delta f_0, \Delta f_1, \Delta f_2)$ refer to the corrected vectors of the affine transformation coefficients in image space, and P_{vcp} represent the weight matrix. The introduction of virtual control points is primarily intended to mitigate the ill-conditioning of the normal equations. Under the premise of unknown a priori information, the weights of virtual control points of different resolution images are not the same. Since the image points of virtual control points and tie points are mutually independent, the weighting of virtual control point observations need not account for any correlation between them. According to the literature [52,53], the weight assigned to a virtual control point can be determined based on prior information about the geometric positioning accuracy of the image to be adjusted without using ground controls. For instance, long-term in-orbit tests of the nadir-view images from the ZY-3 satellite indicate that the root mean square error (RMSE) of their geometric positioning without ground controls is approximately 15 m, which translates to about 7.5 pixels in image space. Accordingly, the weight of the virtual control-point observations can be set to $1/(7.5)^3$. Therefore, one can first use image matching to obtain image tie points for forward intersection of single satellite image, use check points to obtain the coordinate RMSEs of forward intersection in ground space, and convert these RMSEs to image space according to the image resolution, thereby setting the weight of virtual control points in these stereoscopic images to the reciprocal of the cube of the pixel residual value in image space. For example, if the coordinate RMSE obtained from the forward intersection of a single-satellite image is A meters, and the image resolution is B meters, then the weight of the image observation values of virtual control points is set to $1/(A/B)^3$.

3.2.3. Error Equation Construction of Laser Altimetry Points

For the laser altimetry points and their filtered image points, the corresponding error equations can be constructed as follows:

$$\begin{bmatrix} v_{x_lp} \\ v_{y_lp} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{lp}}{\partial e_0} & \frac{\partial x_{lp}}{\partial e_1} & \frac{\partial x_{lp}}{\partial e_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial y_{lp}}{\partial f_0} & \frac{\partial y_{lp}}{\partial f_1} & \frac{\partial y_{lp}}{\partial f_2} \end{bmatrix} \begin{bmatrix} \Delta e_0 \\ \Delta e_1 \\ \Delta e_2 \\ \Delta f_0 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial x_{lp}}{\partial X} & \frac{\partial x_{lp}}{\partial Y} \\ \frac{\partial y_{lp}}{\partial X} & \frac{\partial y_{lp}}{\partial Y} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} - \begin{bmatrix} \Delta x_{lp} \\ \Delta y_{lp} \end{bmatrix}; P_{lp} \quad (4)$$

where (x_{lp}, y_{lp}) represent the corresponding image point coordinates, (X, Y, Z) are the corresponding coordinates in object space, $(\Delta x_{lp}, \Delta y_{lp})$ represent the constant vectors, $(V_{x_{lp}}, V_{y_{lp}})$ refer to the image point residual vectors, $(e_0, e_1, e_2, f_0, f_1, f_2)$ are the affine transformation coefficients in image space, $(\Delta e_0, \Delta e_1, \Delta e_2, \Delta f_0, \Delta f_1, \Delta f_2)$ refer to the corrected vectors of the affine transformation coefficients in image space, and P_{lp} represents the weight matrix. Since the elevation accuracy of laser altimetry points is very high and only provides elevation constraints, the weight of the selected laser altimetry points will be relatively high. The nominal elevation accuracy of ICESAT-1/GLAS data is 0.15 m. Assuming a ground resolution of M meters for the small-satellite imagery, this corresponds to $0.15/M$ pixels in image space, so the weight of the selected laser altimetry points can be set as $1/(0.15/M)^2$.

By combining the error equations in (1)–(3) above, the matrix form of the geometric positioning model for multi-satellite, multi-sensor, and multi-resolution small-satellite imagery assisted by laser altimetry data without introducing a planar constraint is obtained as follows:

$$\begin{cases} V_p = A_p t + B_p X - L_p & P_p \\ V_{vp} = A_{vp} t - L_{vp} & P_{vp} \\ V_{lp} = A_{lp} t + B_{lp} X - L_{lp} & P_{lp} \end{cases} \quad (5)$$

Further, combine the error equation matrices to derive the form of the error matrix equation as follows:

$$V = At + BX - L \quad P \quad (6)$$

where t represents the bias correction vector, X refers to the correction vector of the ground three-dimensional coordinates, P represents the weight matrix, A and B are coefficient matrices, and L is the constant vector.

3.3. Calculation of the Geometric Positioning Model

In Equation (6), the coefficient matrix A contains the random errors of image point observations, and not all elements in A have random errors; some elements are constant elements. When predictors contain known/estimatable measurement noise in part of the coefficient matrix, the partial-EIV model offers many advantages over the conventional Gaussian–Markov model estimated by the Ordinary Least-Squares method (OLS) [54–58]. First, partial nature of the partial-EIV model distinguishes erroneous regressors from error-free controls, applying covariance corrections exclusively to contaminated variables; second, by explicitly modeling the stochastic error structure in coefficient matrix, the partial EIV model generates far more reliable parameter solution with confidence intervals compared to OLS, which fix regressors as deterministic; finally, in practical applications where measurement errors are unavoidable, and replicated data supply reliable error variance estimates, the partial-EIV model can recover the underlying structural linear relation in coefficient matrix. Therefore, this study introduces the total least squares of the partial-EIV model for model calculation, which can be described as follows:

$$V = (t^T \otimes I)(a + \kappa \Delta p_{img}) + BX - L \quad (7)$$

where Δp_{img} represents the correction vector of the image point coordinates in matrix A , I refers to the unit matrix, κ represents the construction matrix, a is the vector after flattening the initial values of matrix A . Equation (7) can be written in the following form:

$$V = (t^T \otimes I)a + (t^T \otimes I)\kappa \Delta p_{img} + BX - L = A_0 t + (t^T \otimes I)\kappa \Delta p_{img} + BX + l^0 \quad (8)$$

where $A_0 t = (t^T \otimes I)(a + \kappa \Delta p_{img})$.

The objective equation is constructed using the Lagrange multiplier method [54], as follows:

$$\min : \phi = V^T Q_V^{-1} V + \Delta p_{img}^T Q_{p_{img}}^{-1} \Delta p_{img} + 2\lambda^T (V - (A_0 t + (t^T \otimes I) \kappa \Delta p_{img} + BX + l^0)) \quad (9)$$

where λ is a Lagrange multiplier vector with $m \times 1$ order, Q_V and $Q_{p_{img}}$ are the co-factor matrix of the correction vector V and Δp_{img} , respectively. The corresponding stochastic model is as follows:

$$\begin{bmatrix} V \\ t^T \otimes I \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} Q_V & 0 \\ 0 & Q_{p_{img}} \end{bmatrix} \right) \quad (10)$$

Then the Lagrange extremum conditions for the objective equation are as follows:

$$\begin{cases} \frac{\partial \phi}{\partial V} = 2V^T Q_V^{-1} + 2\lambda^T = 0 \\ \frac{\partial \phi}{\partial \Delta p_{img}} = 2\Delta p_{img}^T Q_{p_{img}}^{-1} + 2\lambda^T (t^T \otimes I) \kappa = 0 \\ \frac{\partial \phi}{\partial \lambda} = V - (A_0 t + (t^T \otimes I) \kappa \Delta p_{img} + BX + l^0) = 0 \\ \frac{\partial \phi}{\partial t} = A^T \lambda = 0 \\ \frac{\partial \phi}{\partial X} = B^T \lambda = 0 \end{cases} \quad (11)$$

where $A = \begin{bmatrix} A_0 + \begin{bmatrix} \Delta p_{img}^T \kappa_1^T \\ \Delta p_{img}^T \kappa_2^T \\ \vdots \\ \Delta p_{img}^T \kappa_m^T \end{bmatrix} \end{bmatrix}$

According to the constructed Lagrange extremum conditions, it can be calculated that

$$\begin{cases} V = -Q_V \lambda \\ \Delta p_{img} = Q_{p_{img}} ((t^T \otimes I) \kappa)^T \lambda \end{cases} \quad (12)$$

Substituting Equation (12) into the third equation of the constructed Lagrange extremum conditions, we obtain the following:

$$\lambda = (Q_V + (t^T \otimes I) \kappa Q_{p_{img}} ((t^T \otimes I) \kappa)^T)^{-1} (-A_0 t - BX - l^0) \quad (13)$$

Assuming $N = [Q_V + (t^T \otimes I) \kappa Q_{p_{img}} ((t^T \otimes I) \kappa)^T]^{-1}$, substituting Equation (13) into the last two formulas of the constructed Lagrange extremum conditions, we obtain the following:

$$\begin{cases} A^T N^{-1} (-A_0 t - BX - l^0) = 0 \\ B^T N^{-1} (-A_0 t - BX - l^0) = 0 \end{cases} \quad (14)$$

Therefore, the iterative solution for the parameters to be determined is as follows:

$$\begin{bmatrix} A^T N^{-1} A_0 & A^T N^{-1} B \\ B^T N^{-1} A_0 & B^T N^{-1} B \end{bmatrix} \begin{bmatrix} t \\ X \end{bmatrix} = \begin{bmatrix} -A_0^T N^{-1} l^0 \\ -B^T N^{-1} l^0 \end{bmatrix} \quad (15)$$

The iterative solution uses the modified elimination method, the initial values of t and X are solved by least squares, and the initial values of the tie-point coordinates in ground space and the planar coordinates of laser altimetry points are obtained using direct forward intersection. First, eliminate the tie-point coordinate X , and solve the image correction parameter t . The solution is as follows:

$$t = [A^T N^{-1} A_0 - A^T N^{-1} B (B^T N^{-1} B)^{-1} B^T N^{-1} A_0]^{-1} [A^T N^{-1} B (B^T N^{-1} B)^{-1} B^T N^{-1} l^0 - A_0^T N^{-1} l^0] \quad (16)$$

The end-of-iteration condition is set as the difference between the results of two neighboring iterations is less than 0.01 pixels.

3.4. Accuracy Analysis

As a key performance indicator for geometric positioning precision, the determination of accuracy is achieved by leveraging external reference points or GCPs. This method compares positional data against established reference values to quantify the discrepancies identified through RMSEs. Within this investigation, the outcomes were evaluated based on 3D high-resolution points achieved with a precision exceeding 5 cm as measured by GPS. The RMSEs in planar and elevation directions are derived from Equation (17),

utilizing the actual GPS coordinates $(B_{TRUE}^i, L_{TRUE}^i, H_{TRUE}^i)$ ($i=1,2,\dots,n$) and corresponding ground point adjustment values (B^i, L^i, H^i) ($i=1,2,\dots,n$).

$$\left\{ \begin{array}{l} RMSE_{planar} = \sqrt{\frac{\sum_{i=1}^n (B^i - B_{TRUE}^i)^2 + \sum_{i=1}^n (L^i - L_{TRUE}^i)^2}{n}} \\ RMSE_{vertical} = \sqrt{\frac{\sum_{i=1}^n (H^i - H_{TRUE}^i)^2}{n}} \end{array} \right. \quad (17)$$

This paper focuses on the performance of laser altimetry data for elevation accuracy enhancement. Additionally, in order to verify the effectiveness of the algorithm through comprehensive evaluation, this study constructs image combination modalities from 4 aspects: single-satellite single-stereo imagery, single-satellite multiple-stereo imagery, dual-satellite single-stereo imagery, and multiple-satellite multiple-stereo imagery. There are several different image-combination scenarios in each combination mode, enabling a comprehensive geometric positioning analysis of multi-satellite, multi-sensor, and multi-resolution small-satellite imagery.

4. Experiments

4.1. Experiment Datasets

The experimental data used in this study are derived from the Chinese Jilin-1 series of satellites. As a significant source of global aerospace remote sensing information, the Jilin-1 constellation, constructed by Changguang Satellite Technology Co., Ltd. located in Changchun, China, is currently the largest sub-meter commercial remote sensing satellite constellation in the world, capable of providing customers with satellite remote sensing data featuring high temporal, spatial, and spectral resolutions, fast and wide-area coverage, as well as comprehensive application services of space information based on satellite remote sensing data. The datasets employed in this study are all panchromatic images, including the Jilin-1 wide-range 01A satellite imagery (tagged as KF01A satellite), the Jilin-1 high-resolution 02A satellite imagery (tagged as GF02A satellite), and the Jilin-1 high-resolution 02B satellite imagery (tagged as GF02B satellite). The Jilin-1 GF02A and GF02B satellites were launched on 13 November 2019 and 7 December 2019, respectively. Both offer a ground resolution of sub-satellite point of approximately 0.75 m for panchromatic images and 3 m for multispectral images, with a standard scene size of 21.5 km × 21.5 km, an orbital altitude of 535 km, and a planar nominal accuracy (CE90, Circular Error at 90%) of 20 m. The Jilin-1 KF01A satellite was launched on 15 January 2020, and it achieves a ground resolution of approximately 0.75 m for panchromatic images and 3 m for multispectral images. Its width is 136 km, with a single-image field of view of 16.1°, capable of acquiring approximately 1000 km² of sub-meter image data per second. The standard scene size measures 23 km × 23 km, the satellite operates at an orbital altitude of 481.56 km, and it can achieve the plane accuracy of CE90 20 m.

Figure 2 and Table 1 provide detailed locations and parameters of the experimental data. As illustrated in Figure 2, the imagery employed in the comparative experiments covers the Henan region of China and comprises eight scenes acquired at different epochs. In this figure, black borders indicate the KF01A satellite image, pink borders denote the GF02A satellite images, green borders represent the GF02B satellite images, and the selected laser altimetry points are marked as red dots. The overall coverage of the experimental data spans longitudes from 112.9°E to 113.36°E, and latitudes from 34.46°N to 34.83°N. Additionally, a total of 11 distributed ground points, measured by GPS, with an accuracy better than 5 cm, were used for the comparison experiments and accuracy validation, as shown by the red crosses in Figure 2.

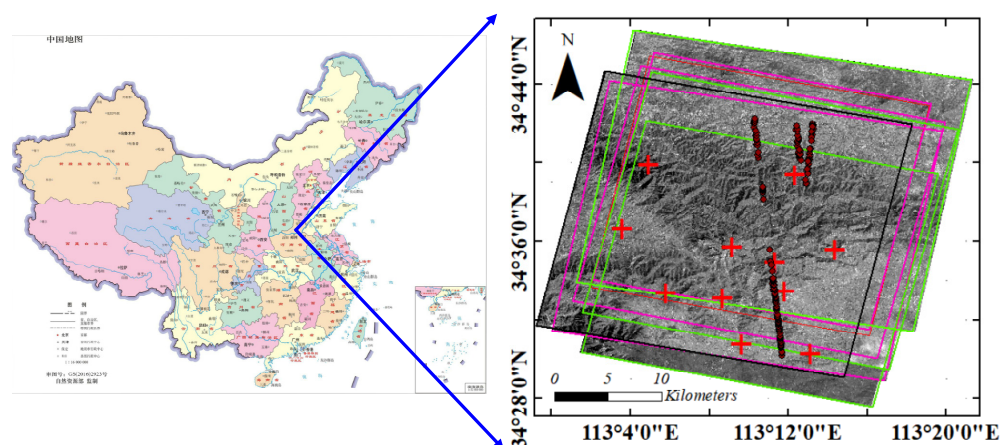


Figure 2. The experimental datasets, with checkpoints denoted by red crosses. (a) The KF01A satellite imagery as shown in the black boxes; (b) the GF02A satellite images as shown in the pink boxes; (c) the GF02B satellite images as shown in the green boxes.

Table 1. The detailed information on the experimental datasets.

Abbreviation	Satellite Platform	Satellite Altitude Angle/Degree	Image Size/Pixel	Resolu- tion/Meter
KF01A-1	KF01A	78.7153	30,666 × 31,140	0.75 × 0.76
GF02A-1	GF02A	70.2285	28,691 × 28,676	0.79 × 0.84
GF02A-2	GF02A	81.6273	28,692 × 28,676	0.76 × 0.77
GF02A-3	GF02A	65.7606	28,692 × 28,676	0.80 × 0.86
GF02A-4	GF02A	88.2947	28,692 × 28,676	0.75 × 0.75
GF02B-1	GF02B	70.467	28,629 × 28,676	0.79 × 0.84
GF02B-2	GF02B	64.0067	28,629 × 28,676	0.79 × 0.84
GF02B-3	GF02B	73.7262	28,630 × 28,676	0.78 × 0.81

4.2. Experimental Design

For small-satellite imagery, the geometric positioning accuracy of stereo image pairs varies considerably, primarily governed by the relative geometric relationship between the constituent images. Under favorable geometric conditions—which encompass more than just the intersection angle—the direct geopositioning accuracy tends to be relatively high, whereas under unfavorable conditions, it degrades markedly. Therefore, to rigorously evaluate the algorithm’s feasibility and performance, this study constructs multiple image combination modes from four distinct categories: single-satellite single-stereo imagery, single-satellite multiple-stereo imagery, dual-satellite single-stereo imagery, and multiple-satellite multiple-stereo imagery, which have different geometric relationships. Within each mode, a variety of distinct image-combination scenarios are implemented. For each scenario, a comparative analysis is carried out with respect to three positioning accuracy metrics regarding laser altimetry-assisted geometric positioning, direct geometric positioning, and positioning constrained by actual ground control points, enabling a comprehensive geometric positioning analysis of multi-satellite, multi-sensor, and multi-resolution small-satellite imagery.

For direct geometric positioning, spatial forward intersection is employed, which determines the three-dimensional coordinates of ground points using vendor-provided RPCs and image-matching tie points. In positioning constrained by actual ground control points, geometric errors are introduced into the RPCs during their generation from rigorous sensor models (RSMs) [59,60]. To compensate for these errors within the rational function model (RFM), three commonly used correction models—namely, the translation

model, the drift model, and the affine model—are employed to enhance positioning accuracy [61,62]. Specifically, the translation model is applied when a single ground control point is available; the drift model is adopted when two control points are used; and the affine model is utilized when three or more control points are employed.

4.2.1. Comparative Test of Single-Satellite Single-Stereo Imagery Combination Scenarios

For the single-satellite single-stereo combination mode, seven distinct image pair scenarios are designed: GF02B-1/GF02B-2, GF02B-2/GF02B-3, GF02A-1/GF02A-3, GF02A-1/GF02A-4, GF02A-2/GF02A-3, GF02A-2/GF02A-4, and GF02A-3/GF02A-4. Owing to the compact size and lightweight nature of small satellites, the initial positional and attitudinal accuracy is relatively limited. Consequently, the image points derived from direct back-projection of laser altimetry data onto small-satellite images are subject to deviations of tens of pixels or more, rendering conventional adjustment positioning infeasible. Furthermore, given that this study primarily investigates the capability of laser altimetry data to improve elevation accuracy, a comparative analysis is conducted across the seven scenarios, focusing on three accuracy metrics: laser-altimetry-assisted geometric positioning based on the proposed image-point-measurement method, direct geometric positioning, and positioning constrained by actual ground control points. We have measured a total of 11 distributed ground points by GPS, with an accuracy better than 5 cm. In this test, for both laser-altimetry-assisted geometric positioning and direct geometric positioning, no actual ground control points are used, and all available measured ground points within the image coverage are employed as checkpoints. Since the number of laser altimetry points serving as elevation control exceeds three, the affine model is adopted in image-space compensation. For positioning constrained by actual ground control points, only a single control point is used as a horizontal and vertical control point for all image-combination scenarios, while the remaining measured ground points within the image area serve as checkpoints. Given that only one control point is utilized for planimetric and height control, the translation model is applied in image-space adjustment.

4.2.2. Comparative Test of Single-Satellite Multiple-Stereo Imagery Combination Scenarios

For the vertical geometric positioning accuracy evaluation in the single-satellite multiple-stereo mode, we design five image-combination scenarios: GF02B-1/GF02B-2/GF02B-3, GF02A-1/GF02A-2/GF02A-3, GF02A-1/GF02A-3/GF02A-4, GF02A-2/GF02A-3/GF02A-4, and GF02A-1/GF02A-2/GF02A-3/GF02A-4. In this experiment, neither laser-altimetry-assisted nor direct geometric positioning uses any ground control points, and all available surveyed points are reserved as checkpoints. Because more than three laser altimetry points are employed for elevation control, we apply the affine model in image space. In contrast, for positioning constrained by actual ground control points, we adopt only a single control point (providing both horizontal and vertical control) for each scenario, using the remaining points as checkpoints. Consequently, the translation model is used in image-space compensation.

4.2.3. Comparative Test of Dual-Satellite Single-Stereo Imagery Combination Scenarios

In the dual-satellite single-stereo experiment, we provide fourteen satellite image-combination scenarios for the vertical geometric positioning accuracy comparison, including KF01A-1/GF02B-1, KF01A-1/GF02B-2, KF01A-1/GF02B-3, KF01A-1/GF02A-1, KF01A-1/GF02A-3, KF01A-1/GF02A-4, GF02B-1/GF02A-2, GF02B-1/GF02A-3, GF02B-1/GF02A-4, GF02B-2/GF02A-1, GF02B-2/GF02A-4, GF02B-3/GF02A-1, GF02B-3/GF02A-3, and GF02B-3/GF02A-4. In all image-combination scenarios, no actual ground control points are used as constraints in either laser altimetry-assisted geometric positioning or direct forward

intersection; consequently, all available ground control points within each scenario are reserved as checkpoints for accuracy validation. For laser altimetry-assisted positioning, the affine model is adopted in image space to correct residual biases. In addition, for positioning constrained by actual ground control points, one control point is selected as a horizontal-and-vertical constraint for bundle adjustment, with the translation model applied for image-space correction, while the remaining surveyed ground points within the image area are used as checkpoints for accuracy assessment.

4.2.4. Comparative Test of Multiple-Satellite Multiple-Stereo Imagery Combination Scenarios

In this test, in order to further comprehensively analyze the performance of the image-point-measurement method of laser altimetry points for improving laser-altimetry-data-assisted positioning accuracy of small-satellite images, we construct multiple-satellite multiple-stereo image combination mode, with thirteen distinct small-satellite image-combination scenarios, including KF01A-1/GF02B-1/GF02A-1, KF01A-1/GF02B-1/GF02A-2, KF01A-1/GF02B-1/GF02A-3, KF01A-1/GF02B-1/GF02A-4, KF01A-1/GF02B-2/GF02A-1, KF01A-1/GF02B-2/GF02A-2, KF01A-1/GF02B-2/GF02A-3, KF01A-1/GF02B-2/GF02A-4, KF01A-1/GF02B-3/GF02A-1, KF01A-1/GF02B-3/GF02A-2, KF01A-1/GF02B-3/GF02A-3, KF01A-1/GF02B-3/GF02A-4, and all images. Consistent with the above, for laser altimetry-assisted geometric positioning, the affine model is applied for image-space bias correction, and all surveyed ground points within the image area are employed as checkpoints for accuracy evaluation. The same checkpoint strategy applies to direct geometric positioning. For control-point-based geometric positioning, one ground control point is selected as a horizontal-and-vertical constraint, with bias correction performed using the translation model in image space, while the remaining surveyed ground points are reserved for accuracy assessment.

4.3. Experimental Results

4.3.1. Experimental Results of Single-Satellite Single-Stereo Imagery Combination Scenarios

The detailed results of the direct positioning accuracy, the laser altimetry data-assisted geometric positioning accuracy based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning accuracy constrained by actual ground points, using the vendor-provided RPCs, are summarized in Table 2, and the geometric positioning accuracy for each imagery combination scenario falls into two directions: planar and vertical. To more objectively highlight the performance of the weighted image-point-measurement method of laser altimetry points in improving the laser-altimetry-data-assisted positioning accuracy of small-satellite images in vertical direction, Figure 3 compared the vertical positioning accuracy of the seven imagery combination scenarios, and Figure 4 comparatively analyzes the elevation accuracy improvement rates of the laser-altimetry-data-assisted geometric positioning based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning constrained by actual ground points.

Table 2. The detailed results of the direct positioning, the laser altimetry data-assisted geometric positioning based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning constrained by actual ground points.

Satellite Image-combination scenarios	Direct Positioning Accuracy			Laser Altimetry Data Assisted Positioning Accuracy				Positioning Accuracy Constrained by Actual Ground Points			
	Planar /m	Height/m		Planar /m	Height/m		Elevation Accuracy Improvement Rate/%	Planar /m	Height/m		Elevation Accuracy Improvement Rate/%
		RMSE	ME		RMSE	ME			RMSE	ME	
GF02B-1 + GF02B-2	27.199	8.782	7.994	27.167	3.843	1.363	56.24%	7.127	3.822	0.565	56.48%
GF02B-2 + GF02B-3	16.205	18.977	18.455	17.059	4.502	3.779	76.28%	8.461	7.411	5.300	60.95%
GF02A-1 + GF02A-3	19.949	46.714	46.683	20.475	2.005	1.120	95.71%	6.575	1.809	0.447	96.13%
GF02A-1 + GF02A-4	20.959	45.144	44.916	13.679	8.022	6.552	82.23%	6.237	6.378	3.803	85.87%
GF02A-2 + GF02A-3	26.461	34.808	34.656	29.260	5.573	4.553	83.99%	6.099	3.880	1.561	88.85%
GF02A-2 + GF02A-4	20.840	10.918	7.561	20.398	7.887	0.394	27.76%	6.492	8.816	0.104	19.25%
GF02A-3 + GF02A-4	21.590	49.273	49.210	29.506	8.705	8.310	82.33%	5.958	5.580	4.403	88.68%

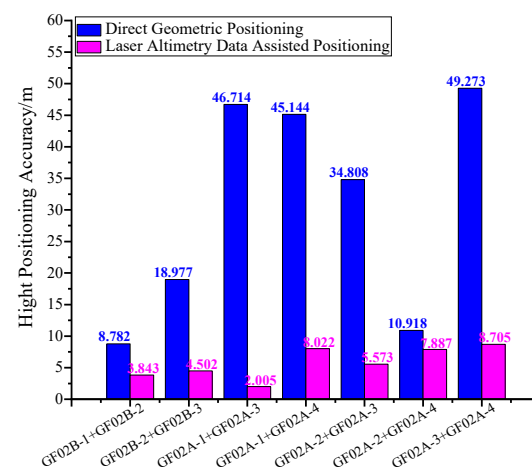


Figure 3. The compared results of the vertical positioning accuracy between the direct geometric positioning and the laser altimetry data-assisted geometric positioning using the proposed method of measuring the corresponding image points of the laser altimetry points.

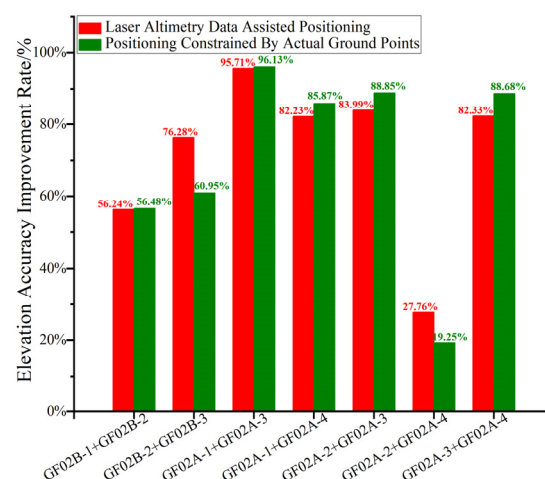


Figure 4. The compared results of the elevation accuracy improvement rates of the laser altimetry data-assisted geometric positioning based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning constrained by actual ground points.

4.3.2. Experimental Results of Single-Satellite Multiple-Stereo Imagery Combination Scenarios

For five satellite image-combination scenarios, Figure 5 gives the geometric positioning error comparative results of the five imagery combination scenarios in the vertical direction between the direct geometric positioning and the laser altimetry data-assisted geometric positioning, and the detailed results are illustrated in Table 3.

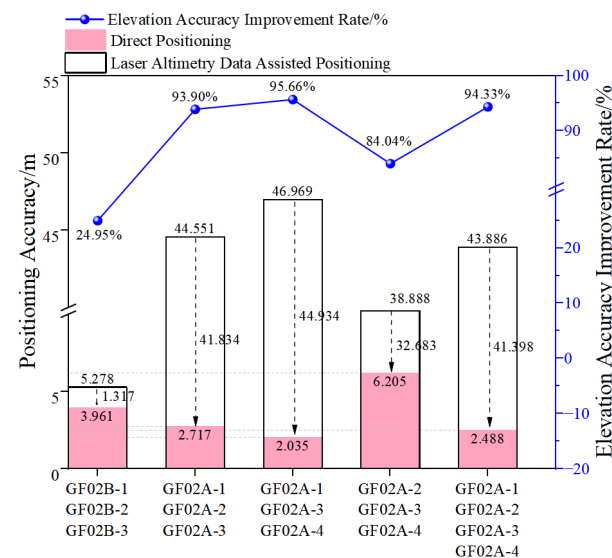


Figure 5. The comparison of the results of the vertical positioning accuracy.

Table 3. The detailed results of the direct positioning, the laser altimetry data-assisted geometric positioning based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning constrained by actual ground points.

Satellite Image-Combination Scenarios	Direct Positioning Accuracy			Laser Altimetry Data Assisted Positioning Accuracy				Positioning Accuracy Constrained by Actual Ground Points			
	Height/m		Planar /m	Height/m		Elevation Accuracy Improvement Rate/%		Height/m		Elevation Accuracy Improvement Rate/%	
	Planar /m	RMSE	ME	Planar /m	RMSE	ME		Planar /m	RMSE	ME	
GF02B-1 GF02B-2 GF02B-3	20.614	5.278	3.661	20.296	3.961	1.115	24.95%	7.736	4.921	2.646	6.76%
GF02A-1 GF02A-2 GF02A-3	23.934	44.551	44.489	22.226	2.717	1.930	93.90%	6.387	2.178	0.704	95.11%
GF02A-1 GF02A-3 GF02A-4	21.479	46.969	46.938	22.253	2.035	1.146	95.66%	6.279	1.818	0.416	96.13%
GF02A-2 GF02A-3 GF02A-4	24.041	38.888	38.523	26.610	6.205	5.505	84.04%	5.979	3.463	1.927	91.09%
GF02A-1 GF02A-2 GF02A-3 GF02A-4	21.339	43.886	43.805	20.396	2.488	1.271	94.33%	6.212	2.321	0.432	94.71%

4.3.3. Experimental Results of Dual-Satellite Single-Stereo Imagery Combination Scenarios

In the dual-satellite single-stereo experiment, fourteen satellite image-combination scenarios are provided for the vertical geometric positioning accuracy comparison. Figure 6 presents the comparative results of geometric positioning errors in the vertical direction for the fourteen imagery combination scenarios, with detailed results depicted in Table 4.

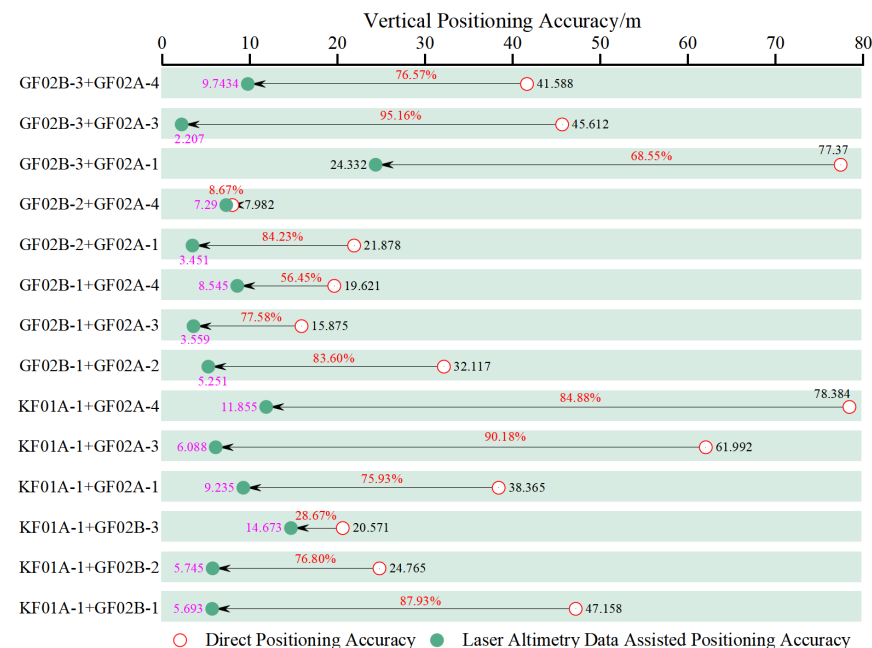


Figure 6. The compared results of the vertical positioning accuracy between the direct geometric positioning and the laser altimetry data-assisted geometric positioning using the proposed method of measuring the corresponding image points of the laser altimetry points.

Table 4. The detailed results of the direct positioning, the laser altimetry data-assisted geometric positioning based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning constrained by actual ground points.

Satellite Image-Combination Scenarios	Direct Positioning Accuracy			Laser Altimetry Data Assisted Positioning Accuracy				Positioning Accuracy Constrained by Actual Ground Points			
	Accuracy			Accuracy				Accuracy			
	Planar /m	Height/m RMSE	ME	Planar /m	Height/m RMSE	ME	Elevation Accuracy Improvement Rate/%	Planar /m	Height/m RMSE	ME	Elevation Accuracy Improvement Rate/%
KF01A-1 + GF02B-1	16.468	47.158	46.816	7.282	5.693	1.640	87.93%	7.575	11.053	8.749	76.56%
KF01A-1 + GF02B-2	11.974	24.765	24.471	11.178	5.745	4.299	76.80%	6.815	4.319	1.141	82.56%
KF01A-1 + GF02B-3	13.503	20.571	14.596	13.631	14.673	2.099	28.67%	8.139	15.760	1.666	23.39%
KF01A-1 + GF02A-1	14.062	38.365	38.155	7.771	9.235	8.372	75.93%	6.589	4.278	0.451	88.85%
KF01A-1 + GF02A-3	16.534	61.992	61.815	22.177	6.088	3.809	90.18%	5.997	6.163	3.285	90.06%
KF01A-1 + GF02A-4	18.757	78.384	77.887	15.715	11.855	7.944	84.88%	6.176	9.992	3.185	87.25%
GF02B-1 + GF02A-2	12.188	32.117	31.778	13.507	5.251	2.292	83.60%	7.010	5.735	2.247	82.14%
GF02B-1 + GF02A-3	16.362	15.875	15.488	16.608	3.559	0.229	77.58%	6.418	5.041	3.115	68.25%
GF02B-1 + GF02A-4	21.231	19.621	19.147	22.869	8.545	7.443	56.45%	6.586	4.471	0.689	77.21%
GF02B-2 + GF02A-1	13.239	21.878	21.624	13.725	3.451	0.983	84.23%	6.933	4.111	1.961	81.21%
GF02B-2 + GF02A-4	8.154	7.982	4.7321	8.071	7.294	3.459	8.67%	6.490	6.948	0.280	12.95%
GF02B-3 + GF02A-1	16.437	77.370	73.930	14.738	24.332	8.513	68.55%	10.432	23.091	7.699	70.16%
GF02B-3 + GF02A-3	22.264	45.612	45.575	23.619	2.207	1.221	95.16%	8.133	3.627	2.841	92.05%
GF02B-3 + GF02A-4	22.689	41.588	41.143	17.177	9.743	7.565	76.57%	6.429	10.231	6.810	75.40%

4.3.4. Experimental Results of Multiple-Satellite Multiple-Stereo Imagery Combination Scenarios

Table 5 shows the results for the stereoscopic positioning accuracy of the thirteen imagery combination scenarios using the direct positioning, the laser altimetry data-assisted positioning, and the positioning constrained by actual ground points, respectively.

Table 5. The detailed results of the direct positioning, the laser altimetry data-assisted geometric positioning based on the proposed method of measuring the corresponding image points of the laser altimetry points, and the positioning constrained by actual ground points.

Satellite Image-Combination Scenarios	Direct Positioning Accuracy			Laser Altimetry Data Assisted Positioning Accuracy				Positioning Accuracy Constrained by Actual Ground Points			
	Planar /m	Height/m		Planar /m	Height/m		Elevation Accuracy Improvement Rate/%	Planar /m	Height/m		Elevation Accuracy Improvement Rate/%
		RMSE	ME		RMSE	ME			RMSE	ME	
KF01A-1 GF02B-1 GF02A-1	11.338	18.489	10.759	10.790	9.184	8.1636	50.323%	6.842	6.803	4.648	63.20%
KF01A-1 GF02B-1 GF02A-2	14.639	27.573	20.759	15.739	12.414	11.054	54.98%	6.691	8.207	5.745	70.23%
KF01A-1 GF02B-1 GF02A-3	24.518	36.558	35.031	23.824	4.505	2.4149	87.68%	6.183	5.469	3.499	85.04%
KF01A-1 GF02B-1 GF02A-4	17.733	17.735	7.6356	17.323	8.514	7.6268	51.99%	6.232	4.125	0.715	76.74%
KF01A-1 GF02B-2 GF02A-1	12.469	25.417	25.203	11.531	3.411	0.7564	86.58%	6.771	3.852	1.544	84.84%
KF01A-1 GF02B-2 GF02A-2	15.762	21.846	20.971	16.417	3.997	0.6210	81.70%	6.525	4.361	1.083	80.04%
KF01A-1 GF02B-2 GF02A-3	8.630	45.689	44.332	8.076	5.376	1.262	88.23%	6.078	5.528	1.870	87.90%
KF01A-1 GF02B-2 GF02A-4	15.914	26.237	25.914	17.941	4.092	0.542	84.40%	6.406	4.414	1.063	83.18%
KF01A-1 GF02B-3 GF02A-1	14.112	22.588	21.669	10.200	8.274	5.315	63.37%	7.830	7.558	2.915	66.54%
KF01A-1 GF02B-3 GF02A-2	12.509	76.049	69.654	9.964	16.057	3.393	78.89%	6.635	16.618	5.603	78.15%
KF01A-1 GF02B-3 GF02A-3	19.559	54.638	54.044	18.912	5.840	3.568	89.31%	6.906	4.965	0.916	90.91%
KF01A-1 GF02B-3 GF02A-4	20.534	45.626	45.250	15.651	9.000	6.733	80.28%	6.416	10.543	7.796	76.89%
All Images	20.527	33.583	31.260	18.956	4.530	1.675	86.51%	6.262	4.130	0.477	87.70%

5. Discussion

5.1. Experimental Analysis of Single-Satellite Single-Stereo Imagery Combination Scenarios

As indicated in Table 2, for the single-satellite single-stereo mode of small-satellite images, there are significant differences in the direct geometric positioning accuracy of the stereoscopic images, which mainly depend on the relative geometric relationship between the images that make up the stereoscopic images. Under strong geometric conditions (geometric conditions refer not only to intersection angles), the direct geometric positioning accuracy of stereoscopic images is also relatively high. Conversely, under weak geometric conditions, the direct geometric positioning accuracy of stereoscopic images is relatively low. For instance, the direct geometric positioning accuracy of the image-combination scenario composed of GF02B-1 and GF02B-2 small-satellite images is approximately 27.199 m in planar direction and about 8.782 m in elevation; For the image-combination scenario of GF02A-2 and GF02A-4 small-satellite images, the direct geometric positioning accuracy is 20.840 m in planimetry and about 10.918 m in height direction. As for the combined scenes of GF02A-1 and GF02A-3 small-satellite images, GF02A-1 and GF02A-4 small-satellite images, and GF02A-3 and GF02A-4 small-satellite images, their direct geometric positioning accuracy in elevation reaches 46.714 m, 45.144 m, and 49.273 m, respectively. It has significant differences from the images captured by three-line or dual-line array mapping satellites. As can be seen from Figure 3, among the seven single-satellite single-stereo image-combination scenarios, compared to the direct geometric positioning accuracy, the laser altimetry-assisted geometric positioning based on the proposed image-point-measurement method achieved significant improvement in elevation accuracy. For instance, in the image-combination scenario of GF02B-2 and GF02B-3, the elevation accuracy was improved from 18.977 m of the direct geometric positioning to 4.502 m, with an improvement rate of 76.28%. In the image-combination scenario of GF02A-2 and GF02A-4, the elevation accuracy was improved from 10.918 m to 7.887 m, with an improvement rate of 27.76%. As is observable from Figure 4, within the seven image-combination scenarios, the laser altimetry-assisted geometric positioning based on the proposed image-point-measurement method achieves an improvement rate of vertical positioning accuracy comparable to or even slightly better than the positioning constrained by actual ground points. For example, in the image-combination scenarios of GF02A-1 and GF02A-3, GF02A-1 and GF02A-4, GF02A-2 and GF02A-3, and GF02A-3 and GF02A-4, both methods achieved an elevation accuracy improvement rate exceeding 80%, with the highest improvement rates reaching 95.71% and 96.13%, respectively. In the image-combination scenarios of GF02B-2 and GF02B-3, and GF02A-2 and GF02A-4, the laser altimetry-assisted geometric positioning based on the proposed image-point-measurement method achieved a higher elevation positioning accuracy improvement rate compared to the positioning constrained by actual ground points. For example, in the image-combination scenario of GF02B-2 and GF02B-3, the improvement rates of the two methods are 76.28% and 60.95%, respectively, with the former demonstrating a significant advantage.

Therefore, for the single-satellite single-stereo mode of small-satellite imagery, the proposed laser altimetry point-measurement method, adapted to small-satellite imagery and the combined geometric positioning model, including virtual control points, laser altimetry points, and image-matching tie points, demonstrates a strong capability for enhancing elevation accuracy.

5.2. Experimental Analysis of Single-Satellite Multiple-Stereo Imagery Combination Scenarios

As seen from Figure 5, in the experimental region, the laser altimetry-assisted geometric positioning based on the proposed image-point-measurement method achieved

significant elevation accuracy enhancement performance for all five single-satellite multiple-stereo imagery combination scenarios. For example, the satellite combination scenario of GF02A-1, GF02A-3, and GF02A-4 achieved the best elevation accuracy improvement rate, approximately 95.66%, from the elevation accuracy of 46.969 m of direct geometric positioning to 2.035 m of the laser altimetry data-assisted geometric positioning. The satellite combination scenario of GF02A-1, GF02A-2, and GF02A-3, and the satellite combination scenario of GF02A-1, GF02A-2, GF02A-3, and GF02A-4 achieve the elevation accuracy improvement rate closer to the satellite combination scenario of GF02A-1, GF02A-3, and GF02A-4, about 93.90% and 94.33%, respectively. The elevation accuracy of the direct geometric positioning in the satellite combination scenario of GF02A-1, GF02A-2, and GF02A-3 is 44.551 m, and the elevation accuracy of the laser altimetry data-assisted geometric positioning is 2.717 m, an improvement of 41.834 m. Similarly, the satellite combination scenario of GF02A-1, GF02A-2, GF02A-3, and GF02A-4 achieved an elevation accuracy improvement from 43.886 m for the direct geometric positioning to 2.488 m for laser altimetry data-assisted positioning, an increase of approximately 41.398 m. As shown in Figure 5 and Table 3, the satellite combination scenario of GF02A-2, GF02A-3, and GF02A-4 achieved a fourth-place ranking in elevation accuracy improvement rate in the five satellite image-combination scenarios, about 84.04%, from the elevation accuracy of 38.888 m of the direct geometric positioning to 6.205 m of the laser altimetry data-assisted geometric positioning. The satellite combination scenario of GF02A-1, GF02A-2, and GF02A-3 achieved the worst improvement rate in elevation accuracy, but the 24.95% improvement was still retained. Table 3 also shows that in the five image-combination scenarios, whether it is direct geometric positioning or laser altimetry data assisted geometric positioning, the positioning accuracy in the planar direction is approximately at the level of 20 m (e.g., 20.614 m in the planar direction of the direct positioning accuracy and 20.296 m in the planar direction of the laser-altimetry-data-assisted positioning in the image-combination scenario of GF02A-1, GF02A-2 and GF02A-3), which is consistent with the nominal planar accuracy of the Jilin-1 images used in this experimental study. Table 3 also indicates that, in terms of elevation accuracy improvement rate, the geometric positioning assisted by laser altimetry data, which is based on the proposed image-point-measurement method, attains an improvement rate of vertical positioning accuracy that is comparable to, or even superior to, the positioning constrained by actual ground points. Among these five image-combination scenarios, all other image-combination scenarios except the image combination scenarios of GF02B-1, GF02B-2, and GF02B-3 improve elevation accuracy exceeding 80% while maintaining comparable performance levels. In the image-combination scenario of GF02B-1, GF02B-2, and GF02B-3, laser altimetry data-assisted positioning also achieved a higher elevation accuracy improvement rate compared to the positioning constrained by actual ground points, reaching 24.95%.

Consequently, in the context of the single-satellite multiple-stereo mode of small-satellite imagery, the proposed laser altimetry-based point-measurement method tailored for small-satellite imagery and the integrated geometric positioning model, which incorporates virtual control points, laser altimetry points, and image-matching tie points, proved effective and exhibits robust capabilities in improving elevation accuracy.

5.3. Experimental Analysis of Dual-Satellite Single-Stereo Imagery Combination Scenarios

As seen from Figure 6 and Table 4, the satellite combination scenario of GF02B-3 and GF02A-3 achieved the best positioning accuracy in the elevation direction after being assisted by laser altimetry data for geometric positioning, approximately 2.207 m. Compared to the elevation accuracy of 45.612 m achieved through direct geometric positioning, the accuracy has been improved by approximately 43 m, with an improvement rate of 95.16%. The satellite combination scenarios of GF02B-2 and GF02A-1, GF02B-1 and

GF02A-3 achieved a height accuracy of 3–4 m of laser altimetry data-assisted positioning, improving from 21.878 m to 3.451 m and from 15.875 m to 3.559 m, respectively. The satellite combination scenario of GF02B-3 and GF02A-1 achieved the worst vertical accuracy of laser altimetry data-assisted geometric positioning, approximately 24.332 m. However, a significant improvement has still been achieved compared to the elevation accuracy of direct geometric positioning of about 77 m, with an improvement rate of approximately 68.55%. Furthermore, among the fourteen satellite image-combination scenarios, the one with the smallest improvement rate is the image-combination scenario of GF02B-2 and GF02A-4, which is approximately 8.67%, improving from 7.982 m to 7.294 m; yet, there is also a significant improvement. When comparing the laser altimetry data-assisted positioning and the positioning constrained by actual ground points, it is evident that both methods can effectively improve elevation accuracy, with the former performing comparably to or even better than the latter in terms of elevation accuracy enhancement. Under the image-combination scenarios of KF01A-1 and GF02B-1, KF01A-1 and GF02B-3, KF01A-1 and GF02A-3, GF02B-1 and GF02A-3, GF02B-2 and GF02A-1, GF02B-3 and GF02A-3, and GF02B-3 and GF02A-4, laser-altimetry-data-assisted positioning demonstrates superior performance compared to the positioning constrained by actual ground points in terms of elevation accuracy improvement rate, achieving 87.93%, 28.67%, 90.18%, 77.58%, 84.23%, 95.16% and 76.57%, respectively. In contrast, the elevation accuracy improvement rates of the positioning constrained by actual ground points in these scenarios are 76.56%, 23.39%, 90.06%, 68.25%, 81.21%, 92.05% and 75.40%, respectively.

In this experimental study, by using the fourteen satellite image-combination scenarios, significant improvement of the vertical geometric positioning accuracy was achieved for laser altimetry-assisted positioning using the proposed image-point-measurement method to obtain the corresponding image points of the laser altimetry points. Consequently, for the dual-satellite single-stereo mode of small-satellite imagery, the proposed laser altimetry point-measurement method adapted for small-satellite imagery and the integrated geometric positioning model, which incorporates virtual control points, laser altimetry points, and image-matching tie points, have demonstrated effectiveness in improving elevation accuracy. These approaches exhibit strong potential for enhancing the accuracy of height measurements in the specific image-combination scenarios.

5.4. Experimental Analysis of Multiple-Satellite Multiple-Stereo Imagery Combination Scenarios

From Table 5, it can be seen that for all thirteen imagery combination scenarios, there is a great improvement in the vertical stereoscopic positioning accuracy between the laser altimetry data-assisted positioning and the direct positioning, which also inversely proves the correctness of the corresponding image points of laser altimetry points, thereby verifying the performance of the proposed method. For example, the elevation accuracy of the satellite combination scenario of KF01A-1, GF02B-2, and GF02A-1 has been improved to 3.411 m, with an improvement rate of 86.58%; the positioning accuracy in height direction in the satellite combination scenario of KF01A-1, GF02B-2, and GF02A-4 was improved from 26.237 m to 4.092 m, with an improvement rate of 84.40%. When performing the direct geometric positioning and the laser altimetry-assisted geometric positioning on eight images with different imaging geometries and phases from three satellite platforms, the elevation accuracy was 33.583 m and 4.530 m, respectively. For all thirteen imagery combination scenarios, both the laser altimetry data-assisted positioning and the positioning constrained by actual ground points demonstrated excellent performance in improving elevation accuracy, with the laser altimetry data-assisted positioning performing better in various image-combination scenarios, which also verifies the effectiveness of the algorithm proposed in this paper. For instance, in the image-combination scenarios of KF01A-1, GF02B-1 and GF02A-3, KF01A-1, GF02B-2 and GF02A-1, KF01A-1, GF02B-2 and

GF02A-2, KF01A-1, GF02B-2 and GF02A-3, KF01A-1, GF02B-2 and GF02A-4, in comparison to the positioning constrained by actual ground points, laser altimetry data-assisted positioning exhibits superior performance in terms of the improvement rate of elevation accuracy, attaining 87.68%, 86.58%, 81.70%, 88.23%, 84.40%, 78.89% and 80.28%, respectively. Conversely, the elevation accuracy improvement rates of the positioning constrained by actual ground points in these scenarios are 85.04%, 84.84%, 80.04%, 87.90%, 83.18%, 78.15% and 76.89%, respectively.

In total, the experimental results indicate that the combined adjustment of small-satellite images with different imaging geometries and laser altimetry control points can significantly improve the elevation positioning accuracy. This verifies the effectiveness of the proposed corresponding image-point-measurement method of laser altimetry points designed for small-satellite images, and the constructed combined geometric positioning models that include virtual control points, laser altimetry points, and image-matching tie points.

6. Conclusions

For small-satellite imagery, the image points obtained by directly back-projecting laser altimetry measurements often exhibit discrepancies of tens of pixels or more, primarily due to the relatively limited attitude and orbit determination accuracy of small-satellite platforms. Moreover, the diversity of imaging sensors, variations in image resolution, and inherently weak image geometric configurations further complicate the accurate acquisition of image-space coordinates for laser altimetry points. To enable the effective use of laser altimetry data for geometric positioning in multi-satellite, multi-sensor, and multi-resolution small-satellite scenarios, this study introduces a measurement method for laser altimetry points and further constructs a combined geometric positioning model that fuses virtual control points, laser altimetry points, and image-matching tie points. The framework consists of four procedural components: (1) an image-point-measurement strategy for laser altimetry points; (2) the construction of a laser altimetry data-assisted geometric positioning model for small-satellite imagery; (3) the solution of the geometric positioning model using a total least squares approach based on the partial-EIV models; and (4) a comprehensive accuracy assessment conducted under multiple image-combination scenarios, including single-satellite single-stereo, single-satellite multi-stereo, dual-satellite single-stereo, and multi-satellite multi-stereo imagery configurations. Experimental validation is conducted using panchromatic images acquired by the Jilin-1 small-satellite constellation (KF01A, GF02A, and GF02B) over the Henan region of China. The results demonstrate that the laser altimetry point-measurement method, together with the combined geometric positioning model, yields substantial improvements in vertical positioning accuracy across all tested scenarios. These collectively attest to the efficacy of the proposed approach in enhancing the elevation performance of stereoscopic small-satellite imagery with heterogeneous platforms, diverse sensors, and variable resolutions over terrain conditions similar to those tested.

Although the current method demonstrates excellent performance on small-satellite images over terrain conditions similar to those tested, the relative geometric relationship between the images that make up the stereoscopic images has a notable impact on the positioning accuracy of small-satellite imagery. In future work, additional validation over mountainous, forested, and urban areas will be essential to assess the method's robustness. Simultaneously, further investigations, such as the influences of this geometric relationship on the performance of the laser-altimetry-based geometric positioning model, and computational schemes based on Helmert variance-component estimation, are warranted.

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