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Article

The Impact of Systemic Risk on the Diversification Benefits of a Risk Portfolio

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Abstract: Risk diversification is the basis of insurance and investment. It is thus crucial to study the effects that could limit it. One of them is the existence of systemic risk that affects all of the policies at the same time. We introduce here a probabilistic approach to examine the consequences of its presence on the risk loading of the premium of a portfolio of insurance policies. This approach could be easily generalized for investment risk. We see that, even with a small probability of occurrence, systemic risk can reduce dramatically the diversification benefits. It is clearly revealed via a non-diversifiable term that appears in the analytical expression of the variance of our models. We propose two ways of introducing it and discuss their advantages and limitations. By using both VaR and TVaR to compute the loading, we see that only the latter captures the full effect of systemic risk when its probability to occur is low.

Keywords: diversification; expected shortfall; investment risk; insurance premium; risk loading; risk measure; risk management; risk portfolio; stochastic model; systemic risk; value-at-risk

MSC classifications: 91B30; 91B70; 62P05

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1. Introduction

Every financial crisis reveals the importance of systemic risk and, as a consequence, the lack of diversification. Diversification is a way to reduce the risk by detaining many different risks, with various probabilities of occurrence and a low probability of happening simultaneously. Unfortunately, in times of crisis, most of the financial assets move together and become very correlated. The 2008/2009 crisis is not an exception. It has highlighted the interconnectedness of financial markets when they all came to a stand still for more than a month waiting for the authorities to restore confidence in the system (see, e.g., the *Systemic Risk Survey of the bank of England*, available on-line). For any financial institution, it is important to be aware of the limits to diversification, while, for researchers in this field, studying the mechanisms that hamper diversification is crucial for the understanding of the dynamics of the system (see, e.g., [1,2] and the references therein). Both risk management and research on risk would enhance our capacity to survive the inevitable failures of diversification. A small fact, like turmoils in the U.S. sub-prime real-estate market, a relatively small market compared to the whole U.S. real-estate market, can trigger a major financial crisis that extends to all markets and all regions in the globe. Systemic risk manifests itself by a breakdown of the diversification benefits and the appearance of dependence structures that were not deemed important during normal times.

Since the crisis, the literature on this subject has been abundant. It mainly centers on two approaches. On one hand, there exist attempts to explain the appearance of systemic risk through structural macro-economic or financial models that include feedback loops and non-linear effects [2–7]. On the other hand, empirical indicators have been proposed to measure the danger of systemic risk in order to provide early-warning systems [8–12]. While these studies are essential to progress in our understanding of the economic factors leading to the emergence of economic crisis due to systemic risk, our approach is to concentrate on the more fundamental underlying mathematical mechanisms that break diversification. In the economic literature, systemic risk is identified with contagion effects and the literature concentrates on finding ways of modeling or measuring it. One consequence of this contagion is the breaking of the law of large numbers and the sudden disappearance of the stabilizing effect of diversification, joining here one of the main questions asked in quantitative risk management (see e.g. [13]). Another important subject of debate in the risk management literature is the controversy about risk measures (see e.g. [14,15] and references therein). The aim of our study is precisely to look at these aspects of the problem through a simple and didactic model. The question we want to answer is: are there simple mathematical mechanisms that break the law of large numbers and destroy the benefits of diversification?

In this study, we introduce a simple stochastic modeling to understand and point out the limitations to diversification and the mechanism leading to the occurrence of systemic risk. The idea is to combine two generating stochastic processes that, through their mixture, produces in the resulting process a non-diversifiable component, which we identify to be a systemic risk. Depending on the way of mixing these processes, the diversification benefit appears with various strengths, due to the emergence of the systemic component. The use of such a model, which is completely specified, allows us to obtain analytical expressions for the variance and then to identify the non-diversifiable term. With the help of

Monte Carlo simulations, we explore the various components of the model and check that we reproduce the analytical results.

The paper is organized as follows: in the first section, we introduce, to measure the effects on diversification, the standard insurance framework for pricing risk and define the risk measures that are used in this study. The second section is dedicated to the mathematical presentation of the model and its various approaches to systemic risk, as well as numerical applications. The obtained results are compared numerically and analytically in the third section, where we also discuss the influence of the choice of the risk measure on the diversification benefits. We conclude the study and suggest new perspectives to extend it.

2. Insurance Framework

Before moving to stochastic modeling, let us introduce the insurance framework in which we are going to compute the risk diversification. It is an example of application, but this study on systemic risk could easily be generalized to any financial institution.

2.1. The Technical Risk Premium

In insurance, risk is priced based on the knowledge of the loss probability distribution. Let L be the random variable (rv) representing a loss defined on a probability space $(\Omega,\ \mathcal{A},\ I\!\!P)$.

2.1.1. One policy case

For any policy incurring a loss $L^{(1)}$, we can define, as in [16], the technical premium, P, that needs to be paid, as:

$$P = I\!\!E[L^{(1)}] + \eta K + e \tag{1}$$

with η: the return expected by shareholders before tax, K: the risk capital assigned to this policy, e: the expenses incurred by the insurer to handle this case.

An insurance company is a company in which we can invest. Therefore, the shareholders that have invested a certain amount of capital in the company expect a return on investment. Therefore, the insurance firm has to make sure that the investors receive their dividends, which corresponds to the cost of capital the insurance company must charge on its premium. This is what we have called η.

We will assume that the expenses are a small portion of the expected loss:

$$e = aI\!\!E[L^{(1)}] \quad \text{with } 0 < a \ll 1$$

which transforms the premium as:

$$P = (1 + a)I\!\!E[L^{(1)}] + \eta K. \tag{2}$$

We can now generalize this premium principle Equation (2) for N similar policies (or contracts).

2.1.2. The case of a portfolio of N policies

The premium for one policy in the portfolio, incurring now a total loss $L^{(N)} = \sum_{i=1}^{N} L_i^{(1)}$ (where the $L_i^{(1)}$'s are N independent copies of $L^{(1)}$), can then be written as

$$P = \frac{(1+a)\mathbb{E}[L^{(N)}] + \eta K_N}{N} = (1+a)\ \mathbb{E}[L^{(1)}] + \eta\ \frac{K_N}{N}$$

where K_N is the capital assigned to the entire portfolio.

2.2. Cost of Capital and Risk Loading

First, we have to point out that the role of capital for an insurance company is to ensure that the company can pay its liability, even in the worst cases, up to some threshold. For this, we need to define the capital that we have to put behind the risk. We are going to use a risk measure, say ρ, defined on the loss distribution. This allows us to estimate the capital needed to ensure the payment of the claim up to a certain confidence level. We then define the risk-adjusted-capital K as a function of the risk measure ρ associated with the risk[1] L as:

$$K = \rho(L) - \mathbb{E}[L] \tag{3}$$

since the risk is defined as the deviation from the expectation.

Note that we could have also defined K as $K = \rho(L) - \mathbb{E}[L] - P$, since the premiums can serve to pay the losses. This would change the premium P defined in Equation (1) into $\hat{P}$ defined by:

$$\hat{P} = \frac{1+a-\eta}{1+\eta}\ \mathbb{E}[L] + \frac{\eta}{1+\eta}\rho(L).$$

Such an alternative definition would reduce the capital, but does not change fundamentally the results of the study.

Consider a portfolio of N similar policies, using the notation for the loss as in Section 2.1. Let R denote the risk loading per policy, defined as the cost of the risk-adjusted capital per policy. Using Equation (3), R can be expressed as a function of the risk measure ρ, namely:

$$R = \eta\ \frac{K_N}{N} = \eta \left(\frac{\rho(L^{(N)})}{N} - \mathbb{E}[L^{(1)}] \right). \tag{4}$$

2.3. Risk Measures

We will consider for ρ two standard risk measures, the value-at-risk (VaR) and the tail value-at-risk (TVaR). Let us recall the definitions of these quantities (see, e.g., [17]).

The value-at-risk with a confidence level α is defined for a risk L by:

$$\mathrm{VaR}_\alpha(L) = inf\{q \in \mathbb{R} : \mathbb{P}(L > q) \le 1 - \alpha\} = inf\{q \in \mathbb{R} : F_L(q) \ge \alpha\} \tag{5}$$

where q is the level of loss that corresponds to a VaR_α (simply, the quantile of L of order α) and F_L the cdf of L.

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1 We use here the word "risk" instead of "loss". In fact, these two words are used for one another in an insurance context.

The tail value-at-risk at a confidence level α of L satisfies:

$$\mathrm{TVaR}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\mathrm{d}u \underset{F_L \text{ contin.}}{=} I\!\!E[L \mid L > VaR_\alpha(L)].$$

When considering a discrete rv L, it can be approximated by a sum, which may be seen as the average over all losses larger than VaR_α:

$$\mathrm{TVaR}_\alpha(L) = \frac{1}{1-\alpha}\sum_{u_i \geq \alpha}^{1} q_{u_i}(L)\Delta u_i \tag{6}$$

where $q_{u_i}(L) = \mathrm{VaR}_{u_i}(L)$ and $\Delta u_i \equiv u_i - u_{i-1}$ corresponds to the probability mass of the particular quantile q_{u_i}. This measure is the only coherent risk measure independently of the underlying distribution (see Artzner *et al.* 1999 [18]) and would always give rise to a diversification benefit, when it exists, as a function of N .

3. Stochastic Modeling

Suppose an insurance company has underwritten N policies of a given risk. To price these policies, the company must know the underlying probability distribution of this risk, as seen in the previous section. In this study, we assume that each policy is exposed n times to this risk.

In a portfolio of N policies, the risk may occur $n \times N$ times. Therefore, we introduce a sequence $(X_i,\ i = 1,\ \ldots, Nn)$ of rvs X_i defined on $(\Omega,\ \mathcal{A},\ I\!\!P)$ to model the occurrence of the risk, with a given severity l. Note that we choose a deterministic severity, but it could be extended to a random one, with a specific distribution.

Hence, the total loss amount, denoted by L, associated with this portfolio is given by:

$$L = l\, S_{Nn} \quad \text{with} \quad S_{Nn} := \sum_{i=1}^{Nn} X_i.$$

3.1. The First Model, under the iid Assumption

We start with a simple model, considering a sequence of independent and identically distributed (iid) rvs X_i's. Let X denote the parent rv, and assume that it is simply Bernoulli distributed, *i.e.*, the loss lX occurs with some probability p:

$$X = \begin{cases} 1 & \text{with probability} \quad p \\ 0 & \text{with probability} \quad 1-p \end{cases}$$

Recall that $I\!\!E[X] = p$ and $var(X) = p(1-p)$.

Hence, the total loss amount $L = l\, S_{Nn}$ of the portfolio is modeled by a binomial distribution $\mathcal{B}(Nn, p)$:

$$I\!\!P[L = k] = l\binom{Nn}{k} p^k (1-p)^{Nn-k}, \quad \text{for } k = 0, \cdots, Nn \tag{7}$$

with $I\!\!E[L] = \sum_{i=1}^{Nn} I\!\!E[X_i] = lNnp$, and by independence, $var(L) = lNn\; var(X) = lnp(1-p)$.

We are interested in knowing the risk premium that the insurance company will ask of a customer if he buys this insurance policy. Therefore, we compute the cost of capital given in Equation (4) for an increasing number N of policies in the portfolio, which becomes for this model:

$$R = \eta \left(\frac{\rho(L)}{N} - lnp \right) \tag{8}$$

since the notation $L^{(N)}$ in Equation (4) has been simplified to L in this section.

Note that the relative risk per policy defined by the ratio $R/\mathbb{E}[L^{(1)}]$ is given by:

$$\frac{R}{\mathbb{E}[L^{(1)}]} = \eta \left(\frac{\rho(L)}{lnp} - 1 \right). \tag{9}$$

3.1.1. Numerical Application

We compute the quantities of interest by fixing the various parameters. We choose, for instance, the number of times one policy is exposed to the risk to be $n = 6$. Then, we take as the cost of capital $\eta = 15\%$, which is a reasonable value, given the fact that the shareholders will obtain a return on investment after taxes of approximately 10%, when considering a standard tax rate of 30%. For a discussion on the choice of the value of η, we refer to [16]. The unit loss l will be fixed to $l = 10$. We choose in the rest of the study $\alpha = 99\%$ for the threshold of the risk measure ρ.

We present in Table 1 an example of the distribution of the loss $L = lS_{1n}$ for one policy ($N = 1$) when taking, e.g., $p = 1/n = 1/6$ (the same probability for each of the n exposures). This would be the typical distribution for the outcome of a particular value when throwing a die (see [19]).

The expected total loss amount is given by $\mathbb{E}[L] = l\mathbb{E}[S_{1n}] = 10$. We see that there is a 26.32% probability (corresponding to $\mathbb{P}[S_{1n} > 10] = 1 - \mathbb{P}[S_{1n} \leq 10]$) that the company will turn out to be paying more than the expectation. Thus, we cannot simply ask the expected loss as the premium. This justifies the premium principle adopted in Section 2.1.

Table 1. The loss distribution for one policy with $n = 6$ and $p = 1/6$.

Number of Losses k	Policy Loss $l\,X(\omega)$	Probability Mass $\mathbb{P}[S_{1n} = k]$	cdf $\mathbb{P}[S_{1n} \leq k]$
0	0	33.490%	33.490%
1	10	40.188%	73.678%
2	20	20.094%	93.771%
3	30	5.358%	99.130%
4	40	0.804%	99.934%
5	50	0.064%	99.998%
6	60	0.002%	100.000%

Now, we compute the cost of capital per policy given in Equation (4) as a function of the number N of policies in the portfolio. The results are displayed in Table 2 for both risk measures VaR and TVaR and when taking $p = 1/6,\ 1/4$ and $1/2$, respectively. The expected total loss amount $\mathbb{E}[L] = l\mathbb{E}[S_{1n}] = nlp$ will change accordingly.

Table 2. The risk loading per policy as a function of the number N of policies in the portfolio (with $n = 6$).

Risk Measure ρ	Number N of Policies	Risk Loading R per Policy with Probability		
		$p = 1/6$	$p = 1/4$	$p = 1/2$
VaR				
	1	3.000	3.750	4.500
	5	1.500	1.650	1.800
	10	1.050	1.200	1.350
	50	0.450	0.540	0.600
	100	0.330	0.375	0.420
	1,000	0.102	0.117	0.135
	10,000	0.032	0.037	0.043
TVaR				
	1	3.226	3.945	4.500
	5	1.644	1.817	1.963
	10	1.164	1.330	1.482
	50	0.510	0.707	0.675
	100	0.372	0.425	0.476
	1,000	0.116	0.134	0.154
	10,000	0.037	0.042	0.049
$\mathbb{E}[L]/N$		*10.00*	*15.00*	*30.00*

Note that, when considering a large number N of policies, the binomial distribution of L could be replaced by the normal distribution $\mathcal{N}(Nnp, Nnp(1-p))$ (for $Nn \geq 30$ and p not close to zero, nor one; e.g., $np > 5$ and $n(1-p) > 5$) using the central limit theorem (CLT). The VaR of order α of L could then be deduced from the αth-quantile, q_α, of the standard normal distribution $\mathcal{N}(0,1)$, as:

$$VaR_\alpha(L) = \sqrt{Nnp(1-p)}\, q_\alpha + Nnp. \tag{10}$$

Thus, the risk loading R would become, in the case of ρ being VaR:

$$R = \eta \times \sqrt{\frac{nlp(1-p)}{N}}\, q_\alpha$$

ever smaller as a function of N.

We can see in Table 2 that the risk loading drops practically by a factor of 100 for a portfolio of 10,000 policies, compared with the one computed for one policy ($N = 1$) that represents 30% of the loss expectation ($\mathbb{E}[L] = 10$ in this case). We notice also numerically that, if R increases with p, the relative risk per policy $R/\mathbb{E}[L^{(1)}]$ decreases when p increases. When considering the Gaussian approximation and the explicit VaR given in Equation (10), the relative risk per policy when choosing $\rho =$ VaR, is, as a function of p, of the order $1/\sqrt{p}$, giving back the numerical result. Finally, it is worth noting that the risk loading with TVaR is always slightly higher than with VaR for the same threshold, as TVaR goes beyond VaR in the tail of the distribution.

In this setting, a fair game is defined by having an equal probability of losing at each exposure: $p = 1/n$. The biased game will be when the probability differs from $1/n$, generally bigger. We can thus define two states, one with a "normal" or equilibrium state ($p = 1/n$) and a "crisis" state with a probability $q >> p$. In the next section, we will introduce this distinction.

3.2. Introducing a Structure of Dependence to Reveal a Systemic Risk

We propose two examples of models introducing a structure of dependence between the risks, in order to explore the occurrence of a systemic risk and, as a consequence, the limits to diversification. We still consider the sequence $(X_i, i = 1, \ldots, Nn)$ to model the occurrence of the risk, with a given severity l, for N policies, but do not assume anymore that the X_is are iid

3.2.1. A Dependent Model, but Conditionally Independent

We assume that the occurrence of the risks X_is depends on another phenomenon, represented by an rv, say U. Depending on the intensity of the phenomenon, *i.e.*, the values taken by U, a risk X_i has more or less chances to occur. Suppose that the dependence between the risks is totally captured by U. Consider, w.l.o.g., that U can take two possible values denoted by one and zero; U can then be modeled by a Bernoulli $\mathcal{B}(\tilde{p})$, $0 < \tilde{p} << 1$. The rv U will be identified with the occurrence of a state of systemic risk. Therefore, $\tilde{p}$ could mathematically take any value between zero and one, but we choose it here to be very small, since we want to explore rare events. We still model the occurrence of the risks with a Bernoulli, but with a parameter depending on U. Since U takes two possible values, the same holds for the parameters of the Bernoulli distribution of the conditionally independent rvs $X_i \mid U$, namely:

$$(X_i \mid U = 1) \sim \mathcal{B}(q) \quad \text{and} \quad (X_i \mid U = 0) \sim \mathcal{B}(p).$$

We choose $q >> p$, so that whenever U occurs (*i.e.*, $U = 1$), it has a big impact in the sense that there is a higher chance of loss. We include this effect in order to have a systemic risk (non-diversifiable) in our portfolio.

Looking at the total amount of losses S_{Nn}, its distribution can then be written, for $k \in \mathbb{N}$, as:

$$\begin{aligned} \mathbb{P}(S_{Nn} = k) &= \mathbb{P}\left[S_{Nn} = k \mid U = 1\right] \mathbb{P}(U = 1) + \mathbb{P}\left[S_{Nn} = k \mid U = 0\right] \mathbb{P}(U = 0) \\ &= \tilde{p}\, \mathbb{P}\left[S_{Nn} = k \mid U = 1\right] \,+\, (1 - \tilde{p})\, \mathbb{P}\left[S_{Nn} = k \mid U = 0\right]. \end{aligned}$$

The conditional and independent variables, $\tilde{S}_q := S_{Nn} | \,(U = 1)$ and $\tilde{S}_p := S_{Nn} | \,(U = 0)$, are distributed as Binomials $\mathcal{B}(Nn, q)$ and $\mathcal{B}(Nn, p)$, with mass probability distributions denoted by $f_{\tilde{S}_q}$ and $f_{\tilde{S}_p}$, respectively. The mass probability distribution f_S of S_{Nn} appears as a mixture of $f_{\tilde{S}_q}$ and $f_{\tilde{S}_p}$ (see, e.g., [20]):

$$f_S = \tilde{p}\, f_{\tilde{S}_q} \,+\, (1 - \tilde{p})\, f_{\tilde{S}_p} \quad \text{with } \tilde{S}_q \sim \mathcal{B}(Nn, q) \text{ and } \tilde{S}_p \sim \mathcal{B}(Nn, p). \tag{11}$$

Note that $\tilde{p} = 0$ gives back the normal state, developed in Section 3.1.

The expected loss amount for the portfolio, denoting $L = L^{(N)}$, is given by:

$$\mathbb{E}[L] = l \times \mathbb{E}[S_{Nn}] = l \times \left(\tilde{p}\, \mathbb{E}[\tilde{S}_q] \,+\, (1 - \tilde{p})\, \mathbb{E}[\tilde{S}_p]\right) = Nnl\, \left(\tilde{p}\, q \,+\, (1 - \tilde{p})\, p\right)$$

whereas for each policy, it is:

$$\frac{l}{N} \mathbb{E}[L] \;=\; l\, n\, \left(\tilde{p}\, q \,+\, (1 - \tilde{p})\, p\right) \tag{12}$$

from which we deduce the risk loading defined in Equations (3) and (4).

Let us evaluate the variance $var(S_{Nn})$ of S_{Nn}. Straightforward computations (see [19]) give:

$$\mathbb{E}[S_{Nn}^2] = Nn\left[\tilde{p}\, q\big(1-q+Nnq\big) \;+\; (1-\tilde{p})\, p\big(1-p+Nnp\big)\right]$$

which, combined with Equation (12), provides:

$$var(S_{Nn}) = Nn\left[q(1-q)\tilde{p} + p(1-p)(1-\tilde{p}) + Nn(q-p)^2\tilde{p}(1-\tilde{p})\right]$$

from which we deduce the variance for the loss of one contract as $\frac{1}{N^2}\, var(L) = \frac{l^2}{N^2}\, var(S_{Nn})$, *i.e.*,

$$\frac{1}{N^2}\, var(L) = \frac{l^2 n}{N}\Big(q(1-q)\tilde{p} + p(1-p)(1-\tilde{p})\Big) \;+\; l^2 n^2 (q-p)^2 \tilde{p}(1-\tilde{p}). \tag{13}$$

Notice that in the variance for one contract, the first term will decrease as the number of contracts increases, but not the second one. It does not depend on N and, thus, represents the non-diversifiable part of the risk.

3.2.2. Numerical Application

For this application, we keep the same parameters $n = 6$ and $p = 1/n$ as in Section 3.1, and we choose the loss probability during the crisis to be $q = 1/2$. We explore different probabilities $\tilde{p}$ of the occurrence of a crisis. The calculation consists in mixing the two Binomial distributions, according to Equation (11), for an increasing number of policies N. Results for the two choices of risk measures are shown in Table 3.

Table 3. For Model Equation (11), the risk loading per policy as a function of the probability of occurrence of a systemic risk in the portfolio using VaR and TVaR measures with $\alpha = 99\%$. The probability of giving a loss in a state of systemic risk is chosen to be $q = 50\%$.

Risk Measure ρ	**Number N of Policies**	**In a Normal State $\tilde{p} = 0$**	**Risk Loading R with Occurrence of a Crisis State** $\tilde{p} = 0.1\%$	$\tilde{p} = 1.0\%$	$\tilde{p} = 5.0\%$	$\tilde{p} = 10.0\%$
VaR						
	1	3.000	2.997	4.469	4.346	5.693
	5	1.500	1.497	2.070	3.450	3.900
	10	1.050	1.047	1.770	3.300	3.450
	50	0.450	0.477	1.410	3.060	3.030
	100	0.330	0.327	1.605	3.000	2.940
	1,000	0.102	0.101	2.549	2.900	2.775
	10,000	0.032	0.029	2.837	2.866	2.724
TVaR						
	1	3.226	3.232	4.711	4.755	5.899
	5	1.644	1.707	2.956	3.823	4.146
	10	1.164	1.266	2.973	3.578	3.665
	50	0.510	0.760	2.970	3.196	3.141
	100	0.372	0.596	2.970	3.098	3.020
	1,000	0.116	0.396	2.970	2.931	2.802
	10,000	0.037	0.323	2.970	2.876	2.732
$\mathbb{E}[L]/N$		*10.00*	*10.02*	*10.20*	*11.00*	*12.00*

In Table 3, we see well the effect of the non-diversifiable risk. As expected, when the probability of occurrence of a crisis is high, the diversification does not play a significant role anymore, already with 100 contracts in the portfolio. The interesting point is that for $\tilde{p} \geq 1\%$, the risk loading barely changes when there is a large number of policies (starting at $N = 1{,}000$) in the portfolio. This is true for both VaR and TVaR. The non-diversifiable term dominates the risk. When looking at a lower probability $\tilde{p}$ of the occurrence of a crisis, we notice that the choice of the risk measure matters. For instance, when choosing $\tilde{p} = 0.1\%$, the risk loading, compared to the normal state, is multiplied by 10 in the case of TVaR, for $N = 10{,}000$ policies, and hardly moves in the case of VaR! This effect remains, but to a lower extent, when diminishing the number of policies. It is clear that the VaR measure does not capture the crisis state well, while TVaR is sensitive to the change of state, even with such a small probability and a high number of policies.

Another interesting effect worth noticing is the fact that the risk loading for any given N should increase with increasing $\tilde{p}$ since the influence of the biased die should increase (as it appears clearly in (12) and (13), the expectation and variance being increasing functions of $\tilde{p}$ (with $\tilde{p} < 1/2$)). Yet, we see that, with VaR and TVaR, it is not the case starting with $N = 50$, although the effect is much smaller for TVaR. It seems to decrease when $\tilde{p}$ increases from 5 to 10%. Pushing $\tilde{p}$ to 15%, we see that the effect of the bias levels off, which could explain the fluctuations around this level and thus this numerical instability.

To explore the increase of the risk loading of the VaR for $\tilde{p} = 1\%$, we redid 1,000 times the 10^6 simulations. It turns out that the obtained values are very unstable, for instance for $N = 10,000$, it jumps from 0.2 to 2.9, without taking any value in between. For the TVaR, it is more stable and we do not see such a behavior; in this case, the variation is less than 1%. When considering values for $\tilde{p}$ close to 1% (0.9 or 1.1%), the variation for the VaR becomes again more or less stable, we do not observe any more such jumps.

3.2.3. A More Realistic Setting to Introduce a Systemic Risk

We adapt further the previous setting to a more realistic description of a crisis. At each of the n exposures to the risk, in a state of systemic risk, the entire portfolio will be touched by the same increased probability of loss, whereas, in a normal state, the entire portfolio will be subject to the same equilibrium probability of loss.

For this modeling, it is more convenient to rewrite the sequence $(X_i,\ i = 1, \ldots, Nn)$ with a vectorial notation, namely $(\mathbf{X}_j,\ j = 1, \ldots, n)$, where the vector $\mathbf{X}_j$ is defined by $\mathbf{X}_j = (X_{1j}, \ldots, X_{Nj})^T$. Hence, the total loss amount S_{Nn} can be rewritten as:

$$S_{Nn} = \sum_{j=1}^{n} \tilde{S}^{(j)} \quad \text{where} \quad \tilde{S}^{(j)} \text{ is the sum of the components of } \mathbf{X}_j : \ \tilde{S}^{(j)} = \sum_{i=1}^{N} X_{ij}.$$

We keep the same notation for the Bernoulli rv U determining the state and for its parameter $\tilde{p}$. However, now, instead of defining a normal ($U = 0$) or a crisis ($U = 1$) state on each element of $(X_i,\ i = 1, \ldots, Nn)$, we do it on each vector $\mathbf{X}_j$, $1 \leq j \leq n$. It comes back to define a sequence of iid rvs $(U_j,\, j=1,\ldots,n)$ with parent rv U. Hence, we deduce that $\tilde S^{(j)}$ follows a Binomial distribution whose probability depends on U_j:

$$\tilde S^{(j)} \mid (U_j=1) \sim \mathcal{B}(N,q) \quad \text{and} \quad \tilde S^{(j)} \mid (U_j=0) \sim \mathcal{B}(N,p).$$

Note that these conditional rvs are independent.

Let us introduce the event A_l defined, for $l=0,\ldots,n$, as:

$$A_l := \{l \text{ vectors } \mathbf{X}_j \text{ are exposed to a crisis state and } n-l \text{ to a normal state}\} = \Big(\sum_{j=1}^n U_j = l\Big)$$

whose probability is given by $\mathbb{P}(A_l) = \mathbb{P}\Big(\sum_{j=1}^n U_j = l\Big) = \binom{n}{l}\, \tilde p^l\, (1-\tilde p)^{n-l}$.

We can then write that:

$$\mathbb{P}(S_{Nn}=k) = \sum_{l=0}^n \mathbb{P}(S_{Nn}=k \mid A_l)\mathbb{P}(A_l) = \sum_{l=0}^n \binom{n}{l}\tilde p^l\,(1-\tilde p)^{n-l}\, \mathbb{P}\big[\tilde S_q^{(l)} + \tilde S_p^{(n-l)} = k\big] \tag{14}$$

with, by conditional independence,

$$\tilde S_q^{(l)} = \sum_{j=1}^l \Big(\tilde S^{(j)} \mid U_j=1\Big) \sim \mathcal{B}(Nl,q) \ \text{ and } \ \tilde S_p^{(n-l)} = \sum_{j=1}^{n-l}\Big(\tilde S^{(j)} \mid U_j=0\Big) \sim \mathcal{B}(N(n-l),p). \tag{15}$$

Expectation and variance are obtained by straightforward computations (see [19]). We have:

$$\mathbb{E}[S_{Nn}] = Nn\,\big((q-p)\tilde p + p\big) \tag{16}$$

and, for one contract,

$$\frac{1}{N}\mathbb{E}[L] = \frac{l}{N}\mathbb{E}[S_{Nn}] = nl\,(\tilde p\, q\, + (1-\tilde p)\, p)$$

which is equal to the expectation Equation (12) obtained with the previous method. The variance can be deduced from Equation (16) and:

$$\mathbb{E}[S_{Nn}^2] = Nn\Big(q(1-q)\tilde p + p(1-p)(1-\tilde p)\Big) + \ N^2n^2\Big(p(1-\tilde p)+q\tilde p\Big)^2 + N^2 n(q-p)^2\tilde p(1-\tilde p)$$

hence:

$$var(S_{Nn}) = Nn\left[q(1-q)\tilde p + p(1-p)(1-\tilde p) + N(q-p)^2\tilde p(1-\tilde p)\right]$$

which is different from the variance $var(S_{Nn})$ obtained with the previous model in Section 3.2.1.

Now, for one contract, we obtain:

$$\frac{1}{N^2}var(L) = \frac{l^2}{N^2}var(S_{Nn}) = \frac{l^2 n}{N}\Big(q(1-q)\tilde p + p(1-p)(1-\tilde p)\Big) \ + \ l^2 n\,(q-p)^2\tilde p(1-\tilde p). \tag{17}$$

Notice that the last term appearing in Equation (17) is only multiplied by n and not n^2, as in Equation (13), and not diversifiable by the number N of policies. It looks like the one of Equation (13); however, its effect is smaller than in the previous model. With this method, we have also achieved producing a process with a non-diversifiable risk.

3.2.4. Numerical Application

Let us revisit our numerical example. In this case, we cannot, contrary to the previous cases, directly use an explicit expression for the distributions. We have to go through Monte Carlo simulations. At each of the n exposures to the risk, we first have to choose between a normal or a crisis state. Since we take here $n = 6$, the chances of choosing a crisis state when $\tilde{p} = 0.1\%$ is very small. To get enough of the crisis states, we need to do enough simulations and then average over all of the simulations. The results shown in Table 4 are obtained with 10 million simulations. We ran it also with one and 20 million simulations to check the convergence. It converges well, as can be seen in Table 5.

Table 4. For Model Equation (14), the risk loading per policy as a function of the probability of occurrence of a systemic risk in the portfolio using VaR and TVaR measures with $\alpha = 99\%$. The probability of giving a loss in a state of systemic risk is chosen to be $q = 50\%$.

Risk Measure ρ	Number N of Policies	In a Normal State $\tilde{p} = 0$	Risk Loading R with Occurrence of a Crisis State $\tilde{p} = 0.1\%$	$\tilde{p} = 1.0\%$	$\tilde{p} = 5.0\%$	$\tilde{p} = 10.0\%$
VaR						
	1	3.000	2.997	2.969	4.350	4.200
	5	1.500	1.497	1.470	1.650	1.800
	10	1.050	1.047	1.170	1.350	1.500
	50	0.450	0.477	0.690	0.990	1.200
	100	0.330	0.357	0.615	0.945	1.170
	1,000	0.102	0.112	0.517	0.882	1.186
	10,000	0.032	0.033	0.485	0.860	1.196
	100,000	0.010	0.008	0.475	0.853	1.199
TVaR						
	1	3.226	3.232	4.485	4.515	4.448
	5	1.644	1.792	1.870	2.056	2.226
	10	1.164	1.252	1.342	1.604	1.804
	50	0.510	0.588	0.824	1.183	1.408
	100	0.375	0.473	0.740	1.118	1.358
	1,000	0.116	0.348	0.605	1.013	1.295
	10,000	0.037	0.313	0.563	0.981	1.276
	100,000	0.012	0.301	0.550	0.970	1.269
$\mathbb{E}[L]/N$		*10.00*	*10.02*	*10.20*	*11.00*	*12.00*

The results shown in Table 4 follow the behavior we expect. The diversification due to the total number of policies is more effective for this model than for the previous one, but we still experience a part which is not diversifiable. We have also computed the case with 100,000 policies, since we used Monte Carlo simulations. It is interesting to note that, as expected, the risk loading in the normal state continues to decrease. In this state, it decreases by $\sqrt{10}$. However, except for $\tilde{p} = 0.1\%$ in the VaR case, the decrease becomes very slow when we allow for a crisis state to occur. The behavior of this model is more complex than the previous one, but more realistic, and we reach also the non-diversifiable part of the risk. For a high probability of occurrence of a crisis (one every 10 years), the limit with VaR is reached already at 100 policies, while, with TVaR, it continues to slowly decrease. The instability with increasing $\tilde{p}$ that we noticed in Table 3 has disappeared here for both VaR and TVaR. The cost of capital always increases with increasing $\tilde{p}$, except for $N = 1$. This could be due to the fact that we consider only one discrete rv and/or to the numerical stability of the results.

Concerning the choice of risk measure, we see a similar behavior as in Table 3 for the case $N =$ 10,000 and $\tilde{p} = 0.1\%$: VaR is unable to catch the possible occurrence of a crisis state, which shows its limitation as a risk measure. Although we know that there is a part of the risk that is non-diversifiable, VaR does not catch it really when $N =$ 10,000 or 100,000, while TVaR does not decrease significantly between 10,000 and 100,000 reflecting the fact that the risk cannot be completely diversified away.

Finally, to explore the convergence of the simulations, we present in Table 5 the results obtained for N= 100 and for various numbers of simulations.

It appears that, already for this number of policies ($N = 100$), the number of simulations has no influence. Obviously, with a lower number of policies, the number of simulations plays a more important role, as one would expect, while for a higher number of policies, it is insensitive to a number of simulations above one million.

Table 5. Testing the numerical convergence: the risk loading as a function of the number of Monte Carlo simulations, for $N = 100$, Model Equation (14) and the same parameters as in Table 4.

Risk Measure ρ	Number N of Policies	In a Normal State $\tilde{p} = 0$	Risk Loading R with Occurrence of a Crisis State $\tilde{p} = 0.1\%$	$\tilde{p} = 1.0\%$	$\tilde{p} = 5.0\%$	$\tilde{p} = 10.0\%$
VaR						
	1 million	0.330	0.357	0.615	0.945	1.170
	10 million	0.330	0.357	0.615	0.945	1.155
	20 million	0.330	0.357	0.615	0.945	1.170
TVaR						
	1 million	0.375	0.476	0.738	1.115	1.358
	10 million	0.374	0.472	0.739	1.117	1.357
	20 million	0.375	0.473	0.740	1.118	1.358
$\mathbb{E}[L]/N$		*10.00*	*10.02*	*10.20*	*11.00*	*12.00*

4. Comparison and Discussion

Let us start with Table 6, presenting a summary of the expectation and the variance of the total loss amount per policy, obtained for each model.

For the first Model Equation (7), we see that the variance decreases with increasing N, while for both of the other models Equations (11) and (14), the variance contains a term that does not depend on N, which corresponds to the presence of a systemic risk and is not diversifiable. Note that the variance for Model Equation (11) contains a non-diversifiable part that corresponds to n-times the non-diversifiable part of the variance for Model Equation (14). This is consistent with the numerical results in Tables 3 and 4; indeed, the smaller the non-diversifiable part, the longer the decrease of the risk loading R (*i.e.*, the effect of diversification) with the increase of the number of policies. The latter model is the most interesting, because it shows both the effect of diversification and the effect of the non-diversifiable term in a more realistic way. It assumes the occurrence of states that are dangerous to the whole portfolio, which is characteristic of a state of crisis in the financial markets. Thus, it is more suitable to explore other properties and the limits of diversification in times of crisis.

Table 6. Summary of the analytical results (expectation and variance per policy) for the three models with the cdf defined by Equations (7), (11) and (14), respectively.

Model	$\mathbb{E}[L]/N$	$var(L)/N^2$
Equation (7)	$ln\, p$	$\frac{l^2 n}{N}\, p(1-p)$
Equation (11)	$ln\left(\tilde{p}\, q\, +\, (1-\tilde{p})\, p\right)$	$\frac{l^2 n}{N}\left(q(1-q)\tilde{p} + p(1-p)(1-\tilde{p})\right) + \mathbf{l^2 n^2 (q-p)^2 \tilde{p}(1-\tilde{p})}$
Equation (14)	$ln\left(\tilde{p}\, q\, +\, (1-\tilde{p})\, p\right)$	$\frac{l^2 n}{N}\left(q(1-q)\tilde{p} + p(1-p)(1-\tilde{p})\right) + \mathbf{l^2 n\, (q-p)^2 \tilde{p}(1-\tilde{p})}$

Here, we should note that there is, in the economic literature, a distinction between "systemic" risk implying contagion effects and "systematic" risk that implies an element that cannot be diversified away. Yet, there is no clear accepted mathematical definition of either phenomenon. As already mentioned above, we see that systemic risk manifests itself by a breakdown of the benefit of diversification. It might be due either to contagion or to the presence of a non-diversifiable risk. In the following, we propose a mechanism that is equivalent to contagion as it happens simultaneously on many risks, but could also be assimilated to a non-diversifiable element coming from the mixture of two distributions. We could interpret our first dependent case as a model for systematic risk in the sense of the economists, while the second one is more a model for systemic risk because once the biased distribution is chosen, it is applied to all the N policies (contagion). However, this model does not contain a time component that the term contagion would imply.

Concerning the choice of risk measure, we have already noticed that there was an issue when evaluating the VaR with small $\tilde{p}$. There are other less obvious stability problems, as, for instance, the VaR in with the cdf Model Equation (11) for $\tilde{p} = 1\%$. It starts to decrease with N increasing, then raises again for large N, while the TVaR decreases with N increasing, then stabilizes to a value whenever $N \geq 50$. For $\tilde{p} = 0.1\%$, in both models with cdf Equations (11) and (14), respectively, VaR is very close to the Case Equation (7), without systemic risk, while TVaR starts to be significantly impacted already with 50 policies, indicating that the systemic risk appears mostly beyond the 99% threshold. Even if there is a part of the risk that is non-diversifiable, VaR, under certain circumstances, might not catch it (see [15], Proposition 3.3).

We have the following property:

If the risk measure ρ is subadditive, then the risk loading R_N defined in (4) satisfies $R_N - R_1 \leq 0$, $\forall N \geq 1$.

The proof is straightforward when noticing that for any model we considered, $\sum_{i=1}^{nN} X_i \overset{d}{=} \sum_{i=k+1}^{k+nN} X_i$, $\forall k \geq 1$, and when using the assumption of subadditvity of ρ, $\rho(L^{(N)}) \leq N\rho(L^{(1)})$. $\square$

This property is satisfied in the three cases. We also see that R_N is a decreasing function of N for TVaR in all tables, but not for VaR. For the latter, we see, in the first dependence model, an increase of VaR for $\tilde{p} = 1\%$ from $N = 100$ onwards and, in the second case, for $\tilde{p} = 10\%$ from $N = 1'000$ onwards. The reason for this increase could be due to numerical instabilities or to a break of subadditivity of VaR.

5. Conclusions

In this study, we have shown the effect of diversification on the pricing of insurance risk through the first simple modeling. Then, for understanding and analyzing possible limitations to diversification benefits, we propose two alternative stochastic models, introducing dependence between risks by assuming the existence of an underlying systemic risk. These models, defined with mixing distributions, allow for a straightforward analytical evaluation of the impact of the non-diversifiable part, which appears in the close form expression of the variance. We have purposely adopted here a probabilistic approach for modeling the dependence and the existence of systemic risk. It could be easily generalized to a time series interpretation by assigning a time step to each exposure n. In the last model, the occurrence of the rv $U = 1$ could then be identified with the time of crisis.

In real life, insurers have to pay special attention to the effects that can weaken the diversification benefits. For instance, in the case of motor insurance, the appearance of a hail storm will introduce a "bias" in the usual risk of accidents, due to a cause independent of the car drivers, which will hit a big number of cars at the same time and, thus, cannot be diversified among the various policies. There are other examples, in life insurance, for instance, with a pandemic or mortality trend that would affect the entire portfolio and that cannot be diversified away. Special care must be given to those risks, as they will affect greatly the risk loading of the premium, as can be seen in our examples. These examples might also find applications for real cases. This approach can be generalized to investments and banking; both are subject to systemic risk, although of a different nature than in the above insurance examples.

The last model we suggested, introducing the occurrence of crisis, may find an interesting application for investment and the change in the risk appetite of investors. It will be the subject of a following paper. Moreover, the models we introduce here allow one to point out explicitly the impact of dependence; they are simple enough to compute an analytic expression and to analyze the impact of the emergence of systemic risks. Yet, they are formulated in such a way that extensions to more sophisticated models are easy and clear. In particular, it makes it possible to obtain an extension to non-identically-distributed rvs or when considering random severity. Another interesting perspective would be to consider econometric models with multiple states.

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Author Contributions

All authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflicts of interest.

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