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Options with Extreme Strikes

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Abstract: In this short paper, we study the asymptotics for the price of call options for very large strikes and put options for very small strikes. The stock price is assumed to follow the Black–Scholes models. We analyze European, Asian, American, Parisian and perpetual options and conclude that the tail asymptotics for these option types fall into four scenarios.

Keywords: option pricing; extreme strikes; Black–Scholes models

1. Introduction

Asymptotics for option prices have been well studied in the finance literature. However, the majority of current research in this area has focused on short maturity asymptotics and implied volatility for European options; see, e.g., [1–5] for the local volatility models, [6–10] for the exponential Lévy models, [11–17] for the stochastic volatility models and [18,19] for model-free frameworks.

Still, there has been work in the direction of extreme strike asymptotics and implied volatility for European options. For example, let I be the implied volatility and x be the log-moneyness. The bounds on the slope $\frac{\partial I}{\partial x}$ were studied in, e.g., Hodges [20] and Gatheral [21]. Assuming only the absence of arbitrage, Hodges [20] used the bounds that if $K_1 < K_2$, then $C(K_1) \geq C(K_2)$ and $P(K_1) \leq P(K_2)$, where $C(K), P(K)$ are call and put prices with strike K, and it follows that:

$$-\frac{N(-d_2)}{\sqrt{T}N'(d_2)} \leq \frac{\partial I}{\partial x} \leq \frac{N(d_2)}{\sqrt{T}N'(d_2)}, \tag{1.1}$$

where $N(\cdot)$ is the cumulative normal distribution function and:

$$d_{1,2} = \frac{\log(S_0 e^{rT}/K)}{I\sqrt{T}} \pm \frac{I\sqrt{T}}{2}. \tag{1.2}$$

By noting that $C(K_1) \geq C(K_2)$ and $\frac{P(K_1)}{K_1} \leq \frac{P(K_2)}{K_2}$, Gatheral [21] improved the bounds on implied volatility and derived that:

$$-\frac{N(-d_1)}{\sqrt{T}N'(d_1)} \leq \frac{\partial I}{\partial x} \leq \frac{N(d_2)}{\sqrt{T}N'(d_2)}. \tag{1.3}$$

Note that these bounds on the slope of I depend on I itself. By solving the corresponding ODEs, one can obtain the bounds on I itself, and Lipton [22] mentioned that $I(x) = O(\sqrt{|x|})$ as $|x| \to \infty$.

In Lee [23], a celebrated moment formula is given. Let the large-strike tail slope be defined as $\beta_R := \limsup_{x \to \infty} \frac{I^2}{|x|/T}$ and the small-strike tail slope be defined as $\beta_L := \limsup_{x \to -\infty} \frac{I^2}{|x|/T}$. Lee [23] showed that in the absence of arbitrage, there is a one-to-one correspondence between the large-strike tail slope and the number of finite moments of the underlying S_T, that is,

$$\frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2} = \sup\left\{p : \mathbb{E}\left[S_T^{1+p}\right] < \infty\right\}, \tag{1.4}$$

and similarly, there is a one-to-one correspondence between the small-strike tail slope and the number of finite moments of S_T^{-1}, that is,

$$\frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2} = \sup\left\{q : \mathbb{E}\left[S_T^{-q}\right] < \infty\right\}. \tag{1.5}$$

In Benaim and Friz [24], they studied the price of the vanilla European options, the implied volatility at extreme strikes for different underlying stock price processes and sharpened Lee's moment formulas [23]. There is a one-to-one correspondence between the tail probabilities of $\mathbb{P}(S_T \geq K)$ for $K \to \infty$ and the asymptotic tails of the option price $C(K) = e^{-rT}\mathbb{E}[(S_T - K)^+]$ and the implied volatility. See also Benaim *et al.* [25]. The natural tool they used is the regular variation theory. Other works concerning the option pricing with extreme strikes include, e.g., Gulisashvili [26], where error estimates for the implied volatility were obtained.

In this paper, we take a different perspective. We concentrate on the Black–Scholes model for the underlying stock price process and study the asymptotic tails for different types of options—European, American, Asian, Parisian and perpetual—and we discover that the options will fall into one of the four scenarios in terms of the asymptotic tails of the option price. The asymptotics for European and perpetual options are straightforward because of the closed-form formulas. For the other types of options that we will study, they lack simple closed-form formulas, which requires some clever estimates to establish the appropriate asymptotics at extreme strikes. Since closed-form formulas are not available for most of the path-dependent options considered in this paper even for the Black–Scholes model, the asymptotics for the price of these options at extreme strikes has its own merits and can serve as analytical approximations as an alternative to the numerical methods. It would be nice to generalize the results from the Black–Scholes model to more general models, e.g., local volatility/stochastic volatility models. The challenge is that unlike the European options, most types of options we consider in this paper are path dependent, which means knowing the marginal distribution of the stock price at the maturity is not sufficient to determine the price of the options. It may not be as clear as the European options how to provide a general framework to discuss the extreme strikes asymptotics for the path-dependent options for local volatility/stochastic volatility models. This can be a future research topic.

The paper is organized as follows. We will state our results in Section 2, discuss some future directions in Section 3 and provide the proofs in Section 4.

2. Main Results

Let the stock price follow the Black–Scholes dynamics:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{2.1}$$

under the risk-neutral measure, where r, q, σ are risk-free rate, dividend yield and volatility and W_t is a standard Brownian motion starting at zero at Time 0.

We are interested in the asymptotics for the call and put options when the strike prices are very large and very small, respectively. For the vanilla European call and put options:

$$C(K) := e^{-rT}\mathbb{E}[(S_T - K)^+], \qquad P(K) := e^{-rT}\mathbb{E}[(K - S_T)^+] \tag{2.2}$$

the famous Black–Scholes formula says that:

$$C(K) = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2) \tag{2.3}$$

$$P(K) = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \tag{2.4}$$

$$d_1 := \frac{\log(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \qquad d_2 := d_1 - \sigma\sqrt{T}, \tag{2.5}$$

where $N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$ is the cumulative distribution of the standard normal random variable, from which it follows that:

$$\lim_{K \to \infty} \frac{\log C(K)}{(\log K)^2} = \lim_{K \to 0} \frac{\log P(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T}. \tag{2.6}$$

Note that (2.6) is implied by Theorem 1 and 2 in Benim and Friz [24]. Formula (2.6) will be used repeatedly in the proofs later in the paper. For completeness, we will give a short proof here.

It is well known that:

$$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) \le N(-x) \le \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}, \qquad x > 0. \tag{2.7}$$

Therefore, as $K \to \infty$,

$$C(K) = S_0 e^{-qT} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}|d_1|}\left(1 + O\left(\frac{1}{d_1^2}\right)\right) - K e^{-rT} \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}|d_2|}\left(1 + O\left(\frac{1}{d_2^2}\right)\right). \tag{2.8}$$

It is straightforward to check that $S_0 e^{-qT} e^{-\frac{d_1^2}{2}} = K e^{-rT} e^{-\frac{d_2^2}{2}}$ and $\frac{d_1}{d_2} \to 1$ as $K \to \infty$. Therefore, as $K \to \infty$,

$$C(K) = S_0 e^{-qT} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\frac{1}{|d_1|} - \frac{1}{|d_2|}\right]\left(1 + O\left(\frac{1}{d_1^2}\right)\right), \tag{2.9}$$

and similarly, we can show that as $K \to 0$,

$$P(K) = S_0 e^{-qT} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\frac{1}{d_2} - \frac{1}{d_1}\right]\left(1 + O\left(\frac{1}{d_1^2}\right)\right). \tag{2.10}$$

Therefore, (2.6) follows from (2.9) and (2.10).

Do these asymptotics in (2.6) hold for other types of options? This is the central question that we are going to investigate in this paper. We will study the Asian, American, Parisian and perpetual options, and we will show that there are four possible scenarios for call options with very large strikes and put options with very small strikes.

2.1. Asian Options

Asian options are widely-traded instruments in the financial markets, which involve the time average of the asset price. Most commonly, the asset price is a stock price or a commodity future price, for example an oil or natural gas price, and the average taken is the arithmetic average. The prices of the Asian call and put options with maturity T and strike price K are given by:

$$C_a(K) := e^{-rT} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] \tag{2.11}$$

$$P_a(K) := e^{-rT} \mathbb{E} \left[\left(K - \frac{1}{T} \int_0^T S_t dt \right)^+ \right] \tag{2.12}$$

We write $C_a(K), C_p(K)$ to emphasize the dependence on strike price K. We are interested in the large strike asymptotics, *i.e.*, when the strike price $K \to \infty$. It is clear that the call option price goes to zero as the strike price goes to infinity, and we are interested in quantifying how fast the option price goes to zero as the strike price goes to infinity.

We have the following asymptotic results:

Proposition 1. *(i)*

$$\lim_{K \to \infty} \frac{\log C_a(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.13}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_a(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.14}$$

Therefore, the asymptotic tails for the Asian options at extreme strikes coincide with the European options.

2.2. American Options

One can also study the price of the American options with large/small strike prices. When there is no dividend, it is well known that the American call option has the same value as its European counterpart. However, when there is a dividend, the American call option does not have a simple closed-form formula. Therefore, there is no simple closed-form formula for the American put option. See, e.g., Hull [27]. Let us recall that the prices of the American call and put options are given by:

$$C_A(K) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} (S_\tau - K)^+ \right] \tag{2.15}$$

$$P_A(K) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} (K - S_\tau)^+ \right] \tag{2.16}$$

where the $\mathcal{T}$ is the set of stopping times less or equal to T.

We have the following asymptotic results:

Proposition 2. *(i)*

$$\lim_{K \to \infty} \frac{\log C_A(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.17}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_A(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.18}$$

Therefore, the asymptotic tails for the Asianoptions at extreme strikes coincide with the European options.

2.3. Parisian Down-And-Out Options

We can also consider the Parisian options. Parisian options are generalizations of the barrier options, in which the option can lose/keep its value if there is an excursion of the underlying asset price that exceeds or falls below a given barrier for a consecutive period of time longer than a fixed number, that is the option window. For basic properties and a summary of Parisian options, we refer to Chesney *et al.* [28].

Let us consider first the down-and-out option. The option loses value if the stock price S_t reaches the level L and remains constantly below this level for a time interval longer than D, a fixed number that is called the option window. Otherwise, the owner will receive $(S_T - K)^+$ for the call option and $(K - S_T)^+$ for the put option at the maturity time T. We assume that $S_0 > L$. Note that the $D = 0$ case reduces to the knock-out options.

We have the following asymptotic results:

Proposition 3. *(i)*

$$\lim_{K \to \infty} \frac{\log C_{DO}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.19}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_{DO}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 D} \tag{2.20}$$

Note that the asymptotics for the call option for a large strike is the same as its European counterpart, but the asymptotics for the put option for a small strike differs from its European counterpart.

2.4. Parisian Up-And-In Options

We can also study the Parisian up-and-in options. The option keeps its value if the stock price S_t reaches the level $L > S_0$ and remains constantly above this level for a time interval longer than D, a fixed number that is called the option window. Otherwise, the owner will receive $(S_T - K)^+$ for the call option and $(K - S_T)^+$ for the put option at the maturity time T.

We have the following asymptotic results:

Proposition 4. *(i)*

$$\lim_{K \to \infty} \frac{\log C_{UI}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.21}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_{UI}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 (T - D)} \tag{2.22}$$

Note that the asymptotics for the call option for a large strike is the same as the European counterpart, but the asymptotics for the put option for a small strike differs from the European counterpart.

2.5. Parisian Up-And-Out Options

The Parisian up-and-out option loses its value if there is an excursion above the level $L > S_0$ that is longer than the option window D. We have the following asymptotic results:

Proposition 5. *(i)*

$$\lim_{K \to \infty} \frac{\log C_{UO}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 D} \tag{2.23}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_{UO}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.24}$$

Note that the asymptotics for the call option for a large strike differs from the European counterpart, but the asymptotics for the put option for a small strike is the same as the European counterpart.

2.6. Parisian Down-And-In Options

The Parisian down-and-in option is worthless unless the stock price falls below level $L < S_0$ for a consecutive time interval, whose length exceeds D, the option window.

We have the following asymptotic results:

Proposition 6. *(i)*

$$\lim_{K \to \infty} \frac{\log C_{DI}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 (T - D)} \tag{2.25}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_{DI}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T} \tag{2.26}$$

Note that the asymptotics for the call option for a large strike differs from the European counterpart, but the asymptotics for the put option for a small strike is the same as the European counterpart.

2.7. Perpetual Options

Even though American options with finite maturity T do not have simple closed-form formulas, it is well known that perpetual American options do have closed-form formulas, e.g., [29–31]. Let:

$$C_P(K) = \sup_{0 \leq \tau \leq \infty} \mathbb{E}\left[e^{-r\tau}(S_\tau - K)^+\right] \tag{2.27}$$

$$P_P(K) = \sup_{0 \leq \tau \leq \infty} \mathbb{E}\left[e^{-r\tau}(K - S_\tau)^+\right] \tag{2.28}$$

where $C_P(K), P_P(K)$ denote the prices of the perpetual American call and put options, respectively.

The quadratic equation:

$$r = (r - q)\gamma + \frac{1}{2}\sigma^2\gamma(\gamma - 1) \tag{2.29}$$

has two solutions $\gamma_0 > 1 > 0 > \gamma_1$, where:

$$\gamma_{0,1} = \frac{-(r - q - \frac{1}{2}\sigma^2) \pm \sqrt{(r - q - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} \tag{2.30}$$

We have the following result:

Proposition 7. *(i)*

$$\lim_{K \to \infty} \frac{\log C_P(K)}{\log K} = 1 - \gamma_0 \tag{2.31}$$

(ii)

$$\lim_{K \to 0} \frac{\log P_P(K)}{\log K} = 1 - \gamma_1 \tag{2.32}$$

3. Conclusions and Future Directions

In this paper, we have obtained asymptotics for call options with large strikes and put options with small strikes for Asian, American, Parisian and perpetual options and showed that they fall into one of four categories. We summarize our results for call options with large strikes in Table 1 and put options with small strikes in Table 2.

Table 1. Summary of four categories for call option prices at extreme strikes, where T is the maturity, D is the option window and $\gamma_0 > 1$ is a constant depending on r, q, σ.

Call Option $C(K)$	Option Types
$e^{-\frac{1}{2\sigma^2 T}(\log K)^2 + o((\log K)^2)}$	European, American, Asian,
	Parisian Down-And-Out, Up-And-In
$e^{-\frac{1}{2\sigma^2 D}(\log K)^2 + o((\log K)^2)}$	Parisian Up-And-Out
$e^{-\frac{1}{2\sigma^2(T-D)}(\log K)^2 + o((\log K)^2)}$	Parisian Down-And-In
$e^{(1-\gamma_0)\log K + o(\log K)}$	Perpetual American

Table 2. Summary of four categories for put option prices at extreme strikes, where T is the maturity, D is the option window and $\gamma_1 < 0$ is a constant depending on r, q, σ.

Put Option $P(K)$	Option Types
$e^{-\frac{1}{2\sigma^2 T}(\log K)^2 + o((\log K)^2)}$	European, American, Asian, Parisian Up-And-Out, Down-And-In
$e^{-\frac{1}{2\sigma^2 D}(\log K)^2 + o((\log K)^2)}$	Parisian Down-And-Out
$e^{-\frac{1}{2\sigma^2 (T-D)}(\log K)^2 + o((\log K)^2)}$	Parisian Up-And-Out
$e^{(1-\gamma_1)\log K + o(\log K)}$	Perpetual American

Note that these asymptotic results are only the first order approximations for extreme strikes. For options with explicit formulas, one can certainly get much better asymptotic approximations. For options without simple closed-form formulas, e.g., Asian options, it will be an interesting future research problem to get more accurate asymptotics at extreme strikes. For this paper, we only consider the first order asymptotics, as an illustration to compare the options of different types at extreme strikes for the Black–Scholes model. It will also be useful to obtain the speed of convergence to these first order asymptotics. Another interesting direction to pursue in the future will be establishing the asymptotics for extreme strikes for various types of options when the underlying stock price follows a more general stochastic process other than the Black–Scholes model.

4. Appendix: Proofs

Proof of Proposition 1. (i) Note that under the risk-neutral measure,

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t + \sigma W_t}, \qquad t \geq 0. \tag{4.1}$$

Therefore,

$$C_a(K) \leq e^{-rT}\mathbb{E}\left[\left(\max_{0 \leq t \leq T} S_t - K\right)^+\right] \tag{4.2}$$

$$\leq e^{-rT}\mathbb{E}\left[\left(S_0 e^{|r-q-\frac{1}{2}\sigma^2|T} e^{\sigma \max_{0 \leq t \leq T} W_t} - K\right)^+\right].$$

By the reflection principle for Brownian motions, $\max_{0 \leq t \leq T} W_t = |W_T|$ in distribution, and therefore,

$$C_a(K) \leq e^{-rT} e^{|r-q-\frac{1}{2}\sigma^2|T}\mathbb{E}\left[\left(S_0 e^{\sigma|W_T|} - \frac{K}{e^{|r-q-\frac{1}{2}\sigma^2|T}}\right)^+\right] \tag{4.3}$$

$$= 2e^{-rT} e^{|r-q-\frac{1}{2}\sigma^2|T}\mathbb{E}\left[\left(S_0 e^{\sigma W_T} - \frac{K}{e^{|r-q-\frac{1}{2}\sigma^2|T}}\right)^+\right],$$

where the last line is due to the symmetry of the Brownian motion. By using the Black–Scholes formula,

$$\mathbb{E}\left[\left(S_0 e^{\sigma W_T} - \frac{K}{e^{|r-q-\frac{1}{2}\sigma^2|T}}\right)^+\right] \tag{4.4}$$

$$= e^{\frac{1}{2}\sigma^2 T}\mathbb{E}\left[\left(S_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - \frac{Ke^{-\frac{1}{2}\sigma^2 T}}{e^{|r-q-\frac{1}{2}\sigma^2|T}}\right)^+\right]$$

$$= S_0 e^{\frac{1}{2}\sigma^2 T} N\left(\frac{\log(S_0 e^{|r-q-\frac{1}{2}\sigma^2|T+\frac{1}{2}\sigma^2 T}/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

$$- \frac{K}{e^{|r-q-\frac{1}{2}\sigma^2|T}} N\left(\frac{\log(S_0 e^{|r-q-\frac{1}{2}\sigma^2|T+\frac{1}{2}\sigma^2 T}/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

$$\leq S_0 e^{\frac{1}{2}\sigma^2 T} N\left(\frac{\log(S_0 e^{|r-q-\frac{1}{2}\sigma^2|T+\frac{1}{2}\sigma^2 T}/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).$$

Therefore, by the property of $N(\cdot)$ (see (2.7)), we get:

$$\limsup_{K\to\infty} \frac{\log C_a(K)}{(\log K)^2} \leq -\frac{1}{2\sigma^2 T}. \tag{4.5}$$

Next, let us turn to the lower bound. For any $0 < \epsilon < T$,

$$C_a(K) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t}dt - K\right)^+\right] \tag{4.6}$$

$$\geq e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_{T-\epsilon}^T S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t}dt - K\right)^+\right]$$

$$= e^{-rT}\mathbb{E}\left[\left(\frac{\epsilon}{T}\frac{1}{\epsilon}\int_{T-\epsilon}^T S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t}dt - K\right)^+\right]$$

$$\geq e^{-rT}\mathbb{E}\left[\left(\frac{\epsilon}{T}S_0 e^{\frac{1}{\epsilon}\int_{T-\epsilon}^T (r-q-\frac{1}{2}\sigma^2)t+\sigma W_t dt} - K\right)^+\right],$$

where the last step used the Jensen's inequality for the integration inside the expectation.

Therefore, we have:

$$C_a(K) \geq e^{-rT}\mathbb{E}\left[\left(\frac{\epsilon}{T}e^{\frac{1}{\epsilon}\int_{T-\epsilon}^T (r-q-\frac{1}{2}\sigma^2)t dt}S_0 e^{\frac{\sigma}{\epsilon}\int_{T-\epsilon}^T W_t dt} - K\right)^+\right] \tag{4.7}$$

$$= e^{-rT}\frac{\epsilon}{T}e^{(r-q-\frac{1}{2}\sigma^2)(T-\frac{\epsilon}{2})}\mathbb{E}\left[\left(S_0 e^{\sigma B_{\epsilon,T}} - \frac{K}{\frac{\epsilon}{T}e^{(r-q-\frac{1}{2}\sigma^2)(T-\frac{\epsilon}{2})}}\right)^+\right],$$

where:

$$B_{\epsilon,T} := \frac{1}{\epsilon}\int_{T-\epsilon}^T W_t dt \tag{4.8}$$

is a Gaussian random variable with mean zero and variance:

$$\mathbb{E}[B_{\epsilon,T}^2] = \frac{1}{\epsilon^2}\mathbb{E}\left[\left(\int_{T-\epsilon}^T W_t dt\right)^2\right] \tag{4.9}$$

$$= \frac{2}{\epsilon^2}\mathbb{E}\left[\int_{T-\epsilon}^T\int_{T-\epsilon}^{t_2} W_{t_1}W_{t_2}dt_1 dt_2\right]$$

$$= \frac{2}{\epsilon^2}\int_{T-\epsilon}^T\int_{T-\epsilon}^{t_2} t_1 dt_1 dt_2$$

$$= T - \frac{2}{3}\epsilon.$$

Therefore, by using the Black–Scholes formula,

$$\liminf_{K\to\infty} \frac{\log C_a(K)}{(\log K)^2} \geq -\frac{1}{2\sigma^2(T-\frac{2}{3}\epsilon)}. \tag{4.10}$$

Since it holds for any $0 < \epsilon < T$, we proved the desired result.

(ii) Next, let us consider the price of the Asian put option with a fixed strike:

$$P_a(K) = e^{-rT}\mathbb{E}\left[\left(K - \frac{1}{T}\int_0^T S_t dt\right)^+\right]. \tag{4.11}$$

It is easy to see that:

$$P_a(K) \geq e^{-rT}\mathbb{E}\left[\left(K - \max_{0\leq t\leq T} S_t\right)^+\right], \tag{4.12}$$

and for any $0 < \epsilon < T$,

$$P_a(K) \leq e^{-rT}\mathbb{E}\left[\left(K - \frac{1}{T}\int_{T-\epsilon}^T S_t dt\right)^+\right], \tag{4.13}$$

and following the similar arguments as before, we can show that:

$$\lim_{K\to 0} \frac{\log P_a(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T}. \tag{4.14}$$

□

Proof of Proposition 2. (i) The price of an American call option is at least as much as the European counterpart:

$$C_A(K) := \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau}(S_\tau - K)^+\right] \geq e^{-rT}\mathbb{E}[(S_T - K)^+], \tag{4.15}$$

and on the other hand,

$$C_A(K) \leq \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-(r-q)\tau}(S_\tau - K)^+\right] = e^{-(r-q)T}\mathbb{E}\left[\left(S_0 e^{(r-q-\frac{1}{2}\sigma^2)T+\sigma W_T} - K\right)^+\right], \tag{4.16}$$

where the equality in (4.16) is due to the fact that an American call option with risk-free rate $r - q$ and zero dividend yield equals the price of the corresponding European call option with risk-free rate $r - q$. Hence, we conclude that:

$$\lim_{K\to\infty} \frac{\log C_A(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T}. \tag{4.17}$$

(ii) The price of the American put option is at least as much as the European counterpart:

$$P_A(K) := \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}(K - S_\tau)^+\right] \geq e^{-rT}\mathbb{E}[(K - S_T)^+], \tag{4.18}$$

and on the other hand,

$$P_A(K) \leq \mathbb{E}\left[\left(K - \min_{0 \leq t \leq T} S_t\right)^+\right] \tag{4.19}$$

$$\leq \mathbb{E}\left[\left(K - S_0 e^{-|r-q-\frac{1}{2}\sigma^2|T} e^{\sigma \min_{0 \leq t \leq T} W_t}\right)^+\right]$$

$$= \mathbb{E}\left[\left(K - S_0 e^{-|r-q-\frac{1}{2}\sigma^2|T} e^{-\sigma|W_T|}\right)^+\right],$$

which implies that:

$$\lim_{K \to 0} \frac{\log P_A(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T}. \tag{4.20}$$

$\square$

Proof of Proposition 3. (i) It is clear that the price of the Parisian down-and-out option is bounded above by the price of the vanilla European option, and it is shown in [28] that we have the lower bound (see Equation (2) in Section 3 of [28]):

$$C_{DO}(K) \geq S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2) \tag{4.21}$$

$$- S_0 \left(\frac{S_0}{L}\right)^{-2\epsilon} N(y) + K e^{-qT} e^{-rT} \left(\frac{S_0}{L}\right)^{2-2\epsilon} N(y - \sigma\sqrt{T}), \tag{4.22}$$

where:

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log(S_0/K) + (r-q)T + \frac{1}{2}\sigma^2 T\right], \qquad d_2 = d_1 - \sigma\sqrt{T}, \tag{4.23}$$

and $\epsilon = \frac{1}{2} + \frac{r}{\sigma^2}$, $y = \frac{1}{\sigma\sqrt{T}}\log\left(\frac{L^2}{S_0 K} + \epsilon\sigma\sqrt{T}\right)$. Hence, we have:

$$\lim_{K \to \infty} \frac{\log C_{DO}(K)}{(\log K)^2} = -\frac{1}{2\sigma^2 T}. \tag{4.24}$$

(ii) We have the upper bound: for any sufficiently small $\delta > 0$,

$$P_{DO}(K) \leq e^{-rT}\mathbb{E}\left[1_{\max_{T-D-\delta \leq t \leq T} S_t \geq L}(K - S_T)^+\right] \tag{4.25}$$

$$= e^{-rT}\mathbb{P}\left(\max_{T-D-\delta \leq t \leq T} S_t \geq L\right)\mathbb{E}\left[(K - S_T)^+ \Big| \max_{T-D-\delta \leq t \leq T} S_t \geq L\right]$$

$$\leq e^{-rT}\mathbb{P}\left(\max_{T-D-\delta \leq t \leq T} S_t \geq L\right)\max_{T-D-\delta \leq t \leq T}\mathbb{E}\left[(K - S_T)^+ | S_t = L\right]$$

$$\leq \mathbb{P}\left(\max_{T-D-\delta \leq t \leq T} S_t \geq L\right)\max_{T-D-\delta \leq t \leq T} e^{-r(T-t)}\mathbb{E}\left[(K - S_T)^+ | S_t = L\right]$$

$$= \mathbb{P}\left(\max_{T-D-\delta \leq t \leq T} S_t \geq L\right) e^{-r(D+\delta)}\mathbb{E}\left[(K - S_T)^+ | S_{T-D-\delta} = L\right],$$

where the last step used the fact that Theta is negative, which implies the upper bound, since it holds for any sufficiently small $\delta > 0$. For the lower bound: for any sufficiently small $\delta > 0$,

$$
\begin{aligned}
P_{DO}(K) &\geq e^{-rT}\mathbb{E}\left[1_{L < S_t < 2L, \forall 0 \leq t \leq T-D+\delta}(K-S_T)^+\right] \\
&\geq e^{-rT}\mathbb{P}\left(L < S_t < 2L, \forall 0 \leq t \leq T-D+\delta\right) \\
&\qquad\qquad \cdot \mathbb{E}\left[(K-S_T)^+ | S_{T-D+\delta} = 2L\right].
\end{aligned}
\tag{4.26}
$$

Since it holds for any sufficiently small $\delta > 0$, we proved the lower bound. $\quad\square$

Proof of Proposition 4. (i) It is clear that the option price $C_{UI}(K)$ is less or equal to its vanilla European counterpart. For the lower bound, notice that:

$$
\begin{aligned}
C_{UI}(K) &\geq e^{-rT}\mathbb{E}\left[1_{S_\delta \geq 2L}1_{\min_{\delta \leq t \leq T} S_t \geq L}(S_T-K)^+\right] \\
&\geq e^{-rT}\mathbb{P}(S_\delta \geq 2L)\mathbb{E}\left[1_{\min_{\delta \leq t \leq T} S_t \geq L}(S_T-K)^+ | S_\delta = 2L\right] \\
&\geq e^{-rT}\mathbb{P}(S_\delta \geq 2L)\mathbb{P}\left(\min_{\delta \leq t \leq T} S_t \geq L | S_\delta = 2L\right)\mathbb{E}[(S_T-K)^+ | S_\delta = 2L],
\end{aligned}
\tag{4.27}
$$

which implies that $\liminf_{K \to \infty} \frac{\log C_{UI}(K)}{(\log K)^2} \geq -\frac{1}{2\sigma^2(T-\delta)}$. Since it holds for any $\delta > 0$, we proved the lower bound.

(ii) For any $\delta > 0$ sufficiently small, we have the lower bound:

$$
\begin{aligned}
P_{UI}(K) &\geq e^{-rT}\mathbb{E}\left[1_{L < S_t < 2L, \forall \delta \leq t \leq D}(K-S_T)^+\right] \\
&\geq e^{-rT}\mathbb{P}\left(L < S_t < 2L, \forall \delta \leq t \leq D\right)\mathbb{E}[(K-S_T)^+ | S_D = 2L],
\end{aligned}
\tag{4.28}
$$

which implies the lower bound. For the upper bound: let τ be the first time that the stock price has exceeded L for a consecutive time of D, *i.e.*,

$$
\tau := \inf\{t \geq D : S_s > L, \forall t-D \leq s \leq t\}.
\tag{4.29}
$$

Then, by the tower property and the strong Markov property,

$$
\begin{aligned}
P_{UI}(K) &= e^{-rT}\mathbb{E}[\mathbb{E}[(K-S_T)^+ | S_\tau]1_{\tau \leq T}] \\
&\leq e^{-rT}\mathbb{E}[\mathbb{E}[(K-S_T)^+ | S_\tau = L]1_{\tau \leq T}] \\
&\leq e^{-rT}\mathbb{P}(\tau \leq T)\sup_{D \leq t \leq T}\mathbb{E}[(K-S_T)^+ | S_t = L] \\
&\leq \mathbb{P}(\tau \leq T)\sup_{D \leq t \leq T}e^{-r(T-t)}\mathbb{E}[(K-S_T)^+ | S_t = L] \\
&= \mathbb{P}(\tau \leq T)e^{-r(T-D)}\mathbb{E}[(K-S_T)^+ | S_D = L].
\end{aligned}
\tag{4.30}
$$

Hence, we proved the upper bound. $\quad\square$

Proof of Proposition 5. (i) For any $\delta > 0$, we have the upper bound:

$$
\begin{aligned}
C_{UP}(K) &\leq e^{-rT}\mathbb{E}[1_{\min_{T-D-\delta \leq t \leq T} S_t \leq L}(S_T-K)^+] \\
&\leq e^{-rT}\sup_{T-D-\delta \leq t \leq T}\mathbb{E}[(S_T-K)^+ | S_t = L] \\
&\leq \sup_{T-D-\delta \leq t \leq T}e^{-rt}\mathbb{E}[(S_T-K)^+ | S_t = L] \\
&\leq e^{-r(T-D-\delta)}\mathbb{E}[(S_T-K)^+ | S_{T-D-\delta} = L],
\end{aligned}
\tag{4.31}
$$

where we used the fact that Thetais negative, that is the option price decreases as the time to maturity decreases. Thus, we have proved that:

$$\limsup_{K\to\infty} \frac{\log C_{UO}(K)}{(\log K)^2} \le -\frac{1}{2\sigma^2(D+\delta)}. \tag{4.32}$$

Since it holds for any $\delta > 0$, the upper bound is proven.

For the lower bound:

$$C_{UO}(K) \ge e^{-rT}\mathbb{E}\left[1_{\frac{1}{2}S_0 < S_t < L, \forall 0 \le t \le T-D}(S_T - K)^+\right] \tag{4.33}$$

$$\ge e^{-rT}\mathbb{P}\left(\frac{1}{2}S_0 < S_t < L, \forall 0 \le t \le T-D\right)\mathbb{E}_{S_{T-D}=\frac{1}{2}S_0}[(S_T - K)^+],$$

which yields that:

$$\liminf_{K\to\infty} \frac{\log C_{UO}(K)}{(\log K)^2} \ge -\frac{1}{2\sigma^2 D}. \tag{4.34}$$

(ii) $P_{UO}(K) \le P(K)$ gives the upper bound. For the lower bound:

$$P_{UO}(K) \ge e^{-rT}\mathbb{E}\left[1_{\max_{0\le t\le T} S_t \le L}(K - S_T)^+\right] \tag{4.35}$$

$$= e^{-rT}\mathbb{P}\left(\max_{0\le t\le T} S_t \le L\right)\mathbb{E}\left[(K - S_T)^+ | \max_{0\le t\le T} S_t \le L\right]$$

$$\ge e^{-rT}\mathbb{P}\left(\max_{0\le t\le T} S_t \le L\right)\mathbb{E}[(K - S_T)^+].$$

$\square$

Proof of Proposition 6. (i) We have the following lower bound, for sufficiently small $\delta > 0$,

$$C_{DI}(K) \ge e^{-rT}\mathbb{E}\left[1_{\frac{L}{2} < S_t < L, \forall \delta \le t \le D+\delta}(S_T - K)^+\right] \tag{4.36}$$

$$\ge e^{-rT}\mathbb{P}\left(\frac{L}{2} < S_t < L, \forall \delta \le t \le D+\delta\right)\mathbb{E}[(S_T - K)^+ | S_{\delta+D} = L/2].$$

Therefore, we have $\liminf_{K\to\infty} \frac{\log C_{DI}(K)}{(\log K)^2} \ge -\frac{1}{2\sigma^2(T-D-\delta)}$. Since it holds for any $\delta > 0$, we proved the lower bound. Next, for the upper bound, for any sufficiently small $\delta > 0$,

$$C_{DI}(K) \le e^{-rT}\mathbb{E}\left[1_{\exists t\in[D-\delta,T]:S_t\le L}(S_T - K)^+\right] \tag{4.37}$$

$$\le e^{-rT}\mathbb{P}\left(\min_{D-\delta\le t\le T} S_t \le L\right)\max_{D-\delta\le t\le T}\mathbb{E}[(S_T - K)^+ | S_t = L]$$

$$\le \mathbb{P}\left(\min_{D-\delta\le t\le T} S_t \le L\right)\max_{D-\delta\le t\le T} e^{-r(T-t)}\mathbb{E}[(S_T - K)^+ | S_t = L]$$

$$= \mathbb{P}\left(\min_{D-\delta\le t\le T} S_t \le L\right)e^{-r(T-D+\delta)}\mathbb{E}[(S_T - K)^+ | S_{D-\delta} = L],$$

where the last step used the fact that Thetais negative for the Black–Scholes model. Therefore, we have $\limsup_{K\to\infty} \frac{\log C_{DI}(K)}{(\log K)^2} \le -\frac{1}{2\sigma^2(T-D+\delta)}$. Since it holds for any $\delta > 0$, we proved the lower bound.

(ii) It is clear that $P_{DI}(K) \leq P(K)$, the price of the vanilla European option, and hence, we have the upper bound. For the lower bound, for any sufficiently small $\delta > 0$,

$$P_{DI}(K) \geq e^{-rT} \mathbb{E} \left[1_{\max_{\delta \leq t \leq T} S_t \leq L} (K - S_T)^+ \right] \tag{4.38}$$

$$= e^{-rT} \mathbb{P} \left(\max_{\delta \leq t \leq T} S_t \leq L \right) \mathbb{E} \left[(K - S_T)^+ | \max_{\delta \leq t \leq T} S_t \leq L \right]$$

$$\geq e^{-rT} \mathbb{P} \left(\max_{\delta \leq t \leq T} S_t \leq L \right) \mathbb{E} \left[(K - S_T)^+ \right],$$

which yields the lower bound. $\quad\square$

Proof of Proposition 7. It is well known that there is an explicit formula (see, e.g., page 259 in [30]) for the price of a perpetual American call option (for sufficiently large K):

$$C_P(K) = \sup_{0 \leq \tau \leq \infty} \mathbb{E} \left[e^{-r\tau} (S_\tau - K)^+ \right] = \frac{S_0^{\gamma_0}}{\gamma_0 \left(\frac{\gamma_0}{\gamma_0 - 1} K \right)^{\gamma_0 - 1}}, \tag{4.39}$$

which implies that:

$$\lim_{K \to \infty} \frac{\log C_P(K)}{\log K} = 1 - \gamma_0. \tag{4.40}$$

The perpetual American put option was studied by McKean [31], and it is well known that (see, e.g., [29]) that for sufficiently small $K > 0$,

$$P_P(K) = \frac{K}{1 - \gamma_1} \left(\frac{-K\gamma_1}{S_0(1 - \gamma_1)} \right)^{-\gamma_1}, \tag{4.41}$$

which implies that:

$$\lim_{K \to 0} \frac{\log P_P(K)}{\log K} = 1 - \gamma_1. \tag{4.42}$$

Other perpetual options include perpetual lookback American options, which have an explicit formula found by Guo and Shepp [32], which exhibits the same asymptotics for extreme strikes. $\quad\square$

Conflicts of Interest

The authors declare no conflict of interest.

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