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Ruin Probabilities in a Dependent Discrete-Time Risk Model With Gamma-Like Tailed Insurance Risks

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Abstract: This paper considered a dependent discrete-time risk model, in which the insurance risks are represented by a sequence of independent and identically distributed real-valued random variables with a common Gamma-like tailed distribution; the financial risks are denoted by another sequence of independent and identically distributed positive random variables with a finite upper endpoint, but a general dependence structure exists between each pair of the insurance risks and the financial risks. Following the works of Yang and Yuen in 2016, we derive some asymptotic relations for the finite-time and infinite-time ruin probabilities. As a complement, we demonstrate our obtained result through a Crude Monte Carlo (CMC) simulation with asymptotics.

Keywords: discrete-time risk model; finite-time and infinite-time ruin probabilities; insurance and financial risks; Gamma-like tail; asymptotics

1. Introduction

Consider a discrete-time risk model, where, for every $i \geq 1$, the insurer's net loss (the aggregate claim amount minus the total premium income) within period i is represented by a real-valued random variable (r.v.) X_i; and the stochastic discount factor from time i to time $i-1$ is denoted by a positive r.v. Y_i. In the terminology in [1], $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ are called the insurance risks and financial risks, respectively. Throughout this paper, we suppose that $\{X_i,\ i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) real-valued r.v.s with a common distribution F; $\{Y_i,\ i \geq 1\}$ is another sequence of i.i.d. positive r.v.s with a common distribution G; and the random vectors $\{(X_i, Y_i),\ i \geq 1\}$ are independent copies of (X, Y) following a certain dependence structure (see (7) below). In this framework, we are interested in the following quantities:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j,\ n \geq 1, \tag{1}$$

with the maxima

$$M_n = \max_{0 \leq k \leq n} S_k,\ n \geq 1, \quad M_\infty = \max_{k \geq 0} S_k,$$

where S_n denotes the stochastic discounted value of aggregate net losses within time n. Then, the two-tail probabilities $\mathbb{P}(M_n > x)$ and $\mathbb{P}(M_\infty > x)$ can be interpreted as the finite-time ruin probability within period n and the infinite-time ruin probability, respectively, where $x \geq 0$ stands for the initial wealth of the insurer. Clearly,

$$0 \leq M_n \leq \sum_{i=1}^{n} \max\{X_i, 0\} \prod_{j=1}^{i} Y_j, \tag{2}$$

as $n \to \infty$. The right-hand side converges almost surely (a.s.), if $-\infty \leq \mathbb{E} \ln Y < 0$ and $\mathbb{E} \ln \max\{X, 1\} < \infty$ (see Theorem 1.6 in [2] and Theorem 1 in [3]). Thus, M_n converges a.s. to the limit M_∞, which has a proper distribution on $[0, \infty)$. In this paper, we aim to investigate the asymptotic behavior of the tail probabilities $\mathbb{P}(S_n > x)$, $\mathbb{P}(M_n > x)$ and $\mathbb{P}(M_\infty > x)$ as $x \to \infty$.

In such a discrete-time risk model, under independence or some certain dependence assumptions imposed on X_is and Y_is, the asymptotic tail behavior of M_n and M_∞ has been extensively studied by many researchers. Notice that the assumption of complete independence is for mathematical convenience, but appears unrealistic in most practical situations. A recent new trend of study is to introduce various dependence structures to describe the insurance and financial risks $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$. One trend is to require the insurance risks $\{X_i,\ i \geq 1\}$ to obey a certain dependence structure (see [4–6] among others). Another trend is to assume that $\{(X_i, Y_i),\ i \geq 1\}$ form a sequence of i.i.d. random vectors, but for each $i \geq 1$, some certain dependence structure exists between X_i and Y_i. Such a work was initially studied by [7]. Chen considered the discrete-time risk model, in which each pair of the insurance and financial risks form the bivariate Farlie–Gumbel–Morgenstern (FGM) distribution. Later, Yang and Konstantinides extended Chen's results in [5], by considering a more general dependence structure than the FGM one. They derived the uniform estimates for the finite-time and infinite-time ruin probabilities, under the assumption that $\{X_i,\ i \geq 1\}$ are of consistent variation. For more details, one can be refereed to [8–11] among others. We remark that all of the above works are studied in the heavy-tailed case, while, in this paper, we consider the light-tailed case, that is, the insurance risks are Gamma-like tailed, thus are light-tailed.

Throughout the paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$; write $f(x) \prec g(x)$ or $g(x) \succ f(x)$ if $\limsup f(x)/g(x) \leq 1$; and write $f(x) = o(g(x))$ if $\lim f(x)/g(x) = 0$. For two real-valued numbers x and y, denote by $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and denote the positive part of x by $x_+ = x \vee 0$. The indicator function of an event A is denoted by $\mathbf{1}_A$.

A distribution F on $\mathbb{R}$ is said to be Gamma-like tailed with shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$ if there exists a slow function $l(\cdot) : (0, \infty) \mapsto (0, \infty)$ such that

$$\overline{F}(x) = 1 - F(x) \sim l(x) x^{\alpha - 1} e^{-\gamma x} \tag{3}$$

(see [12,13]). A larger distribution class is that of generalized exponential distributions. A distribution F on $\mathbb{R}$ is said to belong to the class $\mathcal{L}(\gamma)$ with $\gamma \geq 0$, if for any $y \in \mathbb{R}$,

$$\overline{F}(x - y) \sim e^{\gamma y} \overline{F}(x). \tag{4}$$

Clearly, if $\gamma = 0$, the class $\mathcal{L}(0)$ consists of all long-tailed distributions, which are heavy-tailed. If $\gamma > 0$, then all distributions in the class $\mathcal{L}(\gamma)$ are light-tailed. A class larger than the generalized exponential distribution class $\mathcal{L}(\gamma)$ is that of rapidly varying tailed distributions. A distribution F on $\mathbb{R}$ is said to be rapidly varying tailed, denoted by $F \in \mathcal{R}_{-\infty}$, if $\overline{F}(xy) = o(\overline{F}(x))$ holds for all $y > 1$. For a distribution $F \in \mathcal{R}_{-\infty}$, from Theorem 1.2.2 of [14], it can be seen that for any $\epsilon > 0$ and $\delta > 0$, there exists a sufficiently large constant $D > 0$ such that for all $y \geq x \geq D$,

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq (1 + \epsilon) \left(\frac{y}{x} \right)^{-\delta}.$$

We remark that if a distribution F is Gamma-like tailed with $\alpha > 0$ and $\gamma > 0$, then $F \in \mathcal{L}(\gamma) \subset \mathcal{R}_{-\infty}$ holds.

In the case of light-tailed insurance risks, in [15,16] Tang and his coauthor first established some asymptotic formulas for the finite-time ruin probability $\mathbb{P}(M_n > x)$ under the independence structure, and the conditions that the insurance risks X_is have a common convolution-equivalent or a rapidly varying tail, and the financial risks Y_is have a common distribution G with a finite upper endpoint

$$y_* = y_*(G) = \sup\{y : G(y) < 1\} < \infty. \tag{5}$$

Precisely speaking, consider a discrete-time risk model, in which $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ are two sequences of i.i.d. r.v.s with common distributions F and G, respectively, and these two sequences are mutually independent. If $F \in \mathcal{R}_{-\infty}$ and G have a finite upper endpoint $y_* > 1$ with $p_* = \mathbb{P}(Y_1 = y_*) > 0$, then, for each fixed $n \geq 1$,

$$\mathbb{P}(M_n > x) \sim p_*^n \mathbb{P}\Big(\sum_{i=1}^{n} y_*^i X_i > x\Big). \tag{6}$$

Recently, in [17] Yang and Yuen derived some more precise results than relation (6) in the presence of Gamma-like tailed insurance risks, under the independence structure or a certain dependence structure, where each pair of the insurance risks and the financial risks follow a bivariate Sarmanov distribution (see the definition below). They investigated the asymptotic tail behavior of S_n, M_n and M_∞ in three cases of $0 < y_* < 1$, $y_* = 1$ and $y_* > 1$, respectively, and dropped the condition $p_* > 0$.

In this paper, we restrict ourselves to the framework in which a more general dependence structure exists between each pair of the insurance risks and the financial risks. Precisely speaking, the random vectors of the insurance and financial risks $\{(X_i, Y_i),\ i \geq 1\}$ are assumed to be i.i.d. copies of a generic pair (X, Y) with the dependent components X and Y fulfilling the relation

$$\mathbb{P}(X > x \mid Y = y) \sim \mathbb{P}(X > x)h(y) \tag{7}$$

holding uniformly for all $y \in (0, y_*]$ as $x \to \infty$, i.e.,

$$\lim_{x\to\infty} \sup_{y\in(0,y_*]} \left| \frac{\mathbb{P}(X > x \mid Y = y)}{\mathbb{P}(X > x)h(y)} - 1 \right| = 0.$$

Here, $h(\cdot) : [0, \infty) \mapsto (0, \infty)$ is a positive measurable function, and if y is not a possible value of Y, then the left side of relation (7) consists of the unconditional probability; thus, $h(y)$ equals to 1. Such a dependence structure (7) was introduced by [18], which contains many commonly-used bivariate copulas, such as the Ali–Mikhail–Haq copula, the Farlie–Gumbel–Morgenstern copula, and the Frank copula among others, and allows both positive and negative dependence structures. We remark that if $\mathbb{P}(X > x) > 0$ for all $x \in \mathbb{R}$, then relation (7) leads to $\mathbb{E}h(Y) = 1$. See [19–21] for more details on such a dependence structure. In particular, if X and Y follow a bivariate Sarmanov distribution defined by

$$\mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{0}^{y} 1 + \theta\phi_1(x)\phi_2(y)F(\mathrm{d}u)G(\mathrm{d}v),\ \ x \in \mathbb{R},\ y \in (0, y_*],$$

and assume that the limit $\lim_{x\to\infty} \phi_1(x) = d_1$ exists, then it can be directly verified that relation (7) is satisfied with $h(y) = 1 + \theta d_1 \phi_2(y)$.

Motivated by [17], in this paper, we aim to study the asymptotic tail relations of S_n, M_n and M_∞ in two cases, i.e., $0 < y_* < 1$ and $y_* = 1$, under the assumption that $\{(X_i, Y_i),\ i \geq 1\}$ are i.i.d. random vectors with dependent components fulfilling relation (7). In the case $0 < y_* < 1$, we also obtain a uniform result for both finite-time and infinite-time ruin probabilities, by considering the asymptotic formulas for $\mathbb{P}(M_n > x)$ and $\mathbb{P}(M_\infty > x)$. We still restrict the insurance risks to be Gamma-like tailed. Our obtained results essentially extend the corresponding ones in [17].

The rest of this paper is organized as follows. In Section 2, the main results of the present paper are provided, and Section 3 displays their proofs. In Section 4, we perform a simulation to verify the approximate relationships in the main results by using the Crude Monte Carlo (CMC) method.

2. Main Results

Denote by $p_* = \mathbb{P}(Y_1 = y_*) \geq 0$, which can be equal to 0. The first result investigates the asymptotics for the finite-time ruin probability in two cases of $0 < y_* < 1$ and $y_* = 1$.

Theorem 1. *Consider the above-mentioned discrete-time risk model, where $\{(X_i, Y_i),\ i \geq 1\}$ are a sequence of i.i.d. random vectors with a generic pair (X, Y) satisfying relation (7) uniformly for all $y \in (0, y_*]$. Assume that F is Gamma-like tailed with shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$ defined in relation (3), G has a finite upper endpoint y_* defined in relation (5), and $\inf_{y\in(0,y_*]} h(y) > 0$.*

(1) If $0 < y_* < 1$, then for each fixed $n \geq 1$, it holds that $\mathbb{E}(e^{\gamma S_{n-1}}) < \infty$, $\mathbb{E}(e^{\gamma M_{n-1}}) < \infty$, and

$$\mathbb{P}(S_n > x) \sim \frac{p_* h(y_*)\mathbb{E}(e^{\gamma S_{n-1}})}{y_*^{\alpha-1}} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}, \tag{8}$$

$$\mathbb{P}(M_n > x) \sim \frac{p_* h(y_*)\mathbb{E}(e^{\gamma M_{n-1}})}{y_*^{\alpha-1}} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}. \tag{9}$$

(2) If $y_* = 1$, then for each fixed $n \geq 1$, it holds that

$$\mathbb{P}(M_n > x) \sim \mathbb{P}(S_n > x) \sim \frac{p_*^n h(y_*)\gamma^{n-1}(\Gamma(\alpha))^n}{y_*^{(\alpha-1)n}\Gamma(n\alpha)} (l(x))^n x^{n\alpha-1} e^{-\gamma x}. \tag{10}$$

The condition $y_* \leq 1$ in relations (8)–(10) means that the insurer invests all his/her surpluses into a risk-free market.

The second result gives formula (9) with $n = \infty$, which implies the asymptotic relation for the infinite-time ruin probability. Combining relation (9), a uniform result for both finite-time and infinite-time ruin probabilities is also derived in the case $0 < y_* < 1$.

Theorem 2. *Under the conditions of Theorem 1, if $0 < y_* < 1$, then it holds that*

$$\mathbb{P}(M_\infty > x) \sim \frac{p_* h(y_*)\mathbb{E}(e^{\gamma M_\infty})}{y_*^{\alpha-1}} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}. \tag{11}$$

Furthermore, relation (9) holds uniformly for all $n \geq 1$, that is,

$$\lim_{x\to\infty} \sup_{n\geq 1} \left| \frac{\mathbb{P}(M_n > x)}{p_* h(y_*)\mathbb{E}(e^{\gamma M_{n-1}}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}} - 1 \right| = 0.$$

3. Proofs of Main Results

We start this section by the following lemma, which is initiated by Lemma 2 in [22], where they considered that X and Y are two independent r.v.s and Y is supported on $(0, 1]$, indicating $y_* = 1$.

Lemma 1. *Let X and Y be two dependent r.v.s with distributions F on $\mathbb{R}$ and G on $(0, y_*]$, respectively. If $F \in \mathcal{R}_{-\infty}$ and relation (7) holds uniformly for all $y \in (0, y_*]$, then it holds that*

$$\lim_{x\to\infty} \frac{\mathbb{P}(XY > x)}{\mathbb{P}(X > xy_*^{-1})} = h(y_*)\mathbb{P}(Y = y_*). \tag{12}$$

Proof of Lemma 1. As done in the proof of Lemma 2 in [22], for any $x > 0$ and any $z \in (0, y_*)$, we have that

$$\begin{aligned}\mathbb{P}(XY > x) &= \mathbb{P}(XY > x, Y \in (0,z]) + \mathbb{P}(XY > x, Y \in (z, y_*)) + \mathbb{P}(XY > x, Y = y_*) \\ &=: K_1 + K_2 + K_3.\end{aligned} \tag{13}$$

For K_3, by relation (7), we have that

$$\begin{aligned} K_3 &= \mathbb{P}(XY > x \mid Y = y_*)\mathbb{P}(Y = y_*) \\ &= \mathbb{P}(X > xy_*^{-1} \mid Y = y_*)\mathbb{P}(Y = y_*) \\ &\sim \mathbb{P}(X > xy_*^{-1})h(y_*)\mathbb{P}(Y = y_*). \end{aligned} \tag{14}$$

Again, by relation (7) and $\mathbb{E}h(Y) = 1 < \infty$, we have that

$$\begin{aligned} \lim_{z\uparrow y_*}\limsup_{x\to\infty} \frac{K_2}{\mathbb{P}(X > xy_*^{-1})} &= \lim_{z\uparrow y_*}\limsup_{x\to\infty} \int_z^{y_*} \frac{\mathbb{P}(X > xy^{-1} \mid Y = y)\mathbb{P}(Y \in \mathrm{d}y)}{\mathbb{P}(X > xy_*^{-1})} \\ &= \lim_{z\uparrow y_*}\limsup_{x\to\infty} \int_z^{y_*} \frac{\mathbb{P}(X > xy^{-1})h(y)\mathbb{P}(Y \in \mathrm{d}y)}{\mathbb{P}(X > xy_*^{-1})} \\ &\le \lim_{z\uparrow y_*} \mathbb{E}h(Y)1_{\{Y\in(z,y_*)\}} = 0. \end{aligned} \tag{15}$$

By $F \in \mathcal{R}_{-\infty}$, we have that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(XY > x, Y \in (0,z])}{\mathbb{P}(X > xy_*^{-1})} \le \limsup_{x\to\infty} \frac{\mathbb{P}(X > xz^{-1})}{\mathbb{P}(X > xy_*^{-1})} = 0. \tag{16}$$

Plugging relations (15)–(16) into (13), we can derive the desired relation (12). This ends the proof of Lemma 1. □

Proof of Theorem 1. We firstly prove relation (8) in claim (1). Denote by

$$T_0 = 0,\ \ T_n \overset{\mathrm{d}}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j,\ n \ge 1,$$

where $\overset{\mathrm{d}}{=}$ represents equality in distribution. Clearly, $T_n \overset{\mathrm{d}}{=} S_n,\ n \ge 1$; thus, in order to prove relation (8), we only need to verify the relation

$$\mathbb{P}(T_n > x) \sim \frac{p_* h(y_*)\mathbb{E}(e^{\gamma T_{n-1}})}{y_*^{\alpha-1}} l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}}. \tag{17}$$

Clearly, by relations $\mathbb{P}(T_1 > x) = \mathbb{P}(X_1Y_1 > x)$, relations (12) and (3), (17) hold for $n = 1$, which implies that $\mathbb{E}(e^{\gamma T_1}) < \infty$ and $F_{T_1} \in \mathcal{L}(\gamma/y_*)$. Now, we inductively assume that relation (17) holds for $n = m$ for some integer $m \ge 1$, which implies that $\mathbb{E}(e^{\gamma T_m}) < \infty$ and $F_{T_m} \in \mathcal{L}(\gamma/y_*)$. We aim to show that relation (17) holds for $n = m + 1$. Note that the sequence $\{T_n,\ n \ge 0\}$ satisfies the stochastic equation

$$T_0 = 0,\ \ T_n \overset{\mathrm{d}}{=} (T_{n-1} + X_n)Y_n,\ n \ge 1. \tag{18}$$

We divide the tail probability $\mathbb{P}(T_m + X_{m+1} > x)$ into three parts as

$$\begin{aligned}\mathbb{P}(T_m + X_{m+1} > x) &= \textstyle\sum_{i=1}^{3} \mathbb{P}(T_m + X_{m+1} > x, \Omega_i) \\ &=: I_1 + I_2 + I_3,\end{aligned} \tag{19}$$

where the events $\Omega_1 = (T_m > 0, X_{m+1} > 0)$, $\Omega_2 = (T_m > 0, X_{m+1} \leq 0)$ and $\Omega_3 = (T_m \leq 0, X_{m+1} > 0)$. We firstly deal with I_1. For any $0 < \epsilon < 1$ such that $0 < (1+\epsilon)y_* < 1$, we have that

$$\begin{aligned} I_1 &= \int_0^\infty \mathbb{P}(X_{m+1} > x-u)\mathbb{P}(T_m \in \mathrm{d}u) \\ &= \left(\int_0^{\frac{x}{1+\epsilon}} + \int_{\frac{x}{1+\epsilon}}^\infty\right)\mathbb{P}(X_{m+1} > x-u)\mathbb{P}(T_m \in \mathrm{d}u) \\ &= I_{11} + I_{12}. \end{aligned} \tag{20}$$

By Theorem 1.5.6 (i) in [23], for any $\delta > 0$ and sufficiently large x, we have that

$$\begin{aligned} \frac{l(x-u)(x-u)^{\alpha-1}}{l(x)x^{\alpha-1}} &\leq 2\Big(\frac{x-u}{x}\Big)^{\alpha-1-\delta} \\ &\leq 2\Big(\Big(\frac{\epsilon}{1+\epsilon}\Big)^{\alpha-1-\delta} \vee 1\Big) \end{aligned}$$

with $0 \leq u \leq x/(1+\epsilon)$. Since $l(\cdot)$ is a slowly varying function, according to the dominated convergence theorem, we have that

$$\begin{aligned} I_{11} &\sim \textstyle\int_0^{\frac{x}{1+\epsilon}} l(x-u)(x-u)^{\alpha-1}e^{-\gamma(x-u)}\mathbb{P}(T_m \in \mathrm{d}u) \\ &\sim l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\mathbf{1}_{\{T_m>0\}}\big). \end{aligned} \tag{21}$$

Again by the slow variety of $l(\cdot)$ and the the induction assumption, it holds that

$$\begin{aligned} I_{12} &\leq \mathbb{P}(T_m > \tfrac{x}{1+\epsilon}) \\ &\sim \tfrac{p_* h(y_*)\mathbb{E}\left(e^{\gamma T_{m-1}}\right)}{y_*^{\alpha-1}(1+\epsilon)^{\alpha-1}} l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*(1+\epsilon)}}. \end{aligned} \tag{22}$$

Plugging relations (21) and (22) into relation (21), by $0 < (1+\epsilon)y_* < 1$, we obtain that

$$I_1 \sim l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\mathbf{1}_{\{T_m>0\}}\big). \tag{23}$$

We next deal with I_3. According to relation (3), $F \in \mathcal{L}(\gamma)$ and the dominated convergence theorem, we have that

$$\begin{aligned} I_3 &= \int_{-\infty}^0 \overline{F}(x-u)\mathbb{P}(T_m \in \mathrm{d}u) \\ &\sim \overline{F}(x)\int_{-\infty}^0 e^{\gamma u}\mathbb{P}(T_m \in \mathrm{d}u) \\ &\sim l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\mathbf{1}_{\{T_m \leq 0\}}\big). \end{aligned} \tag{24}$$

As for I_2, by the induction assumption we have that

$$\begin{aligned} I_2 &\leq \mathbb{P}(T_m > x) \\ &\sim \tfrac{p_* h(y_*)\mathbb{E}\left(e^{\gamma T_{m-1}}\right)}{y_*^{\alpha-1}} l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}}. \end{aligned} \tag{25}$$

Thus, by noting $0 < y_* < 1$, we derive from relations (19) and (23)–(25) that

$$\mathbb{P}(T_m + X_{m+1} > x) \sim l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\big). \tag{26}$$

Now, we shall show that if X_{m+1} and Y_{m+1} are dependent according to relation (7) holding uniformly for all $y \in (0, y_*]$, then $T_m + X_{m+1}$ and Y_{m+1} follow the same dependence structure. Clearly, for all $y \in (0, y_*]$, since (X_{m+1}, Y_{m+1}) is independent of T_m, we have that

$$
\begin{aligned}
lll\mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y) &= \int_{-\infty}^{\infty} \mathbb{P}(X_{m+1} > x - u \mid Y_{m+1} = y)\mathbb{P}(T_m \in \mathrm{d}u) \\
&= \left(\int_{-\infty}^{x-a(x)} + \int_{x-a(x)}^{\infty} \right) \mathbb{P}(X_{m+1} > x - u \mid Y_{m+1} = y)\mathbb{P}(T_m \in \mathrm{d}u) \\
&=: L_1 + L_2,
\end{aligned} \tag{27}
$$

where $a(x) < x$ is any infinitely increasing function. For L_1, by relations (7) and (26), it holds uniformly for all $y \in (0, y_*]$ that

$$
\begin{aligned}
L_1 &\sim \int_{-\infty}^{x-a(x)} \mathbb{P}(X_{m+1} > x - u)h(y)\mathbb{P}(T_m \in \mathrm{d}u) \\
&\leq h(y)\mathbb{P}(T_m + X_{m+1} > x) \\
&\sim h(y)l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\big).
\end{aligned} \tag{28}
$$

As for L_2, for the above sufficiently small $0 < \epsilon < 1$ satisfying $0 < (1+\epsilon)y_* < 1$, choose $a(x) = \epsilon(1+\epsilon)^{-1}x$. Then, by the induction assumption and the slow variety of $l(\cdot)$, we have that

$$
\begin{aligned}
L_2 &\leq \mathbb{P}\Big(T_m > \frac{x}{1+\epsilon}\Big) \\
&\sim \frac{p_* h(y_*)\mathbb{E}(e^{\gamma T_{m-1}})}{(y_*(1+\epsilon))^{\alpha-1}} l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*(1+\epsilon)}}.
\end{aligned} \tag{29}
$$

Plugging relations (29) and (29) into relation (27), together with $\inf_{y\in(0,y_*]} h(y) > 0$, yields that uniformly for all $y \in (0, y_*]$,

$$
\mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y) \prec h(y)l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\big). \tag{30}
$$

For the lower bound, by relations (26)–(29), we have that uniformly for all $y \in (0, y_*]$,

$$
\begin{aligned}
\mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y) &\geq L_1 \\
&\sim h(y)\mathbb{P}\Big(T_m + X_{m+1} > x, T_m \leq \frac{x}{1+\epsilon}\Big) \\
&\geq h(y)\Big(\mathbb{P}(T_m + X_{m+1} > x) - \mathbb{P}\Big(T_m > \frac{x}{1+\epsilon}\Big)\Big) \\
&\sim h(y)l(x)x^{\alpha-1}e^{-\gamma x}\mathbb{E}\big(e^{\gamma T_m}\big).
\end{aligned} \tag{31}
$$

Thus, by relations (26), (30) and (31), we derive that uniformly, for all $y \in (0, y_*]$,

$$
\mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y) \sim \mathbb{P}(T_m + X_{m+1} > x)h(y), \tag{32}
$$

which means that $T_m + X_{m+1}$ and Y_{m+1} follow the dependence structure defined in relation (7).

Therefore, according to Lemma 1, we can obtain from relation (26) that

$$
\begin{aligned}
\mathbb{P}(T_{m+1} > x) &\sim \mathbb{P}(T_m + X_{m+1} > xy_*^{-1})h(y_*)\mathbb{P}(Y_{m+1} = y_*) \\
&\sim \frac{p_* h(y_*)\mathbb{E}\big(e^{\gamma T_m}\big)}{y_*^{\alpha-1}} l(x)x^{\alpha-1}e^{-\frac{\gamma x}{y_*}},
\end{aligned} \tag{33}
$$

which also implies that $\mathbb{E}\big(e^{\gamma T_{m+1}}\big) < \infty$ and $F_{T_{m+1}} \in \mathcal{L}(\gamma/y_*)$.

We next prove relation (9). Introduce a Markov chain $\{W_n,\ n \geq 1\}$ satisfying

$$W_0 = 0,\ W_n = (W_{n-1} + X_n)_+ Y_n,\ n \geq 1. \tag{34}$$

By using the identity

$$M_n \stackrel{\mathrm{d}}{=} \bigvee_{k=0}^{n} T_k,\ n \geq 1,$$

and by Theorem 2.1 in [15], similarly to relation (18), we have that

$$M_n \stackrel{\mathrm{d}}{=} W_n,\ n \geq 1.$$

Obviously, the relation (34) holds for $n = 1$. Following the same line of the proof of relation (8), we obtain that

$$\mathbb{P}(W_n > x) \sim \frac{p_* h(y_*) \mathbb{E}\left(e^{\gamma W_{n-1}}\right)}{y_*^{\alpha-1}} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}},$$

which coincides with relation (9). Therefore, we complete the proof of claim (1).

For claim (2), according to the above proof, it suffices to show

$$\mathbb{P}(T_n > x) \sim \frac{p_*^n h(y_*) \gamma^{n-1} (\Gamma(\alpha))^n}{y_*^{(\alpha-1)n} \Gamma(n\alpha)} (l(x))^n x^{n\alpha-1} e^{-\gamma x}. \tag{35}$$

We proceed again by induction on n. Trivially, relation (35) holds for $n = 1$ by Lemma 1. Assume that relation (35) holds for $n = m$ for some integer $m \geq 1$, which implies that $F_{T_m} \in \mathcal{L}(\gamma)$. We aim to prove that relation (35) holds for $n = m + 1$. As done in relation (19), we still divide the tail probability $\mathbb{P}(T_m + X_{m+1} > x)$ into three parts, denoted by I_1, I_2, and I_3, respectively. Starting from I_1, we construct two independent positive conditional r.v.s $X_{m+1}^c = (X_{m+1} \mid X_{m+1} > 0)$ and $T_m^c = (T_m \mid T_m > 0)$, whose tail distributions, by relation (3) and the induction assumption satisfy

$$\mathbb{P}(X_{m+1}^c > x) \sim \frac{1}{\overline{F}(0)} l(x) x^{\alpha-1} e^{-\gamma x},$$

and

$$\mathbb{P}(T_m^c > x) \sim \frac{p_*^m h(y_*) \gamma^{m-1} (\Gamma(\alpha))^m}{\mathbb{P}(T_m > 0) y_*^{(\alpha-1)m} \Gamma(m\alpha)} (l(x))^m x^{m\alpha-1} e^{-\gamma x}.$$

Similarly to the proof of relation (4.13) in [17], we have that

$$I_1 \sim \frac{p_*^m h(y_*) \gamma^m (\Gamma(\alpha))^{m+1}}{y_*^{(\alpha-1)m} \Gamma((m+1)\alpha)} (l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x}. \tag{36}$$

As for I_2, by the induction assumption, we have that

$$\begin{aligned} I_2 &\leq \mathbb{P}(T_m > x) \\ &\sim \frac{p_*^m h(y_*) \gamma^{m-1} (\Gamma(\alpha))^m}{y_*^{(\alpha-1)m} \Gamma(m\alpha)} (l(x))^m x^{m\alpha-1} e^{-\gamma x} \\ &= o(1) (l(x))^{m+1} x^{(m+1)\alpha-1} e^{-\gamma x}. \end{aligned} \tag{37}$$

Similarly, by relation (3), we have that

$$\begin{aligned} I_3 &\leq \overline{F}(x) \\ &\sim l(x)x^{\alpha-1}e^{-\gamma x} \\ &= o(1)(l(x))^{m+1}x^{(m+1)\alpha-1}e^{-\gamma x}. \end{aligned} \tag{38}$$

Then, it follows from relations (19) and (36)–(38) that

$$\mathbb{P}(T_m + X_{m+1} > x) \sim \frac{p_*^m h(y_*)\gamma^m(\Gamma(\alpha))^{m+1}}{y_*^{(\alpha-1)m}\Gamma((m+1)\alpha)}(l(x))^{m+1}x^{(m+1)\alpha-1}e^{-\gamma x}. \tag{39}$$

As done in the proof of claim (1), we next show that $T_m + X_{m+1}$ and Y_{m+1} are dependent according to relation (7) uniformly for all $y \in (0, y_*]$ with $y_* = 1$. As relation (27), we still divide the tail probability $\mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y)$ into two parts L_1 and L_2. On one hand, similarly to relation (29), by relation (39), we have that uniformly for all $y \in (0, y_*]$,

$$\begin{aligned} L_1 &\prec h(y)\mathbb{P}(T_m + X_{m+1} > x) \\ &\sim h(y)\frac{p_*^m h(y_*)\gamma^m(\Gamma(\alpha))^{m+1}}{y_*^{(\alpha-1)m}\Gamma((m+1)\alpha)}(l(x))^{m+1}x^{(m+1)\alpha-1}e^{-\gamma x}. \end{aligned} \tag{40}$$

As for L_2, choose $a(x) = \delta \ln x$ with $0 < \gamma\delta < \alpha$, where $a(x)$ is defined in relation (27). Then, by the induction assumption, similarly to relation (29), we have that

$$\begin{aligned} L_2 &\leq \mathbb{P}(T_m > x - \delta \ln x) \\ &\sim \frac{p_*^m h(y_*)\gamma^{m-1}(\Gamma(\alpha))^m}{y_*^{(\alpha-1)m}\Gamma(m\alpha)}(l(x))^m x^{m\alpha-1}e^{-\gamma(x-\delta \ln x)} \\ &= \frac{p_*^m h(y_*)\gamma^{m-1}(\Gamma(\alpha))^m}{y_*^{(\alpha-1)m}\Gamma(m\alpha)}(l(x))^m x^{m\alpha-1+\gamma\delta}e^{-\gamma x} \\ &= o(1)(l(x))^{m+1}x^{(m+1)\alpha-1}e^{-\gamma x}. \end{aligned} \tag{41}$$

Plugging relations (40) and (41) into relation (27), together with $\inf_{y\in(0,y_*]} h(y) > 0$, leads to

$$\mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y) \prec h(y)\mathbb{P}(T_m + X_{m+1} > x), \tag{42}$$

holding uniformly for all $y \in (0, y_*]$. On the other hand, similarly to relation (31), by relations (39) and (41), we have that uniformly for all $y \in (0, y_*]$,

$$\begin{aligned} \mathbb{P}(T_m + X_{m+1} > x \mid Y_{m+1} = y) &\succ h(y)\big(\mathbb{P}(T_m + X_{m+1} > x) - \mathbb{P}(T_m > x - \delta \ln x)\big) \\ &\succ h(y)\mathbb{P}(T_m + X_{m+1} > x). \end{aligned} \tag{43}$$

Relations (42) and (43) mean that $T_m + X_{m+1}$ and Y_{m+1} follow the dependence (7) holding uniformly for all $y \in (0, y_*]$.

Therefore, combining Lemma 1, we conclude that the desired relation (35) holds for $n = m + 1$. This completes the proof of Theorem 1 (2). □

Remark 1. *Note that [17] considered the three cases of* $0 < y_* < 1$, $y_* = 1$ *and* $y_* > 1$, *respectively, under the conditions that* X *and* Y *are independent or follow the bivariate Sarmanov distribution. However, our Theorem 1 excludes the case* $y_* > 1$ *because, when using the mathematical induction to estimate* $\mathbb{P}(T_n > x)$, *we find no way to prove* $T_m + X_{m+1}$ *and* Y_{m+1} *follow the dependence structure (7); hence, Lemma 1 can not be used.*

Proof of Theorem 2. We firstly prove the asymptotic relation (11). For each $n \geq 0$, define nonnegative r.v.s

$$\xi_n = \sum_{i=n+1}^{\infty} y_*^i X_{i+}. \tag{44}$$

Thus, by relation (2), for all $n \geq 1$, we have that

$$0 \leq M_n \leq \xi_0 - \xi_n. \tag{45}$$

By $\ln y_* < 0$, $\mathbb{E}\ln(X_1 \vee 1) < \infty$ and Theorem 1.6 in [2], we obtain that M_n converges a.s. to a limit M_∞ as $n \to \infty$. Then, in order to verify relation (11), we need to prove that $\mathbb{E}\big(e^{\gamma M_\infty}\big) < \infty$. As done in [17], we introduce a nonnegative r.v. Z, which is independent of $\{(X_i, Y_i),\ i \geq 1\}$, with the tail distribution

$$\overline{F_Z}(x) \sim x^{2\alpha-1} e^{-\frac{\gamma x}{y_*}}.$$

We further construct a nonnegative conditional r.v. $Z^c = (Z \mid Z > x_0)$. It has been proved by [17] that, for every $n \geq 1$, $M_n \overset{\mathrm{d}}{\leq} Z^c$, where the symbol $\overset{\mathrm{d}}{\leq}$ denotes 'stochastically not larger than', that is, for all $x \geq 0$,

$$\mathbb{P}(M_n > x) \leq \mathbb{P}(Z^c > x).$$

Letting $n \to \infty$, we have that

$$M_\infty \leq \xi_0 \overset{\mathrm{d}}{\leq} Z^c,$$

which, together with $\overline{F_{Z^c}}(x) = \overline{F_Z}(x)/\overline{F_Z}(x_0)$ for all $x \geq x_0$ and $\mathbb{E}\big(e^{\gamma Z}\big) < \infty$, yields

$$\mathbb{E}\big(e^{\gamma M_\infty}\big) \leq \mathbb{E}\big(e^{\gamma \xi_0}\big) < \infty. \tag{46}$$

According to the same method of [17], by relation (46), the dominated convergence theorem and the Jensen's inequality, for any $\epsilon > 0$ with $0 < (1+\epsilon)y_* < 1$ and arbitrarily fixed $\bar{y} \in (\sqrt[3]{y_*}, 1)$ (implying $y_* < \bar{y}^3 < 1$), we can choose a sufficiently large integer $n_0 \geq 3$ such that

$$\left| \mathbb{E}\big(e^{\gamma M_{n_0-1}}\big) - \mathbb{E}\big(e^{\gamma M_\infty}\big) \right| \leq \varepsilon, \tag{47}$$

$$\mathbb{E}\big(e^{\gamma \xi_{n_0-1}}\big) \leq 1 + \epsilon, \tag{48}$$

and

$$\sum_{i=n_0+1}^{\infty} \bar{y}^i < 1. \tag{49}$$

For the upper bound of relation (11),

$$\begin{aligned} \mathbb{P}(M_\infty > x) &\leq \mathbb{P}(M_{n_0} + \xi_{n_0} > x) \\ &= \left(\int_0^{\frac{x}{1+\epsilon}} + \int_{\frac{x}{1+\epsilon}}^{\infty} \right) \mathbb{P}(M_{n_0} > x - u)\mathbb{P}(\xi_{n_0} \in \mathrm{d}u) \\ &=: J_1 + J_2. \end{aligned} \tag{50}$$

By relation (9) in Theorem 1, relations (47), (48) and the dominated convergence theorem, we have that

$$\begin{aligned} J_1 &\sim \int_0^{\frac{x}{1+\epsilon}} p_* h(y_*) \mathbb{E}(e^{\gamma M_{n_0 - 1}}) y_*^{-(\alpha-1)} l(x-u)(x-u)^{\alpha-1} e^{-\frac{\gamma(x-u)}{y_*}} \mathbb{P}(\xi_{n_0} \in \mathrm{d}u) \\ &\leq (1+\epsilon) p_* h(y_*) \mathbb{E}(e^{\gamma M_\infty}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}} \int_0^{\frac{x}{1+\epsilon}} \frac{l(x-u)(x-u)^{\alpha-1}}{l(x)x^{\alpha-1}} e^{\frac{\gamma u}{y_*}} \mathbb{P}(\xi_{n_0} \in \mathrm{d}u) \\ &\sim (1+\epsilon) p_* h(y_*) \mathbb{E}(e^{\gamma M_\infty}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}} \mathbb{E}(e^{\gamma \xi_{n_0 - 1}}) \\ &\leq (1+\epsilon)^2 p_* h(y_*) \mathbb{E}(e^{\gamma M_\infty}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}. \end{aligned} \tag{51}$$

As for J_2, it is dealt with along the same line of that in [17], we have that

$$J_2 = o(1) l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}. \tag{52}$$

Plugging relations (51) and (52) into relation (51), we obtain that

$$\mathbb{P}(M_\infty > x) \prec (1+\epsilon)^2 p_* h(y_*) \mathbb{E}(e^{\gamma M_\infty}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}.$$

For the lower bound of relation (11), we derive from relation (9) in Theorem 1 and relation (47) that

$$\begin{aligned} \mathbb{P}(M_\infty > x) &\geq \mathbb{P}(M_{n_0} > x) \\ &\sim p_* h(y_*) \mathbb{E}(e^{\gamma M_{n_0 - 1}}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}} \\ &\geq (1-\epsilon) p_* h(y_*) \mathbb{E}(e^{\gamma M_\infty}) y_*^{-(\alpha-1)} l(x) x^{\alpha-1} e^{-\frac{\gamma x}{y_*}}. \end{aligned}$$

Therefore, the desired relation (11) can be obtained by the arbitrariness of $\epsilon > 0$.

The second part of Theorem 2 can be dealt with the standard argument, which was also shown in [17]. This ends the proof of Theorem 2. □

4. Simulation Study

In this section, we conduct a simulation study through the software MATLAB R2014a (The MathWorks, Inc., Natick, MA, USA) to verify the asymptotic relation (10) for the finite-time ruin probability in the main theoretical result Theorem 1.

Assume that $\{(X_i, Y_i),\ i \geq 1\}$ is a sequence of i.i.d. random vectors with a generic pair (X, Y), where the insurance risk X follows a common exponential distribution with parameters $\lambda > 0$, $\mu \in \mathbb{R}$

$$F(x; \lambda) = 1 - e^{-\lambda(x-\mu)}, \quad x \geq \mu,$$

which satisfies the Gamma-tailed distribution defined in relation (3). Note that if $\alpha = 1$, then relation (3) is the tail distribution of exponential distribution. The financial risk Y follows a common discrete distribution:

Y	0.2	0.6	1
$\mathbb{P}(Y = y)$	0.3	0.4	0.3

We assume X and Y follow a bivariate FGM distribution

$$\mathbb{P}(X \leq x, Y \leq y) = F(x)G(y)(1 + \delta \overline{F}(x)\overline{G}(y)),$$

with parameter $|\delta| \leq 1$. Note that the bivariate FGM distribution is included by the dependence structure defined in relation (7).

The following algorithm is used to generate the component r.v.s X and Y of i.i.d. random vectors $\{(X_i, Y_i),\ i \geq 1\}$, fulfilling the bivariate FGM distribution:

Step a: Generate two i.i.d. r.v.s u and v following the uniform distribution on $(0,1)$;

Step b: Set $X = -\log(1-u)/\lambda + \mu$;

Step c: Set $w = \left(2\delta u - \delta - 1 + ((\delta - 2\delta u + 1)^2 + 4(2\delta u - \delta)v)^{\frac{1}{2}}\right) \Big/ 2(2\delta u - \delta)$, if $w \leq 0.3$, then $Y = 0.2$; if $0.3 < w \leq 0.7$, then $Y = 0.6$; if $w > 0.7$, then $Y = 1$.

Thus, the generated (X, Y) returns the outcome of two dependent r.v.s. fulling the bivariate FGM distribution.

We use the CMC method to perform the simulation. The computation procedure of the estimation of the theoretical finite-time ruin probability $\psi(x; n)$ is listed as the following:

Step 1: Assign a value for the variate x and set $m = 0$;

Step 2: Generate the i.i.d. r.v.s random vectors $\{(X_i, Y_i),\ i \geq 1\}$ satisfying the certain dependence structure according to the above algorithm;

Step 3: Calculate the vector $(Y_1, Y_1Y_2, \ldots, \prod_{j=1}^{n} Y_j)^{\mathrm{T}}$. Then, S_n is equal to the product of the two vectors $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_1Y_2, \ldots, \prod_{j=1}^{n} Y_j)^{\mathrm{T}}$;

Step 4: Select the maximum value from $S_1, \ldots, S_n$ and denote it by M_n, and compare M_n with x: if $M_n > x$; then, $m = m + 1$;

Step 5: Repeat step 2 to step 4 for N times;

Step 6: Calculate $\psi_1(x; n) = m/N$ as the estimate of $\psi(x; n)$.

Step 7: Repeat step 1 to step 6 for l times, and set the mean of all $\psi_1(x; n)$ as the final estimate.

All parameters are set as: $\lambda = 0.1$, $\mu = -16$, $\delta = 1$, $n = 8$, $N = 10,000,000$ and $l = 20$. We set the initial asset x from 100 to 150 and compare the simulated estimate values with the asymptotic values of the finite-time ruin probability in the following Table 1.

Table 1. Comparison between the simulated estimate values and the asymptotic values in Theorems 1-(2) for $N = 1.0 \times 10^7$.

x	Simulated Estimate Values	Asymptotic Values
100	2.06×10^{-5} (1.20451×10^{-6})	1.18×10^{-5}
110	1.16×10^{-5} (3.06396×10^{-7})	8.47×10^{-6}
120	4.85×10^{-6} (2.51962×10^{-7})	5.73×10^{-6}
130	2.05×10^{-6} (1.50271×10^{-7})	3.69×10^{-6}
140	1.08×10^{-6} (1.28344×10^{-7})	2.28×10^{-6}
150	3.92×10^{-7} (2.11470×10^{-7})	1.36×10^{-6}

The standard error of each estimate computed via the CMC method is presented in the bracket behind the estimate. Without surprise, the larger the initial wealth x is, the smaller both the simulated estimate values and the asymptotic values of the ruin probability become, but the more fluctuation their ratio exhibits, the less effective the estimates are. In fact, this is due to the poor performance of the CMC method, which requires a sufficiently large sample size to meet the demands of high accuracy. In order to eliminate the influence of large initial wealth, we repeat the simulation with the sample size N increasing from $10,000,000$ up to $15,000,000$. A significant improvement is observed. See Table 2 below.

Table 2. Comparison between the simulated estimate values and the asymptotic values in Theorem 1-(2) for $N = 1.5 \times 10^7$.

x	Simulated Estimate Values	Asymptotic Values
100	2.03×10^{-5} (6.75224×10^{-7})	1.18×10^{-5}
110	1.23×10^{-5} (3.35691×10^{-7})	8.47×10^{-6}
120	5.04×10^{-6} (2.35759×10^{-7})	5.73×10^{-6}
130	2.27×10^{-6} (1.04678×10^{-7})	3.69×10^{-6}
140	1.45×10^{-6} (6.50245×10^{-8})	2.28×10^{-6}
150	6.98×10^{-7} (3.32866×10^{-8})	1.36×10^{-6}

5. Conclusions

In this paper, we study the asymptotics for finite-time and infinite-time ruin probabilities. We conduct our study in a discrete-time risk model, in which the insurance risks have a common Gamma-like tail the financial risks all have a finite upper endpoint, but a conditionally tailed asymptotic dependence structure exists between each pair of them. We demonstrate through simulations that the approximate relationships obtained in our main results are reasonable.

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