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Bond and CDS Pricing via the Stochastic Recovery Black-Cox Model

Albert Cohen [1,*] and Nick Costanzino [2,*]

1 Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
2 Quantitative Analytics, Barclays Capital, 745 7th Ave, New York, NY 10019, USA
* Correspondence: albert@math.msu.edu (A.C.); Nick.Costanzino@barclayscapital.com (N.C.)

Academic Editor: Mogens Steffensen
Received: 18 January 2017; Accepted: 10 April 2017; Published: 19 April 2017

Abstract: Building on recent work incorporating recovery risk into structural models by Cohen & Costanzino (2015), we consider the Black-Cox model with an added recovery risk driver. The recovery risk driver arises naturally in the context of imperfect information implicit in the structural framework. This leads to a two-factor structural model we call the Stochastic Recovery Black-Cox model, whereby the asset risk driver A_t defines the default trigger and the recovery risk driver R_t defines the amount recovered in the event of default. We then price zero-coupon bonds and credit default swaps under the Stochastic Recovery Black-Cox model. Finally, we compare our results with the classic Black-Cox model, give explicit expressions for the recovery risk premium in the Stochastic Recovery Black-Cox model, and detail how the introduction of separate but correlated risk drivers leads to a decoupling of the default and recovery risk premiums in the credit spread. We conclude this work by computing the effect of adding coupons that are paid continuously until default, and price perpetual (consol bonds) in our two-factor firm value model, extending calculations in the seminal paper by Leland (1994).

1. Background and Motivation

Most legacy credit models assume that recovery is a constant. It is well known, however, that recovery rates are not constant and indeed are correlated to a variety of risk drivers, including default and interest rates. For example, in their study on real-world recovery rates, Altman et al. [1] show that realized recovery rates are inversely proportional to realized default rates. This phenomena has been successfully incorporated into recent economic capital models (c.f. [2–7]) but still hasn't enjoyed mainstream adoption in pricing models. One notable exception has been in CDO pricing, where stochastic recovery has been added to many legacy models. This was necessary after the most recent credit-liquidity crisis, whereby for a period of time it was not possible to calibrate standard CDO models to the complete set of CDX.IG and ITRAXX.IG tranche quotes. The inability to calibrate the standard CDO models has been attributed to the assumption that recovery at default is deterministic, and does not depend on time or state variables. This has led to adding stochastic recovery to CDO models (c.f. [8–13]), which is now standard practice in industry.

However, stochastic recovery has not yet received equally widespread interest and acceptance for other credit products such as defaultable bonds, credit default swaps and credit linked notes. In fact, there are a dearth of pricing formulas for credit products where recovery is modeled explicitly. This has become commonplace even though recovery is clearly a key component in the determination of credit spreads [14]. One recent approach to stochastic recovery modeling can be found in [15]. Take for example bond pricing, where the price of a zero-coupon defaultable bond is given by the risk-neutral expected present value of the payoff Π_τ (where τ is the default time):

$$\Pi_\tau := N\mathbb{1}_{\{\tau>T\}} + R_\tau \mathbb{1}_{\{\tau\le T\}}. \quad (1)$$

The first term describes receiving the full repayment of the notional N at maturity T in the event of no default (i.e., $\tau > T$), while the second term describes receiving a recovery R_τ in the event of default before maturity (i.e., $\tau \le T$). The price of a zero-coupon defaultable bond is therefore given by

$$\begin{aligned} B_{t,T} &= \widetilde{\mathbb{E}}_t[D(t,\tau\wedge T)\Pi_\tau] \\ &= ND(t,T)\widetilde{\mathbb{P}}_t[\tau > T] + \widetilde{\mathbb{E}}_t[D(t,\tau)R_\tau \mathbb{1}_{\{\tau\le T\}}] \end{aligned} \tag{2}$$

where $\wedge$ is the min operator $a \wedge b := \min\{a,b\}$, $\widetilde{\mathbb{P}}_t$ (resp. $\widetilde{\mathbb{E}}_t$) is the risk-neutral probability of default (resp. risk-neutral expectation) conditioned on information about the default and recovery process known at t, and $D(t,s)$ is the present value of a dollar at time t received at time s. To evaluate (2) one needs a model for D, default time τ, and recovery at default R_τ. As noted in [15], it is common in structural models such as the Merton and Black-Cox models to make the simplifying assumption that a single risk factor A_t representing the evolution of the underlying firm assets determines both the default time τ and recovery at default R_τ. Symbolically, for a fixed $f \in [0,1]$ and a filtered probability space $\left(\Omega, \{\mathcal{F}_t^A\}_{t\ge 0}, \mathcal{F}, \widetilde{\mathbb{P}}\right)$,

$$\begin{aligned} \{\tau \le t\} &\in \mathcal{F}_t^A, \quad \forall t \ge 0 \\ R_\tau &:= f A_\tau \end{aligned} \tag{3}$$

where the natural filtration $\mathcal{F}_t^A$ is the minimal $\sigma-$algebra containing $\sigma(A_u)$ for all $u \in [0,t]$. Under this assumption, the price of a defaultable zero-coupon bond simplifies to

$$B_{t,T} = ND(t,T)\widetilde{\mathbb{P}}_t[\tau > T] + f\widetilde{\mathbb{E}}_t[D(t,\tau)A_\tau \mathbb{1}_{\{\tau\le T\}}] \tag{4}$$

which can be computed using only the single risk factor A_t.

The lack of recovery modeling is even more explicit in the case of pricing Credit Default Swaps. For instance, the value of the default protection leg of a CDS per unit notional is given by

$$V_{t,T}^{\text{Protection}} = \widetilde{\mathbb{E}}_t\left[D(t,\tau)(1-R_\tau)\mathbb{1}_{\{\tau\le T\}}\right] \tag{5}$$

which is the expected risk-neutral discounted recovery amount given default. However, it is common to make the simplifying assumption that recovery is a constant so that $R_\tau \equiv R$ in (5) and the recovery term can be taken out of the expectation to obtain the simplified expression

$$V_{t,T}^{\text{Protection}} = (1-R)\widetilde{\mathbb{E}}_t\left[D(t,\tau)\mathbb{1}_{\{\tau\le T\}}\right]. \tag{6}$$

Under this simplifying assumption, which is standard in typical CDS pricing (c.f. [16–18] etc.), one simply needs a model for default τ and need not concern themselves with modeling recovery in the event of default. In this paper, we explicitly model recovery and remove the constant recovery assumption in CDS pricing thereby valuing the protection leg of the CDS using (5) directly rather than (6).

As described by the above examples of bond and CDS pricing, lack of recovery modeling inherent in classical structural models such as Merton and Black-Cox stems from the fact that default and recovery are driven by the same process. This feature makes it impossible to disentangle the effects of default and recovery. For instance, in the Merton model [19], the default time is defined as

$$\tau_{\text{Merton}} := T\mathbb{1}_{\{A_T < N\}} + \infty\mathbb{1}_{\{A_T \ge N\}}. \tag{7}$$

This default time has the benefit that recovery at default is random in the sense that it is defined through the random variable A_T. However, a deficiency in the model is that default is only possible at the maturity T of the bond (i.e., $\tau_{\text{Merton}} \in \{T, \infty\}$). This is in direct violation of empirical evidence showing that firms default on their bonds at times before maturity as well. Black & Cox introduced their model [20] in order to remove this deficiency by allowing default to occur at times up to and including maturity (i.e., $\tau_{\text{BC}} \in \{[0,T],\infty\}$). The authors are able to achieve this by introducing an additional first passage default time τ_{K}, defined as the first time the assets A_t are less than the barrier K:

$$\tau_{\text{K}} := \arg\inf\{t \in [0,\infty) : A_t \le K\} \tag{8}$$

with the usual convention that $\inf\{\emptyset\} = +\infty$. The full Black-Cox default time τ_{BC} is then

$$\tau_{\text{BC}} \quad = \tau_{\text{K}} \mathbb{1}_{\{\tau_{\text{K}} \le T\}} + T \mathbb{1}_{\{\tau_{\text{K}} > T, A_T < N\}} + \infty \mathbb{1}_{\{\tau_{\text{K}} > T, A_T \ge N\}}. \tag{9}$$

However, the very same default mechanism in the Black-Cox model that allows for default before maturity forces recovery at default to be a constant. That is, (9) implies that for default before maturity, the asset value at default is equal to the predetermined constant barrier K (i.e., $A_{\tau_{\text{BC}}} = K$). Hence Black-Cox improves the default modeling of Merton at the expense of constraining recovery to a predetermined known constant K when $\tau_{\text{BC}} < T$.[1] The main reason for this is that the Black-Cox model (as in all one-factor structural models) intertwines default and recovery risks. In one-factor structural models, the same process that determines default, namely A_t, also determines recovery. This structure results in default and recovery rates being multiplicatively linked through the credit spread, which in turn makes the separation of default risk from recovery risk impossible. To disentangle the default risk from the recovery risk in a structural model, one needs to introduce a separate recovery risk driver. This leads us to the Stochastic Recovery Black-Cox model considered herein.

This paper is organized as follows. In Section 2, we review the classic Black-Cox (BC) model as a benchmark for our later results in Section 4 where we include recovery risk. In Section 3, we motivate the need for adding a recovery risk driver into the BC model and present our correlated asset-recovery model (49), which we compare with Moody's PD-LGD model presented in [4]. In Section 4, we define the Stochastic Recovery Black-Cox (SRBC) model and use it to price instruments based on credit quality. Specifically, in Sections 4.1 and 4.4, we price bonds and CDS with the SRBC model. The SRBC setting is essentially a two-factor Black-Cox model where the extra factor is recovery risk. We then compare the prices obtained from the two-factor SRBC model with the original Black-Cox model. In Section 5, we provide an algorithm to compute the market implied recovery rate as well as analytical formulas for the recovery risk premium. Finally, in Section 6, we investigate the addition of coupon payments to bond prices in the stochastic recovery structural model.

2. Review of Credit Risk and Pricing in the Black-Cox Model

In this section, we briefly review bond and CDS pricing in the classical Black-Cox framework. In the original Black-Cox model [20], the default boundary K was taken to be exponential in time, i.e., $K(t) = Ke^{-\beta(T-t)}$. However to simplify the exposition and results, we consider the case of a flat-boundary corresponding to $\beta = 0$, i.e., $K(t) \equiv K$, although the results can be trivially extended to the exponential boundary case as well. The results of this section will mainly serve as a benchmark for which to compare the results in Sections 4.1 and 4.4, where recovery risk is incorporated into Bond and CDS pricing via the Stochastic Recovery Black-Cox model described in Section 3.

We begin by listing the assumptions that form the Black-Cox model.

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[1] This constraint can be improved upon slightly by introducing a curved (time-dependent) boundary $\tau_{\text{K}} := \arg\inf\{t \in [0,T] : A_t \le K(t)\}$ for a suitably chosen $K(t)$. However, the fact remains there is still a single driver for both recovery and default.

Assumption 1. (Constant Rates) *The short rate is assumed to be a constant r so that the discount factor is given by*

$$D(t,s) = e^{-r(s-t)}. \tag{10}$$

Assumption 2. (Asset Dynamics) *The* $\widetilde{\mathbb{P}}_t$*-dynamics of the firms asset value* A_t *at time t are assumed to follow*

$$dA_t = rA_t dt + \sigma_A A_t dW_t^A. \tag{11}$$

where r, σ_A *are constant and* W_t^A *is a standard Brownian process.*

Assumption 3. (Default Time) *Default is defined as*

$$\tau_{\mathrm{BC}} = \tau_{\mathrm{K}} \mathbb{1}_{\{\tau_{\mathrm{K}} \le T\}} + T \mathbb{1}_{\{\tau_{\mathrm{K}} > T, A_T < N\}} + \infty \mathbb{1}_{\{\tau_{\mathrm{K}} > T, A_T \ge N\}}. \tag{12}$$

where τ_{BC} *is explicitly given in* (9).

Assumption 4. (Recovery Dynamics) *Recovery at default is given by the asset value at default,*

$$R_{\tau_{\mathrm{BC}}} = A_{\tau_{\mathrm{BC}}}. \tag{13}$$

The collection of Assumptions 1–4 form the Black-Cox model, and can be used to price defaultable bonds as well as other products such as CDS which we do in Sections 2.1 and 2.2 below.

Assumption 4 isn't often thought of as an explicit assumption of the Black-Cox model, since, as a one-factor model, this assumption is embedded in the default time. This is another example of the intertwining of default and recovery. Furthermore, notice in Assumption 4 that if $\tau_{\mathrm{BC}} < T$ then by definition $A_{\tau_{\mathrm{BC}}} \equiv K$ so recovery given default is constant and deterministic. The main focus of this paper is to remove Assumption 4 and allow for stochastic recovery at default by decoupling the asset risk driver from the recovery risk driver. This should not be confused with adding stochasticity to the default boundary (c.f. [21,22]) since single-factor models with stochastic boundary still cannot disentangle the credit puzzle, as again, the asset value determines both default and recovery amount at default.

The Black-Cox model can be used to price credit products and compute related credit risk measures. For instance, in the case of a zero-coupon bond the price of such a defaultable zero-coupon bond is then given by the risk-neutral pricing formula

$$\begin{aligned} B_{t,T} &= N\widetilde{\mathbb{E}}_t[D(t,T)\, \mathbb{1}_{\{\tau > T\}}] + \widetilde{\mathbb{E}}_t[D(t,\tau) R_\tau\, \mathbb{1}_{\{\tau \le T\}}] \\ &= ND(t,T)\, \widetilde{\mathbb{P}}_t[\tau > T] + \widetilde{\mathbb{E}}_t[D(t,\tau) R_\tau\, \mathbb{1}_{\{\tau \le T\}}]. \end{aligned} \tag{14}$$

Given the price of a defaultable zero-coupon bond via (14), we define the credit spread of the bond $B_{t,T}$ as the spread $S_{t,T}$ over the risk-free rate r which reprices the bond. Thus $S_{t,T}$ is defined implicitly by the solution to

$$B_{t,T} = Ne^{-(r+S_{t,T})(T-t)} \tag{15}$$

which returns the formula

$$S_{t,T} = \frac{1}{T-t} \ln\left(\frac{N}{B_{t,T}}\right) - r. \tag{16}$$

The risk-neutral probability of default $\widetilde{\mathrm{PD}}_{t,T}$ at time t is defined as the probability under the risk-neutral measure $\widetilde{\mathbb{P}}$ that the default event τ occurs at or before maturity T,

$$\widetilde{\mathrm{PD}}_{t,T} = \widetilde{\mathbb{P}}_t\left[\tau \leq T\right]. \tag{17}$$

The risk-neutral expected loss given default $\widetilde{\mathrm{LGD}}_{t,T}$ at time t is defined as

$$\widetilde{\mathrm{LGD}}_{t,T} = 1 - \frac{\widetilde{\mathbb{E}}_t[D(t,\tau)R_\tau \mid \tau \leq T]}{D(t,T)N} \tag{18}$$

The term structure of expected Recovery Given Default $\widetilde{R}_{t,T}$ can easily be inferred from the term structure of Loss Given Default (18) via the relation $\widetilde{\mathrm{LGD}}_{t,T} = 1 - \widetilde{\mathrm{R}}_{t,T}$.

The other credit product we consider is a Credit Default Swap (CDS). In a CDS, one party (the buyer) pays premiums to another party (the seller) to insure against default on a bond (c.f. [16–18]). Pricing consists of separately modeling the present value of the fixed premiums paid by the protection buyer, and the present value of the contingent default payment leg received by the buyer. The difference between the two is then the value of the CDS. If there is no upfront fee at initiation of the contract, then the premium P is given as the value that makes the contract worthless at initiation.

To be more precise, let T be the expiry of the CDS contract and let $\mathbb{T}_n := \{t = t_0, t_1, t_2, ...t_n = T\}$ be the premium payment dates. For $i = 1...n$ we define $\Delta t_i = t_i - t_{i-1}$ to be the time between payments. The premium leg of the transaction is then given by the expected present value of the premium payments $P_{t,T}$ that the buyer pays (and seller receives), namely

$$V_{t,T}^{\text{Premium}} = P_{t,T} N \times \left(\sum_{i=1}^{n} D(t,t_i) \widetilde{\mathbb{P}}_t[\tau > t_i] \Delta t_i + \mathcal{A}_p \right) \tag{19}$$

where $\mathcal{A}_p$ is the accrual payment in case default occurs between two payment dates. Instead of considering premiums paid at discrete dates, we pass to the continuous limit (see [23]) and consider the continuous premium formulation

$$\begin{aligned} V_{t,T}^{\text{Premium}} &= P_{t,T} N \times \left(\widetilde{\mathbb{E}}_t \left[\int_t^T D(t,s) \mathbb{1}_{\{\tau > s\}} ds \right] + \frac{1}{2} \cdot \widetilde{\mathbb{E}}_t \left[D(t,\tau) \mathbb{1}_{\{\tau \leq T\}} \right] \right) \\ &= P_{t,T} N \times \left(\int_t^T e^{-r(s-t)} \widetilde{\mathbb{P}}_t[\tau > s] ds + \frac{1}{2} \cdot \int_t^T e^{-r(s-t)} \widetilde{\mathbb{P}}_t[\tau \in ds] \right) \end{aligned} \tag{20}$$

where the second term is the value of the accrual, i.e.,

$$\mathcal{A}_p := \frac{1}{2} \int_t^T D(t,s) \widetilde{\mathbb{P}}_t\left[\tau \in ds\right]. \tag{21}$$

Under the same continuous premium formulation, the value of the protection (default) leg can then be written as

$$\begin{aligned} V_{t,T}^{\text{Protection}} &= N \times \left(\widetilde{\mathbb{E}}_t \left[D(t,\tau) \left(1 - \frac{R_\tau}{N} \right) \mathbb{1}_{\{\tau \leq T\}} \right] \right) \\ &= N \widetilde{\mathbb{E}}_t\left[D(t, \tau \wedge T)\right] - B_{t,T} \\ &= N \left(D(t,T) \widetilde{\mathbb{P}}_t\left[\tau > T\right] + \widetilde{\mathbb{E}}_t \left[D(t,\tau) \mathbb{1}_{\{\tau \leq T\}} \right] \right) - B_{t,T}. \end{aligned} \tag{22}$$

Using usual no arbitrage principles, the CDS premium $P_{t,T}$ is given as the value that balances these two equations, namely

$$\begin{aligned} P_{t,T} &= \frac{\widetilde{\mathbb{E}}_t\left[D(t,\tau)\left(1-\frac{R_\tau}{N}\right)\mathbb{1}_{\{\tau\le T\}}\right]}{\int_t^T D(t,s)\widetilde{\mathbb{P}}_t[\tau>s]ds+\frac{1}{2}\int_t^T D(t,s)\widetilde{\mathbb{P}}_t\left[\tau\in ds\right]} \\ &= \frac{1}{N}\frac{N\left(D(t,T)\widetilde{\mathbb{P}}_t\left[\tau>T\right]+\widetilde{\mathbb{E}}_t\left[D(t,\tau)\mathbb{1}_{\{\tau\le T\}}\right]\right)-B_{t,T}}{\int_t^T D(t,s)\widetilde{\mathbb{P}}_t[\tau>s]ds+\frac{1}{2}\int_t^T D(t,s)\widetilde{\mathbb{P}}_t\left[\tau\in ds\right]} \\ &= \frac{D(t,T)\widetilde{\mathbb{P}}_t\left[\tau>T\right]+\widetilde{\mathbb{E}}_t\left[D(t,\tau)\mathbb{1}_{\{\tau\le T\}}\right]-\frac{B_{t,T}}{N}}{\widetilde{\mathbb{E}}_t\left[\int_t^T D(t,s)\mathbb{1}_{\{\tau>s\}}ds\right]+\frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[D(t,\tau)\mathbb{1}_{\{\tau\le T\}}\right]} \\ &= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau>T\right]+\widetilde{\mathbb{E}}_t\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau\le T\}}\right]-\frac{B_{t,T}}{N}}{\widetilde{\mathbb{E}}_t\left[\int_t^{\tau\wedge T} e^{-r(s-t)}ds\right]+\frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau\le T\}}\right]}. \end{aligned} \tag{23}$$

To evaluate (23), we need a model for recovery R_t and default τ. A standard assumption [17] in a hazard rate framework is that recovery is a constant, under which our CDS premium (23) reduces to the classical result

$$P_{t,T} = (1-\bar{R})\frac{\widetilde{\mathbb{E}}_t\left[D(t,\tau)\mathbb{1}_{\{\tau\le T\}}\right]}{\widetilde{\mathbb{E}}_t\left[\int_t^T D(t,s)\mathbb{1}_{\{\tau>s\}}ds\right]+\frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[D(t,\tau)\mathbb{1}_{\{\tau\le T\}}\right]} \tag{24}$$

where $\bar{R} = R_\tau/N$.

In Sections 2.1 and 2.2 we present Bond and CDS prices, respectively, for the Black-Cox model which does not account for recovery risk. In Sections 4.1 and 4.4, respectively, we present bond and CDS prices under the Stochastic Recovery Black-Cox model which does take into account recovery risk.

2.1. Bond Pricing with the Black-Cox Model

In this section we price a zero coupon bond with default risk using the Black-Cox model. The structure of the price depends on whether the boundary K is larger or smaller than the notional N.

Definition 1. (Weak and Strong Covenants in Black-Cox Model) *Let K be the default barrier and N be the notional of the zero-coupon bond in the Black-Cox framework. We say the bond has a weak (resp. strong) covenant if $K \le N$ (resp. $K \ge N$).*

Definition 2. (Distances to Default) *Define the following distance to default, which we will use in bond and CDS pricing under both weak and strong covenants.*

• *Weak Covenant* $(K \leq N)$

$$\begin{aligned} d_0^w &= \frac{\ln\left(\frac{A_t}{N}\right) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \\ d_1^w &= \frac{\ln\left(\frac{A_t}{N}\right) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \\ x_0^w &= \frac{\ln\left(\frac{K^2}{NA_t}\right) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \\ x_1^w &= \frac{\ln\left(\frac{K^2}{NA_t}\right) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \end{aligned} \tag{25}$$

• *Strong Covenant* $(K \geq N)$

$$\begin{aligned} d_0^s &= \frac{\ln\left(\frac{A_t}{K}\right) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \\ d_1^s &= \frac{\ln\left(\frac{A_t}{K}\right) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \\ x_0^s &= \frac{\ln\left(\frac{K}{A_t}\right) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \\ x_1^s &= \frac{\ln\left(\frac{K}{A_t}\right) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} \end{aligned} \tag{26}$$

Notice that the weak covenant distances to default d_0^w, d_1^w are the usual Merton distances to default (c.f. [15]).

Lemma 1. (Probability of Hitting the Barrier) *Let τ_K be the first passage time* (8)*. Then for any constant default barrier $K > 0$ and $A_t \geq K$*

$$\widetilde{\mathbb{P}}_t\left[\tau_K \leq T\right] = \Phi(-d_0^s) + \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi(x_0^s) \tag{27}$$

where Φ is the cumulative normal distribution function and d_0^s, x_0^s are given by (26)*.*

Proof. We begin this proof by noting that the value of a bond in the Black-Cox model requires a barrier at covenant value K. One can therefore calculate directly the probability of no-default using barrier option theory. To carry out this computation, we define the function W corresponding to a digital option: t

$$W(A,t) := \widetilde{\mathbb{P}}_t\left[A_T > N\right] = \Phi\left(d_0^w\right) = 1 - \widetilde{\mathrm{PD}}_{t,T}^{\mathrm{Merton}}. \tag{28}$$

From this definition (28) of W, if we add a lower barrier at K (See for example Ch.10 in [24]) then the value $\bar{W}$ of a digital option with a lower barrier at K is

$$\begin{aligned}\bar{W}(A,t) := \widetilde{\mathbb{P}}_0[A_T > N, \tau_K > T] &= W(A,t) - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}-1} W\left(\frac{K^2}{A}, t\right) \\ &= \Phi\left(d_0^w\right) - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi\left(x_0^w\right).\end{aligned} \tag{29}$$

Therefore, by setting $N = K$ in (25) and (29), we obtain the result

$$\widetilde{\mathbb{P}}_t[\tau_K > T] = \widetilde{\mathbb{P}}_t[A_T > K, \tau_K > T] = \Phi\left(d_0^s\right) - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi\left(x_0^s\right) \tag{30}$$

and (27) above follows.
□

Note: The result (27) can also be found in, for instance, Appendix B of [25]. We provide another direct proof within the constructive proof of Theorem 3 below, by computing $\int_K^\infty \widetilde{\mathbb{P}}_t\left[A_T \in da, \tau_K > T\right]$ in the strong covenant case.

Theorem 1. (Bond Pricing under the Black-Cox Model) *Suppose that Assumptions 1–4 are satisfied. Then the price of a defaultable zero-coupon bond* (14) *is given by:*

I. Weak Covenant Case. *If $K \leq N$ the price of a zero-coupon bond is given by*

$$\begin{aligned}B_{t,T}^{\mathrm{BC}}(w) = e^{-r(T-t)} N &\left[\Phi(d_0^w) - \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi(x_0^w)\right] \\ + A_t &\left[\Phi(-d_1^w) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1} \Phi(x_1^w)\right]\end{aligned} \tag{31}$$

and the risk-neutral PD and LGD in the case of a weak covenant are

$$\widetilde{\mathrm{PD}}_{t,T}^{\mathrm{BC}}(w) = \Phi\left(-d_0^w\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi(x_0^w) \tag{32}$$

and

$$\widetilde{\mathrm{LGD}}_{t,T}^{BC}(w) = 1 - e^{r(T-t)} \frac{A_t}{N} \frac{\Phi\left(-d_1^w\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1} \Phi\left(x_1^w\right)}{\Phi\left(-d_0^w\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi\left(x_0^w\right)}. \tag{33}$$

II. Strong Covenant Case. *If $K \geq N$ the price of a zero-coupon bond is given by*

$$\begin{aligned}B_{t,T}^{\mathrm{BC}}(s) = e^{-r(T-t)} N &\left[\Phi(d_0^s) - \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi(x_0^s)\right] \\ + A_t &\left[\Phi(-d_1^s) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1} \Phi(x_1^s)\right]\end{aligned} \tag{34}$$

and the risk-neutral PD and LGD in the case of a strong covenant are

$$\widetilde{\mathrm{PD}}_{t,T}^{\mathrm{BC}}(s) = \Phi\left(-d_0^s\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1} \Phi(x_0^s) \tag{35}$$

and

$$\widetilde{\mathrm{LGD}}_{t,T}^{\mathrm{BC}}(s) = 1 - e^{r(T-t)}\frac{A_t}{N}\frac{\Phi\left(-d_1^s\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi\left(x_1^s\right)}{\Phi\left(-d_0^s\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x_0^s\right)}. \tag{36}$$

Proof. The proof uses the same integral techniques employed in Theorem 3 below. Using the barrier option characterization of the addition of a covenant, the price of a bond is expressed as

$$\begin{aligned} B_{t,T}^{\mathrm{BC}} &= Ne^{-r(T-t)} \cdot \widetilde{\mathbb{P}}_t[A_T \geq N, \tau_{\mathrm{K}} > T] + \widetilde{\mathbb{E}}_t[e^{-r(T-t)}A_T \mathbb{1}_{\{A_T \geq N, \tau_{\mathrm{K}} > T\}^c}] \\ &= Ne^{-r(T-t)} \cdot \widetilde{\mathbb{P}}_t[A_T \geq N, \tau_{\mathrm{K}} > T] + A_t - \widetilde{\mathbb{E}}_t[e^{-r(T-t)}A_T \mathbb{1}_{\{A_T \geq N, \tau_{\mathrm{K}} > T\}}]. \end{aligned} \tag{37}$$

This computation requires the joint density $\tilde{\mathbb{P}}_t[A_T \in da, \tau_{\mathrm{K}} > T]$, which is found in [26]. We compute the associated integrals explicitly in our constructive proof of Theorem 3 for the case of stochastic recovery in the Black-Cox framework.

To carry out these integrals for the standard Black-Cox model, we set $\gamma = \rho_{A,R} = \frac{A_0}{R_0} = 1$ in the integrals constructed for the proof of Theorem 3. The difference between the weak and strong covenant cases (with or without stochastic recovery) is that in the strong covenant case,

$$\{A_T \geq N, \tau_{\mathrm{K}} > T\} = \{A_T \geq K, \tau_{\mathrm{K}} > T\} = \{\tau_{\mathrm{K}} > T\}. \tag{38}$$

This fact is reflected in the appropriate substitution in the lower limits of the integrals against the joint density $\tilde{\mathbb{P}}_t[A_T \in da, \tau_{\mathrm{K}} > T]$.
□

Remark 1. (Special Case where $K = N$) *Note that the special case where $K = N$ is included in both the strong and weak covenant formulas in that*

$$\begin{aligned} &\lim_{K \to N+} B_{t,T}^{\mathrm{BC}}(s) = \lim_{K \to N-} B_{t,T}^{\mathrm{BC}}(w) \\ &= e^{-r(T-t)}\left[N\Phi(-d_0) - \left(\frac{N}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi(x_0)\right] + A_t\left[\Phi(-d_1) + \left(\frac{N}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x_1)\right]. \end{aligned} \tag{39}$$

2.2. CDS Pricing with the Black-Cox Model

We now price CDS premiums using the Black-Cox model. In keeping with the continuous CDS pricing model, we extend the premium rate defined in [23] to include recovery at τ_{BC} while protection is paid as long as τ_{K} has not occurred.

Theorem 2. (CDS Premium under Black-Cox Model) *Suppose that Assumptions 1–4 are satisfied. Then the CDS premium (23) is given by*

$$P_{t,T}^{\mathrm{BC}} = \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}\right] - \frac{B_{t,T}^{\mathrm{BC}}}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}] - e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{BC}} > T]}{r} + \frac{1}{2} \cdot \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}\right]}. \tag{40}$$

In the strong covenant case, where $K \geq N$, this reduces to the closed formula

$$
\begin{aligned}
P^{\mathrm{BC}}_{t,T}(s) &= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}\right] - \frac{B^{\mathrm{BC}}_{t,T}(s)}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}]-e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}}>T]}{r} + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}\right]} \\
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T\right] &= \Phi\left(d_0^s\right) - \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x_0^s\right) = 1 - \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(s) \\
\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}] &= A_t\left[\Phi(-d_1^s) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x_1^s)\right].
\end{aligned}
\tag{41}
$$

In the weak covenant case, where $K \geq N$, this reduces to the closed formula

$$
\begin{aligned}
P^{\mathrm{BC}}_{t,T}(w) &= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\leq T\}}\right] - \frac{B^{\mathrm{BC}}_{t,T}(w)}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\leq T\}}]-e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{BC}}>T]}{r} + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\leq T\}}\right]} \\
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] &= \widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T, A_T \geq N\right] = 1 - \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(w) \\
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T\right] &= \Phi\left(d_0^s\right) - \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x_0^s\right) = 1 - \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(s) \\
\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\leq T\}}\right] &= \widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}] + e^{-r(T-t)}\left(\widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T] - \widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T, A_T \geq N]\right) \\
\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}] &= A_t\left[\Phi(-d_1^s) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x_1^s)\right].
\end{aligned}
\tag{42}
$$

Note

- The numerator in (40) follows from direct substitution of the bond price calculated in Theorem 1 into the numerator in the general CDS formula (23). This direct substitution of the risky bond price also reflects the flexibility in assigning a weak or strong covenant, and will in fact be the only change observed when stochastic recovery is included in Section 5.
- To complete the proof of Theorem 2, we will need to compute the denominator in (23). In the BC, and SRBC model forthcoming, this reduces to computing

$$\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}]. \tag{43}$$

We are able to compute (43) by reducing to something more manageable via the fact that the discounted asset price is a local martingale. The reduced form follows from the optional sampling theorem:

$$\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\leq T\}}] = \frac{A_t - \widetilde{\mathbb{E}}_t[e^{-r(T-t)}A_T\mathbb{1}_{\{\tau_{\mathrm{K}}>T\}}]}{K}. \tag{44}$$

Proof. From Assumption 4, the general CDS premium (23) reduces to

$$
\begin{aligned}
P_{t,T}^{\mathrm{BC}} &= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right] - \frac{B_{t,T}^{\mathrm{BC}}}{N}}{\widetilde{\mathbb{E}}_t\left[\int_t^T e^{-r(T-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}>s\}}ds\right] + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]} \\
&= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right] - \frac{B_{t,T}^{\mathrm{BC}}}{N}}{\widetilde{\mathbb{E}}_t\left[\int_t^{\tau_{\mathrm{BC}}\wedge T} e^{-r(s-t)}ds\right] + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]} \\
&= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right] - \frac{B_{t,T}^{\mathrm{BC}}}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}]-e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{BC}}>T]}{r} + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]}.
\end{aligned} \tag{45}
$$

The main quantity to solve for now is $\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]$, and upon it's computation and insertion into (45), we have the formula for the premium. By the structural definition of τ_{BC}, it follows that

$$
\{\tau_{\mathrm{BC}} \le T\} = \{\tau_{\mathrm{K}} \le T\} \cup \{\tau_{\mathrm{K}} > T, K < A_T < N\} \tag{46}
$$

and so

$$
\begin{aligned}
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] &= \widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T, A_T \ge N\right] \\
\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right] &= \widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\le T\}}] + e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T, K < A_T < N] \\
&= \widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\le T\}}] \\
&\quad + e^{-r(T-t)}\left(\widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T] - \widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T, A_T \ge N]\right).
\end{aligned} \tag{47}
$$

In the case $K \ge N$, the event $\{\tau_{\mathrm{BC}} > T\}$ reduces to the event $\{\tau_{\mathrm{K}} > T\}$. Consequently, we utilize (44), (45), and the integral in (81) below to return the value (41). In the case that $K \le N$, similar calculations lead to (42).

□

3. Modeling Recovery Risk within a Structural Framework

The original Merton and Black-Cox structural models have been extended in several directions by many different authors, including stochastic interest rates, bankruptcy costs, taxes, debt subordination, strategic default, time dependent and stochastic default barrier, jumps in the asset value process, etc. While these extensions are by no means exhaustive (c.f. [16] for a more thorough discussion), they relax several of the main assumptions in the original models. However, none consider recovery as a risk factor, and assume the only risk driver is the asset value itself, or perhaps interest rates if stochastic rates are modeled. Empirical research, however, has shown that recoveries need not be constant in time and that typically the time-series of default probabilities and recoveries are inversely correlated [1,27]. To make matters even more complicated, Hillebrand [28] shows that this correlation does not necessarily have to vary co-monotonously over the whole economic cycle. The correlation between PD and LGD has been investigated by several researchers in the context of credit capital. For instance, Giese [3] incorporates PG-LGD correlations into a single-factor Vasicek framework and finds that capital increases by up to 35% at the 99.9% confidence interval for high-yield credit portfolios. While investigating stressed LGDs, Miu and Ozdemir [6] find that in order to compensate for neglecting the PD-LGD correlation in credit capital modeling, the mean LGD must be increased by about 37% from its unbiased estimate in order to compensate for the lack of correlations.

We incorporate recovery risk into the Black-Cox model for three main reasons. First, as discussed in Section 2, the Black-Cox model has a constant recovery K in the event of default before redemption time. However, there is a large body of empirical evidence showing that recovery rates are inversely correlated with probabilities of default [1,14,27,29–34] and so recovery risk is a real financial phenomena that should be modeled. Second, this correlation can have a very large effect on credit capital [5,6]. Ignoring this effect in a pricing model could potentially lead to large mispricings or significant misestimation of risk. Finally, we show that including recovery risk in a structural framework is a mechanism which allows for larger spreads. This is important because empirical literature suggests that structural models tend to underestimate observed credit spreads by 10–15% on average (c.f. [35–38]). In [39], Gemmill argues that the extra observed spread is explained by other factors such as liquidity risk. However, just as in the Merton case [15], we show in Lemma 9 that adding recovery risk can lead to an increase in the credit spread over constant recovery models, and suggest that recovery risk is another possible mechanism to explain the additional observed spread.

3.1. The Correlated Asset-Recovery Model

Let A_t denote the asset price at time $t > 0$ and let R_t denote the recovery amount at time $t > 0$. The unobservable process R_t is interpreted as the amount that would be recovered if default were to occur at t. The asset and recovery processes are modeled as two correlated geometric Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\begin{aligned} dA_t &= \mu_A A_t dt + \sigma_A A_t dW_t^A \\ dR_t &= \mu_R R_t dt + \sigma_R R_t dW_t^R \\ \rho_{A,R} dt &= dW_t^A dW_t^R \end{aligned} \tag{48}$$

where (W_t^A, W_t^R) are correlated, standard Brownian motions on our probability space.

3.2. Some Preliminary Results

Lemma 2. *(Existence of Risk Neutral Measure) Let (A_t, R_t) be the coupled measurable stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$ given by (48). Then there exists a risk-neutral measure $\widetilde{\mathbb{P}}_t$ such that $(X_t, \mathcal{F}_t^X)$ is martingale under $\widetilde{\mathbb{P}}_t$, where X is the coupled process $X_t := e^{-rt}(A_t, R_t)$. Furthermore, (A_t, R_t) satisfy*

$$\begin{aligned} dA_t &= rA_t dt + \sigma_A A_t dW_t^A \\ dR_t &= rR_t dt + \sigma_R R_t dW_t^R. \end{aligned} \tag{49}$$

Proof. The proof follows from the results in [40] as the pair (A_t, R_t) is a two-dimensional diffusion. □

Lemma 3. *(Solution to the PD-LGD Equations) Let (A_t, R_t) be given by (48). Then, under the physical measure, (A_t, R_t) is given by*

$$\begin{aligned} A_t &= A_0 \exp\left((\mu_A - \frac{1}{2}\sigma_A^2)t + \sigma_A W_t^A \right) \\ R_t &= R_0 \exp\left((\mu_R - \frac{1}{2}\sigma_R^2)t + \sigma_R W_t^R \right) \end{aligned} \tag{50}$$

and under the risk-neutral measure,

$$\begin{aligned} A_t &= A_0 \exp\left((r - \frac{1}{2}\sigma_A^2)t + \sigma_A W_t^A \right) \\ R_t &= R_0 \exp\left((r - \frac{1}{2}\sigma_R^2)t + \sigma_R W_t^R \right). \end{aligned} \tag{51}$$

As an application of Lemma 3, we give a simple proof of Theorems 1–3 in [4]. This result is the first comparison of recovery to asset values when the underlying asset is in state K.

Lemma 4. *(Expected Recovery at Default) Consider the Asset-Recovery model (48) and define*

$$\gamma := \rho_{A,R}\frac{\sigma_R}{\sigma_A} \tag{52}$$

$$\delta^{\mathbb{P}} := (\mu_R - \frac{1}{2}\sigma_R^2) - \gamma(\mu_A - \frac{1}{2}\sigma_A^2) \tag{53}$$

$$\delta^{\widetilde{\mathbb{P}}} := \delta = (r - \frac{1}{2}\sigma_R^2) - \gamma(r - \frac{1}{2}\sigma_A^2). \tag{54}$$

Then

$$\widetilde{\mathbb{E}}[R_s|A_s = K] = \mathbb{E}[R_s|A_s = K]e^{(\delta^{\widetilde{\mathbb{P}}} - \delta^{\mathbb{P}})s} \tag{55}$$

where

$$\mathbb{E}[R_s|A_s = K] = R_0\left(\frac{K}{A_0}\right)^{\gamma} e^{\delta^{\mathbb{P}}s + \frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)s} \tag{56}$$

$$\widetilde{\mathbb{E}}[R_s|A_s = K] = R_0\left(\frac{K}{A_0}\right)^{\gamma} e^{\delta^{\widetilde{\mathbb{P}}}s + \frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)s}. \tag{57}$$

Proof. We consider first the behavior of X under the physical measure, and prove (56). By Lemmas 2 and 3, and by (48), we can consider now the solution form under two standard Brownian motions W_A and W on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ where

$$\begin{aligned} A_t &= A_0 \exp\left((\mu_A - \frac{1}{2}\sigma_A^2)t + \sigma_A W_t^A\right) \\ R_t &= R_0 \exp\left((\mu_R - \frac{1}{2}\sigma_R^2)t + \sigma_R W_t^R\right) \\ W_t^R &= \rho_{A,R}W_t^A + \sqrt{1-\rho_{A,R}^2}W_t \\ \mathbb{P}[dW_t^A dW_t = 0] &= 1. \end{aligned} \tag{58}$$

Using (58) the recovery process R_t under the physical measure can be explicitly written as

$$R_t = R_0\left(\frac{A_t}{A_0}\right)^{\gamma} \exp\left(\delta^{\mathbb{P}}t + \sigma_R\sqrt{1-\rho_{A,R}^2}W_t\right) \tag{59}$$

where γ and $\delta^{\mathbb{P}}$ are defined by (52) and (53) respectively. Then, using (59) we have

$$\mathbb{E}[R_s|A_s = K] = \mathbb{E}\left[R_0\left(\frac{A_s}{A_0}\right)^{\gamma} \exp\left(\delta^{\mathbb{P}}s + \sigma_R\sqrt{1-\rho_{A,R}^2}W_s\right)|A_s = K\right] \tag{60}$$

$$= R_0\left(\frac{K}{A_0}\right)^{\gamma}\mathbb{E}_s\left[\exp\left(\delta^{\mathbb{P}}s + \sigma_R\sqrt{1-\rho_{A,R}^2}W_s\right)\right] \tag{61}$$

$$= R_0\left(\frac{K}{A_0}\right)^{\gamma} e^{\delta^{\mathbb{P}}s + \frac{1}{2}\sigma_R^2(1-\rho_{A,R}^2)s}. \tag{62}$$

To prove (57) we simply set $\mu_A = \mu_R = r$ in (59) to recognize R under the risk-neutral measure as a shift from $\delta^{\mathbb{P}}$ to $\delta^{\widetilde{\mathbb{P}}}$, and (55) follows directly by dividing the closed form solution (57) by (56). □

Under this framework, the price of a zero-coupon bond with face N is

$$B_{t,T} = \widetilde{\mathbb{E}}_t[Ne^{-r(T-t)}\mathbb{1}_{\{\tau > T\}}] + \widetilde{\mathbb{E}}_t[e^{-r(\tau-t)}R_\tau\mathbb{1}_{\{\tau \le T\}}]. \tag{63}$$

where default τ depends on A_t and recovery upon default R_τ depends on R_t through (49), thus correlating default and recovery.

In Sections 4.1 and 4.4 we compute the bond and CDS prices (and related credit measures such as probability of default, loss-given-default and credit spreads) in the Stochastic Recovery Black-Cox default model where default time is defined via (9), but recovery is stochastic and correlated to asset value via (48).

3.3. Comparison of Recovery to Asset Upon Default

One of the features of the stochastic recovery model (48) we work with is that it is possible for the recovery value modeled, R_t, to surpass the asset value A_t at some time $t \leq T$. This is due to the fact that A is the manager's estimated asset value instead of the actual asset value upon recovery. Upon default, it may come to pass that the actual value is higher than the estimated asset value A. Hence, it is possible that when default occurs, the debt holders are paid in full, and there is still some remaining capital. So, $R_t > N > A_t$ is possible, for example. In fact, this is entirely possible when default occurs due to liquidity issues. In this context, A_t is the default driving process, and R_t the actual firm value. Hence, this model we work with proposes that there is less information about the asset than in the classic single-name models of Merton and Black and Cox. We also note here that in the end, it is the firm manager that decides firm default. Whether due to capital structure reasons or more pressingly due to day-to-day operational costs, it is the manager who has final say in this matter. This reinforces the fact that A is used to determine default, but R determines post-default recovery. We note that this allows for the possibility that credit spreads can be negative, at times. Given the correlated asset-recovery process (48), it is natural to estimate the probability that assets priced in recovery exceed the barrier K or notional N . The following Lemma provides such an estimate.

Lemma 5. *At the first passage time τ_K, we have the estimates*

$$\begin{aligned} \widetilde{\mathbb{P}}_t[R_{\tau_K}\mathbb{1}_{\{\tau_K<T\}} > K] &\leq \min\left\{ \frac{R_t}{Ke^{-r(T-t)}}\left[\Phi(-d^s_\gamma) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+(2\gamma-1)}\Phi(x^s_\gamma)\right], 1\right\} \\ \widetilde{\mathbb{P}}_t[R_{\tau_K\wedge T} > N] &\leq \min\left\{\frac{R_t}{Ne^{-r(T-t)}}, 1\right\}. \end{aligned} \tag{64}$$

Proof. By the Markov Inequality and Optional Sampling Theorem, it follows that after once again setting (wlog) $t = 0$,

$$\begin{aligned} \widetilde{\mathbb{P}}_0[R_{\tau_K}1_{\{\tau_K<T\}} > K] &= \widetilde{\mathbb{P}}_0[e^{-r(\tau_K\wedge T)}R_{\tau_K\wedge T}\mathbb{1}_{\{\tau_K<T\}} > Ke^{-r(\tau_K\wedge T)}] \\ &\leq \widetilde{\mathbb{P}}_0[e^{-r(\tau_K\wedge T)}R_{\tau_K\wedge T}\mathbb{1}_{\{\tau_K<T\}} > Ke^{-rT}] \\ &= \widetilde{\mathbb{P}}_0[e^{-r\tau_K}R_{\tau_K}\mathbb{1}_{\{\tau_K<T\}} > Ke^{-rT}] \\ &\leq \frac{\widetilde{\mathbb{E}}_0[e^{-r\tau_K}R_{\tau_K}\mathbb{1}_{\{\tau_K<T\}}]}{Ke^{-rT}} \\ &= \frac{R_0 - \widetilde{\mathbb{E}}_0[e^{-rT}R_T\mathbb{1}_{\{\tau_K\geq T\}}]}{Ke^{-rT}}. \end{aligned} \tag{65}$$

The numerator in (65) is computed using the same integral techniques employed in the proof of Theorem 3, except with a lower limit of K instead of N (i.e., strong covenant.) The second inequality, for the probability that recovered value is above notional, is proved in the same fashion:

$$\begin{aligned} \widetilde{\mathbb{P}}_0[R_{\tau_K \wedge T} > N] &= \widetilde{\mathbb{P}}_0[e^{-r(\tau_K \wedge T)} R_{\tau_K \wedge T} > N e^{-r(\tau_K \wedge T)}] \\ &\leq \widetilde{\mathbb{P}}_0[e^{-r(\tau_K \wedge T)} R_{\tau_K \wedge T} > N e^{-rT}] \\ &\leq \frac{\widetilde{\mathbb{E}}_0[e^{-r(\tau_K \wedge T)} R_{\tau_K \wedge T}]}{N e^{-rT}} \\ &= \frac{R_0}{N e^{-rT}}. \end{aligned} \tag{66}$$

□

3.4. Connection between Recovery Risk and PD-LGD Correlation

In related work, Moody's KMV has recently proposed a PD-LGD Correlation model in the context of credit capital [4]. While the structural model they propose is the same two-factor structural model in (49), the motivation for the model is different. The motivation for Moody's model is the empirically observed PD-LGD correlation presented in Altman et.al. [1], which is justified economically in [27,34,41] among other works. However, the study in [1] was conducted in the physical measure (realized-post-ante default rates) rather than in the risk neutral measure used in pricing, and the economic considerations are different. Nevertheless, in [4], the authors attempt to price a bond with this model by integration. Indeed, we are able price bonds and CDS's in this model via a martingale analysis, completing the analysis initially suggested in [4] by integrating against a joint density for process and first passage time. In particular, in Sections 4.1 and 4.4 below, we explicitly compute the bond price using stochastic calculus and the Optional Sampling Theorem, returning closed form solutions.

4. The Black-Cox Model with Recovery Risk

In this section we introduce the assumptions that define the Stochastic Recovery Black-Cox model (SRBC) and use it to price bonds and CDS in Sections 4.1 and 4.4 respectively. The SRBC model essentially relaxes the recovery assumption, Assumption 4, by replacing it with a weaker assumption on the dynamics of the recovery value, Assumption 6, allowing for randomness in recovery. In particular, the SRBC Model assumes the following:

Assumption 5. (Constant Rates) *The short rate is assumed to be a constant r so that the discount factor is given by*

$$D(t,s) = e^{-r(s-t)}. \tag{67}$$

Assumption 6. (Correlated Asset-Recovery Dynamics) *The $\widetilde{\mathbb{P}}_t$-dynamics of the firms asset value A_t and recovery value R_t at time t are assumed to follow*

$$\begin{aligned} dA_t &= rA_t dt + \sigma_A A_t dW_t^A \\ dR_t &= rR_t dt + \sigma_R R_t dW_t^R \\ \rho_{A,R} dt &= \langle dW_t^A, dW_t^R \rangle \end{aligned} \tag{68}$$

where $r, \sigma_A, \sigma_R, \rho_{A,R}$ are constants and W_t^A, W_t^R are standard Brownian processes.

Assumption 7. (Default Time) *Default is given by the standard Black-Cox (BC) default time:*

$$\tau_{\mathrm{BC}} = \tau_{\mathrm{K}} \mathbb{1}_{\{\tau_{\mathrm{K}} \le T\}} + T \mathbb{1}_{\{\tau_{\mathrm{K}} > T, A_T < N\}} + \infty \mathbb{1}_{\{\tau_{\mathrm{K}} > T, A_T \ge N\}}. \tag{69}$$

where τ_{BC} is also explicitly given in (9).

The collection of Assumptions 5–7 form the Stochastic Recovery Black-Cox model, and can be used to price defaultable bonds as well as other products such as CDS. which we do in Sections 4.1 and 4.4 below. Note that, just as in the original Black-Cox model, the constant interest-rate assumption can be relaxed to include time varying deterministic rates $r = r(t)$ with little effort.

4.1. Bond Pricing with Recovery Risk

If default can occur before the maturity T, say if the bond issuer is forced into default if assets A ever fall below a default point K, then the bond price must reflect this extra possibility. Our main result in this section is the computation of a closed formula for such a price, where a recovery that is correlated to the asset is substituted at default. This incorporates the model first presented by the authors in [4].

Theorem 3. (Bond Price under Stochastic Recovery Black Cox Model) *Suppose Assumptions 5,6,7 hold. Then the general price of a zero-coupon bond and related risk metrics are:*

$$\begin{aligned} B_{t,T}^{\mathrm{SRBC}} &= e^{-r(T-t)} \widetilde{\mathbb{E}}_t \left[B_{T,T}^{\mathrm{SRBC}} \right] \\ &= e^{-r(T-t)} \widetilde{\mathbb{E}}_t \left[N \mathbb{1}_{\{A_T \ge N, \tau_{\mathrm{K}} > T\}} + R_T \mathbb{1}_{\{A_T \ge N, \tau_{\mathrm{K}} > T\}^c} \right] \\ &= N e^{-r(T-t)} \cdot \widetilde{\mathbb{P}}_t[A_T \ge N, \tau_{\mathrm{K}} > T] + \left[R_t - \widetilde{\mathbb{E}}_t [e^{-r(T-t)} R_T \mathbb{1}_{\{A_T \ge N, \tau_{\mathrm{K}} > T\}}] \right] \\ S_{t,T}^{\mathrm{SRBC}} &= \frac{1}{T-t} \ln \left(\frac{1}{1 - \widetilde{\mathrm{PD}}_{t,T}^{\mathrm{BC}} \cdot \widetilde{\mathrm{LGD}}_{t,T}^{\mathrm{SRBC}}} \right). \end{aligned} \tag{70}$$

For sake of consistency, we point out that the SRBC *bond prices reduce to the Stochastic Recovery Merton (*SRM*) bond prices computed in [15] as $K \to 0$. To enable comparison with the BC model, we once again present the result for both weak and strong covenants:*

I. Weak Covenant Case. *If $K \le N$ the price of a zero-coupon bond is given by*

$$B_{t,T}^{\mathrm{SRBC}}(w) = N e^{-r(T-t)} \left[\Phi(d_0^w) - \left(\frac{K}{A_t} \right)^{\frac{2r}{\sigma_A^2} - 1} \Phi(x_0^w) \right] + R_t \left[\Phi(-d_\gamma^w) + \left(\frac{K}{A_t} \right)^{\frac{2r}{\sigma_A^2} + 2\gamma - 1} \Phi(x_\gamma^w) \right]. \tag{71}$$

The risk-neutral PD and LGD in the case of a weak covenant are

$$\begin{aligned} \widetilde{\mathrm{LGD}}_{t,T}^{\mathrm{SRBC}}(w) &= 1 - e^{r(T-t)} \frac{R_t}{N} \frac{\Phi\left(-d_\gamma^w\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2} + 2\gamma - 1} \Phi\left(x_\gamma^w\right)}{\Phi\left(-d_0^w\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2} - 1} \Phi\left(x_0^w\right)} \\ \widetilde{\mathrm{PD}}_{t,T}^{\mathrm{SRBC}}(w) &= \Phi\left(-d_0^w\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2} - 1} \Phi(x_0^w) = \widetilde{\mathrm{PD}}_{t,T}^{\mathrm{BC}}(w). \end{aligned} \tag{72}$$

II. Strong Covenant Case. *If $K \ge N$ the price of a zero-coupon bond is given by*

$$B_{t,T}^{\mathrm{SRBC}}(s) = N e^{-r(T-t)} \left[\Phi(d_0^s) - \left(\frac{K}{A_t} \right)^{\frac{2r}{\sigma_A^2} - 1} \Phi(x_0^s) \right] + R_t \left[\Phi(-d_\gamma^s) + \left(\frac{K}{A_t} \right)^{\frac{2r}{\sigma_A^2} + 2\gamma - 1} \Phi(x_\gamma^s) \right]. \tag{73}$$

The risk-neutral PD and LGD in the case of a strong covenant are

$$\widetilde{\mathrm{LGD}}^{\mathrm{SRBC}}_{t,T}(s) = 1 - e^{r(T-t)}\frac{R_t}{N}\frac{\Phi\left(-d^s_\gamma\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+2\gamma-1}\Phi\left(x^s_\gamma\right)}{\Phi\left(-d^s_0\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x^s_0\right)}$$
$$\widetilde{\mathrm{PD}}^{\mathrm{SRBC}}_{t,T}(s) = \Phi\left(-d^s_0\right) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi(x^s_0) = \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(s) \tag{74}$$

where

$$\begin{aligned} d^w_\gamma &= d^w_0 + \gamma\sigma_A\sqrt{T-t} \\ d^s_\gamma &= d^s_0 + \gamma\sigma_A\sqrt{T-t} \\ x^w_\gamma &= x^w_0 + \gamma\sigma_A\sqrt{T-t} \\ x^s_\gamma &= x^s_0 + \gamma\sigma_A\sqrt{T-t}. \end{aligned} \tag{75}$$

Remark 2. *We remark that the* risk-adjusted *SRBC distances-to-default d_γ, x_γ in (75) reduce to the standard distances-to-default (d_0, x_0) and (d_1, x_1) of the BC model if $\gamma = 0$ or $\gamma = 1$, respectively. This adjustment for gamma reflects the uncertainty of the firm manager in the partial information setting of what the recoverable value of the firm's assets truly are, and affects only the recovery term. It should be noted that the probability of default is the same as in the case of no stochastic recovery. It is only the Loss-Given-Default that is affected by adding R as a recovery driver.*

The price for the zero-coupon bond under the SRBC setting is computed by employing the optional sampling theorem to the local martingale $e^{-rt}R_t$. We use $\tau := \min\{\tau_K, T\}$, a bounded stopping time adapted to the filtration generated by the joint process (A, R). The same technique is also used in the proof of the strong covenant case.

Proof. Without loss of generality, set $t = 0$. Under the risk-neutral measure, the corresponding bond price at issue is

$$\begin{aligned} B^{\mathrm{SRBC}}_{0,T} &= R_0 - \widetilde{\mathbb{E}}_0[e^{-rT}R_T\mathbb{1}_{\{A_T\geq N,\tau_K>T\}}] + Ne^{-rT}\cdot\widetilde{\mathbb{P}}_0[A_T\geq N,\tau_K>T] \\ &= R_0 - \widetilde{\mathbb{E}}_0\left[\exp\left(\delta T + \sigma_R\sqrt{1-\rho^2_{A,R}}W_T\right)\right]\widetilde{\mathbb{E}}_0\left[e^{-rT}R_0\left(\frac{A_T}{A_0}\right)^\gamma\mathbb{1}_{\{A_T\geq N,\tau_K>T\}}\right] \\ &+ Ne^{-rT}\cdot\widetilde{\mathbb{P}}_0[A_T\geq N,\tau_K>T] \\ &= R_0 - \frac{R_0}{A_0^\gamma}e^{-(r-\delta-\frac{1}{2}\sigma_R^2(1-\rho^2_{A,R}))T}\cdot\widetilde{\mathbb{E}}_0\left[A_T^\gamma\mathbb{1}_{\{A_T\geq N,\tau_K>T\}}\right] + Ne^{-rT}\cdot\widetilde{\mathbb{P}}_0[A_T\geq N,\tau_K>T] \\ &= R_0 - \frac{R_0}{A_0^\gamma}e^{[-\gamma r+\frac{1}{2}(\gamma-\gamma^2)\sigma_A^2]T}\int_N^\infty a^\gamma\widetilde{\mathbb{P}}_0\left[A_T\in da,\tau_K>T\right] + Ne^{-rT}\int_N^\infty\widetilde{\mathbb{P}}_0[A_T\in da,\tau_K>T]. \end{aligned} \tag{76}$$

We are able to calculate these two integrals using a result found in [26]:

Theorem 4. (Joint Density for Location and Maximum)

Begin with a standard Brownian motion W on a probability space and define

$$\begin{aligned}
X_t &= \mu t + \sigma W_t \\
\tau_a &= \min\{t : X_t = a\} \\
\underline{X}_t &= \min_{0\le s\le t} X_s \\
g(x,y,t,\mu) &:= \frac{1}{\sigma\sqrt{t}}\phi\Big(\frac{x-\mu t}{\sigma\sqrt{t}}\Big)\bigg(1-\exp\Big(-\frac{4y^2-4xy}{2\sigma^2 t}\Big)\bigg) \\
\phi(x) &:= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = \Phi'(x).
\end{aligned} \tag{77}$$

Then

$$\widetilde{\mathbb{P}}_0\left[X_t \in dx, \underline{X}_t \ge y\right] = g(-x,-y,t,-\mu)dx = g(x,y,t,\mu)dx. \tag{78}$$

Proof. See the proof of Theorem 2.1(i) in [26]. □

We employ the notation of Theorem 2.1(i) in [26] and focus on standard Brownian motions by setting

$$\begin{aligned}
\mu &:= r - \frac{1}{2}\sigma_A^2 \\
\sigma &:= \sigma_A \\
y &:= \ln\frac{K}{A_0} \le 0 \\
w &:= \ln\frac{N}{A_0} \le 0 \\
X_t &:= \ln\frac{A_t}{A_0} \\
\underline{X}_t &:= \min_{0\le s\le t}\ln\frac{A_s}{A_0}.
\end{aligned} \tag{79}$$

From these definitions, it follows that

$$\begin{aligned}
A_t &= A_0 e^{X_t} \\
\{\tau_{\mathrm{K}} > T\} &= \left\{\underline{X}_T > \ln\left(\frac{K}{A_0}\right)\right\} := \left\{\tau_y^X > T\right\}.
\end{aligned} \tag{80}$$

Using this notation, we now compute the remaining integrals in (76):

$$\begin{aligned}
&\int_N^\infty a^\gamma \cdot \widetilde{\mathbb{P}}_0\Big[A_T \in da, \tau_{\mathrm{K}} > T\Big] = \widetilde{\mathbb{E}}_0\Big[A_T^\gamma \mathbb{1}_{\{A_T>N,\tau_{\mathrm{K}}>T\}}\Big] \\
&= \widetilde{\mathbb{E}}_0\bigg[e^{\gamma\Big(\ln(A_0)+X_T\Big)}\mathbb{1}_{\{X_T>w,\tau_y^X>T\}}\bigg] = \int_w^\infty e^{\gamma(x+\ln A_0)}\widetilde{\mathbb{P}}_0\Big[X_T\in dx, \tau_y^X > T\Big] \\
&= \int_w^\infty e^{\gamma(x+\ln A_0)}\frac{1}{\sigma_A\sqrt{T}}\phi\Big(-\frac{(x-\mu T)}{\sigma_A\sqrt{T}}\Big)\bigg(1-\exp\Big(-\frac{4y(y-x)}{2\sigma_A^2 T}\Big)\bigg)dx \\
&= (A_0)^\gamma\int_w^\infty \frac{1}{\sqrt{2\pi\sigma_A^2 T}}\exp\Big(-\frac{(x-\mu T)^2-2\sigma_A^2\gamma T x}{2\sigma_A^2 T}\Big)dx \\
&- (A_0)^\gamma\int_w^\infty \frac{1}{\sqrt{2\pi\sigma_A^2 T}}\exp\Big(-\frac{(x-\mu T)^2+4y(y-x)-2\sigma_A^2\gamma T x}{2\sigma_A^2 T}\Big)dx.
\end{aligned} \tag{81}$$

It follows from completing the square in the exponent of the normal density that

$$
\begin{aligned}
&\int_w^\infty \frac{1}{\sqrt{2\pi\sigma_A^2 T}} \exp\Big(- \frac{(x-\mu T)^2 - 2\sigma_A^2 \gamma T x}{2\sigma_A^2 T}\Big) dx \\
&= e^{(\gamma\mu + \frac{1}{2}\gamma^2\sigma_A^2)T} \Phi\left(- \frac{w - (\mu + \gamma\sigma_A^2)T}{\sqrt{\sigma_A^2 T}} \right) \\
&\int_w^\infty \frac{1}{\sqrt{2\pi\sigma_A^2 T}} \exp\Big(- \frac{(x-\mu T)^2 + 4y(y-x) - 2\sigma_A^2 \gamma T x}{2\sigma_A^2 T}\Big) dx \\
&= e^{\frac{2\mu y}{\sigma_A^2} + 2\gamma y} e^{[\gamma\mu + \frac{1}{2}\gamma^2\sigma_A^2]T} \Phi\left(- \frac{w - 2y - (\mu + \gamma\sigma_A^2)T}{\sqrt{\sigma_A^2 T}} \right).
\end{aligned}
\tag{82}
$$

Substitution of the results of (82) into (81) and leads to

$$
\begin{aligned}
&R_0 - \frac{R_0}{A_0^\gamma} e^{-(\gamma(r - \frac{1}{2}\sigma_A^2) + \frac{1}{2}\gamma^2\sigma_A^2)T} \int_N^\infty a^\gamma \widetilde{\mathbb{P}}_0 \Big[A_T \in da, \tau_K > T \Big] \\
&= R_0 - R_0 \left(\Phi\left(- \frac{w - (\mu + \gamma\sigma_A^2)T}{\sqrt{\sigma_A^2 T}} \right) - e^{2\gamma y + \frac{2\mu y}{\sigma_A^2}} \Phi\left(- \frac{w - 2y - (\mu + \gamma\sigma_A^2)T}{\sqrt{\sigma_A^2 T}} \right) \right) \\
&= R_0 \left[\Phi\left(\frac{w - (\mu + \gamma\sigma_A^2)T}{\sqrt{\sigma_A^2 T}} \right) + e^{2\gamma y + \frac{2\mu y}{\sigma_A^2}} \Phi\left(- \frac{w - 2y - (\mu + \gamma\sigma_A^2)T}{\sqrt{\sigma_A^2 T}} \right) \right].
\end{aligned}
\tag{83}
$$

Similiarily, substituting $\gamma = 0$ into (81) results in

$$
\int_w^\infty \widetilde{\mathbb{P}}_0 \Big[X_T \in dx, \tau_y^X > T \Big] = \Phi\left(- \frac{w - \mu T}{\sqrt{\sigma_A^2 T}} \right) - e^{\frac{2\mu y}{\sigma_A^2}} \Phi\left(- \frac{w - 2y - \mu T}{\sqrt{\sigma_A^2 T}} \right). \tag{84}
$$

Assembling the computations in (83) and (84), along with substitution of w, y in terms of K, N and A_0, and using the form (76) leads to the closed-form solution for the weak covenant case. Note that we have $w = \ln \frac{N}{A_0}$ as our lower limit of integration in the weak covenant case. To switch to the strong covenant case, we substitute $y = \ln \frac{K}{A_0}$ for our lower limit and the result for bond price and associated spreads and loss given default follows.

□

4.2. Consistency and Reduction to Black-Cox Model

As a result of the closed form above for $B_{t,T}^{\text{SRBC}}$, the consistency of the model as $T \to t$ follows quickly for both weak and strong covenants:

Lemma 6. *If $A_t > N$ in the weak covenant case, or if $A_t > K$ in the strong covenant case, then*

$$
\begin{aligned}
\lim_{T \to t^+} B_{t,T}^{\text{SRBC}}(\cdot) &= N \\
\lim_{(\gamma, R_t) \to (1, A_t)} B_{t,T}^{\text{SRBC}}(\cdot) &= B_{t,T}^{\text{BC}}(\cdot).
\end{aligned}
\tag{85}
$$

Proof. Consider that for both weak and strong covenants, the explicit formulae for distances-to-default lead to

$$\begin{aligned}
&\lim_{T\to t^+}\left[\Phi(d_0^{(\cdot)})-\left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi(x_0^{(\cdot)})\right]=1\\
&\lim_{T\to t^+}\left[\Phi(-d_\gamma^{(\cdot)})+\left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+(2\gamma-1)}\Phi(x_\gamma^{(\cdot)})\right]=0.
\end{aligned} \tag{86}$$

Hence, $\lim_{T\to t^+} B_{t,T}^{\mathrm{SRBC}}(\cdot)=N$ follows from the closed-formula given in Theorem 3, as does the reduction to the Black-Cox price when $\gamma\to 1$ and $R_t\to A_t$.
□

4.3. Greeks and Comparison with Standard Black-Cox model

Using our closed formula for bond price, we also compute the Greeks assuming stochastic recovery:

Lemma 7. (Greeks for SRBC). *Let $B_{t,T}^{\mathrm{SRBC}}$ be the zero-coupon bond price from the Black-Cox model with stochastic recovery (70) under a weak covenant. Then the* ***Recovery Greeks*** *are given by*

$$\begin{aligned}
\frac{\partial}{\partial R}B_{t,T}^{\mathrm{SRBC}}(\cdot)&=\Phi(-d_\gamma^{(\cdot)})+\left(\frac{K}{A}\right)^{\kappa_\gamma}\Phi(x_\gamma^{(\cdot)})\\
\frac{\partial}{\partial \gamma}B_{t,T}^{\mathrm{SRBC}}(\cdot)&=R\cdot\left[\sigma_A\sqrt{T-t}\left(-\phi(-d_\gamma^{(\cdot)})+\left(\frac{K}{A}\right)^{\kappa_\gamma}\phi(x_\gamma^{(\cdot)})\right)\right]\\
&\quad+R\cdot\left[2\ln\left(\frac{K}{A}\right)\left(\frac{K}{A}\right)^{\kappa_\gamma}\Phi(x_\gamma^{(\cdot)})\right]\\
\kappa_\alpha&:=\frac{2r}{\sigma_A^2}+(2\alpha-1)\,,\forall\alpha\in\mathbb{R}.
\end{aligned} \tag{87}$$

4.4. CDS Pricing with Recovery Risk

In this section, we price CDS using the SRBC model. We again consider both the weak and strong-covenant cases, and the results below depend on the bond prices in both covenant settings. Because of the general definition of a CDS premium under a structural model in Section 2, the premium here is the same as that computed for the BC model, with the exception that recovery is now defined by the correlated process R:

Theorem 5. (CDS Premium under Stochastic Recovery Black-Cox Model) *Suppose Assumptions 5–7 hold. Then the CDS premium (23) is given by:*

$$P_{t,T}^{\mathrm{SRBC}}=\frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}}>T\right]+\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]-\frac{B_{t,T}^{\mathrm{SRBC}}}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}]-e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{BC}}>T]}{r}+\frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]}. \tag{88}$$

In the strong covenant case, where $K \geq N$, this reduces to the closed formula

$$\begin{aligned}
P^{\mathrm{SRBC}}_{t,T}(s) &= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}} \leq T\}}\right] - \frac{B^{\mathrm{SRBC}}_{t,T}(s)}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}} \leq T\}}]-e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}}>T]}{r} + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}} \leq T\}}\right]} \\
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T\right] &= \Phi\left(d^s_0\right) - \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x^s_0\right) = 1 - \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(s) \\
\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}} \leq T\}}] &= A_t\left[\Phi(-d^s_1) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x^s_1)\right].
\end{aligned} \tag{89}$$

In the weak covenant case, where $K \geq N$, this reduces to the closed formula

$$\begin{aligned}
P^{\mathrm{SRBC}}_{t,T}(w) &= \frac{e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] + \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}\right] - \frac{B^{\mathrm{SRBC}}_{t,T}(w)}{N}}{\frac{1-\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}]-e^{-r(T-t)}\widetilde{\mathbb{P}}_t[\tau_{\mathrm{BC}}>T]}{r} + \frac{1}{2}\cdot\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}\right]} \\
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{BC}} > T\right] &= \widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T, A_T \geq N\right] = 1 - \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(w) \\
\widetilde{\mathbb{P}}_t\left[\tau_{\mathrm{K}} > T\right] &= \Phi\left(d^s_0\right) - \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x^s_0\right) = 1 - \widetilde{\mathrm{PD}}^{\mathrm{BC}}_{t,T}(s) \\
\widetilde{\mathbb{E}}_t\left[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}} \leq T\}}\right] &= \widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}} \leq T\}}] + e^{-r(T-t)}\left(\widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T] - \widetilde{\mathbb{P}}_t[\tau_{\mathrm{K}} > T, A_T \geq N]\right) \\
\widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}} \leq T\}}] &= A_t\left[\Phi(-d^s_1) + \left(\frac{K}{A_t}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x^s_1)\right].
\end{aligned} \tag{90}$$

Proof. Changing to the SRBC model requires only a change to the corresponding bond price in (23), as the default trigger (time) remains τ_{BC}.

□

Because of this closed form, we have consistency in our model as our parameters return to the standard Black-Cox model without stochastic recovery:

Lemma 8. (Premiums Are Model-Consistent) *The SRBC CDS premiums* (88) *are consistent with the BC CDS premiums* (40), *in that*

$$\lim_{(\gamma, R_t)\to(1, A_t)} P^{\mathrm{SRBC}}_{t,T}(\cdot) = P^{\mathrm{BC}}_{t,T}(\cdot). \tag{91}$$

Proof. This result follows directly from the closed form for CDS premiums in Theorems 2 and 5, and the result

$$\lim_{(\gamma, R_t)\to(1, A_t)} B^{\mathrm{SRBC}}_{t,T}(\cdot) = B^{\mathrm{BC}}_{t,T}(\cdot) \tag{92}$$

in Lemma 6.

□

5. The Implied Recovery and Recovery Risk Premium

5.1. Implied Recovery Rates From Observed CDS Premia

We introduce notation in this section only for the function of A and t that defines the probability of default. Specifically, define for any $\alpha \in \mathbb{R}$:

$$\begin{aligned} F_\alpha^{(w)}(A,t) &:= \Phi(-d_\alpha^w) + \left(\frac{K}{A}\right)^{\kappa_\alpha} \Phi(x_\alpha^w) \\ F_\alpha^{(s)}(A,t) &:= \Phi(-d_\alpha^s) + \left(\frac{K}{A}\right)^{\kappa_\alpha} \Phi(x_\alpha^s). \end{aligned} \tag{93}$$

Here, κ_α is the quantity defined in (87). Furthermore, define

$$\begin{aligned} \mathrm{MGF}(A_t,t;w) &:= \widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{BC}}-t)}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}] \\ \mathrm{MGF}(A_t,t;s) &:= \widetilde{\mathbb{E}}_t[e^{-r(\tau_{\mathrm{K}}-t)}\mathbb{1}_{\{\tau_{\mathrm{K}}\le T\}}] \\ \mathrm{Pay}(A_t,t;w) &:= \frac{1-\mathrm{MGF}(A_t,t;w)-e^{-r(T-t)}\left(1-\widetilde{\mathrm{PD}}_{t,T}^{\mathrm{BC}}(w)\right)}{r} + \frac{1}{2}\mathrm{MGF}(A_t,t;w) \\ \mathrm{Pay}(A_t,t;s) &:= \frac{1-\mathrm{MGF}(A_t,t;s)-e^{-r(T-t)}\left(1-\widetilde{\mathrm{PD}}_{t,T}^{\mathrm{BC}}(s)\right)}{r} + \frac{1}{2}\mathrm{MGF}(A_t,t;s) \\ \mathrm{Pro}(A_t,R_t,\gamma,t;w) &:= \mathrm{MGF}(A_t,t;w) - \frac{R_t}{N}F_\gamma^{(w)}(A,t) \\ \mathrm{Pro}(A_t,R_t,\gamma,t;s) &:= \mathrm{MGF}(A_t,t;s) - \frac{R_t}{N}F_\gamma^{(s)}(A,t). \end{aligned} \tag{94}$$

Using the results of Theorem 3 to calculate the quantity $\widetilde{\mathbb{E}}_t\left[D(t,\tau)\frac{R_{\tau_{\mathrm{BC}}}}{N}\mathbb{1}_{\{\tau_{\mathrm{BC}}\le T\}}\right]$ of the protection leg (22) in terms of recovery R_t, we obtain the result

$$\begin{aligned} P_{t,T}^{\mathrm{SRBC}}(w) &= \frac{\mathrm{Pro}(A_t,R_t,\gamma,t;w)}{\mathrm{Pay}(A_t,t;w)} \\ P_{t,T}^{\mathrm{SRBC}}(s) &= \frac{\mathrm{Pro}(A_t,R_t,\gamma,t;s)}{\mathrm{Pay}(A_t,t;s)}. \end{aligned} \tag{95}$$

We also define the recovery rate $\bar{R}_{t,T} = \bar{R}(t,T) := \frac{R_t}{N}$, and solve the Stochastic Recovery Black-Cox CDS premium $P_{t,T}^{\mathrm{SRBC}}$ in (95) for the recovery rate to yield the *Implied Recovery Rate*

$$\begin{aligned} \bar{R}_{t,T}^{\mathrm{Imp}}(w) &= \frac{\mathrm{MGF}(A_t,t;w)-\mathrm{Pay}(A_t,t;w)\bar{P}_{t,T}^{\mathrm{Mkt}}(w)}{F_\gamma^{(w)}(A,t)} \\ \bar{R}_{t,T}^{\mathrm{Imp}}(s) &= \frac{\mathrm{MGF}(A_t,t;s)-\mathrm{Pay}(A_t,t;s)\bar{P}_{t,T}^{\mathrm{Mkt}}(s)}{F_\gamma^{(s)}(A,t)}. \end{aligned} \tag{96}$$

Suppose now there are two CDS on the same obligor, except one references a senior issue with maturity $T_{\text{Sr}} = T$ while the other references a junior issue with maturity $T_{\text{Jr}} = T$. Denote the premiums associated to these two CDS as $P_{\text{Sr}}^{\text{Mkt}}$ and $P_{\text{Jr}}^{\text{Mkt}}$ respectively. Then the market implied recovery ratio is

$$\begin{aligned}
\frac{\bar{R}_{\text{Jr}}^{\text{Imp}}(t,T)(w)}{\bar{R}_{\text{Sr}}^{\text{Imp}}(t,T)(w)} &= \frac{1 - \frac{\text{Pay}(A_t,t;w)}{\text{MGF}(A_t,t;w)} P_{\text{Jr}}^{\text{Mkt}}(t,T)}{1 - \frac{\text{Pay}(A_t,t;w)}{\text{MGF}(A_t,t;w)} P_{\text{Sr}}^{\text{Mkt}}(t,T)} \\
\frac{\bar{R}_{\text{Jr}}^{\text{Imp}}(t,T)(s)}{\bar{R}_{\text{Sr}}^{\text{Imp}}(t,T)(s)} &= \frac{1 - \frac{\text{Pay}(A_t,t;s)}{\text{MGF}(A_t,t;s)} P_{\text{Jr}}^{\text{Mkt}}(t,T)}{1 - \frac{\text{Pay}(A_t,t;s)}{\text{MGF}(A_t,t;s)} P_{\text{Sr}}^{\text{Mkt}}(t,T)}
\end{aligned} \tag{97}$$

where as before $\bar{R}_{\text{Sr}}^{\text{Imp}}(t,T)$ and $\bar{R}_{\text{Jr}}^{\text{Imp}}(t,T)$ are the market implied term structures for recovery rates at time t. Notice that the right hand side of (97) requires the observed CDS premiums as well as the calibrated parameters of the original Black-Cox model, but is independent of the partial information parameter γ. Inverting recovery to obtain implied premiums for junior and senior issues with the same maturity leads to

$$\begin{aligned}
\frac{P_{\text{Jr}}^{\text{Imp}}(t,T)(w)}{P_{\text{Sr}}^{\text{Imp}}(t,T)(w)} &= \frac{1 - \bar{R}_{t,T}^{\text{Jr}} \frac{F_\gamma^{(w)}(A_t,t)}{\text{MGF}(A_t,t;w)}}{1 - \bar{R}_{t,T}^{\text{Sr}} \frac{F_\gamma^{(w)}(A_t,t)}{\text{MGF}(A_t,t;w)}} \\
\frac{P_{\text{Jr}}^{\text{Imp}}(t,T)(w)}{P_{\text{Sr}}^{\text{Imp}}(t,T)(s)} &= \frac{1 - \bar{R}_{t,T}^{\text{Jr}} \frac{F_\gamma^{(s)}(A_t,t)}{\text{MGF}(A_t,t;s)}}{1 - \bar{R}_{t,T}^{\text{Sr}} \frac{F_\gamma^{(s)}(A_t,t)}{\text{MGF}(A_t,t;s)}}
\end{aligned} \tag{98}$$

which also requires knowledge of the recovery process, as well as γ.

Remark 3. *The spread-ratio* (98) *is the Stochastic Recovery Black-Cox model implementation of Equation (6) in* [14] *used to extract recovery risk premiums from empirical data.*

5.2. The Price of Recovery Risk

Lemma 9. (Comparison of the Black-Cox and 2d Black-Cox Model). *Using our notation for F_α defined in* (93) *above, we can write our zero-coupon bond prices as*

$$\begin{aligned}
B_{t,T}^{\text{BC}}(w) &= Ne^{-r(T-t)} \cdot [1 - F_0^{(w)}(A_t,t)] + A_t F_1^{(w)}(A_t,t) \\
B_{t,T}^{\text{BC}}(s) &= Ne^{-r(T-t)} \cdot [1 - F_0^{(s)}(A_t,t)] + A_t F_1^{(s)}(A_t,t) \\
B_{t,T}^{\text{SRBC}}(w) &= Ne^{-r(T-t)} \cdot [1 - F_0^{(w)}(A_t,t)] + R_t F_\gamma^{(w)}(A_t,t) \\
B_{t,T}^{\text{SRBC}}(s) &= Ne^{-r(T-t)} \cdot [1 - F_0^{(s)}(A_t,t)] + R_t F_\gamma^{(s)}(A_t,t).
\end{aligned} \tag{99}$$

Suppressing the (w/s) superscript, we are able to compute the price of Recovery Risk *in our zero coupon bond and associated credit spread:*

$$\begin{aligned}
B_{t,T}^{\text{SRBC}} &= B_{t,T}^{\text{BC}} + RR(B_{t,T}) \\
&= B_{t,T}^{\text{BC}} + [R_t F_\gamma(A_t,t) - A_t F_1(A_t,t)] \\
S_{t,T}^{\text{SRBC}} &= S_{t,T}^{\text{BC}} + RR(S_{t,T}) \\
&= S_{t,T}^{\text{BC}} + \frac{1}{T-t} \ln \left(\frac{1 + e^{r(T-t)} \frac{A_t}{N} \frac{F_1(A_t,t)}{1-F_0(A_t,t)}}{1 + e^{r(T-t)} \frac{R_t}{N} \frac{F_\gamma(A_t,t)}{1-F_0(A_t,t)}} \right).
\end{aligned} \tag{100}$$

6. The Effect of Coupons

If coupons are paid at rate C per unit time, in addition to notional N, then (conditioned on the information $(A_t, R_t) = (A, R)$) the total value of a coupon bond in our model is now derived from the expected present value of coupons paid *until* $\tau_K \wedge T$:

$$\begin{aligned} B_{t,T}^{\mathrm{SRBC},c} &:= B_{t,T}^{\mathrm{SRBC}} + \widetilde{\mathbb{E}}_t[\mathrm{PV}_{t,T}\,[\mathrm{Cpns}]] \\ \widetilde{\mathbb{E}}_t[\mathrm{PV}_{t,T}\,[\mathrm{Cpns}]] &:= \widetilde{\mathbb{E}}_t\left[\int_t^T Ce^{-r(s-t)}\mathbb{1}_{\{\tau_K > s\}}ds\right]. \end{aligned} \tag{101}$$

By using moment-generating functions similar to those employed in calculating the premium-leg of our CDS, we compute the expected present value of coupon income to be

$$\begin{aligned} &\widetilde{\mathbb{E}}_t[\mathrm{PV}_{t,T}\,[\mathrm{Cpns}]] = \widetilde{\mathbb{E}}_t\left[\int_t^{\tau_K \wedge T} Ce^{-r(s-t)}ds\right] \\ &= \frac{C}{r}\left[1 - \widetilde{\mathbb{E}}_t\left[e^{-r(\tau_K - t)}\mathbb{1}_{\{\tau_K \le T\}}\right] - e^{-r(T-t)}\widetilde{\mathbb{P}}_t\left[\tau_K > T\right]\right] \\ &= \frac{C}{r}\left[1 - \frac{A}{K}\left[\Phi(-d_1^s) + \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}+1}\Phi(x_1^s)\right] - e^{-r(T-t)}\left(\Phi\left(d_0^s\right) - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}-1}\Phi\left(x_0^s\right)\right)\right]. \end{aligned} \tag{102}$$

An interesting limit in these calculation is the passage to an infinite horizon for redemption time. In the next section, we calculate the value of perpetual bonds and show consistency with the value (102) computed for coupon bonds under finite horizon.

6.1. Perpetual Bonds

For bondholders interested in long-term bonds, say of the 100-year variety [42,43], an approximation of this long-term bond as a perpetual bond with stochastic recovery is a useful measure of the risk involved with such an instrument. The addition of stochastic recovery, even when the firm-value level that triggers default is known, reflects the uncertainty of what the true value of a firm's assets may be worth upon default when bankruptcy occurs far off into the future.

We begin with some modeling assumptions:

- Managers set a bankruptcy level K and accordingly we recall the definition (8) of τ_K as the first passage time of assets A to level K.
- Coupons are paid continuously at rate C, there is a risk-free interest rate r, and recovery at bankruptcy is R_{τ_K}.

Based on the above assumptions, and conditioned on the information $(A_0, R_0) = (A, R)$, the value of debt at time 0 (wlog) is

$$B_0 = \widetilde{\mathbb{E}}_0\left[\int_0^{\tau_K} Ce^{-rs}ds + e^{-r\tau_K}R_{\tau_K}\mathbb{1}_{\{\tau_K < \infty\}}\right]. \tag{103}$$

Lemma 10. *For market parameters (γ, r, σ_A) such that*

$$\min\left\{\gamma r - \frac{1}{2}\gamma(1-\gamma)\sigma_A^2, \frac{2r}{\sigma_A^2} + 2\gamma - 1\right\} > 0, \tag{104}$$

the value of a perpetual bond under stochastic recovery is

$$B_0 = \frac{C}{r}\left(1 - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}}\right) + R\left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}+2\gamma-1} \tag{105}$$

and

$$B_0 = \lim_{T\to\infty}\left(B^{\text{SRBC}}_{0,T} + \widetilde{\mathbb{E}}_0[PV_{0,T}\,[\text{Cpns}]]\right). \tag{106}$$

Remark 4. *In the Leland model [44] of perpetual bond issuance under a structural model for firm value, the recovery upon default is modeled as*

$$R_{\tau_K} = (1-\alpha)A_{\tau_K} = (1-\alpha)K \tag{107}$$

for fractional bankruptcy cost ratio α. If we substitute $R = (1-\alpha)A$ and $\gamma = 1$ into (105), then we retain the classical Leland [44] formula

$$B_0 = \frac{C}{r}\left(1 - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}}\right) + (1-\alpha)K\left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}}. \tag{108}$$

Proof. We begin by rewriting (103) as

$$\begin{aligned} B_0 &= \widetilde{\mathbb{E}}_0\left[\frac{C}{r}(1-e^{-r\tau_K}) + e^{-r\tau_K}R_{\tau_K}\mathbb{1}_{\{\tau_K<\infty\}}\right] \\ &= \frac{C}{r} + \widetilde{\mathbb{E}}_0\left[e^{-r\tau_K}\left(R_{\tau_K} - \frac{C}{r}\right)\mathbb{1}_{\{\tau_K<\infty\}}\right]. \end{aligned} \tag{109}$$

For $\gamma = \rho_{A,R}\frac{\sigma_R}{\sigma_A}$ and orthogonal decomposition

$$W_t^R = \rho_{A,R}W_t^R + \sqrt{1-\rho_{A,R}^2}\,W_t, \tag{110}$$

Ito-Calculus leads to, $\forall t \le s$,

$$\frac{R_s}{R_t} = \left(\frac{A_s}{A_t}\right)^{\gamma} e^{\left[(r-\frac{1}{2}\sigma_R^2)-\gamma(r-\frac{1}{2}\sigma_A^2)\right](s-t)+\sigma_R\sqrt{1-\rho_{A,R}^2}(W_s-W_t)}. \tag{111}$$

Substituting $t = 0$ and $s = \tau_K$ returns

$$R_{\tau_K} = R_0\left(\frac{K}{A_0}\right)^{\gamma} e^{\left[(r-\frac{1}{2}\sigma_R^2)-\gamma(r-\frac{1}{2}\sigma_A^2)\right]\tau_K+\sigma_R\sqrt{1-\rho_{A,R}^2}W_{\tau_K}}. \tag{112}$$

If we define $f_K(t)$ as the density of our first passage time to level K, then we see that

$$\begin{aligned} B_0 &= \frac{C}{r} + \widetilde{\mathbb{E}}_0\left[e^{-r\tau_K}\left(R_{\tau_K} - \frac{C}{r}\right)\mathbb{1}_{\{\tau_K<\infty\}}\right] \\ &= \frac{C}{r} + \int_0^\infty e^{-rt}\left(\widetilde{\mathbb{E}}_0\left[R_{\tau_K} \mid \tau_K = t\right] - \frac{C}{r}\right) f_K(t)dt. \end{aligned} \tag{113}$$

Assuming that at default, the bondholder receives instead the stochastically varying amount R_{τ_K}, it follows that, conditioned on $(A_0, R_0) = (A, R)$,

$$\begin{aligned} B_0 &= \frac{C}{r} + \widetilde{\mathbb{E}}_0\left[e^{-r\tau_K}\left(R_{\tau_K} - \frac{C}{r}\right)\mathbb{1}_{\{\tau_K<\infty\}}\right] \\ &= \frac{C}{r} + \int_0^\infty e^{-rt}\left(\widetilde{\mathbb{E}}_0\left[R_{\tau_K} \mid \tau_K = t\right] - \frac{C}{r}\right) f_K(t)dt \\ &= \frac{C}{r}\left(1 - \int_0^\infty e^{-rt}f_K(t)dt\right) \\ &\quad + \int_0^\infty e^{-rt}R\left(\frac{K}{A}\right)^{\gamma} e^{\left((1-\gamma)r+\frac{1}{2}\gamma(1-\gamma)\sigma_A^2\right)t} f_K(t)dt. \end{aligned} \tag{114}$$

When rewritten, with $r^* := \gamma r - \frac{1}{2}\gamma(1-\gamma)\sigma_A^2$, the bond value is

$$B_0 = \frac{C}{r}\left(1 - \widetilde{\mathbb{E}}_0[e^{-r\tau_K}\mathbb{1}_{\{\tau_K<\infty\}}]\right) + R\left(\frac{K}{A}\right)^{\gamma}\widetilde{\mathbb{E}}_0[e^{-r^*\tau_K}\mathbb{1}_{\{\tau_K<\infty\}}]. \tag{115}$$

Using well known results for moment-generating functions of first-passage times of Brownian motions with drift, assuming $r^* > 0$, we obtain the closed form

$$\begin{aligned}
\widetilde{\mathbb{E}}_0[e^{-r\tau_K}1_{\{\tau_K<\infty\}}] &= \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}} \\
\widetilde{\mathbb{E}}_0[e^{-r^*\tau_K}1_{\{\tau_K<\infty\}}] &= \left(\frac{K}{A}\right)^{y(r,r^*,\sigma_A)} = \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}+\gamma-1} \\
y(g,z,\sigma) &:= \frac{g - \frac{1}{2}\sigma^2 + \sqrt{(g-\frac{1}{2}\sigma^2)^2 + 2z\sigma^2}}{\sigma^2}
\end{aligned} \tag{116}$$

which results in the closed-form expression for a perpetual bond with stochastic recovery

$$B_0 = \frac{C}{r}\left(1 - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}}\right) + R\left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}+2\gamma-1}. \tag{117}$$

By utilizing (102), (116), and Theorem 3, we see that if in addition to $r^* > 0$, we also have $r > \left(\frac{1}{2} - \gamma\right)\sigma_A^2$, then it follows that $\forall t \geq 0$

$$\begin{aligned}
\lim_{T\to\infty} \widetilde{\mathbb{E}}_t[\mathrm{PV}_{t,T}\,[\mathrm{Cpns}]] &= \frac{C}{r}\left[1 - \left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}}\right] \\
\lim_{T\to\infty} B_{t,T}^{\mathrm{SRBC}} &= R\left(\frac{K}{A}\right)^{\frac{2r}{\sigma_A^2}+2\gamma-1}
\end{aligned} \tag{118}$$

and so we achieve consistency in the passage to infinite horizon.
□

7. Conclusions

In this work we introduced the Stochastic Recovery Black-Cox (SRBC) model which is essentially a Black-Cox model with an extra recovery risk driver. We then consider pricing with recovery risk and in particular explicitly compute closed form prices for both bonds and CDS under this framework, as well as associated risk metrics. This framework allows us to compute the Recovery implies by bond and CDS prices as well as compute the recovery risk premium.

Acknowledgments: The authors would like to thank Joe Campolieti (Wilfrid Laurier) for making us aware of the Reference: Hua He, William P. Keirstead, and Joachim Rebholz (1998). as well as Peter Carr (NYU) and Steven Shreve (Carnegie Mellon) for useful discussions on pricing defaultable bonds and CDS. We are indebted to Harvey Stein (Bloomberg) for introducing us to the partial information perspective for credit modeling which forms the theoretical justification for our much of this work. Finally, we sincerely thank the anonymous reviewers for their careful reading and suggestions for improvement, which we have followed and implemented in this version of our work.

Author Contributions: The two authors contribute equally to this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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