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Potential Densities for Taxed Spectrally Negative Lévy Risk Processes

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Abstract: This paper revisits the spectrally negative Lévy risk process embedded with the general tax structure introduced in Kyprianou and Zhou (2009). A joint Laplace transform is found concerning the first down-crossing time below level 0. The potential density is also obtained for the taxed Lévy risk process killed upon leaving $[0, b]$. The results are expressed using scale functions.

Keywords: spectrally negative Lévy process; general tax structure; first crossing time; joint Laplace transform; potential measure

1. Introduction

The study of loss-carry-forward tax was initiated in Albrecher and Hipp (2007) under the framework of the classical compound Poisson risk model, in which the taxation is imposed at a fixed rate as long as the surplus process of the company stays at the running supremum. In particular, criteria were obtained for the optimal taxation level that maximizes the expected (discounted) accumulated tax payments, and an interesting simple relationship was recovered between the ruin probabilities for scenarios with and without tax. Later, the underlying risk process was generalized to the spectrally negative Lévy process in Albrecher et al. (2008), followed by further generalizations in terms of the tax structures in Kyprianou and Zhou (2009) and Kyprianou and Ott (2012), where new identities were derived for the two-sided exit problem and a generalized version of the Gerber-Shiu function as well as the the net present value of tax payments until ruin. On the other hand, Wei (2009) studied the asymptotic formulas of the ruin probability for the classical compound Poisson risk process with constant credit interest rate and surplus-dependent tax rate. In the mean time, a similar problem for spectrally negative Lévy risk process with periodic tax was investigated in Hao and Tang (2009). In addition, Renaud (2009) obtained the explicit expressions for arbitrary moments of the accumulated discounted tax payments. Moreover, the Gerber-Shiu functions were presented in Wei et al. (2010) and Cheung and Landriault (2012) for Markov-modulated risk models with constant tax rate and for classical compound Poisson risk model with surplus-dependent premium and tax rate, respectively. Several two-sided exit problems for the time-homogeneous diffusion risk processes with surplus-dependent tax rate were investigated in Li et al. (2013). Concerning the related optimization problem, identification of the optimal taxation strategy that maximizes the expected (discounted) accumulated tax pay-out until ruin was addressed in Wang and Hu (2012). More recently, the two-sided exit problem in terms of the linear draw-down time for a spectrally negative Lévy risk process with loss-carry-forward tax was solved in Avram et al. (2017). Another periodic taxation different from that in Hao and Tang (2009) was introduced in Zhang et al. (2017), where the Gerber-Shiu function for the spectrally negative Lévy risk process was studied.

In the present paper, we continue the study of the general tax structure introduced in Kyprianou and Zhou (2009) for the spectrally negative Lévy risk process. Our main goal is to identify the potential density of the taxed Lévy risk process killed upon leaving interval $[0, b]$. To our best knowledge, such a potential density has not been obtained in the literature for taxed processes. In addition, we also find a joint Laplace transform concerning the first down-crossing time of level 0. To this end, we adopt an excursion theory approach, and the aforementioned results are expressed using scale functions for spectrally negative Lévy processes.

This paper is arranged as follows. In Section 2, some preliminary results concerning the spectrally negative Lévy processes and the taxed Lévy risk models with general tax structure are reviewed. The main results on the taxed spectrally negative Lévy risk processes with proofs are included in Section 3.

2. Mathematical Presentation of the Problem

Under probability laws $\{\mathbb{P}_x; x \in \mathbb{R}\}$ and with natural filtration $\{\mathcal{F}_t; t \geq 0\}$, let $X = \{X(t); t \geq 0\}$ be a spectrally negative Lévy process with the usual exclusion of pure increasing linear drift and the negative of a subordinator. Denote by $\{\overline{X}(t) := \sup\limits_{0 \leq s \leq t} X(s); t \geq 0\}$ the running supremum process of X. We assume that, in the case of no control, the risk process evolves according to the law of $\{X(t); t \geq 0\}$.

Define a measurable function $\gamma : [0, +\infty) \to [0, 1)$ satisfying

$$\int_0^{+\infty} (1 - \gamma(z))\, \mathrm{d}z = +\infty, \tag{1}$$

and

$$\overline{\gamma}_x(z) := x + \int_x^z (1 - \gamma(w))\mathrm{d}w$$

for $z \geq x \geq 0$. Intuitively, when an insurer is in a profitable situation at time t, the proportion of the insurer's income that is paid out as tax at time t is equal to $\gamma\left(\overline{X}(t)\right)$, where the insurer is in a profitable situation at time t if $X(t) = \overline{X}(t)$. Therefore, the cumulative tax until time t is given by

$$\int_0^t \gamma\left(\overline{X}(s)\right) \mathrm{d}\overline{X}(s),$$

which is well defined because $\{\overline{X}(t); t \geq 0\}$ is a process of bounded variation (see also, Kyprianou and Ott 2012), and the controlled aggregate surplus process is also well defined by

$$U(t) = X(t) - \int_0^t \gamma\left(\overline{X}(s)\right) \mathrm{d}\overline{X}(s). \tag{2}$$

The taxed risk process given by Equation (2) was first investigated in Kyprianou and Zhou (2009) for $[0, 1)$-valued function γ satisfying Equation (1).

The objective of the present paper is to find the expressions for the joint Laplace transform of the ruin time and the position at ruin, conditioning on the event of ruin occurring before the up-crossing time of b

$$\mathbb{E}_x\left(\mathrm{e}^{-q\sigma_0^- + \theta U(\sigma_0^-)}; \sigma_0^- < \sigma_b^+\right) \tag{3}$$

and the q-potential measuring up to the exit time of $[0, b]$

$$\int_0^{\infty} \mathrm{e}^{-qt}\mathbb{P}_x(U(t) \in \mathrm{d}u, t < \sigma_b^+ \wedge \sigma_0^-)\mathrm{d}t \tag{4}$$

with $b \geq x \geq 0$ and $q, \theta > 0$, where σ_0^- and σ_b^+ are the first down-crossing time of level 0 (or, ruin time) and the first up-crossing time of level b defined, respectively, by

$$\sigma_0^- := \inf\{t \geq 0 : U(t) < 0\} \text{ and } \sigma_b^+ := \inf\{t \geq 0 : U(t) > b\}$$

with the usual convention $\inf \emptyset := \infty$.

The q-potential measure defined in Equation (4) plays an essential role in the theory of temporally homogeneous Markov processes. For example, the q-potential measure has turned out to be helpful in solving the occupation time involved problems (see Landriault et al. 2011 and Li and Zhou 2014), and the fluctuation problems for Lévy processes that are observed at Poisson arrival epochs (see Albrecher et al. 2016). In addition, the fluctuation quantity of Equation (3) involving the first passage times has been extensively investigated for various kinds of stochastic processes. Because σ_0^- can be viewed as the ruin time of a risk process, the study of Equation (3) finds interesting applications in risk theory. Identifying the joint distribution of the ruin time and the deficit at the ruin time, which is the key topic of the well-known Gerber-Shiu function theory, has attracted broad attention in the community of actuarial sciences (see Gerber and Shiu 1998 and the special issue launched in the journal "Insurance: Mathematics and Economics" themed on Gerber-Shiu functions in 2010).

To keep our paper self-contained, we briefly introduce some basic facts of the spectrally negative Lévy process. Let the Laplace exponent of X be defined by

$$\psi(\theta) := \log \mathbb{E}_x[\mathrm{e}^{\theta(X_1 - x)}],$$

which is known to be finite for all $\theta \in [0, \infty)$, and is strictly convex and infinitely differentiable. As defined in Bertoin (1996), for each $q \geq 0$ the scale function $W^{(q)} : [0, \infty) \to [0, \infty)$ is the unique strictly increasing and continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

where $\Phi(q)$ is the largest solution of the equation $\psi(\theta) = q$. For convenience, we extend the domain of $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for all $x < 0$. In particular, write $W = W^{(0)}$ for simplicity. When X has sample paths of unbounded variation, or when X has sample paths of bounded variation and the Lévy measure has no atoms, then the scale function $W^{(q)}$ is continuously differentiable over $(0, \infty)$. Interested readers are referred to Chan et al. (2011) for more detailed discussions on the smoothness of scale functions. In this paper, we only need the scale function $W^{(q)}$ to be right-differentiable over $(0, \infty)$, which is readily satisfied since $W^{(q)}$ is both right- and left-differentiable over $(0, \infty)$ (see, for example, p. 291 of Pistorius 2007 and Lemma 1 of Pistorius 2004). In the sequel, denote by $W_+^{(q)\prime}(x)$ the right-derivative of $W^{(q)}$ in x.

Further, define

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z) \mathrm{d}z, \quad q \geq 0,\, x \geq 0,$$

and

$$Z^{(q)}(x, \theta) = \mathrm{e}^{\theta x} \left(1 + (q - \psi(\theta)) \int_0^x \mathrm{e}^{-\theta z} W^{(q)}(z) \mathrm{d}z\right), \quad q, \theta \geq 0,\, x \geq 0,$$

with $Z^{(q)}(x) = 1$ and $Z^{(q)}(x, \theta) = \mathrm{e}^{\theta x}$ for $x < 0$.

We also briefly recall concepts in excursion theory for the reflected process $\{\overline{X}(t)-X(t);t\geq 0\}$, and we refer to Bertoin (1996) and Kyprianou (2014) for more details. For $x\in\mathbb{R}$, the process $\{L(t):=\overline{X}(t)-x;t\geq 0\}$ serves as a local time at 0 for the Markov process $\{\overline{X}(t)-X(t);t\geq 0\}$ under $\mathbb{P}_x$. Define the corresponding inverse local time as

$$L^{-1}(t):=\inf\{s\geq 0:L(s)>t\}=\sup\{s\geq 0:L(s)\leq t\}.$$

Let $L^{-1}(t-):=\lim\limits_{s\uparrow t}L^{-1}(s)$. The Poisson point process of excursions indexed by this local time is denoted by $\{(t,e_t);t\geq 0\}$

$$e_t(s):=X(L^{-1}(t))-X(L^{-1}(t-)+s),\ s\in(0,L^{-1}(t)-L^{-1}(t-)],$$

whenever $L^{-1}(t)-L^{-1}(t-)>0$. For the case of $L^{-1}(t)-L^{-1}(t-)=0$, define $e_t=\Upsilon$ with Υ being an additional isolated point. Accordingly, we denote a generic excursion as $\varepsilon(\cdot)$ (or, ε for short) belonging to the space $\mathcal{E}$ of canonical excursions. The intensity measure of the Poisson point process $\{(t,\varepsilon_t);t\geq 0\}$ is given by $\mathrm{d}t\times\mathrm{d}n$ where n is a σ-finite measure on the space $\mathcal{E}$. The lifetime of a canonical excursion ε is denoted by ζ, and its excursion height is denoted by $\bar{\varepsilon}=\sup\limits_{t\in[0,\zeta]}\varepsilon(t)$. The first passage time of a canonical excursion ε is defined by

$$\rho_b^+=\rho_b^+(\varepsilon):=\inf\{t\in[0,\zeta]:\varepsilon(t)>b\}$$

with the convention $\inf\emptyset:=\zeta$.

Denote by ε_g the excursion (away from 0) with left-end point g for the reflected process $\{\overline{X}(t)-X(t);t\geq 0\}$, and by ζ_g and $\bar{\varepsilon}_g$ denote its lifetime and excursion height, respectively; see Section IV.4 of Bertoin (1996).

3. Main Results

For the process X, define its first down-crossing time of level 0 and up-crossing time of level b, respectively, by

$$\tau_0^-:=\inf\{t\geq 0:X(t)<0\}\ \text{ and }\ \tau_b^+:=\inf\{t\geq 0:X(t)>b\}.$$

From Kyprianou (2014), the resolvent measure corresponding to X is absolutely continuous with respect to the Lebesgue measure and has a version of density given by

$$\begin{aligned}&q\int_0^\infty e^{-qt}\mathbb{E}_x(f(X(t));t<\tau_0^-\wedge\tau_b^+)\,\mathrm{d}t\\&=q\int_0^b f(y)\left(\frac{W^{(q)}(x)}{W^{(q)}(b)}W^{(q)}(b-y)-W^{(q)}(x-y)\right)\mathrm{d}y,\quad x\in[0,b).\end{aligned}\tag{5}$$

In preparation for showing the main results, we first present the following Lemma 1 which gives the joint Laplace transform of ρ_z^+ and the overshoot at ρ_z^+ of a canonical excursion ε with respect to the excursion measure n, which is a σ-finite measure on the space $\mathcal{E}$ of canonical excursions (see Section 2).

Lemma 1. *For any* $q, z > 0$*, we have*

$$
\begin{aligned}
& n\left(\mathrm{e}^{-q\rho_z^+ + \theta(z - \varepsilon(\rho_z^+))}; \bar{\varepsilon} > z\right) \\
&= \frac{W_+^{(q)\prime}(z)}{W^{(q)}(z)} Z^{(q)}(z,\theta) - \theta Z^{(q)}(z,\theta) - (q - \psi(\theta)) W^{(q)}(z).
\end{aligned} \tag{6}
$$

In particular

$$
n\left(\mathrm{e}^{-q\rho_z^+}; \bar{\varepsilon} > z\right) = \frac{W_+^{(q)\prime}(z)}{W^{(q)}(z)} Z^{(q)}(z) - q W^{(q)}(z),
$$

and

$$
n\left(\mathrm{e}^{\theta(z - \varepsilon(\rho_z^+))}; \bar{\varepsilon} > z\right) = \frac{W_+'(z)}{W(z)} Z(z,\theta) - \theta Z(z,\theta) + \psi(\theta) W(z).
$$

Proof. Taking use of the first result in Proposition 2 of Pistorius (2007), we can prove the desired results following the arguments in Lemma 2.2 of Kyprianou and Zhou (2009). □

Proposition 1 gives the joint Laplace transform of σ_0^- and the position of the process U at σ_0^-. It is similar to Theorem 1.3 of Kyprianou and Zhou (2009) where the Lévy measure is involved in the expression. The following joint Laplace transform is expressed in terms of scale functions.

Proposition 1. *For any* $q, \theta > 0$ *and* $0 \le x \le b$ *we have*

$$
\begin{aligned}
& \mathbb{E}_x\left(\mathrm{e}^{-q\sigma_0^- + \theta U(\sigma_0^-)}; \sigma_0^- < \sigma_b^+\right) \\
&= \int_x^b \frac{1}{(1 - \gamma(\overline{\gamma}_x^{-1}(z)))} \exp\left(-\int_x^z \frac{W_+^{(q)\prime}(w)}{(1 - \gamma(\overline{\gamma}_x^{-1}(w))) W^{(q)}(w)} \mathrm{d}w\right) \\
&\quad \times \left(\frac{W_+^{(q)\prime}(z)}{W^{(q)}(z)} Z^{(q)}(z,\theta) - \theta Z^{(q)}(z,\theta) - (q - \psi(\theta)) W^{(q)}(z)\right) \mathrm{d}z.
\end{aligned} \tag{7}
$$

Proof. By Theorems 1.1 in Kyprianou and Zhou (2009) (with minor adaptation), for $0 \le x \le a$, one has

$$
\mathbb{E}_x\left(\mathrm{e}^{-q\sigma_a^+}; \sigma_a^+ < \sigma_0^-\right) = \exp\left(-\int_x^a \frac{W_+^{(q)\prime}(w)}{(1 - \gamma(\overline{\gamma}_x^{-1}(w))) W^{(q)}(w)} \mathrm{d}w\right). \tag{8}
$$

For $0 \le x \le b$, by Equation (8) and the compensation formula in excursion theory we have

$$
\begin{aligned}
&\mathbb{E}_x\left(\mathrm{e}^{-q\sigma_0^- + \theta U(\sigma_0^-)}; \sigma_0^- < \sigma_b^+\right) \\
=&\,\mathbb{E}_x\left(\sum_g \mathrm{e}^{-qg} \prod_{r<g} \mathbf{1}_{\{\bar{\varepsilon}_r \le \overline{\gamma}_x(x+L(r)), \overline{\gamma}_x(x+L(g)) \le b\}}\right. \\
&\quad \left. \times\, \mathrm{e}^{-q\rho^+_{\overline{\gamma}_x(x+L(g))}(\varepsilon_g) + \theta\left(\overline{\gamma}_x(x+L(g)) - \varepsilon_g(\rho^+_{\overline{\gamma}_x(x+L(g))}(\varepsilon_g))\right)} \mathbf{1}_{\{\bar{\varepsilon}_g > \overline{\gamma}_x(x+L(g))\}}\right) \\
=&\,\mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qt} \prod_{r<t} \mathbf{1}_{\{\bar{\varepsilon}_r \le \overline{\gamma}_x(x+L(r)), \overline{\gamma}_x(x+L(t)) \le b\}}\right. \\
&\quad \left. \times \int_{\mathcal{E}} \mathrm{e}^{-q\rho^+_{\overline{\gamma}_x(x+L(t))} + \theta\left(\overline{\gamma}_x(x+L(t)) - \varepsilon(\rho^+_{\overline{\gamma}_x(x+L(t))})\right)} \mathbf{1}_{\{\bar{\varepsilon} > \overline{\gamma}_x(x+L(t))\}}\, n(\mathrm{d}\varepsilon)\, \mathrm{d}L(t)\right) \\
=&\,\mathbb{E}_x\left(\int_0^{\overline{\gamma}_x^{-1}(b)-x} \mathrm{e}^{-qL^{-1}(t-)} \prod_{r<L^{-1}(t-)} \mathbf{1}_{\{\bar{\varepsilon}_r \le \overline{\gamma}_x(x+L(r))\}}\right. \\
&\quad \left. \times\, n\left(\mathrm{e}^{-q\rho^+_{\overline{\gamma}_x(x+t)} + \theta(\overline{\gamma}_x(x+t) - \varepsilon(\rho^+_{\overline{\gamma}_x(x+t)}))} \mathbf{1}_{\{\bar{\varepsilon} > \overline{\gamma}_x(x+t)\}}\right) \mathrm{d}t\right) \\
=&\int_0^{\overline{\gamma}_x^{-1}(b)-x} \exp\left(-\int_x^{\overline{\gamma}_x(x+t)} \frac{W_+^{(q)\prime}(w)}{(1-\gamma(\overline{\gamma}_x^{-1}(w)))W^{(q)}(w)}\,\mathrm{d}w\right) \\
&\quad \times n\left(\mathrm{e}^{-q\rho^+_{\overline{\gamma}_x(x+t)} + \theta(\overline{\gamma}_x(x+t) - \varepsilon(\rho^+_{\overline{\gamma}_x(x+t)}))}; \bar{\varepsilon} > \overline{\gamma}_x(x+t)\right) \mathrm{d}t \\
=&\int_x^b \exp\left(-\int_x^s \frac{W_+^{(q)\prime}(w)}{(1-\gamma(\overline{\gamma}_x^{-1}(w)))W^{(q)}(w)}\,\mathrm{d}w\right) \frac{n\left(\mathrm{e}^{-q\rho_s^+ + \theta(s-\varepsilon(\rho_s^+))}; \bar{\varepsilon} > s\right)}{1-\gamma(\overline{\gamma}_x^{-1}(s))}\,\mathrm{d}s,
\end{aligned}
$$

which together with Equation (6) yields Equation (7). □

Remark 1. *Let $\gamma \equiv 0$ in Equation (7). Then $U(t) = X(t)$ for $t \ge 0$ and $\overline{\gamma}_x(z) \equiv z$ for $z \ge x$, and by Proposition 1 we have*

$$
\begin{aligned}
&\mathbb{E}_x(\mathrm{e}^{-q\tau_0^- + \theta X(\tau_0^-)}; \tau_0^- < \tau_b^+) \\
=&\int_x^b \frac{W^{(q)}(x)}{W^{(q)}(s)}\left(\frac{W_+^{(q)\prime}(s)}{W^{(q)}(s)} Z^{(q)}(s,\theta) - \theta Z^{(q)}(s,\theta) - (q-\psi(\theta))W^{(q)}(s)\right) \mathrm{d}s \\
=&\,-W^{(q)}(x)\int_x^b \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{Z^{(q)}(s,\theta)}{W^{(q)}(s)}\right) \mathrm{d}s \\
=&\,Z^{(q)}(x,\theta) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b,\theta),
\end{aligned}
$$

which can be found in (8.12) (with an appropriate killing rate added) in Chapter 8 of Kyprianou (2014), or Albrecher et al. (2016).

Let $\gamma \equiv \alpha \in (0,1)$ or $\overline{\gamma}_x(z) = x + (1-\alpha)(z-x)$ in Equation (7), we have for $q, \theta > 0$ and $0 \le x \le b$

$$\begin{aligned}
&\mathbb{E}_x \left(\mathrm{e}^{-q\sigma_0^- + \theta U(\sigma_0^-)}; \sigma_0^- < \sigma_b^+ \right) \\
&= \frac{1}{1-\alpha} \int_x^b \left(\frac{W^{(q)}(x)}{W^{(q)}(z)} \right)^{\frac{1}{1-\alpha}} \left(\frac{W_+^{(q)\prime}(z)}{W^{(q)}(z)} Z^{(q)}(z,\theta) - \theta Z^{(q)}(z,\theta) - (q - \psi(\theta)) W^{(q)}(z) \right) \mathrm{d}z.
\end{aligned}$$

Proposition 2 gives an expression of potential density for the process U.

Proposition 2. *The potential measure corresponding to U is absolutely continuous with respect to the Lebesgue measure with density given by*

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(U(t) \in \mathrm{d}u, t < \sigma_b^+ \wedge \sigma_0^-)\, \mathrm{d}t \\
&= W^{(q)}(0) \frac{1}{1 - \gamma(\overline{\gamma}_x^{-1}(u))} \exp\left(- \int_x^u \frac{W_+^{(q)\prime}(w)}{(1 - \gamma(\overline{\gamma}_x^{-1}(w))) W^{(q)}(w)} \mathrm{d}w \right) \mathbf{1}_{(x,b)}(u)\, \mathrm{d}u \\
&\quad + \int_x^b \frac{1}{1 - \gamma(\overline{\gamma}_x^{-1}(y))} \exp\left(- \int_x^y \frac{W_+^{(q)\prime}(w)}{(1 - \gamma(\overline{\gamma}_x^{-1}(w))) W^{(q)}(w)} \mathrm{d}w \right) \\
&\qquad \times \left(W_+^{(q)\prime}(y-u) - \frac{W_+^{(q)\prime}(y)}{W^{(q)}(y)} W^{(q)}(y-u) \right) \mathbf{1}_{(0,y)}(u)\, \mathrm{d}y\, \mathrm{d}u, \quad x, u \in [0,b],\ q > 0.
\end{aligned} \tag{9}$$

Proof. Let e_q be an exponentially distributed random variable independent of X with mean $1/q$. For any continuous, non-negative and bounded function f, we have

$$\begin{aligned}
&\int_0^\infty q e^{-qt} \mathbb{E}_x(f(U(t)); t < \sigma_b^+ \wedge \sigma_0^-)\, \mathrm{d}t \\
&= \mathbb{E}_x \left(f(U(e_q)) \mathbf{1}_{\{U(e_q) < \overline{U}(e_q),\, e_q < \sigma_b^+ \wedge \sigma_0^-\}} \right) \\
&\quad + \mathbb{E}_x \left(\int_0^\infty q e^{-qt} f(U(t)) \mathbf{1}_{\{U(t) = \overline{U}(t),\, t < \sigma_b^+ \wedge \sigma_0^-\}}\, \mathrm{d}t \right).
\end{aligned} \tag{10}$$

Note that $\int_0^t \mathbf{1}_{\{X(s) = \overline{X}(s)\}} \mathrm{d}s = W^{(q)}(0)\, \overline{X}(t)$, see Corollary 6 in Chapter IV of Bertoin (1996). Recalling that $U(t) = \overline{U}(t)$ is equivalent to $X(t) = \overline{X}(t)$ which implies $t = L^{-1}(L(t))$, we have

$$\begin{aligned}
&\mathbb{E}_x \left(\int_0^\infty q\mathrm{e}^{-qt} f(U(t)) \mathbf{1}_{\{U(t) = \overline{U}(t),\, t < \sigma_b^+ \wedge \sigma_0^-\}}\, \mathrm{d}t \right) \\
&= \mathbb{E}_x \left(\int_0^\infty q\mathrm{e}^{-qL^{-1}(L(t))} f(U(L^{-1}(L(t)))) \mathbf{1}_{\{X(t) = \overline{X}(t),\, L^{-1}(L(t)) < \sigma_b^+ \wedge \sigma_0^-\}} dt \right) \\
&= W(0) \mathbb{E}_x \left(\int_0^\infty q\mathrm{e}^{-qL^{-1}(L(t))} f(U(L^{-1}(L(t)))) \mathbf{1}_{\{L^{-1}(L(t)) < L^{-1}\left(\overline{\gamma}_x^{-1}(b) - x\right) \wedge \sigma_0^-\}}\, \mathrm{d}L_t \right) \\
&= q W(0) \int_0^{\overline{\gamma}_x^{-1}(b) - x} \exp\left(- \int_x^{\overline{\gamma}_x(x+t)} \frac{W_+^{(q)\prime}(w)}{(1 - \gamma(\overline{\gamma}_x^{-1}(w))) W^{(q)}(w)} \mathrm{d}w \right) f(\overline{\gamma}_x(x+t))\, \mathrm{d}t \\
&= q W(0) \int_x^b \frac{1}{(1 - \gamma(\overline{\gamma}_x^{-1}(y)))} \exp\left(- \int_x^y \frac{W_+^{(q)\prime}(w)}{(1 - \gamma(\overline{\gamma}_x^{-1}(w))) W^{(q)}(w)} \mathrm{d}w \right) f(y)\, \mathrm{d}y,
\end{aligned} \tag{11}$$

where Equation (8) is used in the last but one equation.

By the compensation formula in excursion theory and the memoryless property of the exponential random variable, one has

$$
\begin{aligned}
&\mathbb{E}_x\left(f(U(e_q)); U(e_q) < \overline{U}(e_q), e_q < \sigma_b^+ \wedge \sigma_0^-\right) \\
=&\mathbb{E}_x\Bigg(\int_0^\infty \sum_g \mathrm{e}^{-qg} \prod_{r<g} \mathbf{1}_{\{\bar{\varepsilon}_r \le \overline{\gamma}_x(x+L(r)), \overline{\gamma}_x(x+L(g))\le b\}} f\left(\overline{\gamma}_x(x+L(g)) - \varepsilon_g(t-g)\right) \\
&\quad \times q\mathrm{e}^{-q(t-g)} \mathbf{1}_{\{g<t<g+\zeta_g \wedge \rho^+_{\overline{\gamma}_x(x+L(g))}(\varepsilon_g)\}} \mathrm{d}t\Bigg) \\
=&\mathbb{E}_x\Bigg(\int_0^\infty \mathrm{e}^{-qt} \prod_{r<t} \mathbf{1}_{\{\bar{\varepsilon}_r \le \overline{\gamma}_x(x+L(r)), \overline{\gamma}_x(x+L(t))\le b\}} \\
&\quad \times \left(\int_{\mathcal{E}} \int_0^\infty q\mathrm{e}^{-qs} f\left(\overline{\gamma}_x(x+L(t)) - \varepsilon(s)\right) \mathbf{1}_{\{s<\zeta\wedge \rho^+_{\overline{\gamma}_x(x+L(t))}\}} \mathrm{d}s\, n\,(\mathrm{d}\varepsilon)\right) \mathrm{d}L(t)\Bigg) \\
=&q\int_0^{\overline{\gamma}_x^{-1}(b)-x} \exp\left(-\int_x^{\overline{\gamma}_x(x+t)} \frac{W_+^{(q)\prime}(w)}{(1-\gamma(\overline{\gamma}_x^{-1}(w)))W^{(q)}(w)}\mathrm{d}w\right) \\
&\quad \times \int_0^\infty n\left(\mathrm{e}^{-qs} f(\overline{\gamma}_x(x+t) - \varepsilon(s)) \mathbf{1}_{\{s<\zeta\wedge\rho^+_{\overline{\gamma}_x(x+t)}\}}\right) \mathrm{d}s\,\mathrm{d}t \\
=&q\int_x^b \frac{1}{1-\gamma(\overline{\gamma}_x^{-1}(y))} \exp\left(-\int_x^y \frac{W_+^{(q)\prime}(w)}{(1-\gamma(\overline{\gamma}_x^{-1}(w)))W^{(q)}(w)}\mathrm{d}w\right) \\
&\quad \times \int_0^\infty n\left(\mathrm{e}^{-qs} f(y-\varepsilon(s)) \mathbf{1}_{\{s<\zeta\wedge\rho_y^+\}}\right) \mathrm{d}s\,\mathrm{d}y.
\end{aligned} \tag{12}
$$

Applying the same arguments as in Equations (11) and (12), we have

$$
\begin{aligned}
&\mathbb{E}_x\left(f(X(e_q))\mathbf{1}_{\{e_q<\tau_b^+\wedge\tau_0^-\}}\right) \\
=&\mathbb{E}_x\left(f(X(e_q))\mathbf{1}_{\{X(e_q)=\overline{X}(e_q), e_q<\tau_b^+\wedge\tau_0^-\}}\right) + \mathbb{E}_x\left(f(X(e_q))\mathbf{1}_{\{X(e_q)<\overline{X}(e_q), e_q<\tau_b^+\wedge\tau_0^-\}}\right) \\
=&q\int_x^b \frac{W^{(q)}(x)}{W^{(q)}(y)}\left(W(0)f(y) + \int_0^\infty n\left(\mathrm{e}^{-qs} f(y-\varepsilon(s))\mathbf{1}_{\{s<\rho_y^+\wedge\zeta\}}\right)\,\mathrm{d}s\right)\,\mathrm{d}y.
\end{aligned} \tag{13}
$$

Equating the right hand sides of Equations (5) and (13) and then differentiating the resultant equation with respect to b gives

$$
\begin{aligned}
&\frac{W^{(q)}(x)}{W^{(q)}(b)}\left(W(0)f(b) + \int_0^\infty n\left(\mathrm{e}^{-qs} f(b-\varepsilon(s))\mathbf{1}_{\{s<\rho_b^+\wedge\zeta\}}\right)\,\mathrm{d}s\right) \\
=&\frac{W^{(q)}(x)}{W^{(q)}(b)}\left(f(b)W(0) + \int_0^b f(y)\left(W_+^{(q)\prime}(b-y) - \frac{W_+^{(q)\prime}(b)}{W^{(q)}(b)}W^{(q)}(b-y)\right)\,\mathrm{d}y\right),
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&\int_0^\infty n\left(\mathrm{e}^{-qs} f(y-\varepsilon(s))\mathbf{1}_{\{s<\rho_y^+\wedge\zeta\}}\right)\,\mathrm{d}s \\
=&\int_0^y f(w)\left(W_+^{(q)\prime}(y-w) - \frac{W_+^{(q)\prime}(y)}{W^{(q)}(y)}W^{(q)}(y-w)\right)\,\mathrm{d}w,
\end{aligned}
$$

which together with Equation (12) yields

$$
\begin{aligned}
&\mathbb{E}_x\left(f(U(e_q)); U(e_q) < \overline{U}(e_q), e_q < \sigma_b^+ \wedge \sigma_0^-\right) \\
=& q\int_x^b \frac{1}{(1-\gamma(\overline{\gamma}_x^{-1}(y)))} \exp\left(-\int_x^y \frac{W_+^{(q)\prime}(w)}{(1-\gamma(\overline{\gamma}_x^{-1}(w)))W^{(q)}(w)}\,\mathrm{d}w\right) \\
&\times \int_0^y f(w)\left(W_+^{(q)\prime}(y-w) - \frac{W_+^{(q)\prime}(y)}{W^{(q)}(y)}W^{(q)}(y-w)\right)\mathrm{d}w\,\mathrm{d}y,
\end{aligned}
$$

which combined with Equations (10) and (11) yields Equation (9). □

Remark 2. *Letting $\gamma \equiv 0$ in Equation (9), i.e., $U(t) = X(t)$ for $t \ge 0$ and $\overline{\gamma}_x(z) \equiv z$ for $z \ge x$, by Proposition 2 we have for $0 \le x, u \le b$*

$$
\begin{aligned}
&\int_0^\infty e^{-qt} P_x(X(t) \in \mathrm{d}u, t < \sigma_b^+ \wedge \sigma_0^-)\,\mathrm{d}t \\
=& \frac{W^{(q)}(0)W^{(q)}(x)\mathbf{1}_{(x,b)}(u)}{W^{(q)}(u)}\,\mathrm{d}u \\
&+ \int_x^b \frac{W^{(q)}(x)}{W^{(q)}(y)}\left(W_+^{(q)\prime}(y-u) - \frac{W_+^{(q)\prime}(y)}{W^{(q)}(y)}W^{(q)}(y-u)\right)\mathbf{1}_{(0,y)}(u)\,\mathrm{d}y\,\mathrm{d}u \\
=& \frac{W^{(q)}(0)W^{(q)}(x)\mathbf{1}_{(x,b)}(u)}{W^{(q)}(u)}\,\mathrm{d}u + W^{(q)}(x)\int_x^b \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{W^{(q)}(y-u)}{W^{(q)}(y)}\right]\mathbf{1}_{(0,y)}(u)\,\mathrm{d}y\,\mathrm{d}u \\
=& W^{(q)}(x)\left(\frac{W^{(q)}(0)\mathbf{1}_{(x,b)}(u)}{W^{(q)}(u)}\,\mathrm{d}u + \int_x^b \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{W^{(q)}(y-u)}{W^{(q)}(y)}\right]\mathrm{d}y\,\mathrm{d}u\mathbf{1}_{(0,x)}(u)\right. \\
&\left. + \int_u^b \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{W^{(q)}(y-u)}{W^{(q)}(y)}\right]\mathrm{d}y\,\mathrm{d}u\mathbf{1}_{(x,b)}(u)\right) \\
=& \frac{W^{(q)}(b-u)}{W^{(q)}(b)}W^{(q)}(x)\mathbf{1}_{(x,b)}(u)\,\mathrm{d}u + \left(\frac{W^{(q)}(b-u)}{W^{(q)}(b)} - \frac{W^{(q)}(x-u)}{W^{(q)}(x)}\right)W^{(q)}(x)\mathbf{1}_{(0,x)}(u)\,\mathrm{d}u \\
=& \left(\frac{W^{(q)}(x)}{W^{(q)}(b)}W^{(q)}(b-u) - W^{(q)}(x-u)\right)\mathbf{1}_{(0,b)}(u)\,\mathrm{d}u,
\end{aligned}
$$

which recovers Equation (5).

Let $\gamma \equiv \alpha \in (0,1)$ or $\overline{\gamma}_x(z) = x + (1-\alpha)(z-x)$ in Equation (9), we have

$$
\begin{aligned}
&\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U(t) \in \mathrm{d}u, t < \sigma_b^+ \wedge \sigma_0^-)\,\mathrm{d}t \\
=& \frac{W^{(q)}(0)}{1-\alpha}\left(\frac{W^{(q)}(x)}{W^{(q)}(u)}\right)^{\frac{1}{1-\alpha}}\mathbf{1}_{(x,b)}(u)\,\mathrm{d}u + \frac{1}{1-\alpha}\int_x^b \left(\frac{W^{(q)}(x)}{W^{(q)}(y)}\right)^{\frac{1}{1-\alpha}} \\
&\times\left(W_+^{(q)\prime}(y-u) - \frac{W_+^{(q)\prime}(y)}{W^{(q)}(y)}W^{(q)}(y-u)\right)\mathbf{1}_{(0,y)}(u)\,\mathrm{d}y\,\mathrm{d}u, \quad x,u \in [0,b],\ q > 0.
\end{aligned}
$$

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