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A Review of First-Passage Theory for the Segerdahl-Tichy Risk Process and Open Problems

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Abstract: The Segerdahl-Tichy Process, characterized by exponential claims and state dependent drift, has drawn a considerable amount of interest, due to its economic interest (it is the simplest risk process which takes into account the effect of interest rates). It is also the simplest non-Lévy, non-diffusion example of a spectrally negative Markov risk model. Note that for both spectrally negative Lévy and diffusion processes, first passage theories which are based on identifying two "basic" monotone harmonic functions/martingales have been developed. This means that for these processes many control problems involving dividends, capital injections, etc., may be solved explicitly once the two basic functions have been obtained. Furthermore, extensions to general spectrally negative Markov processes are possible; unfortunately, methods for computing the basic functions are still lacking outside the Lévy and diffusion classes. This divergence between theoretical and numerical is strikingly illustrated by the Segerdahl process, for which there exist today six theoretical approaches, but for which almost nothing has been computed, with the exception of the ruin probability. Below, we review four of these methods, with the purpose of drawing attention to connections between them, to underline open problems, and to stimulate further work.

Keywords: Segerdahl process; affine coefficients; first passage; spectrally negative Markov process; scale functions; hypergeometric functions

1. Introduction and Brief Review of First Passage Theory

Introduction. The Segerdahl-Tichy Process Segerdahl (1955); Tichy (1984), characterized by exponential claims and state dependent drift, has drawn a considerable amount of interest—see, for example, Avram and Usabel (2008); Albrecher et al. (2013); Marciniak and Palmowski (2016), due to its economic interest (it is the simplest risk process which takes into account the effect of interest rates—see the excellent overview (Albrecher and Asmussen 2010, Chapter 8). It is also the simplest non-Lévy, non-diffusion example of a spectrally negative Markov risk model. Note that for both spectrally negative Lévy and diffusion processes, first passage theories which are based on identifying two "basic" monotone harmonic functions/martingales have been developed. This means that for these processes many control problems involving dividends, capital injections, etc., may be solved explicitly once the two basic functions have been obtained. Furthermore, extensions to general spectrally negative Markov processes are possible Landriault et al. (2017), Avram et al. (2018); Avram and Goreac (2019); Avram et al. (2019b). Unfortunately, methods for computing the basic functions are still lacking outside the Lévy and diffusion classes. This divergence between theoretical and numerical is strikingly illustrated by the Segerdahl process, for which there exist today six theoretical approaches, but for which almost nothing has been computed, with the exception of the ruin probability Paulsen and Gjessing (1997). Below, we review four of these methods (which apply also to certain generalizations provided in Avram and Usabel (2008); Czarna et al. (2017)), with the purpose of drawing attention to connections between them, to underline open problems, and to stimulate further work.

Spectrally negative Markov processes with constant jump intensity. To set the stage for our topic and future research, consider a spectrally negative jump diffusion on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$, which satisfies the SDE:

$$dX_t = c(X_t)dt + \sigma(X_t)dB_t - dJ_t, \; J_t = \sum_{i=1}^{N_\lambda(t)} C_i, \; \forall X_t > 0 \tag{1}$$

and is absorbed or reflected when leaving the half line $(0, \infty)$. Here, B_t is standard Brownian motion, $\sigma(x) > 0, c(x) > 0, \forall x > 0$, $N_\lambda(t)$ is a Poisson process of intensity λ, and C_i are nonnegative random variables with distribution measure $F_C(dz)$ and finite mean. The functions $c(x)$, $a(x) := \frac{\sigma^2(x)}{2}$ and $\Pi(dz) = \lambda F_C(dz)$ are referred to as the Lévy -Khinchine characteristics of X_t. Note that we assume that all jumps go in the same direction and have constant intensity so that we can take advantage of potential simplifications of the first passage theory in this case.

The Segerdahl-Tichy process is the simplest example outside the spectrally negative Lévy and diffusion classes. It is obtained by assuming $a(x) = 0$ in (1), and C_k to be exponential i.i.d random variables with density $f(x) = \mu e^{-\mu x}$ (see Segerdahl (1955) for the case $c(x) = c + rx, r > 0, c \geq 0$, and Tichy (1984) for nonlinear $c(x)$). Note that, for the case $c(x) = c + rx$, an explicit computation of the ruin probability has been provided (with some typos) in Paulsen and Gjessing (1997). See also Paulsen (2010) and see (Albrecher and Asmussen 2010, Chapter 8) for further information on risk processes with state dependent drift, and in particular the two pages of historical notes and references.

First passage theory concerns the first passage times above and below fixed levels. For any process $(X_t)_{t\geq 0}$, these are defined by

$$\begin{aligned} T_{b,+} &= T^X_{b,+} = \inf\{t \geq 0 : X_t > b\}, \\ T_{a,-} &= T^X_{a,-} = \inf\{t \geq 0 : X_t < a\}, \end{aligned} \tag{2}$$

with $\inf \emptyset = +\infty$, and the upper script X typically omitted. Since a is typically fixed below, we will write for simplicity T instead of $T_{a,-}$.

First passage times are important in the control of reserves/risk processes. The rough idea is that when below low levels a, reserves processes should be replenished at some cost, and when above high levels b, they should be partly invested to yield income—see, for example, the comprehensive textbook Albrecher and Asmussen (2010).

The most important first passage functions are the solutions of the two-sided upward and downward exit problems from a bounded interval $[a, b]$:

$$\begin{cases} \overline{\Psi}^b_q(x,a) := E_x\left[e^{-qT_{b,+}}\mathbf{1}_{\{T_{b,+}<T_{a,-}\}}\right] = P_x\left[T_{b,+} < \min(T_{a,-}, \mathbf{e}_q)\right] \\ \Psi^b_q(x,a) := E_x\left[e^{-qT_{a,-}}\mathbf{1}_{\{T_{a,-}<T_{b,+}\}}\right] = P_x\left[T_{a,-} < \min(T_{b,+}, \mathbf{e}_q)\right] \end{cases} q \geq 0, a \leq x \leq b, \tag{3}$$

where $\mathbf{e}_q$ is an independent exponential random variable of rate q. We will call them (killed) survival and ruin probabilities, respectively[1], but the qualifier killed will be usually dropped below. The absence of killing will be indicated by omitting the subindex q. Note that in the context of potential theory, (3) are called equilibrium potentials Blumenthal and Getoor (2007) (of the capacitors $\{b, a\}$ and $\{a, b\}$).

Beyond ruin probabilities : scale functions, dividends, capital gains, etc. Recall that for "completely asymmetric Lévy " processes, with jumps going all in the same direction, a large variety of first passage problems may be reduced to the computation of the two monotone "scale functions"

[1] See Ivanovs (2013) for a nice exposition of killing.

W_q, Z_q—see, for example , Suprun (1976), Bertoin (1997, 1998), Avram et al. (2004, 2007, 2015, 2016), Ivanovs and Palmowski (2012), Albrecher et al. (2016); Li and Palmowski (2016); Li and Zhou (2017), Avram and Zhou (2017), and see Avram et al. (2019a) for a recent compilation of more than 20 laws expressed in terms of W_q, Z_q.

For example, for spectrally negative Lévy processes, the Laplace transform/killed survival probability has a well known simple factorization[2]:

$$\overline{\Psi}_q^b(x,a) \quad = \frac{W_q(x-a)}{W_q(b-a)}. \tag{4}$$

For a second example, the De-Finetti de Finetti (1957) discounted dividends fixed barrier objective for spectrally negative Lévy processes Avram et al. (2007) has a simple expression in terms of either the W_q scale function or of its logarithmic derivative $\nu_q = \frac{W_q'}{W_q}$[3]:

$$V^b(x) = \begin{cases} \frac{W_q(x)}{W_q'(b)} = e^{-\int_x^b \nu_q(m)dm} \frac{1}{\nu_q(b)} & x \le b \\ V^b(x) = x - b + V^b(b) & x > b \end{cases}. \tag{5}$$

Maximizing over the reflecting barrier b is simply achieved by finding the roots of

$$W_q''(b) = 0 \Leftrightarrow \frac{\partial}{\partial b}\left[\frac{1}{\nu_q(b)}\right] = \frac{\partial}{\partial b}\left[V^b(b)\right] = 1. \tag{6}$$

W, Z **formulas for first passage problems for spectrally negative Markov processes.** Since results for spectrally negative Lévy processes require often not much more than the strong Markov property, it is natural to attempt to extend them to the spectrally negative strong Markov case. As expected, everything worked out almost smoothly for "Lévy -type cases" like random walks Avram and Vidmar (2017), Markov additive processes Ivanovs and Palmowski (2012), etc. Recently, it was discovered that W, Z formulas continue to hold a priori for spectrally negative Markov processes Landriault et al. (2017), Avram et al. (2018). The main difference is that in equations like Equation (4), $W_q(x-a)$ and the second scale function $Z_{q,\theta}(x-a)$ Avram et al. (2015); Ivanovs and Palmowski (2012) must be replaced by two-variable functions $W_q(x,a)$, $Z_{q,\theta}(x,a)$ (which reduces in the Lévy case to $W_q(x,y) = \widetilde{W}_q(x-y)$, with $\widetilde{W}_q$ being the scale function of the Lévy process). This unifying structure has lead to recent progress for the optimal dividends problem for spectrally negative Markov processes (see Avram and Goreac (2019)). However, since the computation of the two-variables scale functions is currently well understood only for spectrally negative Lévy processes and diffusions, AG could provide no example outside these classes. In fact, as of today, we are not aware of any explicit or numeric results on the control of the process (1) which have succeeded to exploit the W, Z formalism.

Literature review. Several approaches may allow handling particular cases of spectrally negative Markov processes:

1. with phase-type jumps, there is Asmussen's embedding into a regime switching diffusion Asmussen (1995)—see Section 5, and the complex integral representations of Jacobsen and Jensen (2007), Jiang et al. (2019).
2. for Lévy driven Langevin-type processes, renewal equations have been provided in Czarna et al. (2017) —see Section 2
3. for processes with affine operator, an explicit integrating factor for the Laplace transform may be found in Avram and Usabel (2008)—see Section 3.

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2 The fact that the survival probability has the multiplicative structure (4) is equivalent to the absence of positive jumps, by the strong Markov property; this is the famous "gambler's winning" formula Kyprianou (2014).

3 ν_q may be more useful than W_q in the spectrally negative Markov framework Avram and Goreac (2019).

4. for the Segerdahl process, the direct IDE solving approach is successful (Paulsen and Gjessing (1997)) —see Section 4.

We will emphasize here the third approach but use also the second to show how the third approach fits within it. The direct IDE solving approach is recalled for comparison, and Asmussen's approach is also recalled, for its generality.

Here is an example of an important problem we would like to solve:

Problem 1. *Find the de Finetti optimal barrier for the Segerdahl-Tichy process, extending the Equations* (5) *and* (6).

Contents. Section 2 reviews the recent approach based on renewal equations due to Czarna et al. (2017) (which needs still be justified for increasing premiums satisfying (8)). An important renewal (Equation (11)) for the "scale derivative" **w** is recalled here, and a new result relating the scale derivative to its integrating factor (16) is offered—see Theorem 1.

Section 3 reviews older computations of Avram and Usabel (2008) for more general processes with affine operator, and provides explicit formulas for the Laplace transforms of the survival and ruin probability (24), in terms of the same integrating factor (16) and its antiderivative.

Section 4 reviews the direct classic Kolmogorov approach for solving first passage problems with phase-type jumps. The discounted ruin probability ($q > 0$) for this process may be found explicitly (33) for the Segerdahl process by transforming the renewal equation (29) into the ODE (30), which is hypergeometric of order 2. This result due to Paulsen has stopped short further research for more general mixed exponential jumps, since it seems to require a separate "look-up" of hypergeometric solutions for each particular problem.

Section 5 reviews Asmussen's approach for solving first passage problems with phase-type jumps, and illustrates the simple structure of the survival and ruin probability of the Segerdahl-Tichy process, in terms of the scale derivative **w**. This approach yields quasi-explicit results when $q = 0$.

Section 6 checks that our integrating factor approach recovers various results for Segerdahl's process, when $q = 0$ or $x = 0$. Section 7 reviews necessary hypergeometric identities. Finally, Section 8 outlines further promising directions of research.

2. The Renewal Equation for the Scale Derivative of Lévy Driven Langevin Processes Czarna et al. (2017)

One tractable extension of the Segerdahl-Tichy process is provided by is the "Langevin-type" risk process defined by

$$X_t = x + \int_0^t c(X_s)\, ds + Y_t, \tag{7}$$

where Y_t is a spectrally negative Lévy process, and $c(u)$ is a nonnegative premium function satisfying

$$\int_{x_0}^{\infty} \frac{1}{c(u)}\, dy = \infty,\ c(u) > 0,\ \forall x_0, u > 0. \tag{8}$$

The integrability condition above is necessary to preclude explosions. Indeed when Y_t is a compound Poisson process, in between jumps (claims) the risk process (7) moves deterministically along the curves x_t determined by the vector field

$$\frac{dx}{dt} = c(x) \Leftrightarrow t = \int_{x_0}^{x} \frac{du}{c(u)} := C(x; x_0), \forall x_0 > 0.$$

From the last equality, it may be noted that if $C(x; x_0)$ satisfies $\lim_{x\to\infty} C(x; x_0) < \infty$, then x_t must explode, and the stochastic process X_t may explode.

The case of Langevin processes has been tackled recently in Czarna et al. (2017), who provide the construction of the process (7) in the particular case of non-increasing functions $c(\cdot)$. This setup can be used to model dividend payments, and other mathematical finance applications.

Czarna et al. (2017) showed that the W, Z scale functions which provide a basis for first passage problems of Lévy spectrally positive negative processes have two variables extensions $\mathcal{W}, \mathcal{Z}$ for the process (7), which satisfy integral equations. The equation for $\mathcal{W}$, obtained by putting $\phi(x) = c(a) - c(x)$ in (Czarna et al. 2017, eqn. (40)), is:

$$\mathcal{W}_q(x,a) = W_q(x-a) + \int_a^x (c(a) - c(z)) W_q(x-z) \mathcal{W}_q'(z;a) dz, \tag{9}$$

where W_q is the scale function of the Lévy process obtained by replacing $c(x)$ with $c(a)$.

It follows that the scale derivative

$$\mathbf{w}_q(x,a) = \frac{\partial}{\partial x} \mathcal{W}_q(x,a)$$

of the scale function of the process (7) satisfies a Volterra renewal equation (Czarna et al. 2017, eqn. (41)):

$$\mathbf{w}_q(x,a) \left(1 + (c(x) - c(a)) W_q(0)\right) = w_q(x-a) + \int_a^x (c(a) - c(z)) w_q(x-z) \mathbf{w}_q(z;a) dz, \tag{10}$$

where w_q is the derivative of the scale function of the Lévy process $Y_t = Y_t^{(a)}$ obtained by replacing $c(x)$ with $c(a)$. This may further be written as:

$$w_q(x-a) + \int_a^x w_q(x-z) \mathbf{w}_q(z;a)(c(a) - c(z)) dz = \begin{cases} \mathbf{w}_q(x,a), & Y_t \text{ of unbounded variation} \\ \mathbf{w}_q(x,a) \frac{c(x)}{c(a)}, & Y_t \text{ of bounded variation} \end{cases} \tag{11}$$

Problem 2. *It is natural to conjecture that the formula (11) holds for all drifts satisfying (8), but this is an open problem for now.*

Remark 1. *Note that renewal equations are a more appropriate tool than Laplace transforms for the general Langevin problem. Indeed, taking "shifted Laplace transform" $\mathcal{L}_a f(x) = \int_a^\infty e^{-s(y-a)} f(y) dy$ of (11), putting*

$$\begin{cases} \widehat{\mathbf{w}}_q(s,a) = \int_a^\infty e^{-s(y-a)} \mathbf{w}_q(y,a) dy, \\ \widehat{\mathbf{w}}_{q,c}(s,a) = \int_a^\infty e^{-s(y-a)} \mathbf{w}_q(y,a) c(y) dy \\ \widehat{w}_q(s) = \int_0^\infty e^{-sy} w_q(y) dy \end{cases},$$

and using

$$\mathcal{L}_a[\int_a^x f(x-y) l(y) dy](s) = \mathcal{L}_0 f(s) \mathcal{L}_a l(s)$$

yields equations with two unknowns:

$$\widehat{w}_q(s)(1 + c(a)\widehat{\mathbf{w}}_q(s,a) - \widehat{\mathbf{w}}_{q,c}(s,a)) = \begin{cases} \widehat{\mathbf{w}}_q(s,a) & \textit{unbounded variation case} \\ \frac{\widehat{\mathbf{w}}_{q,c}(s,a)}{c(a)} & \textit{bounded variation case} \end{cases}, \tag{12}$$

whose solution is not obvious.

The Linear Case $c(x) = rx + c$

To get explicit Laplace transforms , we will turn next to Ornstein-Uhlenbeck type processes[4] $X(\cdot)$, with $c(x) = c(a) + r(x-a)$, which implies

$$\widehat{\mathbf{w}}_{q,c}(s,a) = \int_a^\infty e^{-s(y-a)} \mathbf{w}_q(y,a)(r(y-a) + c(a))dy = -r\widehat{\mathbf{w}}_q'(s,a) + c(a)\widehat{\mathbf{w}}_q(s,a). \tag{13}$$

Equation (12) simplify then to:

$$\widehat{w}_q(s)(1 + r\widehat{\mathbf{w}}_q'(s,a)) = \begin{cases} \widehat{\mathbf{w}}_q(s,a) & \text{unbdd variation case} \\ \widehat{\mathbf{w}}_q(s,a) - \frac{r}{c(a)}\widehat{\mathbf{w}}_q'(s,a) & \text{bdd variation case} \end{cases}. \tag{14}$$

Remark 2. *Note that the only dependence on a in this equation is via $c(a)$, and via the shifted Laplace transform. Since a is fixed, we may and will from now on simplify by assuming w.l.o.g. $a = 0$, and write $c = c(a)$.*

Let now

$$\kappa(s) = \alpha_0 s^2 + cs - s\widehat{\overline{\Pi}}(s) - q, \alpha_0 > 0,$$

denote the Laplace exponent or symbol of the Lévy process $Y_t = \sqrt{2\alpha_0} B_t - J_t + ct$, and recall that

$$w_q(s) = \begin{cases} \frac{s}{\kappa(s)} & \text{unbdd variation case} \\ \frac{s}{\kappa(s)} - \frac{1}{c} & \text{bdd variation case} \end{cases}$$

(where we have used that $W_q(0) = 0(\frac{1}{c})$ in the two cases, respectively).

We obtain now from (14) the following ODE

$$r\widehat{\mathbf{w}}_q'(s,a) - \frac{\kappa(s)}{s}\widehat{\mathbf{w}}_q(s,a) = -1 + \frac{\kappa(s)}{s} W_q(0) = \begin{cases} -1 & \text{unbdd variation case} \\ -1 + \frac{\kappa(s)}{cs} := -\frac{h(s)}{c} & \text{bdd variation case} \end{cases}, \tag{15}$$

where

$$h(s) = \widehat{\overline{\Pi}}(s) + \frac{q}{s}.$$

Remark 3. *The Equation (15) is easily solved multiplying by an integrating factor*

$$I_q(s, s_0) = e^{-\int_{s_0}^s \frac{\kappa(z)/z}{r} dz} = e^{-\int_{s_0}^s \frac{\alpha_0 z + c - \widehat{\overline{\Pi}}(z) - q/z}{r} dz}, \tag{16}$$

where $s_0 > 0$ is an arbitrary integration limit chosen so that the integral converges (the formula (16) appeared first in Avram and Usabel (2008)). To simplify, we may choose $s_0 = 0$ to integrate the first part $\alpha_0 z + c - \widehat{\overline{\Pi}}(z)$, and a different lower bound $s_0 = 1$ to integrate q/z. Putting $\tilde{q} = \frac{q}{r}, \tilde{c} = \frac{c}{r}, \tilde{\alpha}_0 = \frac{\alpha_0}{r}$, we get that

$$I_q(s) = e^{-\int_\cdot^s \frac{\kappa(z)/z}{r} dz} = s^{\tilde{q}} e^{-\left[\left(\frac{\tilde{\alpha}_0}{2}\right)s^2 + \tilde{c}s\right] + \frac{1}{r}\int_0^s \widehat{\overline{\Pi}}(z)dz} := s^{\tilde{q}} I(s) := e^{-\tilde{c}s} i_q(s), \tag{17}$$

where we replaced s_0 by $\cdot$ to indicate that two different lower bounds are in fact used, and we put $I(s) = I_0(s)$ (the subscript 0 will be omitted when $q = 0$).

Solving (15) yields:

[4] For some background first passage results on these processes, see for example Borovkov and Novikov (2008); Loeffen and Patie (2010).

Theorem 1. *Fix a and put* $\overline{I}_q(s) = \int_s^\infty I_q(y)dy$*. Then, the Laplace transform of the scale derivative of an Ornstein-Uhlenbeck type process* (7) *satisfies:*

$$\widehat{\mathbf{w}}_q(s,a) = \frac{\overline{I}_q(s)}{rI_q(s)} - W_q(0) = \begin{cases} \frac{\overline{I}_q(s)}{rI_q(s)}, & \text{in the unbounded variation case} \\ \frac{\overline{I}_q(s)}{rI_q(s)} - \frac{1}{c}, & \text{in the bounded variation case} \end{cases}. \tag{18}$$

Proof. In the unbounded variation case, applying the integrating factor to (15) yields immediately:

$$\widehat{\mathbf{w}}_q(s,a)I_q(s) = r^{-1}\int_s^\infty I_q(y)dy = r^{-1}\overline{I}_q(s).$$

In the bounded variation case, we observe that

$$i_q'(s) = \frac{h(s)}{r}i_q(s),$$

where i_q is defined in (17). An integration by parts now yields

$$\widehat{\mathbf{w}}_q(s,a)I_q(s) = \int_s^\infty \frac{h(y)}{cr}I_q(y)dy = \int_s^\infty \frac{h(y)}{cr}e^{-\tilde{c}y}i_q(y)dy$$
$$= c^{-1}\int_s^\infty e^{-\tilde{c}y}i_q'(y)dy = c^{-1}(-I_q(s) + \tilde{c}\int_s^\infty e^{-\tilde{c}y}i_q(y)dy) = r^{-1}\overline{I}_q(s)dy - c^{-1}I_q(s).$$

□

Remark 4. *The result* (18) *is quite similar to the Laplace transform for the survival and ruin probability (Gerber-Shiu functions) derived in (Avram and Usabel 2008, p. 470)—see* (23), (24) *below; the main difference is that in that case additional effort was needed for finding the values* $\overline{\Psi}(a,a), \Psi(a,a)$*.*

3. The Laplace transform-Integrating Factor Approach for Jump-Diffusions with Affine Operator Avram and Usabel (2008)

We summarize now for comparison the results of Avram and Usabel (2008) for the still tractable, more general extension of the Segerdahl-Tichy process provided by jump-diffusions with affine premium and volatility

$$\begin{cases} c(x) = rx + c \\ \frac{\sigma^2(x)}{2} = \alpha_1 x + \alpha_0, \ \alpha_1, \alpha_0 \geq 0. \end{cases} \tag{19}$$

Besides Ornstein-Uhlenbeck type processes, (19) includes another famous particular case, Cox-Ingersoll-Ross (CIR) type processes, obtained when $\alpha_1 > 0$.

Introduce now a combined ruin-survival expected payoff at time t

$$V(t,u) = \mathbb{E}_{X_0=u}\left[w(X_T)\, 1_{\{T<t\}} + p(X_t)\, 1_{\{T\geq t\}}\right] \tag{20}$$

where w, p represent, respectively:

- A penalty $w(X_T)$ at a stopping time T, $w: \mathbb{R} \to \mathbb{R}$
- A reward for survival after t years: $p(X_t)$, $p: \mathbb{R} \to \mathbb{R}^+$.

Some particular cases of interest are the survival probability for t years, obtained with

$$w(X_T) = 0, \ \ p(X_t) = 1_{\{X_t \geq 0\}}$$

and the ruin probability with deficit larger in absolute value than y, obtained with

$$w(X_T) = 1_{\{X_T < -y\}}, \quad p(X_t) = 0$$

Let

$$V_q(x) = \int_0^\infty qe^{-qt} V(t,x)dt = E_x \left[w(X_T)\, 1_{\{T < \mathbf{e}_q\}} + p(X_{\mathbf{e}_q})\, 1_{\{T \geq \mathbf{e}_q\}} \right], \tag{21}$$

denote a "Laplace-Carson"/"Gerber Shiu" discounted penalty/pay-off.

Proposition 1. *(Avram and Usabel 2008, Lem. 1, Thm. 2) (a) Consider the process (19). Let $V_q(x)$ denote the corresponding Gerber-Shiu function (21), let $w_\Pi(x) = \int_x^\infty w(x-u)\Pi(du)$ denote the expected payoff at ruin, and let $g(x) := w_\Pi(x) + qp(x), \widehat{g}(s)$ denote the combination of the two payoffs and its Laplace transform; note that the particular cases*

$$\widehat{g}(s) = \frac{q}{s}, \; \widehat{g}(s) = \lambda \overline{F}(s)$$

correspond to the survival and ruin probability , respectively Avram and Usabel (2008).

Then, the Laplace transform of the derivative

$$V_*(x) = \int_0^\infty e^{-sx} dV_q(x) = s\widehat{V}_q(s) - V_q(0)$$

satisfies the ODE

$$\begin{aligned} &(\alpha_1 s + r)\, V_*(s)' - (\frac{\kappa(s)}{s} - \alpha_1) V_*(s) = -h(s)V_q(0) - \alpha_0\, V_q'(0) + \widehat{g}(s) \Longrightarrow \\ &V_*(s) I_q(s) = \int_s^\infty I_q(y) \frac{h(y)V_q(0) + \alpha_0\, V_q'(0) - \widehat{g}(y)}{r + \alpha_1 y}\, dy, \end{aligned} \tag{22}$$

where $h(s) = \widehat{\overline{\Pi}}(s) + \frac{q}{s}$ (this corrects a typo in (Avram and Usabel 2008, eqn. (9))), and where the integrating factor is obtained from (16) by replacing c with $c - \alpha_1$ (Avram and Usabel 2008, eqn. (11)). Equivalently,

$$\begin{aligned} &r\left(s\widehat{V}_q(s)\right)' - \frac{\kappa(s)}{s}\, s\widehat{V}_q(s) = -(c + \alpha_0 s)V_q(0) - \alpha_0\, V_q'(0) + \widehat{g}(s) \Longrightarrow \\ &s\widehat{V}_q(s) I_q(s) = \int_s^\infty I_q(y) \frac{(c+\alpha_0 s)V_q(0) + \alpha_0\, V_q'(0) - \widehat{g}(y)}{r + \alpha_1 y}\, dy. \end{aligned} \tag{23}$$

(b) If $\alpha_0 = 0 = \alpha_1$ and $q > 0$, the survival probability satisfies

$$\begin{aligned} &\overline{\Psi}_q(0) = \frac{\tilde{q}\overline{I}_{q-1}(0)}{\tilde{c}\overline{I}_q(0)} \\ &s\widehat{\overline{\Psi}}_q(s) I_q(s) = \int_s^\infty I_q(y)(\tilde{c}\overline{\Psi}_q(0) - \frac{\tilde{q}}{y})dy = \tilde{c}\overline{\Psi}_q(0)\overline{I}_q(s) - \tilde{q}\overline{I}_{q-1}(s) = \tilde{q}\left(\frac{\overline{I}_{q-1}(0)}{\overline{I}_q(0)}\overline{I}_q(s) - \overline{I}_{q-1}(s)\right) \end{aligned} \tag{24}$$

Proof. (b) The survival probability follow from (a), by plugging $\widehat{g}(y) = \frac{q}{y}$. Indeed, the Equation (23) becomes for the survival probability

$$s\widehat{\overline{\Psi}}_q(s) I_q(s) = \int_s^\infty I_q(y)(\tilde{c}\overline{\Psi}_q(0) - \frac{\tilde{q}}{y})\, dy = \tilde{c}\overline{\Psi}_q(0)\overline{I}_q(s) - \tilde{q}\overline{I}_{q-1}(s).$$

Letting $s \to 0$ in this equation yields $\overline{\Psi}_q(0) = \frac{\tilde{q}\overline{I}_{q-1}(0)}{\tilde{c}\overline{I}_q(0)}$.

As a check, let us verify also Equation (23) for the ruin probability, by plugging $\widehat{g}(y) = \lambda \overline{F}(y)$:

$$s\widehat{\Psi}_q(s)I_q(s) = \int_s^\infty I_q(y)(\tilde{c}\Psi_q(0) - \lambda\overline{F}(y))dy = \tilde{c}\Psi_q(0)\bar{I}_q(y) - J(y),$$
$$J(y) = \int_s^\infty y^{\tilde{q}}e^{-\tilde{c}y}j'(y)dy,\ j(y) := e^{\tilde{\lambda}\int_0^y \bar{F}(z)dz}.$$

Integrating by parts, $J(y) = -I_q(s) + \tilde{c}\bar{I}_q(s) - \tilde{q}\bar{I}_{q-1}(s)$. Finally,

$$s\widehat{\Psi}_q(s)I_q(s) = \tilde{c}(1-\overline{\Psi}_q(0))\bar{I}_q(s) - \Big(-I_q(s) + \tilde{c}\bar{I}_q(s) - \tilde{q}\bar{I}_{q-1}(s)\Big) =$$
$$I_q(s) + \tilde{q}\bar{I}_{q-1}(s) - \tilde{c}\overline{\Psi}_q(0)\bar{I}_q(s) = I_q(s) - s\widehat{\overline{\Psi}}_q(s)I_q(s). \tag{25}$$

□

Segerdahl's Process via the Laplace Transform Integrating Factor

We revisit now the particular case of Segerdahl's process with exponential claims of rate μ and $\alpha_0 = \alpha_1 = 0$. Using $\overline{\Pi}(y) = \lambda F_C(y)dy = \frac{\lambda}{y+\mu}$ we find that for Segerdahl's process the integrand in the exponent is

$$\frac{\kappa(s)}{rs} = \tilde{c} - \tilde{\lambda}/(s+\mu) - \tilde{q}/s,$$

and the integrating factor (17) may be taken as

$$I_q(x) = x^{\tilde{q}}e^{-\tilde{c}x}(1+x/\mu)^{\tilde{\lambda}}.$$

The antiderivative $\bar{I}_q(x)$ is not explicit, except for:

1. $x = 0$, when it holds that
$$\bar{I}_q(0) = \mu^{\tilde{q}+1}U(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, \tilde{c}\mu),$$
where (Abramowitz and Stegun 1965, 13.2.5)[5]
$$U[a, a+c, z] = \frac{1}{\Gamma[a]}\int_0^\infty e^{-zt}t^{a-1}(t+1)^{c-1}dt, Re[z] > 0, Re[a] > 0.$$
2. for $q = 0$, when it holds that
$$I(x) = e^{-\tilde{c}x}(1+x/\mu)^{\tilde{\lambda}},\ \bar{I}(x) = \int_x^\infty I(y)dy = \frac{e^{\tilde{c}\mu}(\tilde{c}\mu)^{-\tilde{\lambda}}\Gamma(\tilde{\lambda}+1, \tilde{c}(x+\mu))}{\tilde{c}}.$$

However, the Laplace transforms of the integrating factor $I_q(x)$ and its primitive are explicit:

$$\widehat{I}_q(s) = \int_0^\infty e^{-(s+\tilde{c})x}x^{\tilde{q}}(1+x/\mu)^{\tilde{\lambda}} = \Gamma(\tilde{q}+1)U(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, \mu(\tilde{c}+s)),$$
$$\widehat{\bar{I}}_q(s) = \Gamma(\tilde{q}+1)\frac{U(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, \mu\tilde{c}) - U(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, \mu(\tilde{c}+s))}{s}. \tag{26}$$

Finally, we may compute:

---

5 Note that when $c = 1$, this function reduces to a power: $U(a, a+1, z) = \frac{\int_0^\infty t^{a-1}e^{-zt}dt}{\Gamma(a)} = z^{-a}$.

$$\overline{\Psi}_q(0) = \frac{\tilde{q}\bar{I}_{q-1}(0)}{\tilde{c}\bar{I}_q(0)} = \frac{\tilde{q}U(\tilde{q},\tilde{q}+\tilde{\lambda}+1,\tilde{c}\mu)}{\tilde{c}\mu U(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,\tilde{c}\mu)}$$

$$\Psi_q(0) = 1-\overline{\Psi}_q(0) = 1-\frac{\tilde{q}U(\tilde{q},\tilde{q}+\tilde{\lambda}+1,\tilde{c}\mu)}{\tilde{c}\mu U(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,\tilde{c}\mu)}$$

$$= \frac{\tilde{c}\mu U(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,\tilde{c}\mu)-\tilde{q}U(\tilde{q},\tilde{q}+\tilde{\lambda}+1,\tilde{c}\mu)}{\tilde{c}\mu U(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,\tilde{c}\mu)}$$

$$= \left(\frac{\lambda}{c\mu}\right)\frac{U\left(\tilde{q}+1,\tilde{q}+1+\tilde{\lambda},\mu\tilde{c}\right)}{U\left(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,\mu\tilde{c}\right)}, \tag{27}$$

where we used the identity (Abramowitz and Stegun 1965, 13.4.18)

$$U[a-1,b,z]+(b-a)U[a,b,z] = zU[a,b+1,z], a>1, \tag{28}$$

with $a=\tilde{q}+1, b=\tilde{q}+\tilde{\lambda}+1$. This checks the (corrected) Paulsen result (38) for $x=0$.

Remark 5. *We can now numerically answer Problem 1: (a) obtain the antiderivative $\bar{I}_q(x)$ by numerical integration; (b) compute the Laplace transform of the scale derivative by (18); c) Invert the Laplace transform.*

The example above raises the question:

Problem 3. *Is it possible to compute explicitly the Laplace transforms of the integrating factor $I_q(x)$ and its primitive for affine processes with phase-type jumps?*

4. Direct Conversion to an Ode of Kolmogorov'S Integro-Differential Equation for the Discounted Ruin Probability with Phase-Type Jumps

One may associate to the process (1) a Markovian semi-group with generator

$$\mathcal{G}h(x) = a(x)h''(x)+c(x)h'(x)+\int_{(0,\infty)}[h(x-y)-h(x)]\Pi(dy)$$

acting on $h \in C^2_{(0,\infty)}$, up to the minimum between its explosion and exit time $T_{0,-}$.

The classic approach for computing the ruin, survival, optimal dividends, and other similar functions starts with the well-known Kolmogorov integro-differential equations associated to this operator. With phase-type jumps, one may remove the integral term in Kolmogorov 's equation above by applying to it the differential operator $n(D)$ given by the denominator of the Laplace exponent $\kappa(D)$. For example, with exponential claims, we would apply the operator $\mu+D$.

4.1. Paulsen's Result for Segerdahl's Process with Exponential Jumps Paulsen and Gjessing (1997), ex. 2.1

When $a(x)=0$ and C_k in (1) are exponential i.i.d random variables with density $f(x)=\mu e^{-\mu x}$, the Kolmogorov integro-differential equation for the ruin probability is:

$$c(x)\Psi_q(x,a)'+\lambda\mu\int_a^x e^{-\mu(x-z)}\Psi_q(z,a)dz-(\lambda+q)\Psi_q(x,a)+\lambda e^{-\mu x}=0,\ \Psi_q(b,a)=1, \Psi_q(x,a)=0, x<a. \tag{29}$$

To remove the convolution term $\Psi_q * f_C$, apply the operator $\mu + D$, which replaces the convolution term by $\lambda\mu\Psi_q(x)$[6] yielding finally

$$\left(c(x)D^2 + (c'(x) + \mu c(x) - (\lambda + q))D - q\mu\right)\Psi_q(x) = 0$$

When $c(x) = c + rx, a = 0, b = \infty$, the ruin probability satisfies:

$$\begin{aligned} &\left[(\tilde{c} + x)\,D^2 + (1 + \mu(\tilde{c} + x) - \tilde{q} - \tilde{\lambda})D - \mu\tilde{q}\right]\Psi_q(x) = 0, \\ &(-c\,D + \lambda + q)\Psi_q(0) = \lambda^7, \quad \Psi_q(\infty) = 0 \end{aligned} \tag{30}$$

see (Paulsen and Gjessing 1997, (2.14),(2.15)), where $\tilde{\lambda} = \frac{\lambda}{r}, \tilde{q} = \frac{q}{r}$, and $-\tilde{c} := -\frac{c}{r}$ is the absolute ruin level.

Changing the origin to $-\tilde{c}$ by $z = \mu(x + \tilde{c}), \Psi_q(x) = y(z)$ brings this to the form

$$zy''(z) + (z + 1 - n)y'(z) - \tilde{q}y(z) = 0,\ n = \tilde{\lambda} + \tilde{q}, \tag{31}$$

(we corrected here two wrong minuses in Paulsen and Gjessing (1997)), which corresponds to the process killed at the absolute ruin, with claims rate $\mu = 1$. Note that the (Sturm-Liouville) Equation (31) intervenes also in the study of the squared radial Ornstein-Uhlenbeck diffusion (also called Cox-Ingersoll-Ross process) (Borodin and Salminen 2012, p. 140, Chapter II.8).

Let $K_i(z) = K_i(\tilde{q}, n, z), i = 1, 2, n = \tilde{q} + \tilde{\lambda}$ denote the (unique up to a constant) increasing/decreasing solutions for $z \in (0, \infty)$ of the confluent hypergeometric Equation (31). The solution of (31) is thus

$$c_1K_1(\tilde{q}, n, z) + c_2K_2(\tilde{q}, n, z) = c_1z^n e^{-z}M(\tilde{q} + 1, n + 1, z) + c_2z^n e^{-z}U(\tilde{q} + 1, n + 1, z), \tag{32}$$

where (Abramowitz and Stegun 1965, 13.2.5) $U[a, a + c, z] = \frac{1}{\Gamma[a]}\int_0^\infty e^{-zt}t^{a-1}(t + 1)^{c-1}dt, Re[z] > 0, Re[a] > 0$ is Tricomi's decreasing hypergeometric U function and $M(a, a + c, z) = {}_1F_1(a, a + c; z)$ is Kummer's increasing nonnegative confluent hypergeometric function of the first kind.[8]

The fact that the killed ruin probability must decrease to 0 implies the absence of the function K_1. The next result shows that the function $K_2(\mu(x + c(a)/r))$ is proportional to the ruin probability on an arbitrary interval $[a, \infty), a > -\tilde{c}$, and determines the proportionality constant . K_1 yields the absolute survival probability (and scale function) on $[-\tilde{c}, \infty)$, but over an arbitrary interval we must use a combination of K_1 and K_2.

Theorem 2. *(a) Put $z(x) = \mu(\tilde{c} + x), \tilde{c} = c(a)/r$. The ruin probability on $[a, \infty)$ is*

$$\Psi_q(x, a) = E_x[e^{-qT_{a,-}}] = \frac{\tilde{\lambda}}{\tilde{c}\mu}\,\frac{e^{-\mu x}(1 + x/\tilde{c})^{(\tilde{q}+\tilde{\lambda})}\,U\big(1 + \tilde{q}, 1 + \tilde{q} + \tilde{\lambda}, \mu(\tilde{c} + x)\big)}{U\big(1 + \tilde{q}, 2 + \tilde{q} + \tilde{\lambda}, \mu\tilde{c}\big)} \tag{33}$$

---

6 More generally, for any phase-type jumps C_i with Laplace transform $\widehat{f}_C(s) = \frac{a(s)}{b(s)}$, it may be checked that $\Psi_q * f_C = \widehat{f}_C(D)\Psi_q$ in the sense that $b(D)\Psi_q * f_C = a(D)\Psi_q$, thus removing the convolution by applying the denominator $b(D)$.

7 this is implied by the Kolmogorov integro-differential equation $(\mathcal{G} - \lambda - q)\Psi_q(x) + \lambda\overline{F}(x) = 0, x \geq 0$

8 $M(a, b, z)$ and $U(a, b, z)$ are the increasing/decreasing solutions of the to Weiler's canonical form of Kummer equation $zf''(z) + (b - z)f'(z) - a\,f(z) = 0$, which is obtained via the substitution $y(z) = e^{-z}z^n f(z)$, with $a = \tilde{q} + 1, b = n + 1$. Some computer systems use instead of M the Laguerre function defined by $M(a, b, z) = L_{-a}^{b-1}(z)\frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}$, which yields for natural $-a$ the Laguerre polynomial of degree $-a$.

(when $q=0$, $K_2(0,n,z)=\Gamma(\tilde{\lambda},z)$ *and* (33) *reduces to* $\frac{\Gamma(\tilde{\lambda},\mu(\tilde{c}+x))}{\Gamma(\tilde{\lambda}+1,\mu\tilde{c}))}$*).*[9]

(b) The **scale** *function* $W_q(x,a)$ *on* $[a,\infty)$ *is up to a proportionality constant*

$$K_1(\tilde{q},\tilde{\lambda},z(x))-kK_2(\tilde{q},\tilde{\lambda},z(x)),$$

with k defined in (40).

Proof. (a) Following (Paulsen and Gjessing 1997, ex. 2.1), note that the limit $\lim_{z\to\infty}U(z)=0$ implies

$$\Psi_q(x)=k\,K_2(z)=ke^{-z}z^{\tilde{q}+\tilde{\lambda}}U\left(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+1,z\right),\quad z=\mu(x+\tilde{c}).$$

The proportionality constant k is obtained from the boundary condition (30). Putting $G_b[h](x):=[c(x)(h)'(x)-(\lambda+q)h(x)]_{x=0}$,

$$G_b[\Psi_q](x)+\lambda=0\Longrightarrow k=\frac{\lambda}{-G_b[K_2](z(x))},$$

$$=-e^{-z}z^{\tilde{q}+\tilde{\lambda}+1}U\left(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,z\right)$$

Putting $z_0=\mu c$, we find

$$-G_b[K_2](z(x))=z_0e^{-z_0}z_0^{\tilde{q}+\tilde{\lambda}-1}U\left(\tilde{q},\tilde{q}+\tilde{\lambda},z_0\right)+(\tilde{q}+\tilde{\lambda})e^{-z_0}z_0^{\tilde{q}+\tilde{\lambda}}U\left(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+1,z_0\right)$$
$$=e^{-z_0}z_0^{\tilde{q}+\tilde{\lambda}}(U\left(\tilde{q},\tilde{q}+\tilde{\lambda},z_0\right)+(\tilde{q}+\tilde{\lambda})U\left(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+1,z_0\right)),$$

where we have used the identity (Borodin and Salminen 2012, p. 640)

$$K_2'(z)\quad=\quad-e^{-z}z^{\tilde{q}+\tilde{\lambda}-1}\,U\left(\tilde{q},\tilde{q}+\tilde{\lambda},z\right). \tag{34}$$

This may be further simplified to

$$-G_b[K_2](z(x))=e^{-z_0}z_0^{\tilde{q}+\tilde{\lambda}+1}U\left(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+2,z_0\right)),$$

by using the identity

$$U[a,b,z]+bU[a+1,b+1,z]=zU[a+1,b+2,z],a>1, \tag{35}$$

which is itself a consequence of the identities (Abramowitz and Stegun 1965, 13.4.16,13.4.18)

$$(b-a)U[a,b,z]+zU[a,2+b,z]=(z+b)U[a,1+b,z] \tag{36}$$
$$U[a,b,z]+(b-a-1)U[a+1,b,z]=zU[a+1,b+1,z] \tag{37}$$

(replace a by $a+1$ in the first identity, and subtract the second).

9 Note that we have corrected Paulsen's original denominator by using the identity (Abramowitz and Stegun 1965, 13.4.18) $U[a-1,b,z]+(b-a)U[a,b,z]=zU[a,b+1,z],a>1$.

Finally, we obtain:

$$\Psi_q(x) = \left(\frac{\tilde{\lambda}}{\tilde{c}\mu}\right) \frac{e^{-\mu x}(1+\frac{x}{\tilde{c}})^{\tilde{q}+\tilde{\lambda}} U\left(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, \mu(x+\tilde{c})\right)}{U\left(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}+1, \mu\tilde{c}\right)}$$
$$= \left(\frac{\tilde{\lambda}}{\mu}\right) \frac{\int_x^\infty (s-x)^{\tilde{q}} (s+\tilde{c})^{\tilde{\lambda}-1} e^{-\mu s} ds}{\int_0^\infty s^{\tilde{q}} (s+\tilde{c})^{\tilde{\lambda}} e^{-\mu s} ds},$$

and

$$\Psi_q(0) = \left(\frac{\lambda}{c\mu}\right) \frac{U\left(\tilde{q}+1, \tilde{\lambda}+\tilde{q}+1, \tilde{c}\mu\right)}{U\left(\tilde{q}+1, \tilde{\lambda}+\tilde{q}+2, \tilde{c}\mu\right)} = \left(\frac{\lambda}{c\mu}\right) \frac{\int_0^\infty t^{\tilde{q}} (1+t)^{\tilde{\lambda}-1} e^{-\tilde{c}\mu t} dt}{\int_0^\infty t^{\tilde{q}} (1+t)^{\tilde{\lambda}} e^{-\tilde{c}\mu t} dt}$$
$$= \left(\frac{\tilde{\lambda}}{\mu}\right) \frac{\int_0^\infty s^{\tilde{q}} (s+\tilde{c})^{\tilde{\lambda}-1} e^{-\mu s} ds}{\int_0^\infty s^{\tilde{q}} (s+\tilde{c})^{\tilde{\lambda}} e^{-\mu s} ds} = \left(\frac{\tilde{\lambda}}{\tilde{c}}\right) \frac{\int_0^\infty t^{\tilde{q}} (t+\mu)^{\tilde{\lambda}-1} e^{-\tilde{c}t} dt}{\int_0^\infty t^{\tilde{q}} (t+\mu)^{\tilde{\lambda}} e^{-\tilde{c}t} dt}.$$

For $(a,\infty), a > -\tilde{c}$, the same proof works after replacing $c, z(0)$ by $c(a), z(a)$.

(b) On $(0,\infty)$, we must determine, up to proportionality, a linear combination $W_q(x) = K_1(\tilde{q}, \tilde{\lambda}, z(x)) - kK_2(\tilde{q}, \tilde{\lambda}, z(x))$ satisfying the boundary condition

$$G_b W_q(x) = 0 \Longrightarrow k = \frac{G_b[K_1](0)}{G_b[K_2](0)}, G_b h(x) := [c(x)(h)'(x) - (\lambda+q)h(x)]_{x=0}. \tag{38}$$

Recall we have already computed $G_b[K_2](z(0)) = -e^{-z_0} z_0^{\tilde{q}+\tilde{\lambda}+1} U\left(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, z_0\right)$ in the proof of (a). Similarly, using (Borodin and Salminen 2012, p. 640) (reproduced for convenience in (55) below)

$$-G_b[K_1](z(0)) = (\tilde{\lambda}+\tilde{q}) e^{-z_0} z_0^{\tilde{q}+\tilde{\lambda}} \left[M\left(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+1, z_0\right) - M\left(\tilde{q}, \tilde{q}+\tilde{\lambda}, z_0\right)\right].$$

Hence[10]

$$k = \frac{\tilde{\lambda}+\tilde{q}}{z_0} \frac{M\left(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+1, z_0\right) - M\left(\tilde{q}, \tilde{q}+\tilde{\lambda}, z_0\right)}{U(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, z_0).} \tag{40}$$

□

Remark 6. *Note that on $(-\tilde{c},\infty)$, choosing $-\tilde{c}$ as the origin yields $z_0 = 0 = k$ (since $M\left(\tilde{q}, \tilde{q}+\tilde{\lambda}, 0\right) = 1$ (Abramowitz and Stegun 1965, 13.1.2)) and the scale function is proportional to $K_1(z)$, which follows also from the uniqueness of the nondecreasing solution.*

Corollary 1. *(a) Differentiating the scale function yields*

$$\mathbf{w}_q(x) = e^{-z} z^{\tilde{q}+\tilde{\lambda}-1} \Big((\tilde{q}+\tilde{\lambda}) M(\tilde{q}, \tilde{q}+\tilde{\lambda}, z) + kU(\tilde{q}, \tilde{q}+\tilde{\lambda}, z)\Big).$$

(b) The scale function is increasing.

---

10 Putting $M_{++} = M\left(\tilde{q}+2, \tilde{q}+\tilde{\lambda}+2, \mu\tilde{c}\right), U_{++} = U\left(\tilde{q}+2, \tilde{q}+\tilde{\lambda}+2, \mu\tilde{c}\right)$, we must solve the equation

$$M - lU = \frac{\tilde{q}+1}{\tilde{q}+\tilde{\lambda}+1} M_{++} + l(\tilde{q}+1) U_{++}$$
$$\Leftrightarrow l = \frac{(\tilde{q}+\tilde{\lambda}+1)M - (\tilde{q}+1)M_{++}}{(\tilde{q}+\tilde{\lambda}+1)\left(U + (\tilde{q}+1)U_{++}\right)} = \frac{\tilde{\lambda}}{\tilde{q}+\tilde{\lambda}+1} \frac{M_+}{U_+}, \tag{39}$$

where we put $M_+ = M\left(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, \mu\tilde{c}\right), U_+ = U\left(\tilde{q}+1, \tilde{q}+\tilde{\lambda}+2, \mu\tilde{c}\right)$, and applied the identities (Abramowitz and Stegun 1965, 13.4.3, 13.4.4).

5. Asmussen's Embedding Approach for Solving Kolmogorov's Integro-Differential Equation with Phase-Type Jumps

One of the most convenient approaches to get rid of the integral term in (29) is a probabilistic transformation which gets rid of the jumps as in Asmussen (1995), when the downward phase-type jumps have a survival function

$$\bar{F}_C(x) = \int_x^\infty f_C(u)du = \vec{\beta}e^{Bx}\mathbf{1},$$

where B is a $n \times n$ stochastic generating matrix (nonnegative off-diagonal elements and nonpositive row sums), $\vec{\beta} = (\beta_1, \ldots, \beta_n)$ is a row probability vector (with nonnegative elements and $\sum_{j=1}^n \beta_j = 1$), and $\mathbf{1} = (1, 1, ..., 1)$ is a column probability vector.

The density is $f_C(x) = \vec{\beta}e^{-Bx}\boldsymbol{b}$, where $\boldsymbol{b} = (-B)\mathbf{1}$ is a column vector, and the Laplace transform is

$$\hat{b}(s) = \vec{\beta}(sI - \boldsymbol{B})^{-1}\boldsymbol{b}.$$

Asmussen's approach Asmussen (1995); Asmussen et al. (2002) replaces the negative jumps by segments of slope -1, embedding the original spectrally negative Lévy process into a continuous Markov modulated Lévy process. For the new process we have auxiliary unknowns $A_i(x)$ representing ruin or survival probabilities (or, more generally, Gerber-Shiu functions) when starting at x conditioned on a phase i with drift downwards (i.e., in one of the "auxiliary stages of artificial time" introduced by changing the jumps to segments of slope -1). Let $\mathbf{A}$ denote the column vector with components $A_1, \ldots, A_n$. The Kolmogorov integro-differential equation turns then into a system of ODE's, due to the continuity of the embedding process.

$$\begin{pmatrix} \Psi_q'(x) \\ \mathbf{A}'(x) \end{pmatrix} = \begin{pmatrix} \frac{\lambda+q}{c(x)} & -\frac{\lambda}{c(x)}\vec{\beta} \\ \mathbf{b} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \Psi_q(x) \\ \mathbf{A}(x) \end{pmatrix}, \ x \geq 0. \tag{41}$$

For the ruin probability with exponential jumps of rate μ for example, there is only one downward phase, and the system is:

$$\begin{pmatrix} \Psi_q'(x) \\ A'(x) \end{pmatrix} = \begin{pmatrix} \frac{\lambda+q}{c(x)} & -\frac{\lambda}{c(x)} \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} \Psi_q(x) \\ A(x) \end{pmatrix} \ x \geq 0. \tag{42}$$

For survival probabilities, one only needs to modify the boundary conditions—see the following section.

5.1. Exit Problems for the Segerdahl-Tichy process, with $q = 0$

Asmussen's approach is particular convenient for solving exit problems for the Segerdahl-Tichy process.

Example 1. The eventual ruin probability. When $q = 0$, *the system for the ruin probabilities with* $x \geq 0$ *is:*

$$\begin{cases} \Psi'(x) = \dfrac{\lambda}{c(x)}\left(\Psi(x) - A(x)\right), & \Psi(\infty) = A(\infty) = 0 \\ A'(x) = \mu\left(\Psi(x) - A(x)\right), & A(0) = 1 \end{cases} \tag{43}$$

This may be solved by subtracting the equations. Putting

$$K(x) = e^{-\mu x + \int_0^x \frac{\lambda}{c(v)}dv},$$

we find:

$$\begin{cases} \Psi(x) - A(x) & = (\Psi(0) - A(0))K(x), \\ A(x) & = \mu(A(0) - \Psi(0)) \int_x^\infty K(v)dv, \end{cases} \tag{44}$$

whenever $K(v)$ *is integrable at* ∞.

The boundary condition $A(0) = 1$ *implies that* $1 - \Psi(0) = \frac{1}{\mu \int_0^\infty K(v)dv}$ *and*

$$A(x) = \mu(1 - \Psi(0)) \int_x^\infty K(v)dv = \frac{\int_x^\infty K(v)dv}{\int_0^\infty K(v)dv},$$
$$\Psi(x) - A(x) = -\frac{K(x)}{\mu \int_0^\infty K(v)dv}.$$

Finally,

$$\Psi(x) = A(x) + (\Psi(x) - A(x)) = \frac{\mu \int_x^\infty K(v)dv - K(x)}{\mu \int_0^\infty K(v)dv},$$

and for the survival probability $\overline{\Psi}$,

$$\overline{\Psi}(x) = \frac{\mu \int_0^x K(v)dv + K(x)}{\mu \int_0^\infty K(v)dv} := \overline{\Psi}(0)\mathbf{W}(x) = \frac{\mathbf{W}(x)}{\mathbf{W}(\infty)}, \tag{45}$$
$$\mathbf{W}(x) = \mu \int_0^x K(v)dv + K(x),$$

where $\overline{\Psi}(0) = \frac{1}{\mathbf{W}(\infty)}$ *by plugging* $\mathbf{W}(0) = 1$ *in the first and last terms in* (45).

We may also rewrite (45) *as:*

$$\overline{\Psi}(x) = \frac{1 + \int_0^x \mathbf{w}(v)dv}{1 + \int_0^\infty \mathbf{w}(v)dv} \Leftrightarrow \Psi(x) = \frac{\int_x^\infty \mathbf{w}(v)dv}{1 + \int_0^\infty \mathbf{w}(v)dv}, \mathbf{w}(x) := \mathbf{W}'(x) = \frac{\lambda K(x)}{c(x)} \tag{46}$$

Note that $\mathbf{w}(x) > 0$ *implies that the scale function* $\mathbf{W}(x)$ *is nondecreasing.*

Example 2. *For the two sided exit problem on* $[a, b]$, *a similar derivation yields the scale function*

$$\mathbf{W}(x, a) = \mu \int_a^x \frac{K(v)}{K(a)} dv + \frac{K(x)}{K(a)} = 1 + \frac{1}{K(a)} \int_a^x \mathbf{w}(y)dy,$$

with scale derivative derivative $\mathbf{w}(x, a) = \frac{1}{K(a)}\mathbf{w}(x)$, *where* $\mathbf{w}(x)$ *given by* (46) *does not depend on a.*

Indeed, the analog of (44) *is:*

$$\begin{cases} \overline{\Psi}^b(x, a) - A^b(x) = \overline{\Psi}^b(a, a)\frac{K(x)}{K(a)}, \\ A^b(x) = \mu\overline{\Psi}^b(a, a) \int_a^x \frac{K(v)}{K(a)} dv, \end{cases}$$

implying by the fact that $\overline{\Psi}^b(b,a) = 1$ *that*

$$\begin{aligned}
&\overline{\Psi}^b(x,a) = \overline{\Psi}^b(a,a)\left(\frac{K(x)}{K(a)} + \mu\int_a^x \frac{K(v)}{K(a)}dv\right) = \frac{\mathbf{W}(x,a)}{\mathbf{W}(b,a)} = \frac{1+\frac{1}{K(a)}\int_a^x \mathbf{w}(u)du}{1+\frac{1}{K(a)}\int_a^b \mathbf{w}(u)du} \Leftrightarrow \\
&\Psi^b(x,a) = \frac{\int_x^b \mathbf{w}(u)du}{K(a)+\int_a^b \mathbf{w}(u)du} \Leftrightarrow \\
&\psi^b(x,a) := -(\Psi^b)'(x,a) = \frac{\mathbf{w}(x)}{K(a)+\int_a^b \mathbf{w}(u)du} = \mathbf{w}(x,a)\frac{\overline{\Psi}(a,a)}{\overline{\Psi}(b,a)}.
\end{aligned} \tag{47}$$

Remark 7. *The definition adopted in this section for the scale function* $\mathbf{W}(x,a)$ *uses the normalization* $\mathbf{W}(a,a) = 1$*, which is only appropriate in the absence of Brownian motion.*

Problem 4. *Extend the equations for the survival and ruin probability of the Segerdahl-Tichy process in terms of the scale derivative* $\mathbf{w}_q$*, when* $q > 0$*. Essentially, this requires obtaining*

$$T_q(x) = E_x\left[e^{-q[T_{a,-}\min T_{b,+}]}\right]$$

6. Revisiting Segerdahl's Process via the Scale Derivative/Integrating Factor Approach, When $q = 0$

Despite the new scale derivative/integrating factor approach, we were not able to produce further explicit results beyond (33), due to the fact that neither the scale derivative, nor the integral of the integrating factor are explicit when $q > 0$ (this is in line with Avram et al. (2010)). (33) remains thus for now an outstanding, not well-understood exception.

Problem 5. *Are there other explicit first passage results for Segerdahl's process when* $q > 0$*?*

In the next subsections, we show that via the scale derivative/integrating factor approach, we may rederive well-known results for $q = 0$.

6.1. Laplace Transforms of the Eventual Ruin and Survival Probabilities

For $q = 0$, both Laplace transforms and their inverses are explicit, and several classic results may be easily checked. The scale derivative may be obtained using Proposition 1 and $\Gamma(\tilde{\lambda}+1, v) = e^{-v}v^{\tilde{\lambda}} + \lambda\Gamma(\tilde{\lambda}, v)$ with $v = \tilde{c}(s+\mu)$. We find

$$\begin{aligned}
&\widehat{\mathbf{w}}(s,a) = \frac{e^{\tilde{c}\mu}(\tilde{c}\mu)^{-\lambda}\Gamma(\lambda+1,\tilde{c}(s+\mu))}{e^{-\tilde{c}s}(1+s/\mu)^{\tilde{\lambda}}} - 1 = 1 + \lambda e^{v}(v)^{-\tilde{\lambda}}\Gamma(\tilde{\lambda},v) - 1 = \tilde{\lambda}U(1, 1+\tilde{\lambda}, \tilde{c}(s+\mu)) \\
&\Longrightarrow \mathbf{w}(x,a) = \frac{\tilde{\lambda}}{\tilde{c}}\left(1+\frac{x}{\tilde{c}}\right)^{\tilde{\lambda}-1} e^{-\mu x},
\end{aligned} \tag{48}$$

which checks (46). Using again $\widehat{\mathbf{w}}(s) = \tilde{c}\frac{\bar{I}(y)}{I(y)} - 1$ yields the ruin and survival probabilities:

$$\begin{aligned}
&s\widehat{\overline{\Psi}}(s) = \frac{\int_s^\infty \tilde{c}\overline{\Psi}(0)I(y)dy}{I(s)} = \overline{\Psi}(0)(\widehat{\mathbf{w}}(s)+1) \\
&s\widehat{\Psi}(s) = \frac{\int_s^\infty (\tilde{c}\Psi(0) - \frac{\tilde{\lambda}}{y+\mu})I(y)dy}{I(s)} = \Psi(0)(\widehat{\mathbf{w}}(s)+1) - \widehat{\mathbf{w}}(s).
\end{aligned}$$

Letting $s \to 0$ yields

$$\Psi(0) = \frac{\widehat{\mathbf{w}}(0)}{\widehat{\mathbf{w}}(0)+1} = \frac{\tilde{\lambda} U(1, 1+\tilde{\lambda}, \mu\tilde{c})}{\mu\tilde{c}\, U(1, 2+\tilde{\lambda}, \mu\tilde{c})} = \frac{\tilde{\lambda}\Gamma(\tilde{\lambda}, \tilde{c}\mu)}{\Gamma(\tilde{\lambda}+1, \tilde{c}\mu)} \Leftrightarrow$$
$$\overline{\Psi}(0) = \frac{\lim_{s\to 0} s\widehat{\overline{\Psi}}(s)}{\widehat{\mathbf{w}}(0)+1} = \frac{\overline{\Psi}(\infty)}{1+\tilde{\lambda}\, U(1, 1+\tilde{\lambda}, \mu\tilde{c})} = \frac{1}{\mu\tilde{c}\, U(1, 2+\tilde{\lambda}, \mu\tilde{c})} \tag{49}$$

For the survival probability, we finally find

$$s\widehat{\overline{\Psi}}(s) = \overline{\Psi}(0)(1+\widehat{\mathbf{w}}(s)) = \frac{1+\tilde{\lambda} U(1, 1+\tilde{\lambda}, \mu(\tilde{c}+s))}{1+\tilde{\lambda} U(1, 1+\tilde{\lambda}, \mu\tilde{c})} = \frac{\tilde{c}(\mu+s) U(1, 2+\tilde{\lambda}, \mu(\tilde{c}+s))}{\tilde{c}\mu U(1, 2+\tilde{\lambda}, \mu\tilde{c})},$$

which checks with the Laplace transform of the Segerdahl result (53).

6.2. The Eventual Ruin and survival probabilities

These may also be obtained directly by integrating the explicit scale derivative $\mathbf{w}(x,a) = \frac{\tilde{\lambda}}{\tilde{c}}\left(1+\frac{x}{\tilde{c}}\right)^{\tilde{\lambda}-1} e^{-\mu x}$ (48) Indeed,

$$\int_u^\infty \mathbf{w}(x)dx = \int_u^\infty \frac{\tilde{\lambda}}{\tilde{c}}\left(1+\frac{x}{\tilde{c}}\right)^{\tilde{\lambda}-1} e^{-\mu x}dx = \tilde{\lambda} e^{\mu\tilde{c}} \int_{1+\frac{u}{\tilde{c}}}^\infty y^{\tilde{\lambda}-1} e^{\mu\tilde{c}y} dy$$
$$= \tilde{\lambda} e^{\mu\tilde{c}} \frac{1}{(\mu\tilde{c})^{\tilde{\lambda}}} \int_{\mu(\tilde{c}+u)}^\infty t^{\tilde{\lambda}-1} e^{-t} dt = \tilde{\lambda} e^{\mu\tilde{c}} (\mu\tilde{c})^{-\tilde{\lambda}} \Gamma(\tilde{\lambda}, \mu(\tilde{c}+u)),$$

where $\Gamma(\eta, x) = \int_x^\infty t^{\eta-1} e^{-t} dt$ is the incomplete gamma function. The ruin probability is Segerdahl (1955), (Paulsen and Gjessing 1997, ex. 2.1):

$$\Psi(x) = \tilde{\lambda} \frac{\exp(\mu\tilde{c})(\mu\tilde{c})^{-\tilde{\lambda}}\Gamma(\tilde{\lambda}, \mu(\tilde{c}+x))}{1+\tilde{\lambda}\exp(\mu\tilde{c})(\mu\tilde{c})^{-\tilde{\lambda}}\Gamma(\tilde{\lambda}, \mu\tilde{c})} = \tilde{\lambda} \frac{e^{-\mu x}(1+x/\tilde{c})^{\tilde{\lambda}}\, U(1, 1+\tilde{\lambda}, \mu(\tilde{c}+x))}{1+\tilde{\lambda} U(1, 1+\tilde{\lambda}, \mu\tilde{c})}$$
$$= \frac{\tilde{\lambda}}{\mu\tilde{c}} \frac{e^{-\mu x}(1+x/\tilde{c})^{\tilde{\lambda}}\, U(1, 1+\tilde{\lambda}, \mu(\tilde{c}+x))}{U(1, 2+\tilde{\lambda}, \mu\tilde{c})} = \frac{\tilde{\lambda}\Gamma(\tilde{\lambda}, \mu(\tilde{c}+x))}{\Gamma(\tilde{\lambda}+1, \tilde{c}\mu)}, \tag{50}$$

where we used

$$U(1, 1+\tilde{\lambda}, v) = e^v v^{-\tilde{\lambda}} \Gamma(\tilde{\lambda}, v) \tag{51}$$

and

$$1+\tilde{\lambda} U(1, 1+\tilde{\lambda}, v) = v U(1, 2+\tilde{\lambda}, v), \tag{52}$$

which holds by integration by parts.

A simpler formula holds for the rate of ruin $\psi(x)$ and its Laplace transform

$$\psi(x) = -\Psi'(x) = \frac{\mathbf{w}(x)}{1+\int_0^\infty \mathbf{w}(x)dx} = \frac{\tilde{\lambda}}{\Gamma(\tilde{\lambda}+1, \tilde{c}\mu)} \mu(\mu(\tilde{c}+x))^{\tilde{\lambda}-1} e^{-\mu(\tilde{c}+x)} = e^{-\mu\tilde{c}} \gamma_{\tilde{\lambda},\mu}(x+\tilde{c}) \Leftrightarrow$$
$$\widehat{\psi}(s) = \overline{\Psi}(0)\widehat{\mathbf{w}}(s) = \begin{cases} \frac{\tilde{\lambda} U(1, 1+\tilde{\lambda}, \tilde{c}(s+\mu))}{\tilde{c}\mu U(1, 2+\tilde{\lambda}, \tilde{c}\mu)}, & c > 0 \\ (1+s/\mu)^{-\tilde{\lambda}}, & c = 0 \end{cases}, \tag{53}$$

where γ denotes a (shifted) Gamma density. Of course, the case $c > 0$ simplifies to a Gamma density when moving the origin to the "absolute ruin" point $-\tilde{c} = -\frac{c}{r}$,, i.e., by putting $y = x + \tilde{c}$, $Y_t = X_t + \tilde{c}$, where the process Y_t has drift rate rY_t.

Problem 6. *Find a relation between the ruin derivative $\psi_q(x) = -\Psi_q'(x)$ and the scale derivative $\mathbf{w}_q(x)$ when $q > 0$.*

7. Further Details on the Identities Used in the Proof of Theorem 2

We recall first some continuity and differentiation relations needed here Abramowitz and Stegun (1965)

Proposition 2. *Using the notation $M = M(a,b,z), M(a+) = M(a+1,b,z), M(+,+) = M(a+1,b+1,z)$, and so on, the Kummer and Tricomi functions satisfy the following identities:*

$$bM + (a-b)M\,(b+) = aM\,(a+) \tag{13.4.3}$$

$$b\big(M(a+) - M\big) = zM\,(+,+) \tag{13.4.4}$$

$$(b-a)U + zU(b+2) = (z+b)U(b+1) \tag{13.4.16}$$

$$U + a\,U\,(+,+) = U\,(b+) \tag{13.4.17}$$

$$U + (b-a-1)U(a+1) = zU(+,+) \tag{13.4.18}$$

(see corresponding equations in Abramowitz and Stegun (1965)).

$$U' = -aU\,(+,+)\,, \quad M' = \frac{a}{b}M\,(+,+)\,. \tag{54}$$

Proposition 3. *The functions $K_i(\tilde{q}, \tilde{\lambda}, z)$ defined by (32) satisfy the identities*

$$\begin{aligned} K_1'(\tilde{q}, n, z) &= (\tilde{q}+\tilde{\lambda})e^{-z}z^{\tilde{q}+\tilde{\lambda}-1}\,M\,(\tilde{q}, \tilde{q}+\tilde{\lambda}, z) = (\tilde{q}+\tilde{\lambda})K_1(\tilde{q}-1, \tilde{\lambda}, z) && (55)\\ K_2'(\tilde{q}, n, z) &= -e^{-z}z^{\tilde{q}+\tilde{\lambda}-1}\,U\,(\tilde{q}, \tilde{q}+\tilde{\lambda}, z) = -K_2(\tilde{q}-1, n, z) && (56) \end{aligned}$$

$$K_2(\tilde{q}, n, z) = \int_z^\infty (y-z)^{\tilde{q}}\,(y)^{n-\tilde{q}-1}\,e^{-y}dy \tag{57}$$

Proof: For the first identity, note, using (Abramowitz and Stegun 1965, 13.4.3, 13.4.4), that

$$\begin{aligned} \frac{e^z}{z^{\tilde{q}+\tilde{\lambda}-1}}K_1'(z) &= (\tilde{q}+\tilde{\lambda}-z)M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z) + z\frac{\tilde{q}+1}{\tilde{q}+\tilde{\lambda}+1}M\,(\tilde{q}+2, \tilde{q}+2+\tilde{\lambda}, z)\\ &= (\tilde{q}+\tilde{\lambda})M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z)\\ &+ \frac{z}{\tilde{q}+\tilde{\lambda}+1}\left((\tilde{q}+1)M\,(\tilde{q}+2, \tilde{q}+2+\tilde{\lambda}, z) - (\tilde{q}+\tilde{\lambda}+1)M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z)\right)\\ &= (\tilde{q}+\tilde{\lambda})M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z) - \frac{z}{\tilde{q}+\tilde{\lambda}+1}\tilde{\lambda}M\,(\tilde{q}+1, \tilde{q}+2+\tilde{\lambda}, z)\\ &= (\tilde{q}+\tilde{\lambda})M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z) - \tilde{\lambda}\left(M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z) - M\,(\tilde{q}, \tilde{q}+1+\tilde{\lambda}, z)\right)\\ &= \tilde{q}M\,(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, z) + \tilde{\lambda}M\,(\tilde{q}, \tilde{q}+1+\tilde{\lambda}, z)\,. \end{aligned}$$

The second formula may be derived similarly using 13.4.17, or by considering the function

$${}_z\widetilde{U}(\tilde{q}+1, \tilde{q}+1+\tilde{\lambda}, \mu) := \Gamma(q+1)K_2(z) = \int_z^\infty (s-z)^{\tilde{q}}\,(s)^{\tilde{\lambda}-1}\,e^{-\mu s}ds$$

appearing in the numerator of the last form of (57). An integration by parts yields

$$
\begin{aligned}
&{}_z\widetilde{U}'(\tilde{q}+1,\tilde{q}+1+\tilde{\lambda},1) = \int_z^\infty (s-z)^{\tilde{q}} \frac{d}{dz}[(s)^{\tilde{\lambda}-1} e^{-s}]ds \\
&= (\tilde{\lambda}-1)_z\widetilde{U}(\tilde{q}+1,\tilde{q}+\tilde{\lambda},1) -_z \widetilde{U}(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+1,1), \implies \\
&K_2'(\tilde{q}+1,\tilde{\lambda},z) = e^{-z} z^{\tilde{q}+\tilde{\lambda}-1} \left((\tilde{\lambda}-1)U(\tilde{q}+1,\tilde{q}+\tilde{\lambda},z) - U(\tilde{q}+1,\tilde{q}+\tilde{\lambda}+1,z)\right)
\end{aligned}
$$

and the result follows by (Abramowitz and Stegun 1965, 13.4.18.)[11]

The third formula is obtained by the substitution $y = z(t+1)$.

8. Conclusions and Future Work

Two promising fundamental functions have been proposed for working with generalizations of Segerdahl's process: (a) the scale derivative **w** Czarna et al. (2017) and (b) the integrating factor I Avram and Usabel (2008), and they are shown to be related via Thm. 1.

Segerdahl's process per se is worthy of further investigation. A priori, many risk problems (with absorbtion/reflection at a barrier b or with double reflection, etc.) might be solved by combinations of the hypergeometric functions U and M.

However, this approach leads to an impasse for more complicated jump structures, which will lead to more complicated hypergeometric functions. In that case, we would prefer answers expressed in terms of the fundamental functions **w** or I.

We conclude by mentioning two promising numeric approaches, not discussed here. One due to Jacobsen and Jensen (2007) bypasses the need to deal with high-order hypergeometric solutions by employing complex contour integral representations. The second one uses Laguerre-Erlang expansions—see Abate, Choudhury and Whitt (1996); Avram et al. (2009); Zhang and Cui (2019). Further effort of comparing their results with those of the methods discussed above seems worthwhile.

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[11] See also (Borodin and Salminen 2012, p. 640), where however the first formula has a typo.

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