The Løkka–Zervos Alternative for a Cramér–Lundberg Process with Exponential Jumps

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Abstract: In this paper, we study a stochastic control problem faced by an insurance company allowed to pay out dividends and make capital injections. As in (Løkka and Zervos (2008); Lindensjö and Lindskog (2019)), for a Brownian motion risk process, and in Zhu and Yang (2016), for diffusion processes, we will show that the so-called Løkka–Zervos alternative also holds true in the case of a Cramér–Lundberg risk process with exponential claims. More specifically, we show that: if the cost of capital injections is *low*, then according to a double-barrier strategy, it is optimal to pay dividends and inject capital, meaning ruin never occurs; and if the cost of capital injections is *high*, then according to a single-barrier strategy, it is optimal to pay dividends and never inject capital, meaning ruin occurs at the first passage below zero.

Keywords: stochastic control; optimal dividends; capital injections; bankruptcy; barrier strategies; reflection and absorption; scale functions

1. Introduction

Risk theory initially revolved around minimizing the probability of ruin. However, shareholders are more interested in maximizing the value of the company than minimizing risks. Therefore, (de Finetti 1957) suggested finding the optimal dividend policies which maximize the expected value of the sum of discounted future dividend payments up to the time of ruin; see also (Miller and Modigliani 1961). Another interesting objective, as suggested by (Shreve et al. 1984), is to maximize the expected discounted cumulative dividends while redressing the reserves by injecting capital each time it becomes necessary.

This note is motivated by subsequent results obtained by (Løkka and Zervos 2008; Lindensjö and Lindskog 2019) for a Brownian motion with drift, and by (Zhu and Yang 2016) for diffusions. Their results state that, depending on the size of transaction costs, one of the following strategies is optimal:

1.   if the cost of capital injections is *low*, then according to a double-barrier strategy, it is optimal to pay dividends and to inject capital, meaning ruin never occurs;

2.   if the cost of capital injections is *high*, then according to a single-barrier strategy, it is optimal to pay dividends and never inject capital, meaning ruin occurs at the first passage below zero.

1.1. The Model

In what follows, we will use the following notation: the law of a Markov process X when starting from $X_0 = x$ will be denoted by $\mathbb{P}_x$, and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

To fix ideas, let us start with the Cramér–Lundberg risk model for $t \geq 0$ (see, for example, Dufresne and Gerber 1991; Albrecher and Asmussen 2010):

$$X_t = x + ct - S_t, \quad \text{where } S_t = \sum_{i=1}^{N_t} C_i. \tag{1}$$

Here, $x \geq 0$ is the initial surplus, $c \geq 0$ is the linear premium rate, and $\{C_i, i = 1, 2, \ldots\}$ are independent and identically distributed random variables, with distribution function F and mean $m_1 = \int_0^\infty z F(dz)$ representing non-negative jumps/claims. The inter-arrival times between these jumps are independent and exponentially distributed with mean $1/\lambda$, and N_t denotes the time-t value of the associated Poisson process counting the arrivals of claims on the interval $[0, t]$. We will assume the positive profit condition $p := c - \lambda m_1 > 0$.

The process given in (1) is a particular case of a spectrally negative Lévy process (SNLP), that is, a Lévy process without positive jumps, where in this case there is also a finite mean. More precisely, such a process is defined by adding a Brownian perturbation to (1), and by assuming that S_t is a subordinator with a σ-finite Lévy measure $\Pi(dx)$, having possibly infinite activity near the origin, that is, $\Pi(0, \infty) = \infty$. For a SNLP, the positive profit condition becomes $p = c - \int_0^\infty x \Pi(dx) > 0$. Note that for the SNLP given in (1), we have $\Pi(dx) = \lambda F(dx)$ so $\Pi(0, \infty) = \lambda$. See, for example, (Bertoin 1998) for more details.

The main result of our paper assumes that the claim sizes/jumps are exponentially distributed with mean $1/\mu$, that is, that $F(z) = 1 - e^{-\mu z}$ when $z > 0$. However, as most of our intermediate results hold for a general SNLP, they will be stated in this more general context. Unfortunately, one key fact below holds only for a Cramér–Lundberg process with exponential jumps. Consequently, in the general SNLP case, the Løkka–Zervos alternative is still an open problem.

Recall that a SNLP X is characterized by its Laplace exponent defined by $\psi(\theta) = \ln \mathbb{E}\left[e^{\theta X_1}\right]$. For the Cramér-Lundberg process X given in (1), we have

$$\psi(\theta) = c\theta + \int_0^\infty \left(e^{-\theta z} - 1\right) \lambda F(dz)$$

and, in the case of exponential jumps, we further have

$$\psi(\theta) = \theta \left(c - \frac{\lambda}{\mu + \theta}\right). \tag{2}$$

1.2. The Problem

For the stochastic control problem considered in this paper, an admissible strategy is represented by a pair (C, D) composed of a non-decreasing, left-continuous, and adapted stochastic process $D = \{D_t, t \geq 0\}$ and $C = \{C_t, t \geq 0\}$, where D_t represents the cumulative amount of dividends paid up to time t, while C_t represents the cumulative amount of capital injections made up to time t. We assume $D_0 = 0$ and $C_0 = 0$. For a given strategy (C, D), the corresponding controlled surplus process $U = \{U_t, t \geq 0\}$ is defined by $U_t = X_t - D_t + C_t$. Define also $\tau = \inf\{t > 0 : U_t < 0\}$.

For a given initial surplus $x \geq 0$, let $\mathcal{A}(x)$ be the corresponding set of admissible strategies. Also, let $q > 0$ be the discounting rate and let $k > 1$ be the proportional cost of injecting capital. The objective is to maximize the value of a strategy using the following objective function:

$$J(x, C, D) = \mathbb{E}_x\left[\int_0^\tau e^{-qt}\left(dD_t - k dC_t\right)\right], \tag{3}$$

that is, the goal is to find the optimal value function

$$V_k(x) = \sup_{(C,D) \in \mathcal{A}(x)} J(x, C, D).$$

For a general Markov process X, our problem amounts to solving (in a viscosity sense) the following Hamilton–Jacobi–Bellman (HJB) equation :

$$\begin{cases} \max \left\{ (\mathcal{L} - q) V(x), 1 - V'(x), V'(x) - k, -V(x) \right\} \leq 0, & x \geq 0 \\ \max \left\{ (\mathcal{L} - q) V(x), V'(x) - k, -V(x) \right\} \leq 0, & x < 0 \end{cases} \tag{4}$$

where $\mathcal{L}$ is the infinitesimal generator associated with the underlying uncontrolled process X (see also (Zhu and Yang 2016, sct. 3.6) for the case of diffusions). For the Cramér-Lundberg process with exponential jumps, the operator is

$$\mathcal{L}V(x) = cV'(x) + \lambda\mu \int_0^\infty \left(V(x - z) - V(x) \right) e^{-\mu z} dz. \tag{5}$$

The second part of (4) is associated to the possibility of modifying the surplus by a lump sum dividend payment (see (6) below), and the third part to capital injections.

In the cases already studied, the Løkka–Zervos alternative reduces to the following dilemma: shall we declare bankruptcy at level 0, or shall we use capital injections to maintain the surplus positive?

The classical problems studied by (de Finetti 1957; Shreve et al. 1984) are revisited in Section 2 and new results are obtained. In Section 3, we prove that the Løkka–Zervos alternative holds for a Cramér–Lundberg model with exponential jumps.

2. The Classical Dividend Problems for SNLPs

In this section, we review de Finetti's, as well as Shreve, Lehoczky, and Gaver's optimal dividend problems for general spectrally negative Lévy processes. As is well-known, the value functions can be expressed in terms of scale functions (see, for example, Avram et al. 2004, 2019; Bertoin 1998; Kyprianou 2014).

2.1. De Finetti's Problem

De Finetti's problem corresponds to the case where $k = \infty$, implying that $C \equiv 0$, that is, capital injections cannot be profitable. In this case, the controlled process is ruined as soon as it goes below zero. For this problem, the optimal value function will be denoted by V^{dF}.

It is well-known that for this problem, constant barrier strategies are very important. For $b \geq 0$, the (horizontal) barrier strategy at level b is the strategy with a cumulative amount of dividends paid until time $t > 0$ given by $D_t^b = \left(\sup_{0 < s \leq t} X_s - b \right)_+$. If $X_0 = x > b$, then $D_{0+}^b = x - b$ (a lump sum payment is made). For such a strategy, the value function is such that $J(x, 0, D^b) = V^b(x)$, where

$$V^b(x) := \mathbb{E}_x \left[\int_0^{\tau^b} e^{-qt} dD_t^b \right],$$

where τ^b is the time of ruin for the controlled process $U_t^b = X_t - D_t^b$. In this case, $\mathbb{P}_x \left(\tau^b < \infty \right) = 1$.

It is well-known that, for a SNLP (see, for example, Avram et al. 2007),

$$V^b(x) = \begin{cases} \frac{W_q(x)}{W_q'(b)}, & x \leq b, \\ x - b + \frac{W_q(b)}{W_q'(b)}, & x > b, \end{cases} \tag{6}$$

where the q-scale function W_q (Bertoin 1998) is given through its Laplace transform:

$$\int_0^\infty e^{-\theta x} W_q(x) dx = \frac{1}{\psi(\theta) - q}, \tag{7}$$

for all $\theta > \Phi(q) = \sup\{s \geq 0 : \psi(s) = q\}$.

It is known (see Theorem 1.1 in (Loeffen and Renaud 2010)) that if the tail of the jump distribution is log-convex, then an optimal dividend policy is formed by the barrier strategy at level b^*, where b^* is the last maximum of the *barrier function*

$$H^{dF}(b) = \frac{1}{W_q'(b)}, \quad b > 0. \tag{8}$$

In this case, the optimal value function V^{dF} is given by V^{b*}.

Consequently, for a Cramér–Lundberg risk process with exponentially distributed claims, the optimal value function V^{dF} is equal to the value function of a barrier strategy. More precisely, for X given in (1) with exponential jumps, we have

$$W_q(x) = \frac{A_+ e^{\rho_+ x} - A_- e^{\rho_- x}}{c(\rho_+ - \rho_-)}, \quad x \geq 0,$$

where

$$\rho_\pm = \frac{1}{2c}\left(-(\mu c - \lambda - q) \pm \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c}\right)$$

are such that $\rho_- \leq 0 \leq \rho_+ = \Phi(q)$, and where $A_\pm = \mu + \rho_\pm$. In this case, the barrier function H^{dF} has a unique maximum at level

$$b^* = \begin{cases} \frac{1}{\rho_+ - \rho_-} \log\left(\frac{\rho_-^2 (\mu + \rho_-)}{\rho_+^2 (\mu + \rho_+)}\right) & \text{if } (q + \lambda)^2 - c\lambda\mu < 0, \\ 0 & \text{if } (q + \lambda)^2 - c\lambda\mu \geq 0, \end{cases}$$

and we have $V^{dF} = V^{b*}$.

2.2. Shreve, Lehoczky, and Gaver's Problem

In Shreve, Lehoczky, and Gaver's problem, capital is injected as soon as it is necessary—that is, to keep the controlled process non-negative, so $\tau = \infty$. In this case, the controlled process is never ruined—zero acts as a (lower) reflecting barrier. For this problem, the optimal value function will be denoted by V_k^{SLG}.

It is well-known that for this problem, double-barrier strategies play an important role. For $b \geq 0$, a double-barrier strategy with an upper barrier at level b is such that $0 \leq U_t^{0,b} = X_t - D_t^b + C_t^0 \leq b$ for all $t \geq 0$. As capital injections are now considered, the process D^b here is different from the one in de Finetti's problem; see (Avram et al. 2007) for details. For such a strategy, the value function is such that $J(x, C^0, D^b) = V_k^{0,b}(x)$, where

$$V_k^{0,b}(x) := \mathbb{E}_x\left[\int_0^\infty e^{-qt}\left(dD_t^b - k dC_t^0\right)\right],$$

and the optimal value function is such that $V_k^{SLG}(x) = \sup_{b \geq 0} V_k^{0,b}(x)$.

It is well-known that for a SNLP,

$$V_k^{0,b}(x) = \begin{cases} k\left(\overline{Z}_q(x) + \frac{p}{q}\right) + Z_q(x) H_k^{SLG}(b), & x \leq b, \\ x - b + V_k^{0,b}(b), & x > b, \end{cases} \tag{9}$$

where

$$Z_q(x) = 1 + q \int_0^x W_q(y)\mathrm{d}y \qquad \text{and} \qquad \overline{Z}_q(x) = \int_0^x Z_q(y)\mathrm{d}y, \tag{10}$$

and where, for $x \geq 0$, the *barrier function* is defined by

$$H_k^{SLG}(b) = \frac{1 - kZ_q(b)}{qW_q(b)}, \ b > 0. \tag{11}$$

The next proposition—namely, Proposition 1—contains new results, as well as results taken from Lemma 2 of (Avram et al. 2007). In particular, we provide a new relationship (see (13)) between the value functions of de Finetti's and Shreve, Lehoczky, and Gaver's problems.

The main object in this next proposition is the function $k_f \colon [0, \infty) \to [k_0, \infty)$ defined by

$$k_f(b) := \frac{W_q'(b)}{Z_q(b)W_q'(b) - qW_q^2(b)}, \tag{12}$$

where

$$k_0 := k_f(0_+) = \frac{W_q'(0_+)}{W_q'(0_+) - qW_q^2(0_+))} = \begin{cases} 1, & \text{if } X \text{ is of unbounded variation,} \\ 1 + \frac{q}{\Pi(0,\infty)}, & \text{if } X \text{ is of bounded variation.} \end{cases}$$

The function k_f is increasing, thanks to (Avram et al. 2004, Theorem 1)[1]. Indeed, it is known that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau^b}\right] = Z_q(x) - q\frac{W_q(b)}{W_q'(b)}W_q(x),$$

so $k_f(b) = \frac{1}{\mathbb{E}_b\left[\mathrm{e}^{-q\tau^b}\right]}$. The statement follows from the fact that the map $b \mapsto \mathbb{E}_b\left[\mathrm{e}^{-q\tau^b}\right]$ is decreasing.

The monotonicity allows us to re-parametrize the problem in terms of the optimal barrier, b_k associated to a fixed cost, k.

Proposition 1. *Assume X is a SNLP. We have the following results:*

(a)   *For fixed x and b, the function $k \mapsto V_k^{SLG}(x)$ is non-increasing.*

(b)   *For $k = k_f(b)$, the value function defined in (9) can be written as follows:*

$$V_{k_f(b)}^{0,b}(x) = k_f(b)\left[\overline{Z}_q(x) + \frac{p}{q} - Z_q(x)V^b(b)\right]$$

$$= k_f(b)\left[Z_q^{(1)}(x) + Z_q(x)\left(\frac{p}{q} - V^b(b)\right)\right], \tag{13}$$

where $V^b(x)$ is defined in (6) and

$$Z_q^{(1)}(x) := \int_0^x \left(Z_q(y) - pW_q(y)\right)\mathrm{d}y. \tag{14}$$

(c)   *For fixed k, the barrier function H_k^{SLG} has a unique point of maximum $b_k \geq 0$. It is decreasing, and thus $b_k = 0$ if, and only if $k \in (1, k_0]$. Finally, if $b_k > 0$, then $k = k_f(b_k)$.*

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[1]   Some papers refer to this as the log-convexity of $Z_q(x)$.

Remark 1. *Note that*

$$Z_q^{(1)}(x) = \left. \frac{\partial Z_{q,\theta}(x)}{\partial \theta} \right|_{\theta=0}$$

where $Z_{q,\theta}(x) = (\psi(\theta) - q) \int_0^\infty e^{-\theta y} W_q(x + y) \mathrm{d}y$. *The function* $Z_{q,\theta}$ *was introduced simultaneously in* (*Avram et al. 2015*; *Ivanovs and Palmowski 2012*) *(but was already present implicitly in* (*Avram et al. 2004*, *Theorem 1*)*, where it was presented as an Esscher transform of* $Z_q(x)$*). It was first used as a generating function for Gerber–Shiu penalty functions induced by polynomial rewards* $1, x, x^2$*, which were denoted respectively by* $Z_q, Z_q^{(1)}, Z_q^{(2)}, \ldots$*, and started also being used intensively in exponential Parisian ruin problems following the work of* (*Albrecher et al. 2016*)*. See* (*Avram et al. 2019*) *for more information.*

Proof. (a) The result follows from the fact that $k \mapsto V_k^{0,b}(x)$ is decreasing (by definition) and because $V_k^{SLG}(x)$ is obtained by a maximization of $V_k^{0,b}(x)$ over all barrier levels b (chosen independently of k).

(b) Recalling (9), we need to show that

$$-H_{k_f(b)}^{SLG}(b) = k_f(b)V^b(b). \tag{15}$$

Indeed, it is easy to check that the equality

$$\frac{kZ_q(b) - 1}{qW_q(b)} = k\frac{W_q(b)}{W_q'(b)}$$

holds for $k = k_f(b)$.

(c) It is well-known (see Avram et al. 2007, Lemma 2) that H_k^{SLG} is an increasing-decreasing function in b, with a unique maximum $b_k \geq 0$. For the sake of completeness, let us reproduce this proof. The derivative of the barrier function (11) satisfies

$$q\frac{H'W_q^2}{W_q'}(b) = f(b) := k\frac{\Delta_q^{(ZW)}(b)}{W_q'(b)} - 1 = k\,\mathbb{E}_b[e^{-q\tau^b}] - 1 = \frac{k}{k_f(b)} - 1, \tag{16}$$

where

$$\Delta_q^{(ZW)}(b) := Z^{(q)}(b)W_q'(b) - \left(Z^{(q)}\right)'(b)W_q(b). \tag{17}$$

Therefore, the sign of the derivative of the barrier function (11) coincides with that of f. Clearly, the latter function f is decreasing in b from $\lim_{b\to 0} f(b) = \frac{k}{k_0} - 1$ to -1. □

Remark 2. *In conclusion, if* $k \leq k_0$*, then the barrier function* H_k^{SLG} *reaches its unique maximum at* $b_k = 0$ *and, if* $k > k_0$*, then* b_k *is such that* $k = k_f(b_k)$*.*

Remark 3. *The previous proposition suggests the following new (and short) proof of the Løkka–Zervos alternative in the Brownian motion case. It is easy to verify that*

$$Z_q^{(1)}(x) + Z_q(x)\left(\frac{p}{q} - V^b(b)\right) = Z_q^{(1)}(x) = \frac{\sigma^2}{2}W_q(b),$$

which yields $V_{k_f(b^*)}^{0,b^*}(x) = V^{dF}(x)$*, where* b^* *denotes the optimal barrier level in de Finetti's problem, as defined in Section 2.1. Then, use the monotonicity of* $V_k^{SLG}(x)$ *in* k*.*

Similar computations below will establish the Løkka–Zervos alternative in the Cramér-Lundberg case with exponential jumps.

3. The Løkka–Zervos Alternative for a Cramér–Lundberg Model with Exponential Jumps

Here is our main result.

Theorem 1. *For a Cramér–Lundberg process with exponentially distributed jumps, the Løkka–Zervos alternative holds with two regimes separated by the threshold $k = k_f(b^*)$—that is, for all $x \geq 0$,*

$$V_k^{SLG}(x) \geq V^{dF}(x) \quad \text{if, and only if} \quad k \leq k_f(b^*).$$

Proof. By Proposition 1, we know that for fixed x and b, the function $k \mapsto V_k^{SLG}(x) = \sup_{b \geq 0} V_k^{0,b}(x)$ is non-increasing. One deduces that, for all $x \geq 0$,

$$V_k^{SLG}(x) \geq V_{k_f(b^*)}^{0,b^*}(x) \quad \text{if and only if} \quad k \leq k_f(b^*).$$

Therefore, the Løkka–Zervos alternative follows from Lemma 1, given below, where it is proved that

$$V_{k_f(b^*)}^{0,b^*}(x) = V^{dF}(x). \tag{18}$$

$\square$

Recall that we assume $q > 0$.

Lemma 1. *For a Cramér–Lundberg process with exponentially distributed jumps, we have:*

(a) $Z_q(x) + \mu Z_q^{(1)}(x) = c W_q(x)$, *for all $x > 0$;*

(b) $\dfrac{1}{k_f(b^*)} = \dfrac{c W_q'(b^*)}{\mu}$;

(c) $V_{k_f(b^*)}^{0,b^*}(x) = V^{dF}(x)$, *for all $x > 0$.*

Proof. (a) This can be verified by taking Laplace transforms and using (2). Letting $\widehat{\overline{F}}(s)$ denote the Laplace transform of the tail distribution function $\overline{F}(z) = 1 - F(z)$, it amounts to checking

$$\frac{c}{\psi(s) - q} = \frac{c - \lambda \widehat{\overline{F}}(s)}{\psi(s) - q} + \mu \frac{c - \lambda \widehat{\overline{F}}(s) - p}{s(\psi(s) - q)} \iff \lambda \widehat{\overline{F}}(s) = \mu \lambda \frac{\mu^{-1} - \widehat{\overline{F}}(s)}{s},$$

which holds true for exponential jumps.

(b) Manipulating the Kolmogorov IDE for Z_q, we can reduce it to

$$c Z_q''(x) + (c\mu - \lambda - q) Z_q'(x) - q\mu Z_q(x) = q W_q'(x) \left(c + (c\mu - \lambda - q) \frac{W_q(x)}{W_q'(x)} - \mu \frac{Z_q(x)}{W_q'(x)} \right) = 0.$$

At $x = b^*$, using the fact that

$$V^{dF}(b^*) = \frac{p}{q} - \frac{1}{\mu} \tag{19}$$

(see, for example, Equation (5.24) in Gerber et al. 2006) together with simple algebraic manipulations yields

$$\frac{Z_q(b^*)}{W_q'(b^*)} = q \left(\frac{p}{q} - \frac{1}{\mu} \right)^2 + \frac{c}{\mu}.$$

This is equivalent to the result.

(c) From (13) and part (a), we get

$$V_{k_f(b^*)}^{0,b^*}(x) = \frac{k_f(b^*)}{\mu}\left(\mu Z_q^{(1)}(x) + Z_q(x)\right) = \frac{ck_f(b^*)}{\mu}W_q(x).$$

Then the result follows from (b), as well as the fact that $V^{dF} = V^{b^*}$. $\quad\square$

4. Conclusions and Conjecture

We believe it is important to study the Løkka–Zervos alternative for spectrally negative Lévy processes and for spectrally negative additive Markov processes, both practically and methodologically. We conjecture that in those more general cases, more than two regimes will be involved, giving rise to *Løkka–Zervos alternatives*.

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