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Merton Investment Problems in Finance and Insurance for the Hawkes-Based Models

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Abstract: We show how to solve Merton optimal investment stochastic control problem for Hawkes-based models in finance and insurance (Propositions 1 and 2), i.e., for a wealth portfolio $X(t)$ consisting of a bond and a stock price described by general compound Hawkes process (GCHP), and for a capital $R(t)$ (risk process) of an insurance company with the amount of claims described by the risk model based on GCHP. The main approach in both cases is to use functional central limit theorem for the GCHP to approximate it with a diffusion process. Then we construct and solve Hamilton–Jacobi–Bellman (HJB) equation for the expected utility function. The novelty of the results consists of the new Hawkes-based models and in the new optimal investment results in finance and insurance for those models.

Keywords: Merton investment problem; optimal control; Hawkes process; general compound Hawkes process; LLN and FCLT; risk process; HJB equations; optimal investment in finance; optimal investment in insurance; diffusion approximation

1. Introduction

Merton optimal investment and consumption stochastic problem is one of the most studied classical problem in finance (Merton 1969, 1971, 1990; Bjork 2009; Karatzas and Shreve 1998). In this paper, we will show how to solve the Merton optimal investment stochastic control problem for Hawkes-based models in finance and insurance, i.e., for a wealth portfolio $X(t)$ consisting of a bond and a stock price described by general compound Hawkes process (GCHP) (Swishchuk 2020; Swishchuk and Huffman 2020; Swishchuk 2017b), and for a capital $R(t)$ of an insurance company with the amount of claims described by risk model based on GCHP (Swishchuk 2018; Swishchuk et al. 2020).

Namely, we will show how to solve the following two portfolio investment problems.

(1) Merton portfolio optimization problem in finance (Merton 1969; 1971) aims to find the optimal investment strategy for the investor with those two objects of investment, namely risk-less asset (e.g., investment grade government bonds)), with a fixed rate of interest r, and a number of risky assets (e.g., stocks) whose price GCHP. In this way, in our case, we suppose that B_t and S_t follows the following dynamics, respectively:

$$\begin{cases} B_t &= B_0 \exp\{rt\} \\ S_t &= S_0 \exp\{G(t)\}, \end{cases}$$

where $G(t) := \sum_{i=1}^{N(t)} a(X_i)$ is the GCHP, X_i is a discrete-time Markov chain (MC) with finite or infinite states, $r > 0$ is the interest rate, and $a(x)$ is a continuous and finite function on X. We note, that the justification of using HP in finance may be found in (Da Fonseca and Zaatour 2013), and using GCHP that based on HP $N(t)$ and $a(X_i)$ may be found in (Swishchuk and He 2019; Swishchuk and Huffman 2020). The model for a stock price $S(t)$ that based on GCHP is a new and original in this paper. The investor starts with an initial amount of money, say $X_0 = x$, and wishes to decide how much money to invest in risky and risk-less assets to maximize the final wealth X_t at the maturity T;

(2) Merton portfolio optimization problem in insurance aims to find an optimal investment for the capital $R(t)$ of an insurance company at time t ($R(t)$ is actually the risk model based on general compound Hawkes process (GCHP) (Swishchuk 2018; Swishchuk et al. 2020), when an investor decides to invest some capital $A(t)$ in risky assets (e.g., stocks) and the rest, $(R(t) - A(t))$ in risk-free assets (e.g., bonds or bank account). We note, that the risk model $R(t)$, based on GCHP, has the following representation:

$$R(t) = u + ct - \sum_{i=1}^{N(t)} a(X_i).$$

Here $R(0) = u$ is an insurance company's initial capital, $c > 0$ is the premium rate, $N(t)$ is the Hawkes process, X_i is a discrete-time finite or infinite state Markov chain with state space $X = \{1, 2, 3, ..., N\}$ or $X = \{1, 2, ..., N, ...\}$,, respectively, and $a(x)$ is a continuous and finite function on X. We note, that the justification for the Hawkes-based risk model in the form of the above equation may be found in (Swishchuk et al. 2020).

The investor starts with an initial capital, say $R(0) = u$, and wishes to decide how much money to invest in risky and risk-less assets to maximize the capital $R(t)$.

We solve both problems using diffusion approximation for GCHP $G(t) := \sum_{i=1}^{N(t)} a(X_i)$ (see Sections 4 and 5). We note, that it is not a simplification of the initial models: the resulting models contain all the parameters of the initial models, including the parameters of Hawkes process. Furthermore, significance and insights of the results are discussed in Remark 6 and Remark 7 (Some Insights into the Results). In this case both problems can be solved explicitly. However, we cannot say this if we would like to dal with initial models and going without diffusion approximation (see Section 6. Discussion). We believe that these two problems for those two different models in finance and insurance are considered in the literature for the first time, because none of author's 9 papers in the References contain similar or even close results.

The novelties of the paper are the following ones: (1) we consider a new model for the stock price $S(t)$ in the form $S_t = S_0 \exp\{G(t)\}$, where $G(t) := \sum_{i=1}^{N(t)} a(X_i)$ is the GCHP; we call it Hawkes-based model for the stock price (or *geometric general compound Hawkes process*, similar to geometric Brownian motion); (2) solution of Merton investment problem for this Hawkes-based model; (3) solution of Merton investment problem for the Hawkes-based risk model.

The structure of the paper is the following one. Literature review is presented in Section 2. Section 3 is devoted to the definitions and properties of Hawkes process and general compound Hawkes processes, and LLN (Law of Large Numbers) and FCLT (Functional Central Limit Theorem) for them. Section 4 deals with Merton investment problem in finance for the stock price described by GCHP, and Section 5 deals with Merton investment problem in insurance for the risk model based on GCHP. Section 6 contains some discussions and describes the future work, and Section 7 concludes the paper.

2. Literature Review

The Merton optimal investment and consumption stochastic problem in finance was first considered in the seminal papers of Merton, (Merton 1969, 1971). General description of the problem and coverage of most today's problems and methods may be found in (Fleming and Rishel 1975; Fleming and Soner 1993; Bjork 2009; Merton 1990; Karatzas and Shreve 1998).

The first papers on stochastic optimal control in insurance appeared relatively recently, e.g., we would like to mention the papers written by (Martin-Löf 1994; Brockett and Xia 1995; Browne 1995), to name a few. Since then many papers and books were written on this topic including (Asmussen and Taksar 1997; Højgaard and Taksar 1999; Schmidli 2008). Financial control methods applied in insurance, such as, e.g., in the management and the control of the specific risk insurance companies, are described in (Hipp 2004). The paper (Dufresne and Gerber 1991) considered a model in risk theory based on the

compound Poisson process perturbed by diffusion. The paper (Grandits 2003) studied the Cramer-Lundberg approximation in the optimal investment case. Asymptotic ruin probabilities and optimal investment were investigated in (Gaier et al. 2003). Optimal risk distribution control model with application to insurance was studied in (Taksar 2000). Applications of stochastic processes in insurance and finance may be found in (Rolski et al. 1998). Many mathematical methods and aspects in risk theory may be found in (Bühlmann 1970; Gerber 1979; Grandell 1990).

We note that an alternative approach to consumption-portfolio optimization problem based on martingale methods were developed by (Karatzas et al. 1986; Pliska 1986; Cox and Huang 1989). Applications of martingale methods to the basic optimization problems can be found in (Cox and Huang 1989; Karatzas 1997; Korn and Korn 2000). Investment problem is also closely associated with risk management problems, such as, e.g., insurance/reinsurance and risk prevention. Probably Arrow's 1963 paper (Arrow 1963) was the first one that drown attention to risk management with insurance. How insurance can be used as a risk prevention tool was shown by (Ehrlich and Becker 1972). Some early contributions to insurance/reinsurance problems may be found in Louberge (Louberge 1998; Dionne 2001).

Hawkes process was first introduced in (Hawkes 1971a, 1971b). Good introduction into Hawkes processes and their properties may be found in (Laub et al. 2015). GCHP and regime-switching GCHP were first introduced in (Swishchuk 2017b) and studied in details using real data in (Swishchuk et al. 2019; Swishchuk and He 2019; Swishchuk 2020; Swishchuk and Huffman 2020). Risk model based on GCHP was first introduced in (Swishchuk 2017a) and described in details in (Swishchuk 2018). Applications of the risk model based on GCHP to empirical data and optimal investment problem were considered in (Swishchuk et al. 2020).

3. General Compound Hawkes process

This section contains main definitions and results on one-dimensional Hawkes and general compound Hawkes processes which we will use in our paper. For the completeness, we present them in three subsections below.

3.1. Hawkes Process

Definition 1. (One-dimensional Hawkes Process) (Hawkes 1971a, 1971b). *The one-dimensional Hawkes process is a point process $N(t)$ which is characterized by its intensity $\lambda(t)$ with respect to its natural filtration:*

$$\lambda(t) = \lambda + \int_0^t \mu(t-s)dN(s),$$

where $\lambda > 0$, and the response function or self-exciting function $\mu(t)$ is a positive function and satisfies $\int_0^{+\infty} \mu(s)ds := \hat{\mu} < 1$.

If $(t_1, t_2, ..., t_k)$ denotes the observed sequence of past arrival times of the point process up to time t, the Hawkes conditional intensity is

$$\lambda(t) = \lambda + \sum_{t_k < t} \mu(t - t_k).$$

The function $\mu(t)$ is sometimes also called the excitation function, and the constant λ is called the background intensit .

To avoid the trivial case, we suppose that $\mu(t) \neq 0$,which is, a homogeneous Poisson process. Therefore, the Hawkes process is a non-Markovian extension of the Poisson process.

We note, that the Hawkes process is a self-exciting simple point process first introduced by A. Hawkes in 1971 (Hawkes 1971a, 1971b). Thus, the future evolution of a self-exciting point process is influenced by the timing of past events.

Except for some very special cases (e.g., exponential self-exiting function $\mu(t)$), the Hawkes process is non-Markovian . In this way, the Hawkes process has a long memory and depends on the entire past history .

Among many applications of the Hawkes process, we mention applications in finance, insurance, neuroscience, seismology, genome analysis, to name a few.

The above equation for $\lambda(t)$ has the following interpretation: the events occur according to an intensity with a background intensity λ which increases by $\mu(0)$ at each new event then decays back to the background intensity value according to the function $\mu(t)$.

Therefore, choosing $\mu(0) > 0$ leads to a jolt in the intensity at each new event. This feature is often called a self-exciting feature: an arrival causes the conditional intensity function $\lambda(t)$ in (1) and (2) to increase then the process is said to be self-exciting.

The following LLN and CLT for HP may be found in (Bacry et al. 2013). The convergences are considered in weak sense for the Skorokhod topology.

LLN for HP (Bacry et al. 2013). Let $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$. Then

$$\frac{N(t)}{t} \to_{t \to +\infty} \frac{\lambda}{1 - \hat{\mu}}.$$

Remark 1. *By LLN $N(t) \approx \frac{\lambda}{1-\hat{\mu}}t$ for large t.*

FCLT for HP (Bacry et al. 2013). Under LLN and $\int_0^{+\infty} s\mu(s)ds < +\infty$ conditions

$$P\left(\frac{N(t) - \lambda t/(1 - \hat{\mu})}{\sqrt{\lambda t/(1 - \hat{\mu})^3}} < y\right) \to_{t \to +\infty} \Phi(y),$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.

Remark 2. *By FCLT $N(t) \approx \frac{\lambda}{1-\hat{\mu}}t + \sqrt{\lambda/(1 - \hat{\mu})^3}W(t)$ for large t, where $W(t)$ is a standard Wiener process (see Bacry et al. 2013).*

Remarks 1 and 2 above give the ideas about the averaged and diffusion approximated HP on a large time interval.

3.2. General Compound Hawkes Process

Definition 2. (General Compound Hawkes Process). *General compound Hawkes Process is defined as (Swishchuk 2020; Swishchuk and Huffman 2020; Swishchuk 2017b)*

$$S(t) = S(0) + \sum_{i=1}^{N(t)} a(X_i).$$

Here, X_i is a discrete-time finite or infinite state Markov chain with state space $X = \{1, 2, ..., N\}$, or $X = \{1, 2, ..., N, ...\}$, , respectively, $a(x)$ is a continuous and bounded function on X, and $N(t)$ is a Hawkes process with intensity $\lambda(t) > 0$, independent of X_i.

This general model is rich enough to:
• incorporate non-exponential distribution of inter-arrival times of orders in HFT or claims in insurance (hidden in $N(t)$)
• incorporate the dependence of orders or claims (via MC X_i)
• incorporate clustering of of orders in HFT or claims (properties of $N(t)$)
• incorporate order or claim price changes different from one single number (in $a(X_i)$).
This model is also very general to include:
-**in finance**:
• compound Poisson process: $S_t = S_0 + \sum_{k=1}^{N(t)} X_k$, where $N(t)$ is a Poisson process and $a(X_k) = X_k$ are i.i.d.r.v.

- compound Hawkes process (Swishchuk et al. 2019): $S_t = S_0 + \sum_{k=1}^{N(t)} X_k$, where $N(t)$ is a Hawkes process and $a(X_k) = X_k$ are i.i.d.r.v.
- compound Markov renewal process: $S_t = S_0 + \sum_{k=1}^{N(t)} a(X_k)$, where $N(t)$ is a renewal process and X_k is a Markov chain;
  -in insurance:
- classical Cramer-Lundberg model: $a(X_i) = X_i$, X_i are i.i.d.r.v., and $\mu(t) = 0$ (then $N(t)$ is a poisson process);
- Sparre-Andersen model: $a(X_i) = X_i$, X_i are i.i.d.r.v., $\mu(t) = 0$, and $N(t)$ is a renewal process;
- Markov-modulated model: $a(X_i) = X_i$, X_i are i.i.d.r.v., $\lambda(t) = \lambda(X(t))$, where $X(t)$ is a MC; we call this model regime-switching risk model based on GCHP (Swishchuk 2020; Swishchuk 2017b).

3.3. LLN and FCLT for GCHP

Lemma 1. (LLN for GCHP) (Swishchuk 2020; Swishchuk and Huffman 2020; Swishchuk 2017b). *Let $\hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$, and Markov chain X_i is ergodic with stationary probabilities π_i^*. Then the GCHP S_{nt} satisfies the following weak convergence in the Skorokhod topology:*

$$\frac{S(nt)}{n} \to_{n\to+\infty} a^* \cdot \frac{\lambda}{1-\hat{\mu}} t,$$

or

$$\frac{S(t)}{t} \to_{t\to+\infty} a^* \cdot \frac{\lambda}{1-\hat{\mu}}.$$

Here: a^* is defined as $a^* := \sum_{i\in X} \pi_i^* a(i)$, where π_i are ergodic probabilities for Markov chain X_i.

Theorem 1. (FCLT (or Jump-Diffusion Limit) for GCHP) (Swishchuk 2020; Swishchuk and Huffman 2020; Swishchuk 2017b). *Let X_k be an ergodic Markov chain and with ergodic probabilities $(\pi_1^*, \pi_2^*, ..., \pi_n^*)$. Let also S_t be LGCHP, and $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$ and $\int_0^{+\infty} \mu(s)sds < +\infty$.*

Then

$$\frac{S(nt) - N(nt) \cdot a^*}{\sqrt{n}} \to_{n\to+\infty} \sigma\sqrt{\lambda/(1-\hat{\mu})}W(t),$$

in weak sense for the Skorokhod topology, where $W(t)$ is a standard Wiener process, σ is defined as:
$(\sigma)^2 := \sum_{i\in X} \pi_i^* v(i)$

$$
\begin{aligned}
v(i) &= b(i)^2 + \sum_{j\in X}(g(j)-g(i))^2 P(i,j) \\
&\quad - 2b(i)\sum_{j\in X}(g(j)-g(i))P(i,j), \\
b &= (b(1),b(2),...,b(n))', \\
b(i) :&= a(i) - a^*, \\
g :&= (P + \Pi^* - I)^{-1}b,
\end{aligned}
$$

P is a transition probability matrix for X_k, i.e., $P(i,j) = P(X_{k+1} = j|X_k = i)$, Π^* denotes the matrix of stationary distributions of P, and $g(j)$ is the jth entry of g. The expressions above are valid for both finite and infinite state Markov chain, that is why we used the notation $\sum_{j\in X}$, where X is the states space of a MC X_i.

Remark 3. *The formulas for a^* and σ look much simpler in the case of two-state Markov chain $X_i = \{-\delta, +\delta\}$:*

$$a^* := \delta(2\pi^* - 1) \quad and \quad (\sigma^*)^2 := 4\delta^2 \left(\frac{1 - p' + \pi^*(p' - p)}{(p + p' - 2)^2} - \pi^*(1 - \pi^*) \right),$$

(p, p') are transition probabilities of Markov chain X_k, and $\pi_1^* = \pi^*$, $\quad \pi_2^* = 1 - \pi^*$.

From FCLT for HP, Section 3, and from Theorem 1 above follow the following FCLT for GCHP (pure jump diffusion limit).

Theorem 2. (FCLT (or Pure Diffusion Limit) for GCHP (Swishchuk 2020; Swishchuk and Huffman 2020; Swishchuk 2017b; Swishchuk et al. 2020) . *Let X_k be an ergodic Markov chain and with ergodic probabilities $(\pi_1^*, \pi_2^*, ..., \pi_n^*)$. Let also S_t be LGCHP, and $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1 \quad and \quad \int_0^{+\infty} \mu(s)sds < +\infty$.*

Then

$$\frac{S(t) - a^* \frac{\lambda}{1-\hat{\mu}} t}{\sqrt{t}} \to_{t \to +\infty} \bar{\sigma} N(0, 1),$$

in weak sense for the Skorokhod topology, where $N(0, 1)$ is the standard normal c.d.f., and $\bar{\sigma}$ is defined as:

$$\bar{\sigma} = \sqrt{(\sigma^*)^2 + \left(a^* \sqrt{\frac{\lambda}{(1-\hat{\mu})^3}} \right)^2},$$

where $\sigma^* = \sigma \sqrt{\lambda / (1 - \hat{\mu})}$, σ and a^* are defined in Theorem 1 and Lemma above, respectively.

Remark 4. *From Theorem 2 it follows that $S(t)$ can be approximated by the pure diffusion process:*

$$S(t) \approx S(0) + a^* \frac{\lambda}{1 - \hat{\mu}} t + \bar{\sigma} W(t),$$

where $W(t)$ is a standard Wiener process. This Remark 4 gives the idea about the pure diffusion approximation of GCHP on a large time interval.

Remark 5. *We note, that the rate of convergence in the Theorem 2 is $C(T) / \sqrt{t}$, $0 \le t \le T$, where $C(T) > 0$ is a constant (Swishchuk et al. 2020). Thus, the error of approximation for $S(t)$ in Remark 4 is small for large t.*

4. Merton Investment Problem in Finance for the Hawkes-Based Model

Let us consider Merton portfolio optimization problem. We suppose that B_t and S_t follows the following dynamics, respectively:

$$\begin{cases} B_t &= B_0 \exp\{rt\} \\ S_t &= S_0 \exp\{G(t)\}, \end{cases} \tag{1}$$

where $G(t) := \sum_{i=1}^{N(t)} a(X_i)$ is the GCHP, X_i a discrete-time finite or infinite state Markov chain with state space $X = \{1, 2, 3, ..., N\}$ or $X = \{1, 2, ..., N, ...\}$,, respectively, $r > 0$ is the interest rate.

We note, that the justification of using HP in finance may be found in (Da Fonseca and Zaatour 2013), and using GCHP that based on HP $N(t)$ and $a(X_i)$ may be found in (Swishchuk and He 2019; Swishchuk and Huffman 2020). The model for a stock price $S(t)$ in (1) that based on GCHP is a new and original in this paper.

The investor starts with an initial amount of money, say $X_0 = x$, and wishes to decide how much money to invest in risky and risk-less assets to maximize the expected utility of the terminal wealth X_t at the maturity T,, i.e., X_T.

We denote by $n(t) := (n_B(t), n_S(t))$ an investor portfolio, where $n_B(t)$ and $n_S(t)$ are the amounts in cash invested in the bonds and the risky assets, respectively. The value $X(t)$ at time t of such portfolio is

$$X(t) = n_B(t) + n_S(t).$$

We suppose that our portfolio is admissible, i.e., $X(t) \geq 0$, a.s., $0 \leq t \leq T$, and self-financing, i.e.,

$$dX(t) = n_B(t)\frac{dB(t)}{B(t)} + n_S(t)\frac{dS(t)}{S(t)}.$$

Suppose that $G(t) := \sum_{i=1}^{N(t)} a(X_i)$ follows FCLT when $t \to +\infty$, (Swishchuk 2018; Swishchuk et al. 2020), thus $G(t)$ can be approximated as (see Section 3, Remark 4)

$$G(t) \approx a^* \frac{\lambda}{1 - \hat{\mu}} t + \bar{\sigma} W(t), \tag{2}$$

where $\hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$, a^* is a average of $a(x)$ over stationary distribution of MC X_i, λ is a background intensity, $\bar{\sigma} > 0$ is defined in Section 3, Theorem 2. For exponential decaying intensity $\hat{\mu} = \alpha/\beta$.

Thus, $S(t)$ in (1) can be presented in the following way using (2):

$$S(t) = S(0)e^{a^* \frac{\lambda}{1-\hat{\mu}} t + \bar{\sigma} W(t)}. \tag{3}$$

Using *Itô* formula we can get from (3):

$$dS(t) = S(t)[(a^* \frac{\lambda}{1 - \hat{\mu}} + \frac{\bar{\sigma}^2}{2})dt + \bar{\sigma} dW(t)]. \tag{4}$$

Then the change of the wealth process X_t can be rewritten in the following way, taking into account (1)–(4):

$$
\begin{aligned}
dX_t &= n_B(t)\frac{dB(t)}{B(t)} + n_S(t)\frac{dS(t)}{S(t)} \\
&= n_B(t)rdt + n_S(t)[(a^* \frac{\lambda}{1-\hat{\mu}} + \frac{\bar{\sigma}^2}{2})dt + \bar{\sigma} dW(t)] \\
&= rX(t)dt + n_S(t)(a^* \frac{\lambda}{1-\hat{\mu}} + \frac{\bar{\sigma}^2}{2} - r)dt + n_S(t)\bar{\sigma} dW(t).
\end{aligned}
\tag{5}
$$

Let $\pi(t) := n_S(t)/X(t)$ be the portion of wealth invested in the assets/stocks at time t. Then, from (1)–(5), we have the following expression for $dX(t)$:

$$
\begin{aligned}
dX_t &= rX(t)dt + n_S(t)(a^* \frac{\lambda}{1-\hat{\mu}} + \frac{\bar{\sigma}^2}{2} - r)dt + n_S(t)\bar{\sigma} dW(t). \\
&= dX(t) = X(t)[(r + \pi_t(a^* \frac{\lambda}{1-\hat{\mu}} + \frac{\bar{\sigma}^2}{2} - r))dt + \pi_t\bar{\sigma} dW(t)))]
\end{aligned}
\tag{6}
$$

Finally, after replacing $X(t)$ with $X^{\pi}(t)$, to stress the dependence of $X(t)$ on π_t, from (6) we have the following equation for $dX^{\pi}(t)$:

$$dX^{\pi}(t) = X^{\pi}(t)[(r + \pi(t)(a^* \frac{\lambda}{1 - \hat{\mu}} + \frac{\bar{\sigma}^2}{2} - r))dt + \pi(t)\bar{\sigma} dW(t)))]. \tag{7}$$

Our main goal is to solve the following optimization problem:

$$\max_{\pi} E[U(X_T^{\pi})|X_0 = x],$$

meaning to maximize the wealth/value function or performance criterion $E[U(X_T^{\pi})|X_0 = x]$, where $U(x)$ is a utility function.

To find optimal π, we follow the standard procedure in this case (see Bjork 2009; Karatzas and Shreve 1998). For the utility function we take the logarithmic one, $U(x) = \log(x)$. Therefore, we have to maximize $\max_{\pi} E[\log(X_T^{\pi})|X_0 = x]$, . Solving the Equation (7)

and maximizing non-martingale term in the exponent for the solution, we can find the optimal investment solution $\pi^*(t)$:

$$\pi^*(t) = \frac{a^* \frac{\lambda}{1-\hat{\mu}} + \frac{\bar{\sigma}^2}{2} - r}{\bar{\sigma}^2}, \tag{8}$$

where

$$\bar{\sigma} = \sqrt{(\sigma^*)^2 + \left(a^* \sqrt{\frac{\lambda}{(1-\hat{\mu})^3}}\right)^2}, \tag{9}$$

and σ^* and a^* are defined in Section 3, Theorem 2.

Thus, we have arrived to the following proposition:

Proposition 1. *Let the conditions of Theorem 2, Section 3, are satisfied. Then the optimal investment solution for the Merton portfolio optimization problem is presented by $\pi^*(t)$ in (8) with $\bar{\sigma}$ in (9).*

Remark 6. *(Some Insights into the Results). As we can see from the expression for $\pi^*(t)$, the optimal investment solution depends on all parameters of the Hawkes-based model, namely, Hawkes process's parameters λ and $\hat{\mu}$, Markov chain X_i and function $a(x)$ through a^*. For example, if a^* increases then $\pi^*(t)$ increase, and if $\bar{\sigma}$ increases then $\pi^*(t)$ decreases, which follows from (8). The latter is obvious: in a very volatile market we should avoid a risk associated with investing in stocks. Furthermore, also obvious that if r increases then $\pi^*(t)$ decreases: it's better to invest in bonds than in stocks.*

5. Merton Investment Problem in Insurance for the Hawkes-Based Risk Model

Let us consider $R(t)$ as the risk model based on GCHP, namely,

$$R(t) = u + ct - \sum_{i=1}^{N(t)} a(X_i). \tag{10}$$

Here, X_i (claim sizes) is a discrete-time finite or infinite state Markov chain with state space $X = \{1, 2, 3, ..., N\}$ or $X = \{1, 2, ..., N, ...\}$,, respectively, $a(x)$ is a bounded and continuous function on $X = \{1, 2, ..., N\}$- space state for X_i, and $N(t)$ is a Hawkes process with intensity $\lambda(t) > 0$, independent of X_i, and satisfying:

$$\lambda(t) = \lambda + \int_0^t \mu(t-s)dN(t).$$

Here, $\mu(t)$ is self-exiting function.

We note, that the justification for the Hawkes-based risk model in the form of the above Equation (10) may be found in (Swishchuk et al. 2020).

As long as we will consider optimization for a first insurer, thus we will focus on problems with infinite planning horizon.

Let $A(t)$ be an amount invested in a risky asset, and suppose that the price $S(t)$ of the risky asset follows GBM, i.e.,

$$dS(t) = S(t)(adt + bdW(t)),$$

where a is a real constant, $b > 0$.

Further, the leftover, $R(t) - A(t) > 0$, is invested in a bank account (or bonds) with interest rate $r > 0$, thus

$$d(R(t) - A(t)) = r(R(t) - A(t))dt.$$

By $\beta(t) := A(t)/S(t)$ we define the number of assets held at time t. Thus, the position of the insurer has the following evolution:

$$dR(t) = rR(t)dt + \beta(t)dS(t) - r\beta(t)S(t)dt + cdt - d(G(t)).$$

Therefore, the dynamics for $R(t)$ is (taking into account all above equations for $S(t)$, $d(R(t) - A(t))$ and $dR(t)$):

$$dR(t) = rR(t)dt + A(t)(adt + bdW(t)) - rA(t)dt + cdt - d(G(t))$$

Here, $G(t) = \sum_{i=1}^{N(t)} a(X_i)$.

Let $\pi(t) := A(t)/R(t)$ be the fraction of the total wealth $R(t)$ invested in the risky assets.

Then, we can rewrite the equation for $dR(t)$ in the following way (we use notation $R^\pi(t)$ to stress dependence of $R(t)$ from π):

$$
\begin{aligned}
dR^\pi(t) &= rR^\pi(t)dt + A(t)(adt + bdW(t)) - rA(t)dt + cdt - d(G(t)) \\
&= rR^\pi(t)dt + A(t)((a-r)dt + bdW(t)) + cdt - d(G(t)) \\
&= R^\pi(t)[(r + \pi(t)(a-r))dt + \pi(t)bdW(t)] + cdt - d(G(t)),
\end{aligned}
\tag{11}
$$

where $W(t)$ is a standard Brownian motion.

As for the control at time t we will take the function $\pi(t)$,, i.e., the fraction of the total wealth $R^\pi(t)$ which should be invested in risky assets.

We will show how to find the optimal strategy $\pi(t)$, which maximizes our expected utility function, $E[U(R^\pi(t))|R^\pi(0) = r]$, where $U(r)$ is a special utility function.

We suppose that $G(t) = \sum_{i=1}^{N(t)} a(X_i)$ follows FCLT when $t \to +\infty$, (Swishchuk 2018; Swishchuk et al. 2020), thus $G(t)$ can be approximated as (see Section 3, Remark 4)

$$G(t) \approx a^* \frac{\lambda}{1 - \hat{\mu}} t + \bar{\sigma} W_1(t), \tag{12}$$

where $\hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$, a^* is a average of $a(x)$ over stationary distribution of MC X_i, λ is a background intensity, $\bar{\sigma}$ is defined in Section 3. For exponential decaying intensity $\hat{\mu} = \alpha/\beta$.

Here, $W_1(t)$ is a Wiener process independent of $W(t)$ (the case for correlated $W_1(t)$ with $W(t)$, i.e., such that $[W(t), W_1(t)] = \rho dt$, can be considered as well with some modifications).

We suppose that

$$c > a^* \frac{\lambda}{1 - \hat{\mu}}$$

(*safety loading condition* (SLC)).

After substituting (12) into (11) for dR(t) we get:

$$
\begin{aligned}
dR^\pi(t) &= [R^\pi(t)(r + (a-r)\pi(t)) + (c - a^*\frac{\lambda}{1-\hat{\mu}})]dt \\
&+ \sqrt{R^2(t)b^2\pi^2(t) + \bar{\sigma}^2}dW_2(t),
\end{aligned}
\tag{13}
$$

where $W_2(t)$ is a standard Wiener process independent of $W(t)$ and $W_1(t)$.

Generator for $R^\pi(t)$ in (13) is (here, $R^\pi(0) = x$)

$$
\begin{aligned}
A^\pi &= [x(r + (a-r)\pi(t)) + (c - a^*\frac{\lambda}{1-\hat{\mu}})]\frac{\partial}{\partial x} \\
&+ [(x^2b^2\pi^2(t) + \bar{\sigma}^2)/2]\frac{\partial}{\partial x^2}
\end{aligned}
$$

Thus, we have to maximize $E_r[U(R^\pi(t))]$, where $U(r)$ is a utility function.

The HJB equation has the following form:

$$\frac{\partial}{\partial r} v(t,r) + \sup_{\pi}[A^{\pi} v(t,r)] = 0,$$

where $v(t,r) = \sup_{\pi} E_r[U(R^{\pi}(t))]$.

We take the exponential utility function $U(r) = -e^{-pr}, \quad p > 0$.

Solving the HJB equation we get the optimal control $\pi(t)$:

$$\pi(t) = \frac{(a-r)}{xpb^2}$$

where $x = R^{\pi}(0) > 0$, and $\pi(t)$ depends on p. After finding

$$p = \frac{\theta + \sqrt{\theta^2 + \bar{\sigma}^2(a-r)^2/b^2}}{\bar{\sigma}^2},$$

where $\theta := xr + (c - a^* \frac{\lambda}{1-\hat{\mu}})$, we can finally find that

$$\pi(t) = \frac{\bar{\sigma}^2(a-r)}{xb(\theta b + \sqrt{\theta^2 b^2 + \bar{\sigma}^2(a-r)^2})}, \tag{14}$$

where

$$\bar{\sigma} = \sqrt{(\sigma^*)^2 + \left(a^*\sqrt{\frac{\lambda}{(1-\hat{\mu})^3}}\right)^2}, \tag{15}$$

$\sigma^* = \sigma\sqrt{\lambda/(1-\hat{\mu})}$, σ and a^* are defined in Theorem 1 and Lemma, Section 3, respectively.
Thus, we arrived to the following proposition:

Proposition 2. *Let the conditions of Theorem 2, Section 3, are satisfied. Then the optimal investment solution for the Merton investment problem in insurance is presented by $\pi(t)$ in (14) with $\bar{\sigma}$ in (15).*

As we can see, the optimal control $\pi(t) = \pi$ does not depend on t, thus is a constant, and contains all initial parameters of the risk model based on GCHP.

Remark 7. *(Some Insights into the Results). As we can see from the expression for $\pi(t)$, the optimal control depends not only from interest rate r, but also from all parameters of the Hawkes-based model, namely, Hawkes process's parameters λ and $\hat{\mu}$, Markov chain X_i and function $a(x)$ through a^*, and the asset's parameters a and b. For example, from (14) we can see that $\pi(t)$ decreases if $\bar{\sigma}$ increases, where $\bar{\sigma}$ is, as we could call it, a 'volatility of all volatilities', and thus it is not a good idea to invest in stocks in highly volatile market; also, if, for example, self-exiting function $\mu(t) = \beta \exp(-\alpha t)$ is exponential, then we can see from (14) that $\pi(t)$ depends on parameters α and β in the following way (here: $\hat{\mu} = \alpha/\beta$): if α increases then $\pi(t)$ is also increases, and when β increases then $\pi(t)$ decreases. Furthermore, $\pi(t)$ has a significant dependence on parameter θ, as we could call it, 'rate of increase of company's capital', $\theta := xr + (c - a^* \frac{\lambda}{1-\hat{\mu}})$, ($\theta > 0$, which follows from SLC): if θ increases, then $\pi(t)$ decreases (see (14)). The latter is understandable: if the capital of a company increases due to interest r and premium c, then there is no need to take a risk investing in stocks, and as a result, $\pi(t)$ should be decreased.*

Corollary 1. *Merton Investment Problem for Poisson-based Risk Model in Insurance.*

The optimal control $\pi_P(t)$ for Poisson Risk Model (in this case, $N(t)$ is a Poisson process, $a(x) = x$, X_i are i.i.d.r.vs) is (which follows from (14)):

$$\pi_P(t) = \frac{\lambda E[X_i]^2(a-r)}{xb(b(xr+(c-\lambda EX_i)) + \sqrt{b^2(xr+(c-\lambda EX_i)) + \lambda E[X_i]^2(a-r)^2)})}.$$

Here: $\bar{\sigma} = \sqrt{\lambda E[X_i]^2}$.

6. Discussion

Even though there is the advantage of the solution of Merton problems in finance and insurance that associated with the approximation of the initial complicated model by much simpler diffusion model, there are some limitations related to proposed portfolio design and to improving the proposed approximation method. Thus, the future work will be devoted to numerical example based on real data, simulations and considering the case *without diffusion approximation* for $R(t)$, by creating HJB equation for initial risk model $R(t) = R(0) + ct - \sum_{i=1}^{N(t)} a(X_i)$ and by solving this HJB equation for initial risk model base on GCHP. Probably we cannot avoid here numerical methods, because the HJB equation in this case cannot be solved exactly with a close form solution.

7. Conclusions

We described in this paper how to solve Merton optimal investment stochastic control problem for Hawkes-based models in finance and insurance (Propositions 1 and 2), i.e., for a wealth portfolio $X(t)$ consisting of a bond and a stock price described by general compound Hawkes process (GCHP), and for a capital $R(t)$ of an insurance company with the amount of claims described by the risk model based on GCHP. The novelty of the results consists of the new Hawkes-based models and in the new optimal investment results in finance and insurance for those models. Specifically: (1) we considered a new model for the stock price $S(t)$ in the form of $S_t = S_0 \exp\{G(t)\}$, where $G(t) := \sum_{i=1}^{N(t)} a(X_i)$ is the GCHP; we call it Hawkes-based model for the stock price (or *geometric general compound Hawkes process*, similar to the geometric Brownian motion); (2) solution of Merton investment problem for this Hawkes-based model; (3) solution of Merton investment problem for the Hawkes-based risk model. We also gave some insights into the obtained results (see Remarks 6 and 7).

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

LLN	Law of Large Numbers
FCLT	Functional Central Limit Theorem
GCHP	General Compound Hawkes Process
HJB	Hamilton–Jaocobi–Bellman Equation

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