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Pricing and Hedging Bond Power Exchange Options in a Stochastic String Term-Structure Model

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Abstract: We study power exchange options written on zero-coupon bonds under a stochastic string term-structure framework. Closed-form expressions for pricing and hedging bond power exchange options are obtained and, as particular cases, the corresponding expressions for call power options and constant underlying elasticity in strikes (CUES) options. Sufficient conditions for the equivalence of the European and the American versions of bond power exchange options are provided and the put-call parity relation for European bond power exchange options is established. Finally, we consider several applications of our results including duration and convexity measures for bond power exchange options, pricing extendable/accelerable maturity zero-coupon bonds, options to price a zero-coupon bond off of a shifted term-structure, and options on interest rates and rate spreads. In particular, we show that standard formulas for interest rate caplets and floorlets in a LIBOR market model can be obtained as special cases of bond power exchange options under a stochastic string term-structure model.

Keywords: stochastic string process; term-structure model; bond option pricing; Malliavin calculus

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1. Introduction

Since the publication of the seminal paper of Margrabe (1978), *exchange options*, i.e., options to exchange one risky asset for another, have attracted the interest of derivatives researchers. Some examples are Carr (1995); Li (2008); Cheang and Chiarella (2011) and Chen and Suchanecki (2011). A particular class of exchange options that has generated special interest is that of *power exchange options*, namely, exchange options in which the assets to be exchanged are raised to different powers. Power exchange options are introduced in Blenman and Clark (2005b) which is an extension of their earlier work on options with constant underlying elasticity in strikes (CUES) (Blenman and Clark (2005a)). Subsequently, several studies have extended power exchange options to include additional risk factors in the underlying securities, such as correlated jump risk (Wang (2016)), counterparty risk (Wang et al. (2017)), and stochastic volatility (Xia (2019); Lee et al. (2020); Yue et al. (2021)); different mathematical tools, such as the Shanon wavelet inverse Fourier technique (Huang et al. (2022)) and uncertain fractional differential Equations (Yang and Zhu (2021)); or its application to exotic derivatives such as geometric Asian options (Zhang et al. (2018); Shokrollahi (2018)). Nevertheless, to the best of our knowledge, there has been no contribution in the literature extending power exchange options to the case in which the assets to be exchanged are zero-coupon bonds with different maturities. In the present paper we will consider such options, referring to them as *bond power exchange options*.

Bond power exchange options cannot be priced directly under the standard framework of Blenman and Clark (2005a) because it is valid only for assets whose values depend solely on the current time. In the case of bond power exchange options the values of the underlying assets (zero-coupon bonds) depend on both the current time and on the maturities of the bonds. Thus, in order to price these options, we need a model of the dynamics for bond prices derived from the term structure of interest rates (TSIR).

A very general continuous-time model for the TSIR, the *stochastic string* model, is introduced in Santa-Clara and Sornette (2001) and reformulated in Bueno-Guerrero et al. (2015). Bueno-Guerrero et al. (2016) prove that the stochastic string model generalizes the Heath et al. (1992) model, even in its infinite-dimensional version. As most of the standard continuous-time models for the dynamics of the TSIR are particular cases of the HJM model, and as the HJM model is a particular case of the stochastic string model, we will adopt the stochastic string model as our model for pricing bond power exchange options.

The key feature of stochastic string term structure models is that the source of randomness generating the dynamics of the forward curve, the *stochastic string process*, is not the same across different maturities (as in the HJM model for each factor), but rather varies point by point along the whole TSIR. The only condition that is imposed is that shocks for different maturities are imperfectly correlated with maintain continuity in the forward curve. This apparently simple generalization produces several advantages for stochastic string models compared to traditional HJM models:

- Recalibration is unnecessary (Goldstein (2000); McDonald and Beard (2002) and Kimmel (2004)). The HJM models allow a perfect fit to the current TSIR but they are inconsistent with the innovations in the forward curve as, in general, the realizations from the N Brownian motions in a N-factor HJM model are incompatible with the possible innovations along the whole TSIR. Thus, HJM models require continuous recalibration to fit the current curve. For stochastic string models, such recalibration is not necessary as we can always find a path for the stochastic string shock to go from the initial to the final forward curve.
- The best instrument to hedge a bond is another bond with a close maturity (Goldstein (2000); Carmona and Tehranchi (2004) and Cont (2005)). The N-factor HJM models have the property that any interest rate derivative can be hedged with N bonds with arbitrary maturities chosen a priori and independently from the bonds underlying the derivative. This fact is inconsistent with the usual practice of market participants who hedge interest rate derivatives using bonds of similar maturities. This practice suggests the existence of a specific risk at maturity, not considered in the factor models. String models incorporate this risk in the stochastic string shock and predict that the best instrument to hedge a bond is another bond with a close maturity.
- We do not need to include the error term when estimating the model (Santa-Clara and Sornette (2001); Bester (2004)). In the N-factor models, any sample of $L(>N)$ forward rates has a covariance matrix whose rank is not greater than N. In this case, error terms must be introduced in the econometric specification of the model. In stochastic string models, we can always find a realization from the shock on a time interval to go from the initial forward curve to the final one. Thus, these models are compatible with any sample of forward rates and there is no need to include error terms in econometric specifications.
- Stochastic string models are more parsimonious than factor models (Goldstein (2000); Santa-Clara and Sornette (2001)). The number of extra parameters in string models with respect to a one-factor model depends on the parameter specification of the correlation function between shocks. If we choose a one-parameter specification for such a function, there is just one more parameter to estimate than in the one-factor HJM model. Thus, string models are more parsimonious than the corresponding N-factor models, that consider a large number of factors (and, then, of parameters) to obtain realistic correlations.

With regard to hedging bond power exchange options, i.e., finding a self-financing portfolio that replicates the option value at expiration, the stochastic string framework is a good choice to work with. In fact, under stochastic string dynamics, Bueno-Guerrero et al. (2022) show that the bond market is complete. That is, any contingent claim written on zero-coupon bonds can be hedged. Moreover, and also within the stochastic string framework, in Bueno-Guerrero et al. (2017), a closed-form expression for the hedging portfolio is obtained in terms of the Malliavin derivative of the discounted payoff of the claim; and in Bueno-Guerrero (2019), necessary and sufficient conditions are established so that the hedging portfolio does not have a bank account part.

Margrabe (1978) shows the equivalence between the American and European versions of exchange options on stocks and obtains a put-call parity result. Pricing and hedging results for power exchange options on lognormally distributed underlying assets, as well as sufficient conditions under which European and American power exchange options on stocks are equivalent are derived in Blenman and Clark (2005b). The objective of this paper is to establish analogous results for pricing and hedging bond power exchange options under the stochastic string framework. In doing so, we also aim to demonstrate that the power exchange option is a unifying concept that generalizes several standard fixed-income derivatives. We even suggest some interesting novel applications for bond power exchange options.

The remainder of the paper is organized as follows. Section 2 includes the fundamental results for the stochastic string modeling that are necessary for the rest of the paper. Section 3 states the main theorem for pricing bond power exchange options and, as corollaries, expressions for the value of a call option and a call power option. A proposition with a put-call parity type result for this kind of option and a theorem with sufficient conditions for the equivalence of American and European exercise styles are also proved. In Section 4, the expression of the hedging portfolio for a bond power exchange option is derived. Section 5 applies the previous results to obtain duration and convexity measures for bond power exchange options and to price extendable/accelerable maturity zero-coupon bonds, options to price a zero-coupon bond off of a shifted term-structure and options on interest rates. Section 6 summarizes and concludes.

2. Preliminary Results

In this section, we review those results from the stochastic string framework that we will need in what follows. We present them without proofs, which can be found in the original papers.

We assume the existence of a market in which a continuum of zero-coupon bonds with any maturity is traded, together with a riskless asset. Regarding the probabilistic framework, we consider a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual hypotheses. We also assume that $\mathcal{F} = \mathcal{F}_{\mathrm{Y}}$ where Y denotes the finite time horizon for trading zero-coupon bonds. The specific form of the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq \mathrm{Y}}$ will be determined later.

The source of randomness for the model is the infinite-dimensional stochastic process (or random field) $Z(t,x,\omega)$ (the *stochastic string process*), consisting of a continuum of adapted stochastic processes $Z(\cdot,x,\omega)$ indexed by time to maturity. Concretely,

$$\begin{array}{rccc} Z: & \Delta^2 \times \Omega & \to & \mathbb{R} \\ & (t,x,\omega) & \mapsto & Z(t,x,\omega) \end{array}$$

where $\Delta^2 = \left\{(t,x) \in \mathbb{R}^2 : 0 \leq t \leq \mathrm{Y},\ x \geq 0\right\}$. From now on we will drop explicit mention of the dependence on ω.

The infinite dimensional process Z is assumed to satisfy the following properties:

(a) The stochastic processes $Z(\cdot,x)$ and $Z(t,\cdot)$ are continuous for each $x \geq 0$ and for each $t \in [0,\mathrm{Y}]$, respectively.
(b) The process $Z(\cdot,x)$ is a martingale for each $x \geq 0$.
(c) The process $Z(t,\cdot)$ is differentiable for each $t \in [0,\mathrm{Y}]$.
(d) For each $x,y \geq 0$, it is the case that

$$d[Z(\cdot,x), Z(\cdot,y)]_t = c(t,x,y)dt \tag{1}$$

where $c(t,x,y)$ is an admissible, continuous, and differentiable correlation function for each t.[1]

For each fixed time to maturity $x \geq 0$, the dynamics of the instantaneous forward interest rate $f(t,x)$ is given by

$$df(t,x) = \alpha(t,x)dt + \sigma(t,x)dZ(t,x)$$

where, for each x, $\alpha(\cdot,x)$ and $\sigma(\cdot,x) > 0$ are continuous adapted stochastic processes and, for each t, $\alpha(t,\cdot)$ and $\sigma(t,\cdot) > 0$ are continuous and differentiable adapted stochastic processes. The available information at any time $t \geq 0$ is given by the filtration

$$\mathcal{F}_t = \sigma\{Z(s,x) : 0 \leq s \leq t,\ x \geq 0\},\ t \leq \Upsilon$$

The short-term interest rate, r_t, can be obtained as $r_t = f(t,0)$, and the bank-account process, B_t, is given by $B_t = e^{\int_0^t r_s ds}$. We denote the time t price of a zero-coupon bond maturing at time $T > t$ as $P(t,T)$.

In order to guarantee the absence of arbitrage opportunities, we posit the existence of a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted price process for any asset is a martingale under $\mathbb{Q}$. The probability measure $\mathbb{Q}$ is known as the *equivalent martingale measure*. Specifically, under the $\mathbb{Q}$ measure, the dynamics of the instantaneous forward interest rate can be written as

$$df(t,x) = \alpha(t,x)dt + \sigma(t,x)d\widetilde{Z}(t,x)$$

with the no-arbitrage condition (Santa-Clara and Sornette (2001); Bueno-Guerrero et al. (2015))

$$\alpha(t,x) = \frac{\partial f(t,x)}{\partial x} + \sigma(t,x)\Big[\int_0^x c(t,x,u)\sigma(t,u)\,du + \int_0^\infty c(t,x,u)\lambda(t,u)\,du\Big],$$

where $d\widetilde{Z}(t,x)$ is the stochastic string shock under $\mathbb{Q}$, and $\lambda(t,u)$ is the market price of risk associated with the time to maturity u.

Bueno-Guerrero et al. (2016) develop a Gaussian stochastic string framework for option pricing. The following result corresponds to Lemma 6.4 in that paper and provides the conditional expectations and variances of log-prices under forward measures. Appearing in the lemma are the *forward measures*, $\mathbb{Q}_T$, which are equivalent to $\mathbb{Q}$, and are defined by their Radon-Nikodym derivatives $\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = [P(0,T)B_T]^{-1}$.

Lemma 1. *If $\sigma(t,x)$ and $c(t,x,y)$ are deterministic, then, under the T_j-forward measure $\mathbb{Q}_{T_j}$, the bond price has a conditioned lognormal distribution with mean*

$$\mathbb{E}^{\mathbb{Q}_{T_j}}[\ln P(T_0,T_i)|\mathcal{F}_s] = \ln\frac{P(s,T_i)}{P(s,T_0)} + \Delta_{ij}(s,T_0) - \frac{1}{2}\Delta_{ii}(s,T_0)$$

and variance $Var\,[\ln P(T_0,T_i)|\mathcal{F}_s] = \Delta_{ii}(s,T_0)$, that is independent of the probability measure, where

$$\begin{aligned}\Delta_{ij}(s,T_0) &= Cov\big[\ln P(T_0,T_i), \ln P\big(T_0,T_j\big)|\mathcal{F}_s\big] \\ &= \int_{t=s}^{T_0}\left[\int_{y=T_0-t}^{T_i-t}\int_{u=T_0-t}^{T_j-t} c(t,u,y)\sigma(t,u)\sigma(t,y)dudy\right]dt\end{aligned}$$

and satisfies $\Delta_{ij}(s,T_0) = \Delta_{ji}(s,T_0)$ and $\Delta_{0j}(s,T_0) = 0$.

For the present study, we adopt the stochastic string model for bond portfolios introduced by Bueno-Guerrero et al. (2022) for which the market is complete. We recall here the key definitions and results from that framework.

Definition 1. *A* ***portfolio*** *in the bond market is a pair* $\{g_t, h(t,\cdot)\}$ *where*

(a) g *is a predictable process.*
(b) *For each* ω, t, $h(\omega, t, \cdot)$ *is a generalized function in* (t, ∞).
(c) *For each* T, *the process* $h(t, T)$ *is predictable.*

The process g_t represents the number of units of the risk-free asset in the portfolio at time t, while $h(t,T)dT$ represents the "number" of bonds with maturities between T and $T + dT$ in the same portfolio at time t.

Definition 2. *The* ***value process****,* V*, of a portfolio* $\{g, h\}$ *is defined by*

$$V_t = g_t B_t + \int_{T=t}^{\infty} h(t,T) P(t,T) dT$$

Definition 3. *A portfolio is* ***self-financing*** *if its value process satisfies*

$$dV_t = g_t dB_t + \int_{T=t}^{\infty} h(t,T) dP(t,T) dT$$

Henceforth, discounting with respect to the risk-free asset B_t will be denoted by the overline symbol "$\overline{}$".

Definition 4. *Consider a discounted contingent claim* $\overline{X} \in L^{\infty}(\mathcal{F}_{T_0})$*. We say that* $\overline{X}$ *can be* ***replicated*** *or that we can* ***hedge*** *against* $\overline{X}$ *if there exists a self-financing portfolio with bounded, discounted value process* $\overline{V}$*, such that* $\overline{V}_{T_0} = \overline{X}$.

The following theorem, giving an explicit expression for the bond part of the hedging portfolio, is the main result in Bueno-Guerrero et al. (2022).

Theorem 1. *In the stochastic string model of Bueno-Guerrero et al. (2022), the market is complete and the generalized function* $h(t,\cdot)$ *in the hedging portfolio is given by*

$$h(t,T) = \frac{1}{\overline{P}(t,T)} \left[\frac{j(t,T-t)}{\sigma(t,T-t)} \right]'$$

where the symbol $'$ *means derivative with respect to* T *in the sense of distributions,* $j(t,\cdot)$ *is given by the martingale representation of* $\overline{V}_t$

$$d\overline{V}_t = \int_{u=0}^{\infty} j(t,u) d\widetilde{Z}(t,u) du \tag{2}$$

and $\widetilde{Z}(t,u)$ *is the stochastic string process with respect to the equivalent martingale measure.*

The problem that arises with the application of Theorem 1 is that usually the process $j(t,u)$ in the martingale representation of $\overline{V}_t$ is not known. However, using the Malliavin calculus valid for stochastic string models developed in Bueno-Guerrero et al. (2017), it is possible to obtain the martingale representation (2) in terms of the Malliavin derivative of the payoff.[2] Proceeding in this way, we can rewrite Theorem 1 as follows (Theorem 5 of Bueno-Guerrero et al. (2017)).

Theorem 2. *In the Gaussian stochastic string model, the generalized function* $h(t,\cdot)$ *in the hedging portfolio is given by*

$$h(t,T) = \frac{1}{\overline{P}(t,T)} \left[\frac{\mathbb{E}^{\mathbb{Q}}[D_{t,T-t}\overline{X}|\mathcal{F}_t]}{\sigma(t,T-t)} \right]' \tag{3}$$

whenever the discounted payoff $\overline{X}$ is Malliavin differentiable and where $D_{t,T-t}$ is the Malliavin derivative for stochastic strings and $\mathbb{Q}$ is the equivalent martingale measure.

All of the general results from the Malliavin calculus that are not applicable specifically to Brownian motion can be applied to the stochastic string framework. To apply expression (3) we will need to know the Malliavin derivative of discounted bond prices. The following result corresponds to Proposition 6 of Bueno-Guerrero et al. (2017).

Proposition 1. *In the Gaussian stochastic string framework, and working under the equivalent martingale measure, $\overline{P}(\nu, T_1)$ is Malliavin differentiable and*

$$D_{t,T-t}\overline{P}(\nu, T_1) = -\overline{P}(\nu, T_1)\sigma(t, T-t)\mathbf{1}_{T<T_1} \tag{4}$$

3. Pricing Bond Power Exchange Options

In this section, we will price bond power exchange options under the Gaussian stochastic string framework. A bond power exchange option gives the holder the option to exchange the value of a zero-coupon bond raised to a power for the value of another zero-coupon bond raised to another power. Concretely, we consider payoffs of the form

$$X_{T_0} = [\lambda_1 P^{\alpha_1}(T_0, T_1) - \lambda_2 P^{\alpha_2}(T_0, T_2)]^+ \tag{5}$$

where T_0 is the exercise time of the option, and for $i = 1, 2$, $P(T_0, T_i)$ is the time-T_0 price of a zero-coupon bond maturing at T_i, α_i, $\lambda_i \in \mathbb{R}$ are constants, and $[\cdot]^+$ is the positive part function. The next result is the pricing formula for bond power exchange options.

Theorem 3. *Under the Gaussian stochastic string framework, the price, $\mathbf{BPE}(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$, of the bond power exchange option with payoff (5) is given by*

$$\begin{aligned}&\mathbf{BPE}(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)\\ &= \lambda_1 P^{1-\alpha_1}(t, T_0)P^{\alpha_1}(t, T_1)e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1) - \lambda_2 P^{1-\alpha_2}(t, T_0)P^{\alpha_2}(t, T_2)e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2)\end{aligned}$$

with

$$d_1 = \frac{\ln\left[\frac{\lambda_1}{\lambda_2}\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1}\left(\frac{P(t,T_0)}{P(t,T_2)}\right)^{\alpha_2}\right] + \frac{1}{2}(\alpha_2\Delta_{22} - \alpha_1\Delta_{11}) + \alpha_1^2\Delta_{11} - \alpha_1\alpha_2\Delta_{12}}{\sqrt{\alpha_1^2\Delta_{11} + \alpha_2^2\Delta_{22} - 2\alpha_1\alpha_2\Delta_{12}}}$$

$$d_2 = d_1 - \sqrt{\alpha_1^2\Delta_{11} + \alpha_2^2\Delta_{22} - 2\alpha_1\alpha_2\Delta_{12}}$$

Proof. The price of the bond power exchange option can be obtained as

$$\begin{aligned}\mathbf{BPE}(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2) &= \mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_t^{T_0} r_s ds}[\lambda_1 P^{\alpha_1}(T_0, T_1) - \lambda_2 P^{\alpha_2}(T_0, T_2)]^+|\mathcal{F}_t\right\}\\ &= P(t, T_0)\Big\{\lambda_1\mathbb{E}^{\mathbb{Q}_{T_0}}\left[P^{\alpha_1}(T_0, T_1)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right]\\ &\quad - \lambda_2\mathbb{E}^{\mathbb{Q}_{T_0}}\left[P^{\alpha_2}(T_0, T_2)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right]\Big\}\end{aligned} \tag{6}$$

where in the last step, we have passed to the T_0-forward measure $\mathbb{Q}_{T_0}$. Applying Lemma 1, we have that under $\mathbb{Q}_{T_0}$, $\ln P(T_0, T_i)$ has a conditioned normal distribution with

$$\mathbb{E}^{\mathbb{Q}_{T_0}}[\ln P(T_0, T_i)|\mathcal{F}_t] = \ln\frac{P(t, T_i)}{P(t, T_0)} - \frac{1}{2}\Delta_{ii}(t, T_0)$$

and

$$Var^{\mathbb{Q}_{T_0}}[\ln P(T_0, T_i)|\mathcal{F}_t] = \Delta_{ii}(t, T_0)$$

Therefore, by defining

$$x_i(t, T_0) = \frac{\ln P(T_0, T_i) - \ln \frac{P(t,T_i)}{P(t,T_0)} + \frac{1}{2}\Delta_{ii}(t, T_0)}{\sqrt{\Delta_{ii}(t, T_0)}}$$

we can write

$$P^{\alpha_i}(T_0, T_i) = \left(\frac{P(t, T_i)}{P(t, T_0)}\right)^{\alpha_i} \exp\left\{\alpha_i\left(\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}\right)\right\} \tag{7}$$

with $x_i(t, T_0) \sim \mathcal{N}(0, 1)$, $i = 1, 2$ under $\mathbb{Q}_{T_0}$ and conditioned to $\mathcal{F}_t$.

Using these transformations, the expectations in (6) can be written as

$$\mathbb{E}^{\mathbb{Q}_{T_0}}\left[P^{\alpha_i}(T_0, T_i)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1) - \lambda_2 P^{\alpha_2}(T_0,T_2) > 0}|\mathcal{F}_t\right] = \left(\frac{P(t, T_i)}{P(t, T_0)}\right)^{\alpha_i} \int_\Omega g(x_1, x_2; M) e^{\alpha_i\left[\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}\right]} d\mathbf{x} \tag{8}$$

where $g(x_1, x_2; M)$ is the density function of a bivariate normal distribution with $(M)_{kl} = \dfrac{\Delta_{kl}}{\sqrt{\Delta_{kk}}\sqrt{\Delta_{kk}}}$ and $\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : \lambda_1 P^{\alpha_1}(T_0, T_1) - \lambda_2 P^{\alpha_2}(T_0, T_2) > 0\right\}$.

In order to obtain the expectation for the case $i = 1$, we need the limits of the integral for x_1. We have the following

$$\begin{aligned}
&\lambda_1 P^{\alpha_1}(T_0, T_1) - \lambda_2 P^{\alpha_2}(T_0, T_2) > 0 \\
&\quad \Leftrightarrow x_1 > \frac{\ln\left[\frac{\lambda_2}{\lambda_1}\left(\frac{P(t,T_0)}{P(t,T_1)}\right)^{\alpha_1}\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2}\right] + \alpha_2\left[\sqrt{\Delta_{22}}x_2 - \frac{1}{2}\Delta_{22}\right] + \frac{\alpha_1}{2}\Delta_{11}}{\alpha_1\sqrt{\Delta_{11}}} =: \Theta(x_2)
\end{aligned}$$

Thus, defining $\rho = \dfrac{\Delta_{12}}{\sqrt{\Delta_{11}}\sqrt{\Delta_{22}}}$, we have

$$\begin{aligned}
&\int_\Omega g(x_1, x_2; M) e^{\alpha_1\left[\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}\right]} d\mathbf{x} \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{x_2=-\infty}^{+\infty} \int_{x_1=\Theta(x_2)}^{+\infty} e^{-\frac{1}{2(1-\rho^2)}\left(x_1^2 + x_2^2 - 2\rho x_1 x_2\right)} e^{\alpha_1\left[\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}\right]} dx_1 dx_2
\end{aligned} \tag{9}$$

Performing the calculations we arrive at

$$\int_\Omega g(x_1, x_2; M) e^{\alpha_1\left[\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}\right]} d\mathbf{x} = \frac{e^{\frac{\alpha_1(\alpha_1 - 1)}{2}\Delta_{11}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2}} \Phi(\delta + \lambda w) dw$$

where

$$\delta = \frac{\alpha_1^2\Delta_{11} - \ln\left[\frac{\lambda_2}{\lambda_1}\left(\frac{P(t,T_0)}{P(t,T_1)}\right)^{\alpha_1}\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2}\right] - \rho\alpha_1\alpha_2\sqrt{\Delta_{11}\Delta_{22}} + \frac{\alpha_2}{2}\Delta_{22} - \frac{\alpha_1}{2}\Delta_{11}}{\alpha_1\sqrt{\Delta_{11}(1-\rho^2)}}$$

and

$$\lambda = \frac{\alpha_1\rho\sqrt{\Delta_{11}} - \alpha_2\sqrt{\Delta_{22}}}{\alpha_1\sqrt{\Delta_{11}(1-\rho^2)}}$$

Using the formula $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\Phi[a + bx]dx = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$, we obtain

$$\int_\Omega g(x_1, x_2; M) e^{\alpha_1\left[\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}\right]} d\mathbf{x} = e^{\frac{\Delta_{11}}{2}\alpha_1(\alpha_1 - 1)} \Phi\left(\frac{\delta}{\sqrt{1+\lambda^2}}\right)$$

Replacing this expression in (8) with $i = 1$, using $\rho = \frac{\Delta_{12}}{\sqrt{\Delta_{11}}\sqrt{\Delta_{22}}}$ and reducing, we get

$$\mathbb{E}^{\mathbb{Q}_{T_0}}\left[P^{\alpha_1}(T_0,T_1)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right] = \left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1) \quad (10)$$

Following a similar procedure it is not difficult to obtain for the second expectation in (6) the value

$$\mathbb{E}^{\mathbb{Q}_{T_0}}\left[P^{\alpha_2}(T_0,T_2)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right] = \left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2) \quad (11)$$

and replacing these two expectations in (6) we arrive at the desired result. □

As a consistency test, in the following corollary, whose proof is immediate, we restrict the result of Theorem 1 to the case of a standard call option obtaining the price of this option in the Gaussian stochastic string framework (Bueno-Guerrero et al. (2015)).

Corollary 1. *Under the conditions of Theorem 3, the price of a standard call option with payoff* $X_{T_0} = [P(T_0,T_1) - K]^+$, $\mathbf{C}(t,T_0,T_1,K)$*, is given by*

$$\mathbf{C}(t,T_0,T_1,K) = \mathbf{BPE}(t,T_0,T_1,T_0,1,0,1,K) = P(t,T_1)\Phi\left(\overline{d}_1\right) - KP(t,T_0)\Phi\left(\overline{d}_2\right)$$

where

$$\overline{d}_1 = \frac{\ln\left(\frac{P(t,T_1)}{KP(t,T_0)}\right) + \frac{1}{2}\Delta_{11}}{\sqrt{\Delta_{11}}}$$

$$\overline{d}_2 = \overline{d}_1 - \sqrt{\Delta_{11}}$$

The following result is also an immediate consequence of Theorem 3 and gives the price of a standard bond call power option.

Corollary 2. *Under the conditions of Theorem 3, the price of the standard call power option with payoff* $X_{T_0} = [P^\alpha(T_0,T_1) - K]^+$, $\mathbf{CP}(t,T_0,T_1,\alpha,K)$*, is given by*

$$\begin{aligned}\mathbf{CP}(t,T_0,T_1,\alpha,K) &= \mathbf{BPE}(t,T_0,T_1,T_0,\alpha,0,1,K)\\ &= P^{1-\alpha}(t,T_0)P^\alpha(t,T_1)e^{\frac{\alpha(\alpha-1)}{2}\Delta_{11}}\Phi\left(\widehat{d}_1\right) - KP(t,T_0)\Phi\left(\widehat{d}_2\right)\end{aligned}$$

where

$$\widehat{d}_1 = \frac{\ln\left(\frac{P^\alpha(t,T_1)}{KP^\alpha(t,T_0)}\right) + \left(\alpha^2 - \frac{\alpha}{2}\right)\Delta_{11}}{\alpha\sqrt{\Delta_{11}}}$$

$$\widehat{d}_2 = \widehat{d}_1 - \alpha\sqrt{\Delta_{11}}$$

In Blenman and Clark (2005a), a class of options with constant underlying elasticity in strikes (CUES) is introduced. To end this section we apply Theorem 3 to CUES options written on zero-coupon bonds. The proof is immediate from the theorem.

Corollary 3. *Under the conditions of Theorem 3, the price of a CUES bond option with payoff* $X_{T_0} = [P(T_0, T_1) - \lambda P^{\alpha}(T_0, T_1)]^+$, $\mathbf{CUES}(t, T_0, T_1, \alpha, \lambda)$, *with* $\alpha < 1$ *is given by*

$$\begin{aligned}\mathbf{CUES}(t, T_0, T_1, \alpha, \lambda) &= \mathbf{BPE}(t, T_0, T_1, T_1, 1, \alpha, 1, \lambda) \\ &= P(t, T_1)\Phi(d_1^*) - \lambda P^{1-\alpha}(t, T_0)P^{\alpha}(t, T_1)e^{\frac{\alpha(\alpha-1)}{2}\Delta_{11}}\Phi(d_2^*)\end{aligned}$$

with

$$d_1^* = \frac{\ln\left[\frac{1}{\lambda}\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{1-\alpha}\right] + \frac{1}{2}(1-\alpha)\Delta_{11}}{(1-\alpha)\sqrt{\Delta_{11}}}$$

$$d_2^* = d_1^* - (1-\alpha)\sqrt{\Delta_{11}}$$

3.1. American Power Exchange Options

In Margrabe (1978), the equivalence of American and European stock exchange options is proved. For the case of power exchange options on stocks, sufficient conditions for the equivalence can be found in Blenman and Clark (2005b). In a similar way, we can obtain the conditions under which bond power exchange options are equivalent in the American and European cases. The result is the following.

Theorem 4. *If* $\alpha_1 \geq \frac{2\ln P(t,T_0)}{\Delta_{11}}$ *and* $\alpha_2 \leq \frac{2\ln P(t,T_0)}{\Delta_{22}}$, *then, under the Gaussian stochastic string framework, the value of the European bond power exchange option* $\mathbf{BPE}(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ *is the same as that of its American version.*

Proof. Passing to the T_0-forward measure, we can write for $t \leq T_0$

$$\begin{aligned}\mathbf{BPE}(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2) &= P(t, T_0)\mathbb{E}^{\mathbb{Q}_{T_0}}\left\{[\lambda_1 P^{\alpha_1}(T_0, T_1) - \lambda_2 P^{\alpha_2}(T_0, T_2)]^+ | \mathcal{F}_t\right\} \\ &\geq P(t, T_0)\left[\mathbb{E}^{\mathbb{Q}_{T_0}}\{[\lambda_1 P^{\alpha_1}(T_0, T_1) - \lambda_2 P^{\alpha_2}(T_0, T_2)] | \mathcal{F}_t\}\right]^+\end{aligned} \tag{12}$$

where in the last line we have applied Jensen's inequality.

On the other hand, using Equation (7) and working as in the proof of Theorem 3, we get for $i = 1, 2$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_{T_0}}\{P^{\alpha_i}(T_0, T_i) | \mathcal{F}_t\} &= \left(\frac{P(t, T_i)}{P(t, T_0)}\right)^{\alpha_i}\mathbb{E}^{\mathbb{Q}_{T_0}}\left\{\exp\left\{\alpha_i\left(\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}\right)\right\} | \mathcal{F}_t\right\} \\ &= \left(\frac{P(t, T_i)}{P(t, T_0)}\right)^{\alpha_i}\int_{\mathbb{R}^2} g(x_1, x_2; M)e^{\alpha_i[\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}]}d\mathbf{x} \\ &= \left(\frac{P(t, T_i)}{P(t, T_0)}\right)^{\alpha_i} e^{\frac{\alpha_i(\alpha_i-1)}{2}\Delta_{ii}}\end{aligned} \tag{13}$$

Replacing these expectations in (12), we arrive at

$$\begin{aligned}\mathbf{BPE}(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2) \geq &\Big[\lambda_1 P^{\alpha_1}(t, T_1)P^{1-\alpha_1}(t, T_0)e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}} \\ &-\lambda_2 P^{\alpha_2}(t, T_2)P^{1-\alpha_2}(t, T_0)e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Big]^+ \\ \geq &[\lambda_1 P^{\alpha_1}(t, T_1) - \lambda_2 P^{\alpha_2}(t, T_2)]^+\end{aligned}$$

where in the last step we have applied that $\alpha_1 \geq \frac{2\ln P(t,T_0)}{\Delta_{11}}$ and $\alpha_2 \leq \frac{2\ln P(t,T_0)}{\Delta_{22}}$. Thus, when these inequalities are satisfied, early exercise of an American option is never optimal and the values of the European and the American options coincide. □

If we restrict Theorem 4 to the case of standard call options, we obtain the following corollary, whose proof is straightforward, and which is in line with the results obtained in Merton (1973) related to the pricing of American options.

Corollary 4. *If* $P(t,T_0) \leq \sqrt{e^{\Delta_{11}}}$, *then an American standard* T_0*-expiry call option on a* T_1*-maturity* ($T_1 \geq T_0$) *zero-coupon bond should not be exercised prior to maturity.*

3.2. Put-Call Parity

As is the case with every exchange option, we have seen that the bond power exchange option can be interpreted either as a call or as a put option. Nevertheless, it is reasonable to wonder if there exists a similar result to the well-known put-call parity for these options. Specifically, is there an expression that relates the price of the options with payoffs $[\lambda_1 P^{\alpha_1}(T_0,T_1) - \lambda_2 P^{\alpha_2}(T_0,T_2)]^+$ and $[\lambda_2 P^{\alpha_2}(T_0,T_2) - \lambda_1 P^{\alpha_1}(T_0,T_1)]^+$? The answer is affirmative and the expression is stated in the following result.

Proposition 2. *Under the Gaussian stochastic string framework, it is the case that*

$$\begin{aligned}\mathbf{BPE}(t,T_0,T_1,T_2,\alpha_1,\alpha_2,\lambda_1,\lambda_2) &= \mathbf{BPE}(t,T_0,T_2,T_1,\alpha_2,\alpha_1,\lambda_2,\lambda_1)\\ &\quad + \lambda_1 P^{1-\alpha_1}(t,T_0)P^{\alpha_1}(t,T_1)e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\\ &\quad - \lambda_2 P^{1-\alpha_2}(t,T_0)P^{\alpha_2}(t,T_2)e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\end{aligned}$$

Proof. Taking into account the identity $[x-y]^+ - [y-x]^+ = x-y$ we can write

$$\begin{aligned}\mathbf{BPE}(t,T_0,T_1,T_2,\alpha_1,\alpha_2,\lambda_1,\lambda_2) &= \mathbf{BPE}(t,T_0,T_2,T_1,\alpha_2,\alpha_1,\lambda_2,\lambda_1)\\ &\quad + \mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_t^{T_0} r_s ds}[\lambda_1 P^{\alpha_1}(T_0,T_1) - \lambda_2 P^{\alpha_2}(T_0,T_2)]|\mathcal{F}_t\right\}\\ &= \mathbf{BPE}(t,T_0,T_2,T_1,\alpha_2,\alpha_1,\lambda_2,\lambda_1)\\ &\quad + P(t,T_0)\mathbb{E}^{\mathbb{Q}_{T_0}}\{[\lambda_1 P^{\alpha_1}(T_0,T_1) - \lambda_2 P^{\alpha_2}(T_0,T_2)]|\mathcal{F}_t\}\\ &= \mathbf{BPE}(t,T_0,T_2,T_1,\alpha_2,\alpha_1,\lambda_2,\lambda_1)\\ &\quad + \lambda_1 P^{1-\alpha_1}(t,T_0)P^{\alpha_1}(t,T_1)e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\\ &\quad - \lambda_2 P^{1-\alpha_2}(t,T_0)P^{\alpha_2}(t,T_2)e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\end{aligned}$$

where in the last step we have used expression (13). □

A direct application of the put-call parity allows us to obtain the put version of Theorem 3.

Corollary 5. *The price,* $\mathbf{BPE}(t,T_0,T_2,T_1,\alpha_2,\alpha_1,\lambda_2,\lambda_1)$, *of the bond power exchange option with payoff* $[\lambda_2 P^{\alpha_2}(T_0,T_2) - \lambda_1 P^{\alpha_1}(T_0,T_1)]^+$ *is given by*

$$\begin{aligned}&\mathbf{BPE}(t,T_0,T_2,T_1,\alpha_2,\alpha_1,\lambda_2,\lambda_1)\\ &= -\lambda_1 P^{1-\alpha_1}(t,T_0)P^{\alpha_1}(t,T_1)e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(-d_1) + \lambda_2 P^{1-\alpha_2}(t,T_0)P^{\alpha_2}(t,T_2)e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(-d_2)\end{aligned}$$

4. Hedging Bond Power Exchange Options

In this section, we will apply the framework of Bueno-Guerrero et al. (2017) to obtain the hedging portfolio for the bond power exchange option. The main result is the following.

Theorem 5. *Under the Gaussian stochastic string framework, the hedging portfolio for the option with payoff (5) is given by*

$$\begin{array}{lll} \lambda_1(1-\alpha_1)\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1) & - & \lambda_2(1-\alpha_2)\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2) \\ & & \textit{units of } T_0\textit{-bond}, \\ \lambda_1\alpha_1\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1-1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1) & & \textit{units of } T_1\textit{-bond}, \\ -\lambda_2\alpha_2\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2-1} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2) & & \textit{units of } T_2\textit{-bond}, \end{array} \tag{14}$$

and it has no bank account part.

Proof. Taking $P(T_0, T_0) = 1$ we have $B^{-1}(T_0) = \overline{P}(T_0, T_0)$ and we can write the payoff of Equation (5) in discounted terms as

$$\begin{aligned} \overline{X}_{T_0} &= \overline{P}(T_0,T_0)[\lambda_1 P^{\alpha_1}(T_0,T_1) - \lambda_2 P^{\alpha_2}(T_0,T_2)]^+ \\ &= \overline{P}(T_0,T_0)\left[\lambda_1\left(\frac{\overline{P}(T_0,T_1)}{\overline{P}(T_0,T_0)}\right)^{\alpha_1} - \lambda_2\left(\frac{\overline{P}(T_0,T_2)}{\overline{P}(T_0,T_0)}\right)^{\alpha_2}\right]^+ \\ &= \left[\lambda_1\overline{P}^{1-\alpha_1}(T_0,T_0)\overline{P}^{\alpha_1}(T_0,T_1) - \lambda_2\overline{P}^{1-\alpha_2}(T_0,T_0)\overline{P}^{\alpha_2}(T_0,T_2)\right]^+ \end{aligned}$$

Taking Malliavin derivatives and using (4) we obtain

$$\begin{aligned} D_{t,T-t}\overline{X}_{T_0} = \Big\{ & \left[-\lambda_1(1-\alpha_1)\overline{P}^{1-\alpha_1}(T_0,T_0)\overline{P}^{\alpha_1}(T_0,T_1) + \lambda_2(1-\alpha_2)\overline{P}^{1-\alpha_2}(T_0,T_0)\overline{P}^{\alpha_2}(T_0,T_2)\right]\mathbf{1}_{T<T_0} \\ & -\lambda_1\alpha_1\overline{P}^{1-\alpha_1}(T_0,T_0)\overline{P}^{\alpha_1}(T_0,T_1)\mathbf{1}_{T<T_1} + \lambda_2\alpha_2\overline{P}^{1-\alpha_2}(T_0,T_0)\overline{P}^{\alpha_2}(T_0,T_2)\mathbf{1}_{T<T_2}\Big\} \\ & \times \mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0} \end{aligned}$$

Applying expression (3) and using $\left(\mathbf{1}_{T<T_i}\right)' = -\delta(T-T_i)$, we have

$$\begin{aligned} h(t,T) &= \frac{1}{\overline{P}(t,T)}\left[\frac{\mathbb{E}^{\mathbb{Q}}\left[D_{t,T-t}\overline{X}_{T_0}|\mathcal{F}_t\right]}{\sigma(t,T-t)}\right]' \\ &= \frac{1}{\overline{P}(t,T)}\Big\{\Big[\lambda_1(1-\alpha_1)\mathbb{E}^{\mathbb{Q}}\left[\overline{P}^{1-\alpha_1}(T_0,T_0)\overline{P}^{\alpha_1}(T_0,T_1)\mathbf{1}_{\lambda_1 P^{\alpha_1}-\lambda_2 P^{\alpha_2}>0}|\mathcal{F}_t\right]\Big] \\ &\quad - \lambda_2(1-\alpha_2)\mathbb{E}^{\mathbb{Q}}\left[\overline{P}^{1-\alpha_2}(T_0,T_0)\overline{P}^{\alpha_2}(T_0,T_2)\mathbf{1}_{\lambda_1 P^{\alpha_1}-\lambda_2 P^{\alpha_2}>0}|\mathcal{F}_t\right]\delta(T-T_0) \\ &\quad + \lambda_1\alpha_1\mathbb{E}^{\mathbb{Q}}\left[\overline{P}^{1-\alpha_1}(T_0,T_0)\overline{P}^{\alpha_1}(T_0,T_1)\mathbf{1}_{\lambda_1 P^{\alpha_1}-\lambda_2 P^{\alpha_2}>0}|\mathcal{F}_t\right]\delta(T-T_1) \\ &\quad -\lambda_2\alpha_2\mathbb{E}^{\mathbb{Q}}\left[\overline{P}^{1-\alpha_1}(T_0,T_0)\overline{P}^{\alpha_1}(T_0,T_1)\mathbf{1}_{\lambda_1 P^{\alpha_1}-\lambda_2 P^{\alpha_2}>0}|\mathcal{F}_t\right]\delta(T-T_2)\Big\} \end{aligned} \tag{15}$$

The expectations in the last expression can be written for $i = 1,2$ as

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\overline{P}^{1-\alpha_i}(T_0,T_0)\overline{P}^{\alpha_i}(T_0,T_i)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\overline{P}(T_0,T_0)P^{\alpha_i}(T_0,T_i)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right] \\ &= \overline{P}(t,T_0)\mathbb{E}^{\mathbb{Q}_{T_0}}\left[P^{\alpha_i}(T_0,T_i)\mathbf{1}_{\lambda_1 P^{\alpha_1}(T_0,T_1)-\lambda_2 P^{\alpha_2}(T_0,T_2)>0}|\mathcal{F}_t\right] \end{aligned}$$

Replacing them in (15) and using expressions (10) and (11), we arrive at

$$h(t,T) = \frac{P(t,T_0)}{P(t,T)}\Bigg\{\Bigg[\lambda_1(1-\alpha_1)\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1)$$
$$-\lambda_2(1-\alpha_2)\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2)\Bigg]\delta(T-T_0)$$
$$+\lambda_1\alpha_1\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1)\delta(T-T_1)$$
$$-\lambda_2\alpha_2\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2)\delta(T-T_2)\Bigg\}$$

from which we obtain the composition of the bond part in the hedging portfolio. The value of this bond part is

$$\int_{T=t}^{\infty} h(t,T)P(t,T)dT = P(t,T_0)\Bigg\{\lambda_1(1-\alpha_1)\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1)$$
$$-\lambda_2(1-\alpha_2)\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2)$$
$$+\lambda_1\alpha_1\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha_1} e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1)$$
$$-\lambda_2\alpha_2\left(\frac{P(t,T_2)}{P(t,T_0)}\right)^{\alpha_2} e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2)\Bigg\}$$
$$= \lambda_1 P^{1-\alpha_1}(t,T_0)P^{\alpha_1}(t,T_1)e^{\frac{\alpha_1(\alpha_1-1)}{2}\Delta_{11}}\Phi(d_1)$$
$$-\lambda_2 P^{1-\alpha_2}(t,T_0)P^{\alpha_2}(t,T_2)e^{\frac{\alpha_2(\alpha_2-1)}{2}\Delta_{22}}\Phi(d_2)$$
$$= \mathbf{BPE}(t,T_0,T_1,T_2,\alpha_1,\alpha_2,\lambda_1,\lambda_2)$$

and thus, the hedging portfolio has no bank account part. □

Now, following the lines of Section 3, we obtain as particular cases of Theorem 5 the hedging portfolios for standard call, power call and CUES options. All the results can be obtained easily just by using the appropriate parameters in (14).

Corollary 6. *Under the conditions of Theorem 5 we have:*
(i) The hedging portfolio for a standard call option with payoff $X_{T_0} = [P(T_0,T_1)-K]^+$ is given by

$$\begin{array}{rl} -K\Phi\left(\bar{d}_2\right) & \textit{units of } T_0\textit{-bond} \\ \Phi\left(\bar{d}_1\right) & \textit{units of } T_1\textit{-bond} \end{array}$$

(ii) The hedging portfolio for a standard call power option with payoff $X_{T_0} = [P^\alpha(T_0,T_1)-K]^+$ is given by

$$\begin{array}{rl} (1-\alpha)\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha} e^{\frac{\alpha(\alpha-1)}{2}\Delta_{11}}\Phi\left(\hat{d}_1\right) - K\Phi\left(\hat{d}_2\right) & \textit{units of } T_0\textit{-bond} \\ \alpha\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha-1} e^{\frac{\alpha(\alpha-1)}{2}\Delta_{11}}\Phi\left(\hat{d}_1\right) & \textit{units of } T_1\textit{-bond} \end{array}$$

(iii) The hedging portfolio for a CUES bond option with payoff $X_{T_0} = [P(T_0,T_1)-\lambda P^\alpha(T_0,T_1)]^+$ with $\alpha<1$ is given by

$$\begin{array}{rl} -\lambda(1-\alpha)\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha} e^{\frac{\alpha(\alpha-1)}{2}\Delta_{11}}\Phi(d_2^*) & \textit{units of } T_0\textit{-bond} \\ \Phi(d_1') - \lambda\alpha\left(\frac{P(t,T_1)}{P(t,T_0)}\right)^{\alpha-1} e^{\frac{\alpha(\alpha-1)}{2}\Delta_{11}}\Phi(d_2^*) & \textit{units of } T_1\textit{-bond} \end{array}$$

5. Applications

In this section, we consider several interesting applications for bond power exchange options. Some of these applications will make reference to the zero-coupon term structure, or as it is often called, the zero curve. Rates on the time t zero curve are defined in terms of the zero-coupon bond prices. That is, the time t zero rate for maturity $T > 0$ is given by

$$y(t,T) = \eta\left[P(t,T)^{-\frac{1}{\eta(T-t)}} - 1\right],$$

where η is the number of compounding periods per year. The time t forward rate in effect on $[T_0, T_1]$, $t \leq T_0 \leq T_1$ is given by

$$f(t,T_0,T_1) = \eta\left[\frac{P(t,T_0)}{P(t,T_1)}^{\frac{1}{\eta(T_1-T_0)}} - 1\right]. \tag{16}$$

Unless otherwise stated, in the following we assume $\eta = 2$, that is, semi-annual compounding as is the convention in the U.S. Treasuries market.

5.1. Pricing Extendable/Accelerable Maturity Zero-Coupon Bonds

Consider a derivative contract under which, at time T_0, the holder chooses a long position in either a T_1-maturity or a T_2-maturity zero-coupon bond ($T_0 \leq T_1 \leq T_2$). In a strictly positive interest rate environment, the T_1-maturity bond would always be more valuable than the T_2-maturity bond at time T_0. However if interest rates can become negative, then the bond with longer maturity could be more valuable. Such a contract is replicated by a portfolio consisting of a T_1-maturity zero-coupon bond and an option to exchange the T_1-maturity bond for a T_2-maturity bond, or equivalently, by a portfolio consisting of a T_2-maturity zero-coupon bond and an option to exchange the T_2-maturity for a T_1-maturity. That is, the payoff of the contract can be written as

$$\begin{aligned} X_{T_0} &= \max\{P(T_0,T_1), P(T_0,T_2)\} \\ &= P(T_0,T_1) + [P(T_0,T_2) - P(T_0,T_1)]^+ \\ &= P(T_0,T_2) + [P(T_0,T_1) - P(T_0,T_2)]^+ \end{aligned}$$

and by Theorem 3, the value at time $t \leq T_0$ of the contract, $\mathbf{EAM}(t,T_0,T_1,T_2)$, is given by

$$\mathbf{EAM}(t,T_0,T_1,T_2) = P(t,T_1) + \mathbf{BPE}(t,T_0,T_2,T_1,1,1,1,1) = P(t,T_2) + \mathbf{BPE}(t,T_0,T_1,T_2,1,1,1,1)$$

This makes it clear that this contract can be thought of as either an extendable or an accelerable maturity zero-coupon bond.

5.2. Option to Price a Zero-Coupon Bond off of a Shifted Term-Structure

The bond power exchange option with payoff

$$X_{T_0} = \left[P(T_0,T_2) - P(T_0,T_1)^{\frac{T_2-T_0}{T_1-T_0}}\right]^+ \tag{17}$$

for $T_2 > T_1 > T_0$ gives the holder the option to buy a T_2-maturity zero-coupon bond priced off of the zero curve shifted $T_2 - T_1$ units to the right. To see this, note that the exercise price is

$$P(T_0,T_1)^{\frac{T_2-T_0}{T_1-T_0}} = \frac{1}{\left(1 + \frac{y(T_0,T_1)}{2}\right)^{2(T_2-T_0)}},$$

which is, at time T_0, the price of a T_2-maturity zero coupon bond priced using the T_1-maturity yield $y(T_0, T_1)$. Using Corollary 5, it is not difficult to obtain the expression for the price, $\mathbf{STS}(t, T_1, T_2)$ of this option as

$$\mathbf{STS}(t, T_1, T_2) = P(t, T_2)\Phi(-d_2) - P(t, T_0)^{-\frac{T_2-T_1}{T_1-T_0}} P(t, T_1)^{\frac{T_2-T_0}{T_1-T_0}} e^{\frac{(T_2-T_0)(T_2-T_1)}{2(T_1-T_0)^2}\Delta_{11}} \Phi(-d_1)$$

5.3. Bond Option Duration and Convexity

The results of Theorem 5 can be applied to derive duration and convexity formulas for bond power exchange options.[3] For $i = 0, 1, 2$, define $\theta_i(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ to be the units of the T_i-maturity zero-coupon bond in the hedging portfolio from (14). Since the duration at time t of a T-maturity zero-coupon bond is $T - t$, the duration of the bond power exchange option is

$$Dur_{BPE} = \sum_{i=0}^{2} \frac{\theta_i(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)P(t, T_i)}{BPE(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)}(T_i - t).$$

Similarly, the convexity at time t of a T-maturity zero-coupon bond is

$$Conv = \frac{(T-t)^2 + \frac{T-t}{2}}{\left(1 + \frac{y(t,T)}{2}\right)^2},$$

and the convexity of the bond power exchange option is

$$Conv_{BPE} = \sum_{i=0}^{2} \frac{\theta_i(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)P(t, T_i)}{BPE(t, T_0, T_1, T_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2)} \frac{(T_i - t)^2 + \frac{T_i - t}{2}}{\left(1 + \frac{y_i(t,T_i)}{2}\right)^2}.$$

Thus, it is straightforward to employ bond power exchange options for standard interest rate hedging based on duration and convexity.

5.4. Options on Interest Rates

For the parameter values $\alpha_1 = -\frac{1}{2(T_1-T_0)}$, $\alpha_2 = -\frac{1}{2(T_2-T_0)}$, $\lambda_1 = \lambda_2 = 2$, the payoff in (5) becomes

$$[y(T_0, T_1) - y(T_0, T_2)]^+.$$

In this case, the bond power exchange option is a derivative on the T_1,T_2 interest rate spread. For the parameter values $\alpha_1 = -\frac{1}{2(T_1-T_0)}$, $\alpha_2 = 0$, $\lambda_1 = 2$, and $\lambda_2 = 2 + K$ for $K \geq 0$, the payoff in (5) becomes

$$[y(T_0, T_1) - K]^+,$$

so that, in this case, the bond power exchange option becomes a standard call option on an interest rate.

In fact, we can use bond power exchange options to derive formulas for interest rate caps and floors.[4] In the interest rate swaps market, the relevant zero curve is the LIBOR swap curve and the convention is to use simple interest. With this in mind we define the time t forward LIBOR rate over the time interval $[T_{j-1}, T_j]$, $0 \leq t \leq T_{j-1} \leq T_j$ as a forward rate with compounding parameter η in (16) equal to $\frac{1}{T_j - T_{j-1}}$,

$$L(t, T_{j-1}, T_j) = \frac{1}{T_j - T_{j-1}}\left[\frac{P(t, T_{j-1})}{P(t, T_j)} - 1\right].$$

For $t = T_{j-1}$, we have

$$L(T_{j-1}, T_{j-1}, T_j) = \frac{1}{T_j - T_{j-1}} \Big[P(T_{j-1}, T_j)^{-1} - 1 \Big]. \tag{18}$$

Consider a caplet with payoff at time T_j given by

$$\mathbf{Cpl}_{T_{j-1},T_j}(T_j) = (T_j - T_{j-1})[L(T_{j-1}, T_{j-1}, T_j) - K]^+$$

As the LIBOR rate is determined at time T_{j-1}, discounting with the LIBOR rate from T_j to T_{j-1} we have

$$\mathbf{Cpl}_{T_{j-1},T_j}(T_{j-1}) = (T_j - T_{j-1}) \left[\frac{L(T_{j-1}, T_{j-1}, T_j) - K}{1 + (T_j - T_{j-1}) L(T_{j-1}, T_{j-1}, T_j)} \right]^+$$

that, using (18), can be rewritten as

$$\mathbf{Cpl}_{T_{j-1},T_j}(T_{j-1}) = [1 + (T_j - T_{j-1})K] \left[\frac{1}{1 + (T_j - T_{j-1})K} P(T_{j-1}, T_{j-1}) - P(T_{j-1}, T_j) \right]^+$$

Thus, the caplet is a special case of the payoff in (5) with parameter values $\alpha_1 = 1$, $\alpha_2 = 1$, $\lambda_1 = 1$, and $\lambda_2 = \big[1 + \big(T_j - T_{j-1}\big)K\big]$. So a caplet can be priced as a bond power exchange option. Specifically,

$$\mathbf{Cpl}_{T_{j-1},T_j}(t) = P\big(t, T_{j-1}\big)\Phi(d_1) - \big[1 + \big(T_j - T_{j-1}\big)K\big] P\big(t, T_j\big)\Phi(d_2).$$

A floorlet can then be priced using the put/call parity result (Proposition 2),

$$\mathbf{Frl}_{T_{j-1},T_j}(t) = \big[1 + \big(T_j - T_{j-1}\big)K\big] P\big(t, T_j\big)\Phi(-d_2) - P\big(t, T_{j-1}\big)\Phi(-d_1).$$

Formulas for prices of interest rate caps and floors can then be expressed as sums of caplets and floorlets.

6. Conclusions

Power exchange options, introduced by Blenman and Clark (2005b) for lognormally distributed underlying asset prices, feature payoff functions that nest the payoffs of several other options including standard calls and puts, power options, CUES options (Blenman and Clark (2005a)) and exchange options (Margrabe (1978)).

In this paper, we study power exchange options on zero-coupon bonds under a stochastic string model of the term-structure of interest rates. We obtain closed-form expressions for pricing and hedging bond power exchange options in the Gaussian case, and as particular instances, we obtain the corresponding expressions for standard calls and puts, power options, CUES options, and exchange options. Moreover, we state a put-call parity result and indicate sufficient conditions for the equivalence between American and European bond power exchange options. As a consequence, we obtain a new result with a sufficient condition for the equality of the price of American and European standard call options on zero-coupon bonds.

We discuss several potential applications for bond power exchange options. For an interest rate environment in which negative yields are possible, an option to extend or accelerate the maturity of a zero-coupon bond is potentially valuable. We show how to replicate such an option using bond power exchange options. Secondly, since raising a zero-coupon bond price to a power is equivalent to pricing a bond with the same yield but a different maturity, bond power exchange options can be parameterized to be options to price bonds off of a shifted term structure. Thirdly, bond power exchange options can be replicated by a portfolio of zero-coupon bonds, and it is straightforward to calculate the sensitivities of option prices to interest rate changes. Therefore it is straightforward to use bond power exchange options for bond portfolio hedging. Finally, bond power exchange options can be parameterized to be options on interest rates and rate spreads. In particular, we show that standard formulas for interest rate caplets and floorlets in a LIBOR market model can be obtained as special cases of bond power exchange options under a stochastic string term-structure model.

Although our approach in this paper is completely general, it has the limitation that the analytical treatment that leads to obtaining the results has been carried out under a Gaussian framework. Thus, a possible avenue for future research would be to consider other types of dynamics for bond prices, such as those with stochastic volatility or driven by fractional Brownian motion.

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Abbreviations

The following abbreviations are used in this manuscript:

BPE	Bond Power Exchange Option
$Conv_{BPE}$	Convexity of the bond power exchange option
CP	Call Power Option
Cpl	Caplet
CUES	Constant Underlying Elasticity in Strikes Option
Dur_{BPE}	Duration of the bond power exchange option
EAM	Extendable/Accelerable Maturity zero-coupon bond
Frl	Floorlet
HJM	Heath, Jarrow, and Morton Model
STS	Option to price a zero-coupon bond off of a Shifted Term-Structure
TSIR	Term Structure of Interest Rates

Notes

1. By admissibility we mean that for each t, $c(t, \cdot, \cdot)$ is symmetric, positive semidefinite and satisfies $|c(t, x, y)| \leq 1$ and $c(t, x, x) = 1$, $\forall x, y \geq 0$ (Santa-Clara and Sornette (2001)).
2. We refer the reader to Nualart (2006) for the general theory of Malliavin calculus and to Bueno-Guerrero et al. (2017) for the specific issues related to stochastic strings.
3. The duration of an interest rate sensitive asset is $-\frac{(1+\frac{y}{2})}{P}\frac{\partial P}{\partial y}$ where y is the rate or yield to maturity to which it is exposed. For assets or portfolios of assets with exposures to multiple rates in a term structure, duration is a measure of price sensitivity to a parallel shift of the term structure.
4. Bueno-Guerrero et al. (2020) study the valuation of caplets, caps, and swaptions under a stochastic string model, however they do not consider caplet pricing as an application of bond power exchange options.

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