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The Convergence Rate of Option Prices in Trinomial Trees

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Abstract: We study the convergence of the binomial, trinomial, and more generally m-nomial tree schemes when evaluating certain European path-independent options in the Black–Scholes setting. To our knowledge, the results here are the first for trinomial trees. Our main result provides formulae for the coefficients of $1/\sqrt{n}$ and $1/n$ in the expansion of the error for digital and standard put and call options. This result is obtained from an Edgeworth series in the form of Kolassa–McCullagh, which we derive from a recently established Edgeworth series in the form of Esseen/Bhattacharya and Rao for triangular arrays of random variables. We apply our result to the most popular trinomial trees and provide numerical illustrations.

Keywords: option pricing; trinomial tree; asymptotic expansion; Edgeworth series

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1. Introduction

In this article, we assume that the stock price S_t follows the Black–Scholes model, that is,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ is the drift and σ is the volatility. In order to price options, the martingale measure is introduced under which μ is replaced by the risk-free interest rate r. After the Black–Scholes model was introduced, the binomial model appeared and it is shown by Cox and Rubinstein (1985) that the European call and put option price P_n, calculated by their binomial model, converges to the Black–Scholes price P_{BS} as the number of periods (or time steps) $n \to \infty$. Later (see the literature discussion below), scholars studied the rate of the convergence of P_n to P_{BS} and found that for certain binomial models there was a bounded coefficient C_n, such that $P_n = P_{BS} + C_n/n + O(n^{-3/2})$. Now, trinomial models have been studied by many authors. However, as far as we are aware, there are no similar results for trinomial prices. The main objective of this paper is to fill this gap. In fact, our study comprises general self-similar m-nomial models, that is, at any positive time step, the stock price changes to one of m prices at the next period, where the mechanism and probabilities of these changes are independent of the value of the stock and the time of the change may depend on n, which is the number of periods. The trinomial case corresponds to $m = 3$. m-nomial models, where $m > 3$, have been rarely used, but it turned out that our results were as easily proved for general m as for $m = 3$. The trees we study are recombining, as models which are not recombining are not interesting from the computational point of view. As far as we know, the models we study include all self-similar binomial and trinomial models studied in the literature, except the somewhat pathological cases, where the convergence of call and put option prices occurs at a speed of only $1/\sqrt{n}$.

In this paper, under general conditions which ensure that the moments of the stock price in the m-nomial model behave like the moments in the Black–Scholes model, we demonstrate in Theorem 1 that the price P_n of a European put in the m-nomial model satisfies the relation

$$P_n = P_{BS} + \frac{C_n}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

where P_{BS} is the Black–Scholes price and C_n is a bounded sequence which we determine explicitly. A similar result for calls is obtained by using a put-call parity. Of course, C_n depends on the particular model being used. For digital options, we obtain an analogous formula, where, however, the coefficient of $1/\sqrt{n}$ is not zero. Note that we obtain the results of Chang and Palmer (2007) as special cases.

The proof of Theorem 1 uses an Edgeworth series. Bock and Korn (2016) were the first to extend the Edgeworth series to triangular arrays of random variables, and they applied their results to binomial models. Here, we demonstrate that their analysis can also be applied to m-nomial models. The form of Edgeworth series used by Bock and Korn was inspired by Bhattacharya and Rao (2010) and Esseen (1945). We are able to simplify their expansion using the ideas of Kolassa and McCullagh (1990).

In the case of European path-independent options, the following results have been obtained for binomial trees. If the price P_n of an option calculated in a tree model with n time steps converges to the Black–Scholes price P_{BS} of the same option, we say that there exists an asymptotic expansion of the error $P_n - P_{BS}$ of order $O(n^{-(i_0+1)/2})$ in the powers of $1/\sqrt{n}$ if there exist bounded coefficients $c_{n,k}$, such that

$$P_n = P_{BS} + \sum_{k=1}^{i_0} c_{n,k} n^{-k/2} + O(n^{-(i_0+1)/2}) \tag{1}$$

for some integer $i_0 > 1$. Using Skorokhod embedding, Walsh (2003) found an explicit expansion of the error of order $O\left(n^{-3/2}\right)$ for European path-independent options subject to a general class of payoff functions, but in the specific case of where the discounted process satisfies the Cox–Ross–Rubinstein (CRR) scheme. Diener and Diener (2004) used an integral expression for the price of call options in a general class of binomial models to demonstrate how an expansion in the powers of $1/\sqrt{n}$ of the price of call options can be obtained up to an arbitrary order of $n^{-(i_0+1)/2}$ using a Computer Algebra System (CAS) such as Maple. Chang and Palmer (2007) introduced a general class of binomial models with an additional drift parameter λ_n that smooths the convergence of the option prices. Using a result by Uspensky (1937), they provided an explicit formula for the coefficient of $1/\sqrt{n}$ and $1/n$ in the expansion of the error for digital call and call options. In Joshi (2009a), Joshi showed that when n is odd and the terminal layer of the tree is centered around the strike, the coefficients $c_{n,k}$ in the expansion (1) of European call and put options are independent of n. In a follow-up paper, Joshi (2010) constructed binomial trees with an arbitrarily fast convergence for vanilla European options. Korn and Müller (2013) found an expression for the optimal drift λ_n in Chang and Palmer's general class of binomial models. Using a localization of the error, Leduc (2013) found an explicit expansion of the error of order $O\left(n^{-3/2}\right)$ in the case of general payoff functions for the general class of binomial models introduced by Chang and Palmer. Using the expansion in Diener and Diener (2004), Leduc (2016) showed how the drift λ_n can be chosen to reach an arbitrarily fast convergence in Chang and Palmer's model. Bock and Korn (2016) developed a formula for the Edgeworth expansion of the cumulative distribution of

$$S_n = \frac{1}{\sqrt{n}}(X_{n,1} + \ldots + X_{n,n} - n\mu_n)$$

for independent and identically distributed $\mathbb{Z}^d$-valued triangular arrays $X_{n,1}, \ldots, X_{n,n}$, $n = 1, 2, \ldots$, of random variables with mean μ_n, and they used it to improve the convergence of option prices. Using the expansion in Leduc (2013), Leduc and Nurkanovic Hot (2020) found an explicit expression for the coefficient of $1/n$ (the coefficient of $1/\sqrt{n}$ being zero) in the expansion in powers of $1/\sqrt{n}$ for the price of a European put option calculated using a two-parameter non self-similar split tree introduced by Joshi (2009b), which was designed to improve the convergence for the American put option.

To further motivate this paper, let us mention that, due to their simplicity and flexibility, binomial and multinomial trees are used in the pricing of a broad class of options such as barrier options (Appolloni et al. 2014; Leduc and Palmer 2020; Lin and Palmer 2013), lookback options (Grosse-Erdmann and Heuwelyckx 2016; Heuwelyckx 2014; Leduc and Palmer 2019), Asian options (Gambaro et al. 2020; Hsu and Lyuu 2011; Klassen 2001), Parisian and ParAsian options (Gaudenzi and Zanette 2017). Binomial and multinomial trees are also used for pricing options in several models, such as Levy models (Maller et al. 2006), stochastic volatility models (Akyıldırım et al. 2014) and regime switching models (Leduc and Zeng 2017; Liu 2010). In Muroi (2020), the discrete Malliavin calculus is developed for option sensitivity with binomial tree models, and spectral binomial trees are used to price double barrier options. A discrete cosine transform approach for the binomial tree is developed in (Muroi and Suda 2022). In spite of (Lamberton 1998, 2002, 2020; Leisen 1998; Liang et al. 2007), it remains an open problem to establish a sharp convergence speed for the price of the American put options evaluated in a binomial tree approximation of the Black–Scholes model. In Li and Zhang (2018), it is pointed out that "a particularly interesting work would be to provide an error analysis for numerical methods based on trees for general diffusion models". The techniques developed for vanilla options in the Black–Scholes model have often been central in papers dealing with more complex options, and we believe that Edgeworth price expansions based on moments and cumulants, such as the one developed in this paper, will find applications in a future work dealing with more complex options.

Now, we summarize the contents of the paper. In Section 2, we define m-nomial models and find an expression for the m-nomial prices of put options in terms of the prices of two digital put options using a change of numeraire. In Section 3, we state our main theorem, Theorem 1, providing the coefficients of $1/\sqrt{n}$ and $1/n$ in the expansion (1) for digital put and standard put options. Next, in Section 4, we verify that our result coincides with the result of Chang and Palmer (2007) for binomial trees. Then, we use Theorem 1 to find expressions for the coefficients of $1/\sqrt{n}$ and $1/n$ in the expansion of the error in four well-known trinomial models, and we run some simulations to support our result numerically. In Section 5, we prove Theorem 1. The proof is based on a theorem (Theorem 2) for the expansion of the m-nomial prices of digital put options. Theorem 2 is proved in Section 6. Theorem 2 follows in turn from Theorem 3, which is proved in Section 7. Theorem 3 derives from the Edgeworth series in the form of Kolassa and McCullagh (1990), using the result of Korn and Müller (2013). The proof of Theorem 3 depends on the technical results that we put in the Appendix A.

2. *M*-Nomial Models

First, we define what we mean by an m-nomial model.

Definition 1. *Given initial stock price $S_0 > 0$ and maturity $T > 0$, we say that $S_t^{(n)}$, $t = t_{n,k} = kT/n = k\Delta t$, $k = 0, 1, \dots, n$ with $S_0^{(n)} = S_0$ is an n-period m-nomial model if at time $t_{n,k+1}$, the price $S_{t_{n,k+1}}^{(n)}$ can take any of the m values $S_{t_{n,k}}^{(n)} u_{n,i}$ with*

$$u_{n,i} = e^{(-\Lambda_n + (i-1)\Delta_n)\sqrt{\Delta t}}, \quad i = 1, \dots, m,$$

where $\Lambda_n, \Delta_n > 0$. The condition $\Lambda_n > 0$ implies that $u_{n,1} < 1$. We further assume that $-\Lambda_n + (m-1)\Delta_n > 0$ so that $u_{n,m} > 1$. The probabilities $p_{n,i} > 0$ that $S_{t_{n,k+1}}^{(n)} = S_{t_{n,k}}^{(n)} u_{n,i}$ satisfy $p_{n,1} + \dots + p_{n,m} = 1$. We denote such a model by $(\Lambda_n, \Delta_n, p_n)$ where $p_n = [p_{n,1}, \cdots, p_{n,m}]$. The model is risk neutral if

$$\sum_{i=1}^{m} p_{n,i} u_{n,i} = e^{r\Delta t}. \tag{2}$$

Note that $u_{n,i+1}/u_{n,i} = e^{\Delta_n\sqrt{\Delta t}}$ does not depend on i and this ensures that the tree is recombining. This is obviously satisfied when $m = 2$, that is, for binomial models, and it seems to be satisfied for all self-similar trinomial models $m = 3$ in the literature. Note that once the $u_{n,i}$ are determined, there may not exist probabilities $p_{n,i} > 0$ such that $\sum_{i=1}^{m} p_{n,i}u_{n,i} = e^{r\Delta t}$ and if they do exist, they may not be unique (except in the binomial case). For example, in the trinomial case $m = 3$, such probabilities exist if and only if $u_{n1} < e^{r\Delta t} < u_{n3}$ and p_{n3} can be chosen as any number in the interval

$$\left(\max\left\{0, \frac{e^{r\Delta t} - u_{n2}}{u_{n3} - u_{n2}}\right\}, \frac{e^{r\Delta t} - u_{n1}}{u_{n3} - u_{n2}}\right)$$

with

$$p_{n1} = \frac{u_{n2} - e^{r\Delta t} + (u_{n3} - u_{n2})p_{n3}}{u_{n2} - u_{n1}}, \quad p_{n2} = \frac{e^{r\Delta t} - u_{n1} - (u_{n3} - u_{n1})p_{n3}}{u_{n2} - u_{n1}}.$$

When the probabilities $p_{n,i}$ are risk neutral, there are no arbitrage opportunities in the m-nomial model. However, except in the binomial case, the risk-neutral probabilities are not unique and an option does not have a uniquely defined price. In any case, relative to a given set of probabilities, we take the price of an option with payoff $f(S_T^{(n)})$, where $S_T^{(n)}$ is the terminal stock price, to be $e^{-rT}E(f(S_T^{(n)}))$, where the expectation is with respect to the given probabilities, and we do this even when the probabilities are not risk neutral.

In the risk neutral case, put–call parity holds with the above definition of the price since the payoff to a long call and a short put with exercise price K and maturity T is $S_T^{(n)} - K$ and, under risk neutrality, $e^{-rT}E(S_T^{(n)} - K) = S_0 - Ke^{-rT}$. Thus, we can derive the price of a call option from that of a put option.

Next, we demonstrate that when the probabilities $p_{n,i}$ are risk neutral, the formula for the m-nomial price of a put option can be written as a combination of the formulas for two digital put options. In the Black–Scholes world with initial stock price S_0, volatility σ and interest rate r, the price of a put option with strike K and maturity T is given by

$$P_{BS} = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}. \tag{3}$$

Here, $\Phi(-d_2)$ is the probability that $S_T \leq K$ is under the risk neutral measure so that $e^{-rT}\Phi(-d_2)$ is the price of a digital put with strike K. $\Phi(-d_1)$ is the probability that $S_T \leq K$ under the stock measure and similarly $e^{-rT}\Phi(-d_1)$ is the price of a digital put with strike K under the stock measure. We show that a similar result holds for a risk-neutral m-nomial model. Our argument adapts Cox and Rubinstein's argument in (Cox and Rubinstein 1985) for the binomial model.

The possible values of the terminal stock price $S_T^{(n)}$ are

$$S_0u_{n,k_1}\cdots u_{n,k_n} = S_0\lambda_n^n\delta_n^{k_1+\cdots+k_n-n},$$

where $\lambda_n = e^{-\Lambda_n\sqrt{\Delta t}}$, $\delta_n = e^{\Delta_n\sqrt{\Delta t}}$ and $1 \leq k_i \leq m$. Thus, the terminal stock prices are

$$S_0\lambda_n^n\delta_n^k = S_0e^{(-n\Lambda_n + k\Delta_n)\sqrt{\Delta t}}, \tag{4}$$

where $0 \le k \le n(m-1)$. Let $\Pi_{n,k}$ be the probability of reaching the price $S_0\lambda_n^n\delta_n^k$. Then, the price of a put option with strike K is

$$P_n = e^{-rT}\sum_{k=0}^{n(m-1)}\Pi_{n,k}\max\{K - S_0\lambda_n^n\delta_n^k, 0\} = e^{-rT}\sum_{k=0}^{\ell_n}\Pi_{n,k}(K - S_0\lambda_n^n\delta_n^k),$$

where ℓ_n is such that

$$S_0\lambda_n^n\delta_n^{\ell_n} \le K < S_0\lambda_n^n\delta_n^{\ell_n+1}.$$

Then

$$P_n = Ke^{-rT}\sum_{k=0}^{\ell_n}\Pi_{n,k} - S_0\lambda_n^n e^{-rT}\sum_{k=0}^{\ell_n}\Pi_{n,k}\delta_n^k.$$

Now

$$\Pi_{n,k} = \sum_{k_1+\cdots+k_n=k+n} p_{n,k_1}\cdots p_{n,k_n},$$

where $1 \le k_i \le m$. Then

$$\lambda_n^n e^{-rT}\Pi_{n,k}\delta_n^k = \sum_{k_1+\cdots+k_n=k+n} q_{n,k_1}\cdots q_{n,k_n},$$

where

$$q_{n,i} = e^{-r\Delta t}\lambda_n\delta_n^{i-1}p_{n,i} = e^{-r\Delta t}u_{n,i}p_{n,i}, \quad i = 1,\ldots,m. \tag{5}$$

Note that

$$\sum_{i=1}^m q_{n,i} = e^{-r\Delta t}\sum_{i=1}^m u_{n,i}p_{n,i} = 1 \quad \text{by risk neutrality (see (2))},$$

and that

$$\sum_{i=1}^m q_{n,i}\frac{e^{r\Delta t}}{S_0 u_{n,i}} = \sum_{i=1}^m \frac{p_{n,i}}{S_0} = \frac{1}{S_0}$$

so that the $q_{n,i}$'s can be thought of as the stock measure corresponding to the risk neutral measure defined by the p_{ni}, that is under the probability measure defined by $q_n = [q_{n,1},\cdots,q_{n,m}]$ we have $E\big(e^{r\Delta t}/S_{\Delta t}\big) = 1/S_0$. Then

$$P_n = Ke^{-rT}\sum_{k=0}^{\ell_n}\Pi_{n,k} - S_0\sum_{k=0}^{\ell_n}Q_{n,k}, \tag{6}$$

where

$$Q_{n,k} = \sum_{k_1+\cdots+k_n=k+n} q_{n,k_1}\cdots q_{n,k_n}, \quad 1 \le k_i \le m.$$

Note that $\sum_{k=0}^{\ell_n}\Pi_{n,k}$ is the probability of arriving at a stock price $\le K$ under the risk neutral probabilities $p_{n,i}$, whereas $\sum_{k=0}^{\ell_n}Q_{n,k}$ is the probability of arriving at a stock price $\le K$ under the probabilities $q_{n,i}$. Thus, the problem of pricing a put option is reduced to pricing two digital put options.

3. The Main Theorem

Now, we state our main theorem. We assume we are in the Black–Scholes world with an initial stock price S_0, volatility σ and interest rate r, and that the options under consideration have maturity T. Then, we require the following hypotheses on our m-nomial model $S_t^{(n)}$ with parameters $(\Lambda_n, \Delta_n, p_n)$.

(H1): Λ_n is bounded, Δ_n has a positive (>0) limit and

$$\inf_n p_{n,i} > 0, \quad i = 1,\ldots,m.$$

(H2): Define X_n as the random variable $\log\left(S_{\Delta t}^{(n)}/S_0\right)/\sqrt{\Delta t}$ which takes the value $x_{n,i} = -\Lambda_n + (i-1)\Delta_n$ with probability $p_{n,i}$, $i = 1, \ldots, m$. Note that X_n is bounded. Then, we assume

$$
\begin{aligned}
E(X_n) &= \theta\sqrt{\Delta t} + D_n(\Delta t)^{3/2} + O((\Delta t)^2), \\
E(X_n^2) &= \sigma^2 + F_n\Delta t + O((\Delta t)^{3/2}), \\
E(X_n^3) &= G_n\sqrt{\Delta t} + O(\Delta t), \\
E(X_n^4) &= H_n + O(\sqrt{\Delta t}),
\end{aligned}
$$

where $\theta = r - \sigma^2/2$, D_n, F_n, G_n, H_n are bounded functions of n, and where we observe that $E(X_n^k) = \sum_{i=1}^m p_{n,i}x_{n,i}^k$.

Remark 1. *Note that in our m-nomial model, we want $S_{\Delta t}^{(n)}$ to be an approximation to $S_{\Delta t}$, where S_t is the stock price under the Black–Scholes model, when the stock price at time 0 is S_0, the interest rate is r and the volatility is σ. Thus, we want X_n to be an approximation to $Y = \log(S_{\Delta t}/S_0)/\sqrt{\Delta t}$, which is normally distributed with mean $\theta\sqrt{\Delta t}$ and variance σ^2. Then,*

$$
\begin{aligned}
E(Y) &= \theta\sqrt{\Delta t}, \\
E(Y^2) &= \sigma^2 + \theta^2\Delta t, \\
E(Y^3) &= 3\sigma^2\theta\sqrt{\Delta t} + \theta^3(\Delta t)^{3/2}, \\
E(Y^4) &= 3\sigma^4 + 6\theta^2\sigma^2\Delta t + \theta^4(\Delta t)^2.
\end{aligned}
$$

Under (H1), it can be demonstrated that (H2) is equivalent to the condition that all moments of the terminal stock price $S_T^{(n)}$ in the m-nomial model converge at a rate of $1/n$ to the corresponding moments of the terminal stock price S_T in the Black–Scholes model.

Now, we state the main theorem.

Theorem 1. *Suppose $S_t^{(n)}$ is an n-period m-nomial model with parameters $(\Lambda_n, \Delta_n, p_n)$, time steps $t_k^n = k\Delta t = kT/n$ and initial stock price S_0 for which (H1) and (H2) hold. We define the price of an option in this model with payoff $f(S_T^{(n)})$ to be $e^{-rT}E(f(S_T^{(n)}))$, where the expectation is taken with respect to the measure defined by the probabilities $p_{n,i}$. Then,*

(i) the price $P_d(n)$ of a digital put option with strike K and maturity T in the n-period m-nomial model satisfies

$$
P_d(n) = P_{BS} + e^{-rT}\phi(d_2)\left[\frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{d_2\bar{\Delta}_n^2}{2n} - \frac{B_n}{n}\right] + O\left(\frac{1}{n^{3/2}}\right),
$$

where P_{BS} is the Black–Scholes price, $\phi(\cdot)$ is the standard normal density function,

$$
\bar{\Delta}_n = \frac{\Delta_n(1 - 2\,\mathrm{frac}(a_n))}{2\sigma}, \quad a_n = \frac{\log(K/S_0) + n\Lambda_n\sqrt{\Delta t}}{\Delta_n\sqrt{\Delta t}},
$$

and

$$
\begin{aligned}
B_n \quad &= \frac{d_2Tr^2}{2\sigma^2} + \frac{(1 - d_1d_2)\sqrt{T}r}{2\sigma} + \frac{d_1^2d_2 - 2d_1 - d_2}{8} + \frac{d_2\Delta_n^2}{24\sigma^2} \\
&\quad + \frac{T^{3/2}D_n}{\sigma} - \frac{d_2TF_n}{2\sigma^2} + \frac{(d_2^2 - 1)\sqrt{T}G_n}{6\sigma^3} + \frac{(3d_2 - d_2^3)H_n}{24\sigma^4}.
\end{aligned}
$$

(ii) If, in addition, the model is risk-neutral, the price $P(n)$ of a put option with strike K and maturity T in the n-period m-nomial model satisfies

$$P(n) = P_{BS} + \sigma\sqrt{T}S_0\phi(d_1)\left[-\frac{\bar{\Delta}_n^2}{2n} + \frac{C_n}{n}\right] + O\left(\frac{1}{n^{3/2}}\right),$$

where P_{BS} is the Black–Scholes price and

$$\begin{aligned} C_n &= -\frac{Tr^2}{2\sigma^2} + \frac{d_2\sqrt{T}r}{2\sigma} + \frac{1-d_1d_2}{8} + \frac{\Delta_n^2}{24\sigma^2} + \frac{TF_n}{2\sigma^2} \\ &+ \frac{(d_1-2d_2)\sqrt{T}G_n}{6\sigma^3} + \frac{(d_1^2-3d_1d_2+3d_2^2-1)H_n}{24\sigma^4}. \end{aligned}$$

The price of a call option satisfies the same equation.

Remark 2. *Note that $K = S_0e^{(-n\Lambda_n+a_n\Delta_n)\sqrt{\Delta t}}$, so that*

$$S_0e^{(-n\Lambda_n+k_n\Delta_n)\sqrt{\Delta t}} \le K < S_0e^{(-n\Lambda_n+(k_n+1)\Delta_n)\sqrt{\Delta t}},$$

where $k_n = \text{floor}(a_n)$. Thus, a_n describes the position of the strike relative to the terminal stock prices (see (4)). Since, in general, $\text{frac}(a_n)$ oscillates between 0 and 1, the quantity $\bar{\Delta}_n$ also oscillates.

Remark 3. *The risk neutral condition in (ii) is not needed. Under the hypotheses of the theorem, we find that*

$$\sum_{i=1}^{m} p_{n,i}u_{n,i} = e^{r\Delta t} + \gamma_n(\Delta t)^2 + O((\Delta t)^{5/2}),$$

where $\gamma_n = D_n + F_n/2 + G_n/6 + H_n/24 - r^2/2$. Then, it turns out that, for the put price, (ii) still holds if we add $-\gamma_n T^{3/2}\Phi(-d_1)/(\sigma\phi(d_1))$ to the C_n, which would be obtained if the model were risk neutral. For the call price, an additional term $\gamma_n T^{3/2}/(\sigma\phi(d_1))$ has to be added to C_n.

4. Verification of the Result

We test our result in two ways. First, we check that in the case of the flexible binomial model of Chang and Palmer (2007), (which includes the CRR model as a special case), our results reduce to those in Chang and Palmer. Then, we apply Theorem 1 to four trinomial models and test our result numerically on these trinomial models.

4.1. Comparison with Chang and Palmer's Binomial Model

First, we compare our results with the binomial model in Chang and Palmer, where in their notation,

$$u = e^{\sigma\sqrt{\Delta t}+\lambda_n\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t}+\lambda_n\sigma^2\Delta t}, \quad p = \frac{e^{r\Delta t}-d}{u-d}.$$

The authors considered digital calls but, by modifying their proof, we can demonstrate that the binomial price of a digital put satisfies

$$P_d(n) = e^{-rT}\Phi(-d_2) + e^{-rT}\phi(d_2)\left[\frac{\tilde{\Delta}_n}{\sqrt{n}} + \frac{d_2\tilde{\Delta}_n^2}{2n} - \frac{A_n}{n}\right] + O(n^{-3/2}),$$

where

$$\tilde{\Delta}_n = 1 - 2\,\text{frac}(a_n), \quad a_n = \frac{\log(K/S_0) - n\log(d)}{\log(u/d)} \tag{7}$$

and A_n is

$$\frac{Td_1}{2\sigma^2}(r-\lambda_n\sigma^2)^2 + \frac{(2-d_1d_2-d_1^2)\sqrt{T}}{6\sigma}(r-\lambda_n\sigma^2) + \frac{d_1^3+d_1d_2^2+2d_2-4d_1}{24}.$$

In this risk neutral model, (H1) and (H2) are satisfied with $m = 2$ and

$$\Delta_n = 2\sigma, \quad \Lambda_n = \sigma - \lambda_n \sigma^2 \sqrt{\Delta t}, \quad p_{n,1} = \frac{u - e^{r\Delta t}}{u - d}, \quad p_{n,2} = 1 - p_{n,1},$$
$$D_n = 2\sigma\beta_n,\ F_n = \lambda_n\sigma^2(2\theta - \lambda_n\sigma^2),\ G_n = \sigma^2(\theta + 2\lambda_n\sigma^2),\ H_n = \sigma^4,$$

where

$$\theta = r - \sigma^2/2, \quad \beta_n = \frac{\sigma^4(4\lambda_n + 1) - 4\sigma^2 r + 12(r - \lambda_n\sigma^2)^2}{48\sigma}.$$

Applying Theorem 1, we observe first that $\bar{\Delta}_n = \dfrac{\Delta_n(1 - 2\,\mathrm{frac}(a_n))}{2\sigma} = \tilde{\Delta}_n$. Some algebra yields

$$\begin{aligned}
B_n \quad &= \frac{d_2}{2\sigma^2}Tr^2 + \frac{1 - d_1 d_2}{2\sigma}\sqrt{T}r + \frac{d_1^2 d_2 - 2d_1 - d_2}{8} + \frac{d_2}{6} + 2\beta_n T^{3/2} \\
&\qquad - \frac{d_2 T}{2}\lambda_n(2\theta - \lambda_n\sigma^2) + \frac{3d_2 - d_2^3}{24} + \frac{(d_2^2 - 1)\sqrt{T}(\theta + 2\lambda_n\sigma^2)}{6\sigma} \\
&= \frac{d_1}{2\sigma^2}Tr^2 + \left(\frac{1 - d_1 d_2 - d_1^2}{6\sigma} - \lambda_n d_1\sqrt{T}\right)\sqrt{T}r + \frac{\sigma^2 T}{2}d_1\lambda_n^2 \\
&\qquad + \frac{\sigma\sqrt{T}}{6}(d_1^2 + d_1 d_2 - 2)\lambda_n + \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24}.
\end{aligned}$$

This coincides with A_n in the Chang–Palmer result. Thus, in the case of digital puts, Theorem 1 gives a result consistent with that of Chang and Palmer.

For the Chang–Palmer model, applying put–call parity to the Chang–Palmer result for calls, we find that after some rearrangement, the price of a put option satisfies

$$P(n) = P_{BS} + S_0\sigma\sqrt{T}\phi(d_1)\left(-\frac{\tilde{\Delta}_n^2}{2n} + \frac{A_n}{n}\right) + O(n^{-3/2}),$$

where $\tilde{\Delta}_n$ is as in (7) and

$$A_n = -\frac{T(r - \lambda_n\sigma^2)^2}{2\sigma^2} + \frac{d_1 + d_2}{6\sigma}\sqrt{T}(r - \lambda_n\sigma^2) + \frac{6 - d_1^2 - d_2^2}{24}.$$

Now, from Theorem 1, since, as above, $\bar{\Delta}_n = \tilde{\Delta}_n$,

$$P(n) = P_{BS} + \sigma\sqrt{T}S_0\phi(d_1)\left[-\frac{\tilde{\Delta}_n^2}{2n} + \frac{C_n}{n}\right] + O\left(n^{-3/2}\right),$$

where, again after some algebra, we find that

$$\begin{aligned}
C_n &= -\frac{Tr^2}{2\sigma^2} + \frac{d_2\sqrt{T}r}{2\sigma} + \frac{1 - d_1 d_2}{8} + \frac{\Delta_n^2}{24\sigma^2} + \frac{TF_n}{2\sigma^2} \\
&\qquad + \frac{(d_1 - 2d_2)\sqrt{T}G_n}{6\sigma^3} + \frac{(d_1^2 - 3d_1 d_2 + 3d_2^2 - 1)H_n}{24\sigma^4} \\
&= -\frac{Tr^2}{2\sigma^2} + \left(\frac{d_1 + d_2}{6\sigma} + \lambda_n\sqrt{T}\right)\sqrt{T}r - \frac{\sigma^2 T}{2}\lambda_n^2 - \frac{d_1^2 - d_2^2}{6}\lambda_n + \frac{2 - d_1^2 - d_2^2}{24}.
\end{aligned}$$

Again, this coincides with A_n in the Chang–Palmer result. Thus, also in the case of puts, Theorem 1 gives a result consistent with that of Chang and Palmer.

4.2. Application of Theorem 1 to Trinomial Models

Next, we calculate B_n and C_n in Theorem 1 for five trinomial (that is, $m = 3$) models, the first four of which are risk-neutral. These models satisfy (H1) and (H2). In fact, D_n, F_n,

G_n, H_n are constants and Δ_n = constant + $O(\sqrt{\Delta t})$, all of which have the consequence that B_n and C_n are constants. In all these models, we write

$$u_{n,1} = d,\ u_{n,2} = m,\ u_{n,3} = u,\ p_{n,1} = p_d,\ p_{n,2} = p_m,\ p_{n,3} = p_u.$$

(1) First, we study Tian's (1993) equal probability tree, where

$$p_u = p_m = p_d = 1/3, \quad m = M(3-V)/2,\ u = X + \sqrt{X^2 - m^2},\ d = X - \sqrt{X^2 - m^2},$$

with $M = e^{r\Delta t}$, $V = e^{\sigma^2 \Delta t}$ and $X = M(V+3)/4$. For this model, we find that

$$\Delta_n = \frac{\sqrt{3}\sigma}{\sqrt{2}} + O(\sqrt{\Delta t}),\ D_n = -\frac{3\sigma^4}{8},\ F_n = \theta^2 + \frac{7\sigma^4}{8},\ G_n = 3\sigma^2\theta,\ H_n = \frac{3\sigma^4}{2},$$

where $\theta = r - \sigma^2/2$ and hence that

$$B_n = -\frac{6d_1^3 - 11d_1^2 d_2 + 4d_1 d_2^2 + 2d_2}{16},\ C_n = \frac{6d_1^2 - 11d_1 d_2 + 4d_2^2 + 2}{16}.$$

(2) Next, we study Tian's (1993) fourth-order moment matching model, where the first four moments of $S_{\Delta t}^{(n)}$ match the moments of $S_{\Delta t}$ in the Black–Scholes model. In this model,

$$m = MV^2, \quad u = X - \sqrt{X^2 + m^2}, \quad d = X - \sqrt{X^2 - m^2},$$
$$p_d = \frac{um - M(u+m) + M^2 V}{(u-d)(m-d)}, \quad p_u = \frac{md - M(m+d) + M^2 V}{(u-d)(u-m)},$$

where $M = e^{r\Delta t}$, $V = e^{\sigma^2 \Delta t}$ and $X = M(V^4 + V^3)/2$. We find that

$$\Delta_n = \sqrt{3}\sigma + O(\sqrt{\Delta t}),\ D_n = 0,\ F_n = \theta^2,\ G_n = 3\sigma^2\theta,\ H_n = 3\sigma^4$$

and hence that

$$B_n = \frac{d_2}{8}, \quad C_n = \frac{1}{8}.$$

(3) Next, we study the adjusted trinomial tree (Chan et al. 2009), where the tree is centered on the strike in the log space. In this model, the probabilities p_u, p_m and p_d are defined as in the previous model but now $m = (K/S_0)^{1/n}$ and

$$X = \frac{V}{2}(MV + m) + \frac{m}{2M}(m - M).$$

We find, as in the previous model, that

$$\Delta_n = \sqrt{3}\sigma + O(\sqrt{\Delta t}),\ D_n = 0,\ F_n = \theta^2,\ G_n = 3\sigma^2\theta,\ H_n = 3\sigma^4$$

and hence that

$$B_n = \frac{d_2}{8}, \quad C_n = \frac{1}{8}.$$

(4) Next, we study Boyle's tree (Boyle 1986), with parameter λ, in which

$$d = e^{-\lambda\sigma\sqrt{\Delta t}}, \quad m = 1, \quad u = d^{-1},$$
$$p_d = \frac{u - M(1+u) + M^2 V}{(u-d)(1-d)}, \quad p_u = \frac{d - M(1+d) + M^2 V}{(u-d)(u-1)},$$

where $M = e^{r\Delta t}$ and $V = e^{\sigma^2 \Delta t}$. Then

$$\Delta_n = \lambda\sigma, \quad D_n = \frac{(\lambda^2-3)\sigma^2(\sigma^2+4r)}{12},$$
$$F_n = \theta^2 - \frac{(\lambda^2-3)\sigma^2(\sigma^2+12r)}{12}, \quad G_n = \lambda^2\sigma^2\theta, \quad H_n = \lambda^2\sigma^4$$

and hence

$$B_n = \frac{d_2}{8} + (\lambda^2-3)\left(\frac{2d_1^2 - d_1 d_2 - 1}{6\sigma}\sqrt{T}r + \frac{2d_1^3 - 5d_1^2 d_2 + 2d_1 d_2^2 + 2d_1 + 2d_2}{24}\right)$$
$$C_n = \frac{1}{8} + (\lambda^2-3)\left(\frac{d_2 - 2d_1}{6\sigma}\sqrt{T}r - \frac{2d_1^2 - 5d_1 d_2 + 2d_2^2}{24}\right).$$

(5) Finally, we study the Kamrad–Ritchken model (Kamrad and Ritchken 1991) with parameter λ, in which

$$d = e^{-\lambda\sigma\sqrt{\Delta t}}, \quad m = 1, \quad u = d^{-1}, \quad p_d = 1 - \frac{1}{\lambda^2} - \frac{\theta\sqrt{\Delta t}}{2\lambda\sigma}, \quad p_u = 1 - \frac{1}{\lambda^2} + \frac{\theta\sqrt{\Delta t}}{2\lambda\sigma}.$$

This model is not risk neutral, since

$$p_u u + p_m m + p_d d = e^{r\Delta t} + \gamma(\Delta t)^2 + O((\Delta t)^{5/2}),$$

where

$$\gamma = -\frac{1}{24}(12r^2 - 4\lambda^2\sigma^2 r + \lambda^2\sigma^4).$$

All other hypotheses of Theorem 1 are satisfied with

$$\Delta_n = \lambda\sigma, \quad D_n = F_n = 0, \quad G_n = \lambda^2\sigma^2\theta, \quad H_n = \lambda^2\sigma^4.$$

For the digital put, we find that B_n is

$$\frac{d_2 T r^2}{2\sigma^2} + \frac{1 - d_1 d_2}{2\sigma}\sqrt{T}r + \frac{d_1^2 d_2 - 2d_1 - d_2}{8} + \lambda^2\left(\frac{d_2^2 - 1}{6\sigma}\sqrt{T}r - \frac{2d_1 d_2^2 - d_2^3 - 2d_1 - 2d_2}{24}\right).$$

For the put option, it follows from Remark 3 that C_n is

$$-\frac{\gamma T^{3/2}\Phi(-d_1)}{\sigma\phi(d_1)} - \frac{Tr^2}{2\sigma^2} + \frac{d_2}{2\sigma}\sqrt{T}r + \frac{1 - d_1 d_2}{8} + \lambda^2\left(\frac{d_1 - 2d_2}{6\sigma}\sqrt{T}r - \frac{d_1^2 - 3d_1 d_2 + d_2^2}{24}\right).$$

Notice that all values coincide for models (2) and (3). When $\lambda^2 = 3$, the moments in Boyle's model match those in these models up to negligible terms, so that B_n and C_n reduce to those in models (2) and (3).

4.3. Numerical Results for the Trinomial Models

Finally, numerical results for the trinomial models described above are displayed in Figures 1–4 when $S = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. We label Tian's equal probability model as 'EqualProb', Joshi's adjusted trinomial model as 'Adjusted', Boyle's model with parameter $\lambda = 1.1$ as 'Boyle', Kamrad-Ritchken's model with parameter $\lambda = \sqrt{1.5}$ as 'KR', and Tian's fourth-order moment matching model as 'Tian4'.

For the put, the Black–Scholes price $P_{BS} = 7.900442$, and we denote by $P(n)$ the n-period price calculated in the trinomial models. For the digital put, the Black–Scholes price $P_{BS} = 0.511215$, and we denote by $P_d(n)$ the n-period price calculated in the trinomial models. Letting $n = 100, 200, \dots, 2000$, Figure 1 illustrates the convergence of $P(n)$ to P_{BS}, while Figure 2 illustrates the convergence of $P_d(n)$ to P_{BS}.

Figures 1 and 2 show that the price's convergence to the Black–Scholes limit for Joshi's adjusted model is far less oscillatory than that of the other models. This is because $m = (K/S_0)^{1/n}$, so that the strike K is the terminal node of the tree. Consequently, $\text{frac}(a_n) = 0$, because $d = e^{-\Lambda_n\sqrt{\Delta t}}$ and $m = de^{\Delta_n\sqrt{\Delta t}}$ so that $a_n = n$. It follows that the coefficients of $1/\sqrt{n}$ and $1/n$ in the expansions of the prices are constant. As a result, the convergence is smoother than it is for the other models, where oscillations triggered by $\text{frac}(a_n)$ cause the coefficients of $1/\sqrt{n}$ and $1/n$ in the price expansion to oscillate.

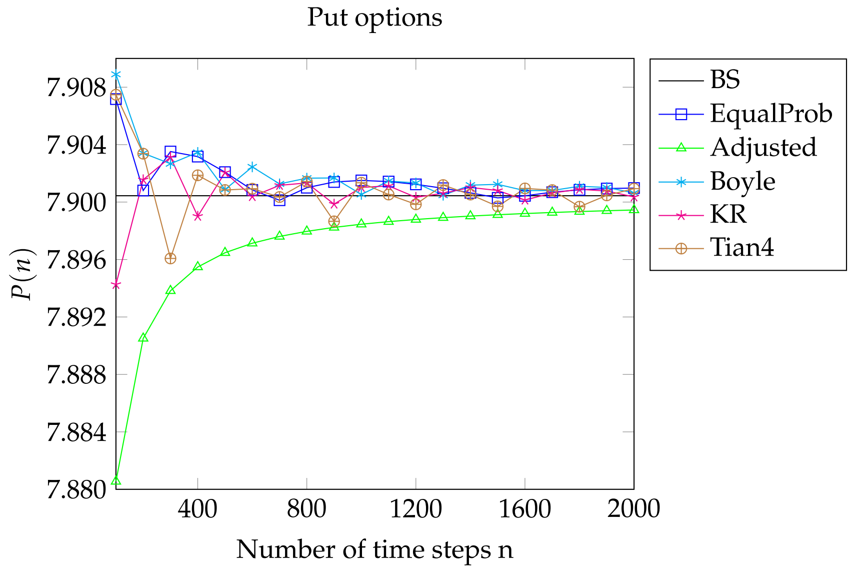

Figure 1. Here, $S = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. We plot the prices $P(n)$ of put options in various trinomial models against the Black–Scholes price $P_{BS} = 7.900442$.

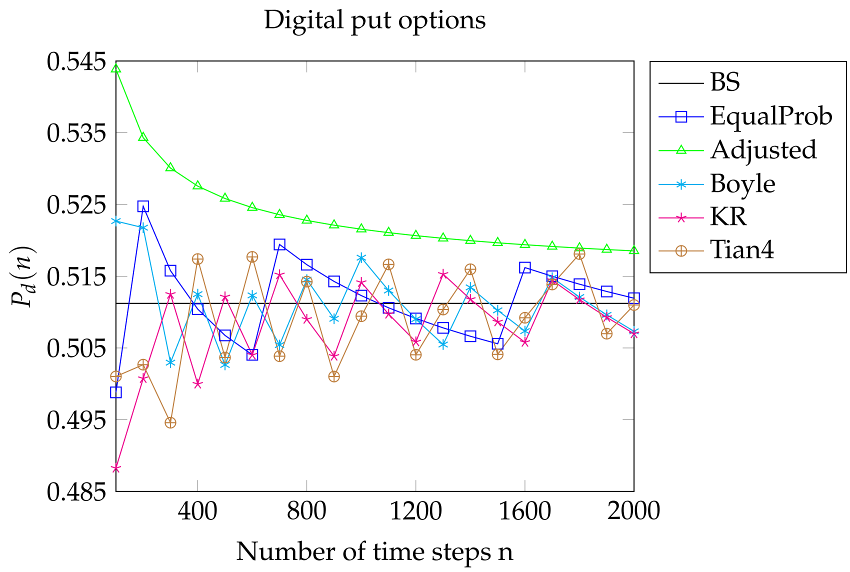

Figure 2. Here, $S = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. We plot the prices $P_d(n)$ of digital put options in various trinomial models against the Black–Scholes price $P_{BS} = 0.511215$.

For the put option, we define 'error' as

$$\text{error} = P(n) - P_{BS} - \sigma\sqrt{T}S_0\phi(d_1)\left[-\frac{\bar{\Delta}_n^2}{2n} + \frac{C_n}{n}\right].$$

For the digital put option, we set

$$\text{error} = P_d(n) - P_{BS} - e^{-rT}\phi(d_2)\left[\frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{d_2\bar{\Delta}_n^2}{2n} - \frac{B_n}{n}\right].$$

We expect $n^{1.5}\times$ error to be bounded and this seems to be the case, as illustrated in Figure 3 for the put option and in Figure 4 for the digital put option.

Figures 3 and 4 show that the values of $n^{1.5}\times$ error for Joshi's adjusted model are far less oscillatory than those for the other models. This suggests that the coefficient of $1/n^{1.5}$ in the price expansion is constant. In the case of the digital put, the coefficient appears to be 0. This could be verified using Edgeworth expansions with more coefficients. We leave this to the interested reader.

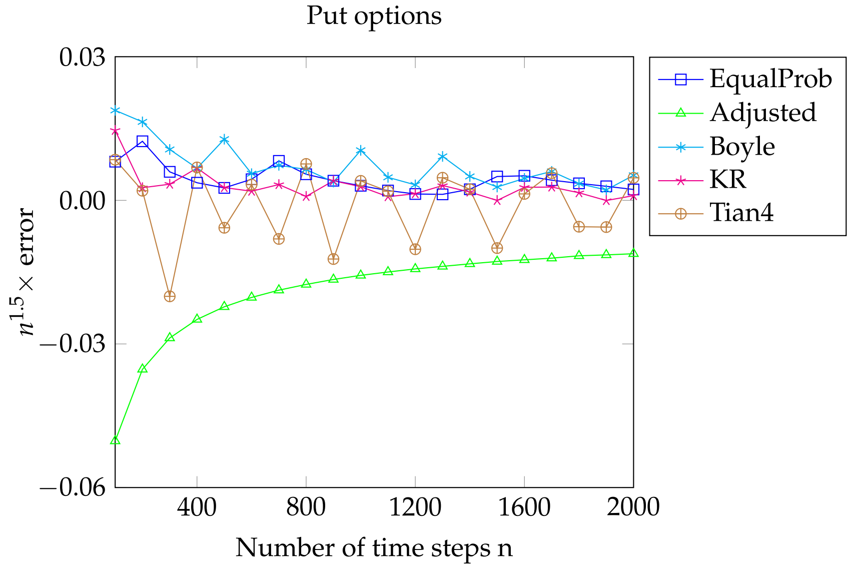

Figure 3. Here, $S = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. For the put, the Black–Scholes price $P_{BS} = 7.900442$. $P(n)$ is the n-period price calculated by various models, and error $= P(n) - P_{BS} - \sigma\sqrt{T}S_0\phi(d_1)\left[-\frac{\bar{\Delta}_n^2}{2n} + \frac{C_n}{n}\right]$. We expect $n^{1.5}\times$ error to be bounded and this appears to be the case.

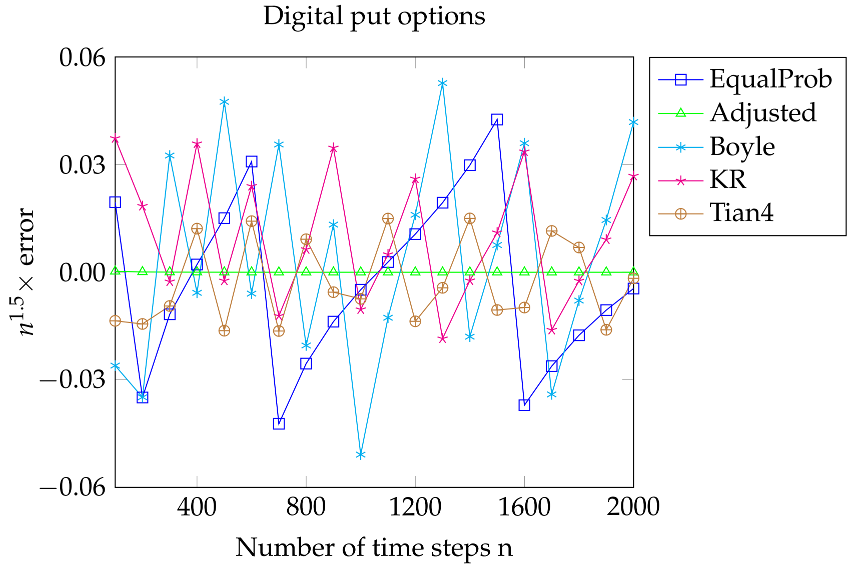

Figure 4. Here, $S = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. For the digital put, the Black–Scholes price $P_{BS} = 0.511215$. $P_d(n)$ is the n-period price calculated by various models, and error $= P_d(n) - P_{BS} - e^{-rT}\phi(d_2)\left[\frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{d_2\bar{\Delta}_n^2}{2n} - \frac{B_n}{n}\right]$. We expect $n^{1.5}\times$ error to be bounded and this again appears to be the case.

5. Proof of Theorem 1

The main tool in the proof is the following theorem, which gives an Edgeworth expansion for the cdf of the terminal stock price in an m-nomial model. We defer the proof to later. First, we define an Edgeworth expansion.

Let $\gamma_n = (\gamma_{n,2}, \gamma_{n,3}, \ldots)$, $n = 1, 2, \ldots$, be any sequence of sequences of real numbers such that $\gamma_{n,2} > 0$ for all n. Set $\sigma_n := \sqrt{\gamma_{n,2}}$. This notation is used because $\gamma_{n,2}$ usually corresponds to a variance. Then, for each integer $i_0 \geq 3$, the Edgeworth expansion $E_{i_0}(x, \gamma_n)$ is defined as

$$E_{i_0}(x, \gamma_n) = \sum_{j=0}^{3i_0-6} (-1)^j \nu_{n,j} \frac{d^j}{dx^j} \Phi\left(\frac{x}{\sigma_n}\right), \tag{8}$$

where Φ is the standard normal cdf and $\nu_n = (\nu_{n,0}, \nu_{n,1}, \ldots)$, $n = 1, 2, \ldots$, is defined by the relation

$$\sum_{j=0}^{\infty} \nu_{n,j} s^j = \exp\left(\sum_{j=3}^{i_0} \frac{\gamma_{n,j}}{j!} s^j\right). \tag{9}$$

Clearly, $\nu_{n,0} = 1$, $\nu_{n,1} = \nu_{n,2} = 0$. Note, we can also write

$$E_{i_0}(x, \gamma_n) = \Phi\left(x\sigma_n^{-1}\right) - \phi(x\sigma_n^{-1}) \sum_{j=3}^{3i_0-6} h_{j-1}(x\sigma_n^{-1}) \frac{\nu_{n,j}}{\sigma_n^j}, \tag{10}$$

where ϕ is the standard normal pdf and h_j is the jth Hermite polynomial. This is a consequence of the fact that

$$\frac{d^j}{dx^j} \Phi\left(x\sigma_n^{-1}\right) = \frac{d^{j-1}}{dx^{j-1}} \sigma_n^{-1} \phi\left(x\sigma_n^{-1}\right) = (-1)^{j-1} \sigma_n^{-j} h_{j-1}(x\sigma_n^{-1}) \phi\left(x\sigma_n^{-1}\right). \tag{11}$$

Theorem 2. *Suppose $S_t^{(n)}$ is an n-period m-nomial model with parameters $(\Lambda_n, \Delta_n, p_n)$, time steps $t_k^n = k\Delta t = kT/n$ and initial stock price S_0 for which (H1) holds and $Var(X_n) \to V > 0$. Let b_j be the j^{th} Bernoulli number. Denote by $\kappa_{n,j}$ the jth cumulant of X_n, and set*

$$\gamma_{n,j} = \frac{\kappa_{n,j}}{\sqrt{\kappa_{n,2}}^j \sqrt{n}^{j-2}} - \frac{b_j}{j} \left(\frac{\Delta_n}{\sqrt{n}\sqrt{\kappa_{n,2}}}\right)^j, \quad j > 1,$$

$$-d_n = \frac{\ln(K/S_0) - \sqrt{Tn}\kappa_{n,1}}{\sqrt{\kappa_{n,2}}\sqrt{T}} + \frac{\Delta_n(1 - 2\,\mathrm{frac}(a_n))}{2\sqrt{n}\sqrt{\kappa_{n,2}}},$$

where

$$a_n = \frac{\ln(K/S_0) + n\Lambda_n\sqrt{\Delta t}}{\Delta_n\sqrt{\Delta t}}.$$

Then, for every integer $i_0 \geq 3$, and every $K > 0$,

$$P\left(S_T^{(n)} \leq K\right) = E_{i_0}(-d_n, \gamma_n) + O\left(n^{-\frac{i_0-1}{2}}\right).$$

Remark 4. *d_n is not to be confused with d_1 and d_2 in (3) when $n = 1, 2$, though it does turn out that d_n is related to d_1 and d_2. a_n is as in Theorem 1.*

Remark 5. *X_n takes the values $-\Lambda_n + (i-1)\Delta_n$. Its standardization $Y_n = (X_n - \kappa_{n,1})/\sqrt{\kappa_{n,2}}$ takes values $\alpha_n + (i-1)\Delta_n/\sqrt{\kappa_{n,2}}$, where $\alpha_n = -(\Lambda_n + \kappa_{n,1})/\sqrt{\kappa_{n,2}}$. In the proof of Theorem 2, we consider $S_n = \sum_{k=1}^n Y_{n,k}/\sqrt{n}$, where $Y_{n,1}$,...,$Y_{n,n}$ are independent copies of Y_n. Note that the jth cumulant of S_n is $\kappa_{n,j}/(\sqrt{\kappa_{n,2}}^j \sqrt{n}^{j-2})$. Then, $\gamma_{n,j}$ is the jth Sheppard-corrected cumulant of S_n.*

First, we prove the part of Theorem 1 for the digital put. We apply Theorem 2 with $i_0 = 4$. Thus,

$$P\left(S_T^{(n)} \le K\right) = E_4(-d_n, \gamma_n) + O\left(n^{-3/2}\right) \tag{12}$$

First, we observe what consequences (H2) has for the cumulants $\kappa_{n,j}$. According to Smith (1995), these are related to the moments $\mu_{n,j} = E(X_n^j)$ according to

$$\kappa_{n,j} = \mu_{n,j} - \sum_{\ell=1}^{j-1} \binom{j-1}{\ell-1} \kappa_{n,\ell}\mu_{n,j-\ell}, \quad j \ge 1.$$

Then, it follows from (H2) that

$$\begin{aligned}
\kappa_{n,1} &= \mu_{n,1} = \theta\sqrt{\Delta t} + D_n(\Delta t)^{3/2} + O((\Delta t)^2), \\
\kappa_{n,2} &= \mu_{n,2} - \mu_{n,1}\kappa_{n,1} = \sigma^2 + (F_n - \theta^2)\Delta t + O((\Delta t)^{3/2}), \\
\kappa_{n,3} &= \mu_{n,3} - \kappa_{n,1}\mu_{n,2} - 2\kappa_{n,2}\mu_{n,1} = (G_n - 3\theta\sigma^2)\sqrt{\Delta t} + O(\Delta t), \\
\kappa_{n,4} &= \mu_{n,4} - \kappa_{n,1}\mu_{n,3} - 3\kappa_{n,2}\mu_{n,2} - 3\kappa_{n,3}\mu_{n,1} = H_n - 3\sigma^4 + O(\sqrt{\Delta t}).
\end{aligned} \tag{13}$$

Now, $E_4(-d_n, \gamma_n)$ is equal to

$$\Phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right) - \sum_{j=3}^{6} \phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right) h_{j-1}\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right) \frac{\nu_{n,j}}{(\gamma_{n,2})^{j/2}} + O(n^{-2}), \tag{14}$$

where $\nu_{n,j}$ is as in (9) with $i_0 = 4$. We analyze the terms $\nu_{n,j}$ for $j = 3, \dots, 6$. From (9), we observe that $\nu_{n,1} = \nu_{n,2} = 0$. From Lemma A1 in the Appendix A, we observe that for $j \ge 3$

$$\nu_{n,j} = \frac{\gamma_{n,j}}{j!} + \sum_{\ell=3}^{j-1} \frac{\ell}{j} \frac{\gamma_{n,\ell}}{\ell!} \nu_{n,j-\ell},$$

where, when using Lemma A1, we understand that $\gamma_{n,j} = 0$ if $j = 1, 2$ or $j > i_0 = 4$. So

$$\nu_{n,3} = \gamma_{n,3}/6, \quad \nu_{n,4} = \gamma_{n,4}/24, \quad \nu_{n,5} = 0, \quad \nu_{n,6} = (\gamma_{n,3})^2/72.$$

Recalling that $b_j = 0$ for odd $j > 1$ and using (13), we obtain

$$\begin{aligned}
\gamma_{n,3} &= \frac{\kappa_{n,3}}{\sqrt{n}\kappa_{n,2}^{3/2}} = \frac{\sqrt{T}(G_n - 3\theta\sigma^2)}{\sigma^3 n} + O(n^{-3/2}), \\
\gamma_{n,4} &= \frac{\kappa_{n,4}}{n\kappa_{n,2}^2} - \frac{b_2}{2}\frac{\Delta_n^2}{n^2\kappa_{n,2}^2} = \frac{H_n - 3\sigma^4}{n\sigma^4} + O\left(n^{-3/2}\right).
\end{aligned}$$

We deduce that

$$\begin{aligned}
\nu_{n,3} &= \frac{\sqrt{T}(G_n - 3\theta\sigma^2)}{6\sigma^3 n} + O(n^{-3/2}), \\
\nu_{n,4} &= \frac{H_n - 3\sigma^4}{24n\sigma^4} + O\left(n^{-3/2}\right), \quad \nu_{n,5} = 0, \quad \nu_{n,6} = O(n^{-2}).
\end{aligned} \tag{15}$$

Next, we note, using (13), that

$$\gamma_{n,2} = 1 - \frac{b_2\Delta_n^2}{2n\kappa_{n,2}} = 1 - \frac{b_2\Delta_n^2}{2n\sigma^2} + O\left(n^{-3/2}\right). \tag{16}$$

It follows from (16) and (15) that

$$\frac{\nu_{n,3}}{\gamma_{n,2}^{3/2}} = \frac{\sqrt{T}(G_n - 3\theta\sigma^2)}{6\sigma^3 n} + O(n^{-3/2}), \ \frac{\nu_{n,4}}{\gamma_{n,2}^2} = \frac{H_n - 3\sigma^4}{24n\sigma^4} + O\Big(n^{-3/2}\Big),$$
$$\frac{\nu_{n,5}}{\gamma_{n,2}^{5/2}} = 0, \quad \frac{\nu_{n,6}}{\gamma_{n,2}^3} = O(n^{-2}).$$

Then using the boundedness of the functions $\phi(x)h_j(x)$, we deduce from these relations and (14) that

$$E_4(-d_n,\gamma_n) = \Phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right) - \phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)h_2\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)\frac{\sqrt{T}(G_n - 3\theta\sigma^2)}{6\sigma^3 n}$$
$$- \phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)h_3\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)\frac{H_n - 3\sigma^4}{24n\sigma^4} + O(n^{-3/2}). \tag{17}$$

Next, using (13) and (16), we obtain

$$\frac{-d_n}{\sqrt{\gamma_{n,2}}} = -d_2 + \frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{Q_n}{n} + O(n^{-3/2}), \tag{18}$$

where

$$d_2 = \frac{\log(S_0/K) + \theta T}{\sigma\sqrt{T}}, \quad Q_n = \frac{d_2\big(2(F_n - \theta^2)T - \Delta_n^2 b_2\big)}{4\sigma^2} - \frac{D_n T^{3/2}}{\sigma}.$$

Since the derivative of $\phi(x)h_j(x)$ is bounded for each j and in view of (18), it follows for each j that

$$\phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)h_j\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right) = \phi(-d_2)h_j(-d_2) + O(n^{-1/2}).$$

Using this, we conclude from (17) that

$$E_4(-d_n,\gamma_n) = \Phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right) - \phi(-d_2)h_2(-d_2)\frac{\sqrt{T}(G_n - 3\theta\sigma^2)}{6\sigma^3 n}$$
$$- \phi(-d_2)h_3(-d_2)\frac{H_n - 3\sigma^4}{24n\sigma^4} + O(n^{-3/2}). \tag{19}$$

Next, we consider the term $\Phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)$. Using Taylor expansion about $-d_2$, we obtain that

$$\Phi\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}}\right)$$
$$= \Phi(-d_2) + \phi(-d_2)\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}} + d_2\right) + \frac{1}{2}d_2\phi(-d_2)\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}} + d_2\right)^2$$
$$+O\left(\left(\frac{-d_n}{\sqrt{\gamma_{n,2}}} + d_2\right)^3\right)$$

and, using (18), we continue with

$$= \Phi(-d_2) + \phi(-d_2)\left(\frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{Q_n}{n}\right) + \frac{1}{2}d_2\phi(-d_2)\frac{\bar{\Delta}_n^2}{n} + O(n^{-3/2})$$
$$= \Phi(-d_2) + \phi(-d_2)\frac{\bar{\Delta}_n}{\sqrt{n}} + \phi(-d_2)\left(\frac{1}{2}d_2\bar{\Delta}_n^2 + Q_n\right)\frac{1}{n} + O(n^{-3/2}).$$

Then, combining this with (19), we obtain

$$E_4(-d_n,\gamma_n)=\Phi(-d_2)+\phi(-d_2)\left[\frac{\bar{\Delta}_n}{\sqrt{n}}+\left(\frac{1}{2}d_2\bar{\Delta}_n^2-B_n\right)\frac{1}{n}\right]+O(n^{-3/2}),$$

where

$$B_n=-Q_n+\frac{h_2(-d_2)(G_n-3\theta\sigma^2)\sqrt{T}}{6\sigma^3}+\frac{h_3(-d_2)(H_n-3\sigma^4)}{24\sigma^4}.$$

Then, using (12),

$$P(S_T^{(n)}\le K)=\Phi(-d_2)+\phi(-d_2)\left[\frac{\bar{\Delta}_n}{\sqrt{n}}+\left(\frac{1}{2}d_2\bar{\Delta}_n^2-B_n\right)\frac{1}{n}\right]+O(n^{-3/2}) \tag{20}$$

and since $P_d(n)=e^{-rT}P(S_T^{(n)}\le K)$ and $P_{BS}=e^{-rT}\Phi(-d_2)$,

$$P_d(n)=P_{BS}+e^{-rT}\phi(-d_2)\left[\frac{\bar{\Delta}_n}{\sqrt{n}}+\left(\frac{1}{2}d_2\bar{\Delta}_n^2-B_n\right)\frac{1}{n}\right]+O(n^{-3/2}).$$

All that remains is to show that B_n is as stated in the theorem. Using $h_2(x)=x^2-1$ and $h_3(x)=x^3-3x$,

$$B_n=I_1+I_2, \tag{21}$$

where, using $b_2=1/6$,

$$\begin{aligned}I_1&=\frac{d_2\theta^2T}{2\sigma^2}-\frac{3d_2-d_2^3}{8}-\frac{\theta\sqrt{T}(d_2^2-1)}{2\sigma},\\ I_2&=\frac{d_2\Delta_n^2}{24\sigma^2}+\frac{D_nT^{3/2}}{\sigma}-\frac{d_2F_nT}{2\sigma^2}+\frac{(d_2^2-1)G_n\sqrt{T}}{6\sigma^3}+\frac{(3d_2-d_2^3)H_n}{24\sigma^4}.\end{aligned}$$

Using $\sigma\sqrt{T}=d_1-d_2$ and $\theta=r-\sigma^2/2$, we obtain

$$I_1=\frac{d_2}{2\sigma^2}Tr^2+\frac{1-d_1d_2}{2\sigma}\sqrt{T}r+\frac{d_1^2d_2-2d_1-d_2}{8}.$$

Thus, B_n is as stated in the theorem and the proof of (i) is finished.

As shown in (6), in a risk-neutral m-nomial model, the price of a put with strike K and maturity T is given by

$$\begin{aligned}P(n)&=Ke^{-rT}P(S_T^{(n)}\le K)-S_0Q(S_T^{(n)}\le K)\\ &=KP_d(n)-S_0Q(S_T^{(n)}\le K).\end{aligned}$$

We can determine $Q(S_T^{(n)}\le K)$ in a way similar to that with which we determined $P(S_T^{(n)}\le K)$. The difference is that the probabilities $p_{n,i}$ are now replaced by (see (5))

$$q_{n,i}=e^{-r\Delta t+\sqrt{\Delta t}x_{n,i}}p_{n,i},$$

where $x_{n,i}=-\Lambda_n+(i-1)\Delta_n$ is the ith value of X_n. We observe that

$$\begin{aligned}q_{n,i}=&\left(1+x_{n,i}\sqrt{\Delta t}+\left(\frac{1}{2}x_{n,i}^2-r\right)\Delta t+\left(\frac{1}{6}x_{n,i}^3-rx_{n,i}\right)(\Delta t)^{3/2}\right)p_{n,i}\\ &+O\Big((\Delta t)^2\Big).\end{aligned}$$

Hence, the moments $\tilde{\mu}_{n,k} = E(X_n^k)$ of X_n corresponding to $q_{n,i}$ satisfy

$$\begin{aligned}\tilde{\mu}_{n,k} &= \sum_{i=1}^{m} q_{n,i} x_{n,i}^k \\ &= \mu_{n,k} + \mu_{n,k+1}\sqrt{\Delta t} + \left(\frac{1}{2}\mu_{n,k+2} - r\mu_{n,k}\right)\Delta t \\ &\quad + \left(\frac{1}{6}\mu_{n,k+3} - r\mu_{n,k+1}\right)(\Delta t)^{3/2} + O\Big((\Delta t)^2\Big),\end{aligned}$$

where $\mu_{n,k}$ are the moments of X_n corresponding to $p_{n,i}$. Then, using (H2),

$$\begin{aligned}\tilde{\mu}_{n,1} &= (\theta + \sigma^2)\sqrt{\Delta t} + \tilde{D}_n(\Delta t)^{3/2} + O((\Delta t)^2), \\ \tilde{\mu}_{n,2} &= \sigma^2 + \tilde{F}_n\Delta t + O((\Delta t)^{3/2}) \\ \mu_{n,3} &= (G_n + H_n)\sqrt{\Delta t} + O(\Delta t), \\ \tilde{\mu}_{n,4} &= H_n + O(\Delta t),\end{aligned} \tag{22}$$

where

$$\tilde{D}_n = D_n + F_n + \frac{G_n}{2} + \frac{H_n}{6} - r(\theta + \sigma^2), \quad \tilde{F}_n = F_n + G_n + \frac{H_n}{2} - r\sigma^2. \tag{23}$$

From these relations, we observe that (H1) and (H2) hold with $p_{n,i}$ replaced by $q_{n,i}$, $E(X_n^k)$ by $\tilde{\mu}_{n,k}$, θ replaced by $\tilde{\theta} = r + \sigma^2/2$ and D_n, F_n, G_n replaced by $\tilde{D}_n, \tilde{F}_n, G_n + H_n$.

Then, it follows from (20), (21) and

$$d_1 = \frac{\log(S_0/K) + \tilde{\theta}T}{\sigma\sqrt{T}},$$

that

$$Q(S_T^{(n)} \le K) = \Phi(-d_1) + \phi(-d_1)\left[\frac{\bar{\Delta}_n}{\sqrt{n}} + \left(\frac{1}{2}d_1\bar{\Delta}_n^2 - \tilde{B}_n\right)\frac{1}{n}\right] + O(n^{-3/2}),$$

where

$$\tilde{B}_n = \tilde{I}_1 + \tilde{I}_2,$$

with

$$\begin{aligned}\tilde{I}_1 &= \frac{d_1\tilde{\theta}^2 T}{2\sigma^2} - \frac{3d_1 - d_1^3}{8} - \frac{\tilde{\theta}\sqrt{T}(d_1^2 - 1)}{2\sigma}, \\ \tilde{I}_2 &= \frac{d_1\Delta_n^2}{24\sigma^2} + \frac{\tilde{D}_n T^{3/2}}{\sigma} - \frac{d_1 T\tilde{F}_n}{2\sigma^2} + \frac{(d_1^2 - 1)\tilde{G}_n\sqrt{T}}{6\sigma^3} + \frac{(3d_1 - d_1^3)\tilde{H}_n}{24\sigma^4}.\end{aligned}$$

After some algebra, we find that

$$\tilde{B}_n - B_n = \tilde{I}_1 - I_1 + \tilde{I}_2 - I_2 = \sigma\sqrt{T}C_n,$$

where

$$\begin{aligned}C_n = &-\frac{Tr^2}{2\sigma^2} + \frac{d_2}{2\sigma}\sqrt{T}r + \frac{1 - d_1 d_2}{8} + \frac{\Delta_n^2}{24\sigma^2} + \frac{T}{2\sigma^2}F_n \\ &+ \frac{(d_1 - 2d_2)\sqrt{T}}{6\sigma^3}G_n + \frac{d_1^2 - 3d_1 d_2 + 3d_2^2 - 1}{24\sigma^4}H_n.\end{aligned}$$

Then, using $S_0\phi(-d_1) = Ke^{-rT}\phi(-d_2)$, the price of the put is

$$
\begin{aligned}
&P(n) \\
&= KP_d(n) - S_0 Q(S_n \le K) \\
&= K\left[e^{-rT}\Phi(-d_2) + e^{-rT}\phi(-d_2)\left(\frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{1}{2}d_2\bar{\Delta}_n^2 - B_n\right)\frac{1}{n}\right] \\
&\quad - S_0\left[\Phi(-d_1) + \phi(-d_1)\left(\frac{\bar{\Delta}_n}{\sqrt{n}} + \frac{1}{2}d_1\bar{\Delta}_n^2 - \tilde{B}_n\right)\frac{1}{n}\right] + O(n^{-3/2}) \\
&= P_{BS} + S_0\phi(-d_1)\left(-\frac{1}{2}(d_1 - d_2)\bar{\Delta}_n^2 + \tilde{B}_n - B_n\right)\frac{1}{n} + O(n^{-3/2}) \\
&= P_{BS} + S_0\sigma\sqrt{T}\phi(-d_1)\left(-\frac{1}{2}\bar{\Delta}_n^2 + C_n\right)\frac{1}{n} + O(n^{-3/2}).
\end{aligned}
$$

The result for a call follows by using put–call parity.

6. Proof of Theorem 2

To prove Theorem 2, we use the following theorem, the proof of which we defer to later.

Theorem 3 (Edgeworth expansion for triangular arrays)**.** *Let $Y_{n,k}$, $k = 1, 2, \ldots, n$ be independent and identically distributed versions of some random variable Y_n. Assume that Y_n is supported by some lattice $\alpha_n + \Delta_n\mathbb{Z}$, where α_n is bounded and $\Delta_n > 0$ has a positive limit, and there exists a positive integer m, such that the set*

$$
\{x \in \alpha_n + \Delta_n\mathbb{Z} : P(Y_n = x) > 0\}
$$

consists of m distinct points $\alpha_n + \Delta_n x_i$, $i = 1, \ldots, m$, where $x_i \in \mathbb{Z}$. Moreover, for each i, $\inf_n p_{n,i} > 0$ *where $p_{n,i} = P(Y_n = \alpha_n + \Delta_n x_i)$, and $Var(Y_n) \to V > 0$. Let*

$$
S_n = \sum_{k=1}^{n} \frac{Y_{n,k} - E(Y_{n,k})}{\sqrt{n}}.
$$

Then, for all $i_0 \ge 3$,

$$
\sup_{x\in\mathbb{R}}\left|P(S_n \le x) - E_{i_0}(x_+, \gamma_n)\right| = O\left(n^{\frac{1-i_0}{2}}\right), \tag{24}
$$

where $\gamma_n = (\gamma_{n,2}, \gamma_{n,3}, \ldots)$, $\gamma_{n,j}$ being the Sheppard-corrected cumulant of S_n of order j, that is

$$
\gamma_{n,j} = \frac{\rho_{n,j}}{\sqrt{n}^{j-2}} - \frac{b_j}{j}\frac{\Delta_n^j}{n^{j/2}}, \quad j > 1,
$$

$\rho_{n,j}$ is the jth cumulant of Y_n, b_j is the j^{th} Bernoulli number, and x_+ is the continuity corrected point in the lattice space $\mathcal{L}_n = \sqrt{n}(\alpha_n - \rho_{n,1}) + (\Delta_n/\sqrt{n})\mathbb{Z}$, that is,

$$
x_+ = \sup\{y \in \mathcal{L}_n : y \le x\} + \frac{\Delta_n}{2\sqrt{n}}.
$$

Remark 6. *In particular, $\gamma_{n,2}$ is called the Sheppard-corrected variance of S_n.*

Proof of Theorem 2. By definition, we have

$$
S_T^{(n)} = S_0 \exp\left(\sqrt{\Delta t}\sum_{k=1}^{n} X_{n,k}\right)
$$

where $X_{n,k}$ are independent versions of the random variable X_n, which takes the value $-\Lambda_n + (i-1)\Delta_n$ with probability $p_{n,i}$, such that Λ_n is bounded, Δ_n has a positive limit, $\inf_n p_{n,i} > 0$ and $Var(X_n) \to V > 0$.

Now, simple algebraic manipulations give

$$P(S_T^{(n)} \leq K) = P\left(S_0 \exp\left(\sqrt{\Delta t}\sum_{k=1}^{n} X_{n,k}\right) \leq K\right) = P\left(S_n \leq -\hat{d}_n\right),$$

where

$$S_n := \sum_{k=1}^{n} \frac{Y_{n,k}}{\sqrt{n}}, \quad Y_{n,k} := \frac{X_{n,k} - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}, \quad -\hat{d}_n := \frac{\ln(K/S_0) - \sqrt{Tn}\kappa_{n,1}}{\sqrt{T}\sqrt{\kappa_{n,2}}},$$

where $\kappa_{n,j}$ denotes the jth cumulant of X_n. For simplicity, we set $Y_n := Y_{n,1}$. Note that Y_n takes the value

$$\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}} + \frac{\Delta_n}{\sqrt{\kappa_{n,2}}}(i-1) \in \frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}} + \frac{\Delta_n}{\sqrt{\kappa_{n,2}}}\mathbb{Z}$$

with probability $p_{n,i}$, for $i = 1, \ldots, m$. Note that $\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}$ is bounded because $\Lambda_n, \kappa_{n,1} = E(X_n)$ are bounded and $\kappa_{n,2} = \mathrm{Var}(X_n) \to V > 0$. Next, $\frac{\Delta_n}{\sqrt{\kappa_{n,2}}}$ has a positive limit because both Δ_n and $\kappa_{n,2}$ have one. Finally, $\mathrm{Var}(Y_n) = 1$. Hence, Y_n satisfies the conditions of Theorem 3 and $E(Y_n) = 0$. We denote now by $\rho_{n,j}$ the jth cumulant of Y_n. Clearly, $\rho_{n,1} = 0$, $\rho_{n,2} = 1$, since $\rho_{n,1} = E(Y_n)$ and $\rho_{n,2} = \mathrm{Var}(Y_n)$. For $j \geq 2$, $\rho_{n,j} = \kappa_{n,j}/\kappa_{n,2}^{j/2}$. Then, applying Theorem 3 to Y_n, for $i_0 \geq 3$,

$$\sup_{x \in \mathbb{R}} \left| P(S_n \leq x) - E_{i_0}(x_+, \gamma_n) \right| = O\left(n^{\frac{1-i_0}{2}}\right), \tag{25}$$

where $\gamma_n = (\gamma_{n,2}, \gamma_{n,3}, \ldots)$ is given by

$$\gamma_{n,j} = \frac{\rho_{n,j}}{\sqrt{n}^{j-2}} - \frac{b_j}{j}\left(\frac{\Delta_n}{\sqrt{\kappa_{n,2}}}\right)^j \frac{1}{n^{j/2}} = \frac{\kappa_{n,j}}{\kappa_{n,2}^{j/2}\sqrt{n}^{j-2}} - \frac{b_j}{j}\frac{\Delta_n^j}{\kappa_{n,2}^{j/2} n^{j/2}}, \tag{26}$$

and x_+ is the continuity corrected point in the lattice space

$$\mathcal{L}_n = \sqrt{n}\left(\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}\right) + \frac{\Delta_n}{\sqrt{n}\sqrt{\kappa_{n,2}}}\mathbb{Z},$$

that is,

$$x_+ = \sup\{y \in \mathcal{L}_n : y \leq x\} + \frac{\Delta_n}{2\sqrt{n}\sqrt{\kappa_{n,2}}}.$$

Now if

$$\sqrt{n}\left(\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}\right) + \frac{\Delta_n(k)}{\sqrt{n}\sqrt{\kappa_{n,2}}} \leq -\hat{d}_n < \sqrt{n}\left(\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}\right) + \frac{\Delta_n(k+1)}{\sqrt{n}\sqrt{\kappa_{n,2}}},$$

then

$$(-\hat{d}_n)_+ = \sqrt{n}\left(\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}\right) + \frac{\Delta_n}{\sqrt{n}\sqrt{\kappa_{n,2}}}\left(k + \frac{1}{2}\right).$$

We observe that $k = \text{floor}(a_n) = a_n - \text{frac}(a_n)$, so that

$$\begin{aligned}(-\hat{d}_n)_+ &= \sqrt{n}\left(\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}\right) + \frac{\Delta_n}{\sqrt{n}\sqrt{\kappa_{n,2}}}\left(a_n - \text{frac}(a_n) + \frac{1}{2}\right) \\ &= \sqrt{n}\left(\frac{-\Lambda_n - \kappa_{n,1}}{\sqrt{\kappa_{n,2}}}\right) + \frac{\Delta_n}{\sqrt{n}\sqrt{\kappa_{n,2}}}a_n + \frac{\Delta_n(1 - 2\,\text{frac}(a_n))}{2\sqrt{n}\sqrt{\kappa_{n,2}}} \\ &= -\hat{d}_n + \frac{\Delta_n(1 - 2\,\text{frac}(a_n))}{2\sqrt{n}\sqrt{\kappa_{n,2}}} \\ &= -d_n.\end{aligned}$$

This completes the proof of Theorem 2. □

7. Proof of Theorem 3

First, assume that $\alpha_n = 0$ and $\Delta_n = 1$, so that Y_n takes the value x_i in $\mathbb{Z}$ with probability $p_{n,i} = P(Y_n = x_i)$. Then, for each $j \geq 0$

$$E(|Y_n^j|) = \sum_{i=1}^{m} p_{n,i}|x_i^j|$$

is bounded. It is clear that the moment generating function $M_n(t) = E(e^{tY_n})$ and the cumulant generating function $K_n(t) = \log M_n(t)$ exist and can be written as a power series. This guarantees (Smith 1995) that for $j \geq 1$, the cumulants $\rho_{n,j}$ of Y_n are related to the moments $\mu_{n,j} = E(Y_n^j)$, according to

$$\rho_{n,j} = \mu_{n,j} - \sum_{\ell=1}^{j-1} \binom{j-1}{\ell-1} \rho_{n,\ell}\mu_{n,j-\ell}.$$

Hence, for each j, $\rho_{n,j}$ is also bounded.

Then, we want to apply the case $d = 1$ of Theorem A1 in Bock and Korn (2016). Clearly, the first three of the conditions (A1) are satisfied. The fourth follows from the fact that $E(|Y_n^j|)$ is bounded for each j. Moreover, we have $V_n = Var(Y_n) \to V > 0$. Next, note that it follows from Lemma A2 in Bock and Korn (2016) that condition (A2) of Theorem A1 is also satisfied. Thus, we conclude that

$$\sup_{x \in \mathbb{R}} |P(S_n \leq x) - F_n(x)| = O\left(n^{\frac{1-i_0}{2}}\right), \tag{27}$$

where $F_n(x)$ is defined as

$$\sum_{r=0}^{i_0-2} n^{-r/2} \sum_{j=0}^{i_0-r-2} n^{-j/2}(-1)^j S_j(n\mu_n + \sqrt{n}x) \frac{d^j}{dx^j} P_r(-\Phi_{0,V_n}, \{\rho_{n,\nu}\})(x)$$

with

$$S_j(x) = \frac{B_j(x - \lfloor x \rfloor)}{j!},$$

where $\lfloor x \rfloor = \text{floor}(x)$, B_j is the jth order Bernoulli polynomial (note $S_j(x)$ is not to be confused with S_n), $\mu_n = \mu_{n,1} = E(Y_n)$, and according to Bock (2014),

$$P_r(-\Phi_{0,V_n}, \{\rho_{n,\nu}\})(x) = \tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x), \tag{28}$$

with $\tilde{P}_{n,0} = 1$ (note that $\Phi_{0,V_n}(x) = \Phi(x/\sqrt{V_n})$) and for $r \geq 1$, the polynomial $\tilde{P}_{n,r}(z)$ is defined by the relation

$$1 + \sum_{r=1}^{\infty} \tilde{P}_{n,r}(z)u^r = \exp\left(\sum_{r=1}^{\infty} \frac{\rho_{n,r+2}z^{r+2}}{(r+2)!}u^r\right). \tag{29}$$

Interchanging the order of summation and using (28), we find that $F_n(x)$ is

$$\sum_{j=0}^{i_0-2} n^{-j/2}(-1)^j S_j(n\mu_n + \sqrt{n}x)\frac{d^j}{dx^j}\left(\sum_{r=0}^{i_0-j-2} n^{-r/2}\tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x)\right). \tag{30}$$

In Lemma A2 in the Appendix A, we show that for $r \geq 0$ and $s \geq 0$,

$$\sup_{x\in\mathbb{R}}\left|\frac{d^s}{dx^s}\tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x)\right| \leq M_r(s), \tag{31}$$

for some real-valued number $0 < M_r(s) < \infty$. For $0 \leq j \leq i_0 - 2$ and $i_0 - j - 1 \leq r \leq i_0 - 2$, we have $n^{-r/2-j/2} \leq n^{(1-i_0)/2}$. Since $S_j(x)$ is bounded, this means we can replace $F_n(x)$ in (30) by

$$\sum_{j=0}^{i_0-2} n^{-j/2}(-1)^j S_j(n\mu_n + \sqrt{n}x)\frac{d^j}{dx^j}\left(\sum_{r=0}^{i_0-2} n^{-r/2}\tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x)\right)$$

and (27) still holds. Using Proposition A1 in the Appendix A, we obtain that for $j = 0, \ldots, i_0 - 2$,

$$\sup_{x\in\mathbb{R}}\left|\frac{d^j}{dx^j}\sum_{r=0}^{i_0-2} n^{-r/2}\tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x) - \frac{d^j}{dx^j}E_{i_0}(x,\bar{\rho}_n)\right| = O(n^{\frac{1-i_0}{2}}), \tag{32}$$

where $\bar{\rho}_n = (\bar{\rho}_{n,2}, \bar{\rho}_{n,3}, \ldots)$ and

$$\bar{\rho}_{n,j} = \frac{\rho_{n,j}}{n^{(j-2)/2}}$$

is the jth cumulant of S_n. Hence, in (27), we may replace $F_n(x)$ by

$$F_n(x) = \sum_{j=0}^{i_0-2} n^{-j/2}(-1)^j S_j(n\mu_n + \sqrt{n}x)\frac{d^j}{dx^j}E_{i_0}(x,\bar{\rho}_n) \tag{33}$$

and the inequality still holds.

Now S_n takes values in the lattice $\mathcal{L}_n = -\mu_n\sqrt{n} + \frac{1}{\sqrt{n}}\mathbb{Z}$. Let $k \in \mathbb{Z}$ be such that

$$-\mu_n\sqrt{n} + \frac{1}{\sqrt{n}}k \leq x < -\mu_n\sqrt{n} + \frac{1}{\sqrt{n}}(k+1).$$

Then

$$x_+ = -\mu_n\sqrt{n} + \frac{1}{\sqrt{n}}(k + 1/2)$$

so that

$$n\mu_n + \sqrt{n}x_+ = k + 1/2$$

and hence

$$S_j(n\mu_n + \sqrt{n}x_+) = S_j(1/2) = \frac{B_j(1/2)}{j!}.$$

Then, it follows from (33) that

$$F_n(x_+) = \sum_{j=0}^{i_0-2} \frac{(-1)^j B_j(1/2)}{n^{j/2} j!} \frac{d^j}{dx^j} E_{i_0}(x_+, \bar{\rho}_n). \tag{34}$$

Note that

$$|P(S_n \le x) - F_n(x_+)| = |P(S_n \le x_+) - F_n(x_+)|$$

so that, by (27),

$$\sup_{x \in \mathbb{R}} |P(S_n \le x) - F_n(x_+)| = O\left(n^{\frac{1-i_0}{2}}\right), \tag{35}$$

where $F_n(x_+)$ is as in (34).

From this point onwards, we are essentially following Kolassa and McCullagh (1990). However, we are using a somewhat different definition of the Edgeworth expansion and, in order to be complete, the proof requires a number of additional steps.

First, we introduce a definition suggested by a definition of Kolassa and McCullagh. Let $i_0 \ge 3$, let $a = (a_0, a_1, \ldots)$ be a sequence of real numbers and let λ be a positive number. Then, we define

$$\psi_{i_0,x,\lambda}(a) = \sum_{j=0}^{3i_0-6} (-1)^j a_j \frac{d^j}{dx^j} \Phi\left(\frac{x}{\sqrt{\lambda}}\right). \tag{36}$$

Now, from the definition,

$$E_{i_0}(x_+, \bar{\rho}_n) = \psi_{i_0,x_+,\sigma_n^2}(\alpha_n),$$

where

$$\sigma_n = \sqrt{\bar{\rho}_{n,2}} = \sqrt{\rho_{n,2}} = \sqrt{V_n} \tag{37}$$

and

$$\sum_{j=0}^{\infty} \alpha_{n,j} s^j = \exp\left(\sum_{j=3}^{i_0} \frac{\bar{\rho}_{n,j}}{j!} s^j\right). \tag{38}$$

It follows from Lemma A3 in the Appendix A, using $\bar{\rho}_{n,j} = O(n^{(2-j)/2})$, that for each j

$$\alpha_{n,j} = O(n^{-j/6}). \tag{39}$$

Then, since also $\sigma_n \to \sqrt{V}$, it follows from Lemma A4(ii) in the Appendix A that

$$\frac{d^j}{dx^j} E_{i_0}(x_+, \bar{\rho}_n) = \frac{d^j}{dx^j} \psi_{i_0,x_+,\sigma_n^2}(\alpha_n) = (-1)^j \psi_{i_0,x_+,\sigma_n^2}(T^j \alpha_n) + O(n^{\frac{j+1-i_0}{2}}),$$

where

$$T^j \alpha_n = (\overbrace{0, \ldots, 0}^{j \text{ times}}, \alpha_{n,0}, \alpha_{n,1}, \ldots) \text{ for } j \ge 0,$$

so that

$$\sup_{x \in \mathbb{R}} \left| \frac{(-1)^j}{n^{j/2}} \frac{d^j}{dx^j} E_{i_0}(x_+, \bar{\rho}_n) - \frac{1}{n^{j/2}} \psi_{i_0,x_+,\sigma_n^2}(T^j \alpha_n) \right| = O(n^{\frac{1-i_0}{2}}).$$

Hence, we may replace the expression in (34) by

$$F_n(x_+) = \sum_{j=0}^{i_0-2} \frac{B_j(1/2)}{n^{j/2} j!} \psi_{i_0,x_+,\sigma_n^2}(T^j \alpha_n)$$

and (35) still holds. Moreover, because

$$\psi_{i_0,x_+,\sigma_n^2}(T^j \alpha_n) = \sum_{k=j}^{3i_0-6} (-1)^k \alpha_{n,k-j} \frac{d^k}{dx^k} \Phi\left(\frac{x_+}{\sigma_n}\right)$$

is uniformly bounded in x and n by (A2) in Lemma A2 in the Appendix A and by (39), we may even take

$$F_n(x_+) = \sum_{j=0}^{3i_0-6} \frac{B_j(1/2)}{n^{j/2} j!} \psi_{i_0,x_+,\sigma_n^2}(T^j \alpha_n),$$

and (35) still holds. Then, using Lemma A4(i) in the Appendix A, we have

$$F_n(x_+) = \psi_{i_0,x_+,\sigma_n^2}(\beta_n), \tag{40}$$

where

$$\beta_n = \sum_{j=0}^{3i_0-6} \frac{B_j(1/2)}{n^{j/2} j!} T^j \alpha_n.$$

Here, $\beta_n = (\beta_{n,0}, \beta_{n,1}, \ldots)$ with

$$\beta_{n,k} = \sum_{j=0}^{3i_0-6} \frac{B_j(1/2)}{n^{j/2} j!} (T^j \alpha_n)_k.$$

so that

$$\sum_{k=0}^{\infty} \beta_{n,k} s^k = \sum_{k=0}^{\infty} \sum_{j=0}^{3i_0-6} \frac{B_j(1/2)}{n^{j/2} j!} (T^j \alpha_n)_k s^k.$$

Now,

$$\sum_{k=0}^{\infty} (T^j \alpha_n)_k s^k = \sum_{k=j}^{\infty} \alpha_{n,k-j} s^k = s^j \sum_{k=j}^{\infty} \alpha_{n,k-j} s^{k-j} = s^j \sum_{k=0}^{\infty} \alpha_{n,k} s^k$$

so that, using (38),

$$\sum_{k=0}^{\infty} (T^j \alpha_n)_k s^k = s^j \exp\left(\sum_{k=3}^{i_0} \frac{\bar{\rho}_{n,k}}{k!} s^k \right).$$

Hence, β_n is determined by

$$\sum_{k=0}^{\infty} \beta_{n,k} s^k = \sum_{j=0}^{3i_0-6} \frac{B_j(1/2)}{n^{j/2} j!} s^j \exp\left(\sum_{k=3}^{i_0} \frac{\bar{\rho}_{n,k}}{k!} s^k \right).$$

If now $\tilde{\beta}_n = (\tilde{\beta}_{n,0}, \tilde{\beta}_{n,1}, \ldots)$ is determined by

$$\sum_{k=0}^{\infty} \tilde{\beta}_{n,k} s^k = \sum_{j=0}^{\infty} \frac{B_j(1/2)}{n^{j/2} j!} s^j \exp\left(\sum_{k=3}^{i_0} \frac{\bar{\rho}_{n,k}}{k!} s^k \right),$$

then $\tilde{\beta}_{n,j} = \beta_{n,j}$ for $j = 0, \ldots, 3i_0 - 6$ and thus, from the definition (36) of ψ,

$$F_n(x_+) = \psi_{i_0,x_+,\sigma_n^2}(\beta_n) = \psi_{i_0,x_+,\sigma_n^2}(\tilde{\beta}_n). \tag{41}$$

Now, by an equation in Kolassa and McCullagh (1990, p. 984),

$$\sum_{j=0}^{\infty} \frac{B_j(1/2)}{n^{j/2} j!} s^j = \exp\left(\sum_{k=2}^{\infty} \frac{-b_k}{n^{k/2} k\, k!} s^k \right).$$

Hence,

$$\sum_{k=0}^{\infty} \tilde{\beta}_{n,k} s^k = \exp\left(\sum_{k=2}^{\infty} \frac{-b_k}{n^{k/2} k\, k!} s^k + \sum_{k=3}^{i_0} \frac{\bar{\rho}_{n,k}}{k!} s^k \right).$$

Next, if $\Gamma_n = (\Gamma_{n,0}, \Gamma_{n,1}, \ldots)$ is determined by

$$\sum_{k=0}^{\infty} \Gamma_{n,k} s^k = \exp\left(\sum_{k=2}^{i_0} \frac{-b_k}{n^{k/2} k\, k!} s^k + \sum_{k=3}^{i_0} \frac{\bar{\rho}_{n,k}}{k!} s^k \right),$$

then, from Lemma A4(iv) in Appendix A,

$$\sup_{x \in \mathbb{R}} \left| \psi_{i_0, x_+, \sigma_n^2}(\tilde{\beta}_n) - \psi_{i_0, x_+, \sigma_n^2}(\Gamma_n) \right| = O(n^{(1-i_0)/2})$$

so that we may replace $F_n(x_+)$ in (41) by

$$F_n(x_+) = \psi_{i_0, x_+, \sigma_n^2}(\Gamma_n) \tag{42}$$

and (35) still holds. Finally note that

$$\sum_{k=0}^{\infty} \Gamma_{n,k} s^k = \exp\left(\frac{-b_2}{4n} s^2 + \sum_{k=3}^{i_0} \gamma_{n,k} \frac{s^k}{k!} \right)$$

where, from the statement of the theorem,

$$\gamma_{n,k} = \bar{\rho}_{n,k} - \frac{b_k}{n^{k/2} k}, \quad k > 1,$$

is $O(n^{(2-k)/2})$ since $\bar{\rho}_{n,k} = O(n^{(2-k)/2})$. Let ν_n be defined by

$$\sum_{k=0}^{\infty} \nu_{n,k} s^k = \exp\left(\sum_{k=3}^{i_0} \gamma_{n,k} \frac{s^k}{k!} \right).$$

Lemma A3 in Appendix A guarantees that $\nu_{n,j} = O(n^{-j/6})$ for each j. Then, according to Proposition A2 in the Appendix A applied to $a_n = \nu_n$, $\bar{a}_n = \Gamma_n$, and $c = -b_2/4$, we may replace σ_n^2 as in (37) by the Sheppard-corrected variance $\gamma_{n,2} = \sigma_n^2 - \frac{b_2}{2n}$ in the sense that

$$\sup_{x \in \mathbb{R}} \left| \psi_{i_0, x, \sigma_n^2}(\Gamma_n) - \psi_{i_0, x, \gamma_{n,2}}(\nu_n) \right| = O(n^{(1-i_0)/2}). \tag{43}$$

Thus, we may replace $F_n(x_+)$ in (42) by $\psi_{i_0, x_+, \gamma_{n,2}}(\nu_n)$ and (35) still holds. However, from the definitions (8) and (36), $\psi_{i_0, x_+, \gamma_{n,2}}(\nu_n) = E(x_+, \gamma_n)$, and so we conclude that

$$\sup_{x \in \mathbb{R}} |P(S_n \leq x) - E(x_+, \gamma_n)| = O\left(n^{\frac{1-i_0}{2}} \right).$$

This proves the case $\alpha_n = 0$, $\Delta_n = 1$.

We remain to consider the case when Y_n takes values in the lattice $\alpha_n + \Delta_n \mathbb{Z}$. We define the random variables $\bar{Y}_n = \frac{1}{\Delta_n}(Y_n - \alpha_n)$, which take values in $\mathbb{Z}$. Since $\bar{Y}_n$ takes the value x_i with probability $p_{n,i}$, where $\inf_n p_{n,i} > 0$, and $\mathrm{Var}(\bar{Y}_n) = \mathrm{Var}(Y_n)/\Delta_n^2$ has a positive limit, we may apply the first part of this proof to $\bar{Y}_n$ in order to complete the proof.

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Appendix A

In this Appendix, we prove lemmas and propositions needed for the proof of Theorem 3. We make extensive use of the following fact, which we state as a lemma (essentially the same as Equation (6) in Smith (1995)).

Lemma A1. *If $f(x) = \sum_{j=0}^{\infty} a_j x^j$ and $g(x) = \sum_{j=1}^{\infty} c_j x^j$ are power series related by $f(x) = \exp(g(x))$, then $a_0 = 1$ and for $j \geq 1$,*

$$a_j = c_j + \sum_{m=1}^{j-1} \frac{m}{j} c_m a_{j-m}.$$

Appendix A.1. Properties of $\tilde{P}_{n,r}(z)$

Recall from (29) that the polynomials $\tilde{P}_{n,r}(z)$ are defined by the relation

$$\sum_{r=0}^{\infty} \tilde{P}_{n,r}(z) u^r = \exp\left(\sum_{r=1}^{\infty} \rho_{n,r+2} u^r \frac{z^{r+2}}{(r+2)!} \right). \tag{A1}$$

We verify (31) in the following lemma.

Lemma A2. *Assume that $V_n \geq V > 0$ for $n \geq 1$. Then for each non-negative integer s, there exist a real number $N(s)$ such that*

$$\sup_{x \in \mathbb{R}} \left| \frac{d^s}{dx^s} \Phi_{0,V_n}(x) \right| \leq N(s). \tag{A2}$$

If, additionally, $\rho_{n,j}$ is bounded as a function of n for $j \geq 3$, then for each pair of non-negative integers r, s there exists a real number $M_r(s)$, such that

$$\sup_{x \in \mathbb{R}} \left| \frac{d^s}{dx^s} \tilde{P}_{n,r}\left(-\frac{d}{dx} \right) \Phi_{0,V_n}(x) \right| \leq M_r(s).$$

Proof. First, we take

$$N(s) = \sup_x \left| \frac{d^s}{dx^s} \Phi(x) \right| V^{-s}.$$

Then, for each s, we define $M_r(s)$ recursively, according to

$$M_1(s) = k(3) N(s+3)/6$$

and for $r > 1$:

$$M_r(s) = \frac{k(r+2) N(r+s+2)}{(r+2)!} + \sum_{m=1}^{r-1} \frac{k(m+2)}{(m+2)!} \frac{m}{r} M_{r-m}(m+s+2),$$

where $k(j)$ is a bound on $|\rho_{n,j}|$. The proof uses the recurrence relation

$$\tilde{P}_{n,r}(z) = \frac{\rho_{n,r+2} z^{r+2}}{(r+2)!} + \sum_{m=1}^{r-1} \frac{\rho_{n,m+2} z^{m+2}}{(m+2)!} \frac{m}{r} \tilde{P}_{n,r-m}(z).$$

which follows from (A1), using Lemma A1. We leave the details to the reader. □

Next, we verify (32) in the following proposition.

Proposition A1. *Let $\bar{\rho}_n = (\bar{\rho}_{n,2}, \bar{\rho}_{n,3}, \ldots)$, $\bar{\rho}_{n,j} = \rho_{n,j}/n^{(j-2)/2}$, and $V_n = \rho_{n,2}$, where $\rho_{n,j}$ is bounded as a function of n for $j \geq 2$, and $V_n \geq V > 0$ for $n \geq 1$. Then, for every integer $j \geq 0$,*

$$\sup_{x\in\mathbb{R}}\left|\frac{d^j}{dx^j}E_{i_0}(x,\bar{\rho}_n) - \frac{d^j}{dx^j}\sum_{r=0}^{i_0-2} n^{-r/2}\tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x)\right| = O(n^{\frac{1-i_0}{2}}).$$

Proof. We define

$$f_n(u,z) = \exp\left(\sum_{r=1}^{i_0-2}\rho_{n,r+2}u^r\frac{z^{r+2}}{(r+2)!}\right) = \sum_{\ell=0}^{\infty}Q_{n,\ell}(u)\frac{z^\ell}{\ell!},$$

where

$$Q_{n,\ell}(u) = \frac{\partial^\ell f_n}{\partial z^\ell}(u,0) = \sum_{r=0}^{\infty}\frac{A_{nr\ell}}{r!}u^r, \tag{A3}$$

where

$$A_{nr\ell} = \frac{d^r Q_{n,\ell}}{du^r}(0) = \frac{\partial^{r+\ell}f_n}{\partial u^r\partial z^\ell}(0,0).$$

Note that

$$\frac{\partial^r f_n}{\partial u^r}(0,z) = \sum_{\ell=0}^{\infty}\frac{A_{nr\ell}}{\ell!}z^\ell. \tag{A4}$$

$f_n(u,z)$ is related to both $\tilde{P}_{n,r}(z)$ and $E_{i_0}(x,\bar{\rho}_n)$ in a way explained below. However, first we establish the following claims:

(i) $Q_{n,0}(u) = 1$, $Q_{n,1}(u) = Q_{n,2}(u) = 0$, *and for* $\ell \geq 3$, $Q_{n,\ell}(u)$ *is a polynomial of degree at most* $\ell - 2$*; in fact,* $A_{nr\ell} = 0$ *for* $0 \leq r < \ell/3$ *and* $r > \ell - 2$*;*
(ii) *the* $A_{nr\ell}$ *are bounded as functions of n.*

To prove claim (i), we write

$$f_n(u,z) = 1 + \sum_{\ell=1}^{\infty}Q_{n,\ell}(u)\frac{z^\ell}{\ell!},$$

where $Q_{n,0}(u) = 1$ since $f_n(u,0) = 1$. Now,

$$f_n(u,z) = \exp\left(\sum_{r=3}^{i_0}\rho_{n,r}u^{r-2}\frac{z^r}{r!}\right),$$

and it follows from Lemma A1 that

$$Q_{n,1}(u) = 0, \quad Q_{n,2}(u) = 0, \quad Q_{n,3}(u) = \rho_{n,3}u, \tag{A5}$$

and for $\ell \geq 3$, $Q_{n,\ell}(u)$ satisfies the relation

$$Q_{n,\ell}(u) = \rho_{n,\ell}u^{\ell-2} + \sum_{m=3}^{\ell-1}\binom{\ell-1}{m-1}\rho_{n,m}u^{m-2}Q_{n,\ell-m}(u). \tag{A6}$$

To finish the proof of claim (i), we must show that for $\ell \geq 3$, $Q_{n,\ell}(u)$ is a polynomial of a degree at most $\ell - 2$ and the coefficients of u^i, where $0 \leq i < \ell/3$ are zero (note that $\ell/3 \leq \ell - 2$, since $\ell \geq 3$). The claim we just made is true for $\ell = 3$. Suppose it is true for $Q_{n,r}(u)$, where $3 \leq r < \ell$. Then, from (A6), $Q_{n,\ell}(u)$ is a polynomial and the possible powers of u occurring in $Q_{n,\ell}(u)$ are $\ell - 2$ and $m - 2 + i$, where u^i is a power occurring in $Q_{n,\ell-m}(u)$ for some m, such that $3 \leq m \leq \ell - 3$ ($m = \ell - 1$ and $\ell - 2$ can be excluded because $Q_{n,1}(u) = 0$ and $Q_{n,2}(u) = 0$). Now, by the induction hypothesis and since $\ell - m \geq 3$, $(\ell - m)/3 \leq i \leq \ell - m - 2$. Then,

$$m - 2 + i \leq m - 2 + \ell - m - 2 = \ell - 4 \; < \ell - 2,$$

and since $m \geq 3$,

$$m - 2 + i \geq m - 2 + (\ell - m)/3 = 2(m/3 - 1) + \ell/3 \geq \ell/3.$$

Thus, claim (i) is proved.

We prove claim (ii) by induction on ℓ. Note that $A_{n00} = 1$ and $A_{nr0} = 0$ for $r \geq 1$ since $Q_{n,0}(u) = 1$. Then, $A_{nr\ell} = 0$ for $r \geq 0$ and $\ell = 1, 2$ since $Q_{n,1}(u) = Q_{n,2}(u) = 0$ and by (A5), $A_{nr3} = 0$ unless $r = 1$ when it is $\rho_{n,3}$. Thus, $A_{nr\ell}$ is bounded as a function of n when $\ell \leq 3$ and $r \geq 0$. Suppose now that $\ell \geq 4$ and A_{nrs} is bounded as a function of n when $0 \leq s < \ell$, $r \geq 0$. Recall that $A_{nr\ell} = 0$ if $r > \ell - 2$. We obtain from (A6) that when $r = \ell - 2$,

$$\frac{A_{nr\ell}}{r!} = \rho_{n,\ell} + \sum_{m=3}^{\ell-1} \binom{\ell-1}{m-1} \rho_{n,m} \frac{A_{n,\ell-m,\ell-m}}{(\ell-m)!}$$

and when $r < \ell - 2$,

$$\frac{A_{nr\ell}}{r!} = \sum_{m=3}^{r+2} \binom{\ell-1}{m-1} \rho_{n,m} \frac{A_{n,r-m+2,\ell-m}}{(r-m+2)!}.$$

It follows then that $A_{nr\ell}$ is also bounded and the induction proof is complete. This finishes the proof of claim (ii).

Now, we explain the relationship between $f_n(u, z)$ and $\tilde{P}_{n,r}(z)$ and $E_{i_0}(x, \bar{\rho}_n)$. From (A1),

$$\tilde{P}_{n,r}(z) = \frac{1}{r!} \frac{\partial^r}{\partial u^r} \exp\left(\sum_{r=1}^{\infty} \rho_{n,r+2} u^r \frac{z^{r+2}}{(r+2)!} \right) \bigg|_{u=0}$$

and so, for $r = 0, \ldots, i_0 - 2$,

$$\tilde{P}_{n,r}(z) = \frac{1}{r!} \frac{\partial^r f_n}{\partial u^r}(0, z). \tag{A7}$$

Next note from (8) and (9) that

$$E_{i_0}(x, \bar{\rho}_n) = \Phi_{0,V_n}(x) + \sum_{r=3}^{3i_0-6} \alpha_{n,r} \left(-\frac{d}{dx} \right)^r \Phi_{0,V_n}(x), \tag{A8}$$

where $\bar{\rho}_{n,j} = \rho_{n,j}/n^{(j-2)/2}$, $V_n = \rho_{n,2} = \bar{\rho}_{n,2}$, $\alpha_{n,r}$ is defined by

$$\sum_{r=0}^{\infty} \alpha_{n,r} z^r = \exp\left(\sum_{r=1}^{i_0-2} \bar{\rho}_{n,r+2} \frac{z^{r+2}}{(r+2)!} \right) = f_n(1/\sqrt{n}, z)$$

and, as observed previously, $\Phi_{0,V_n}(x) = \Phi(x/\sqrt{V_n})$. Thus, for $r \geq 0$,

$$\alpha_{n,r} = \frac{1}{r!} \frac{\partial^r f_n}{\partial z^r}(1/\sqrt{n}, 0). \tag{A9}$$

Now, we proceed with the rest of the proof. Suppose that $3 \leq \ell \leq 3i_0 - 6$. From claim (i) above, $A_{nr\ell} = 0$ if $r < \ell/3$ or $r > \ell - 2$ and so certainly if $r = 0$ or $r > 3i_0 - 6$. Hence, using (A3),

$$\frac{\partial^\ell f_n}{\partial z^\ell}(u, 0) = \sum_{r=0}^{\infty} A_{nr\ell} \frac{u^r}{r!} = \sum_{r=1}^{3i_0-6} A_{nr\ell} \frac{u^r}{r!}. \tag{A10}$$

Suppose, next, that $1 \le r \le i_0 - 2$. Then, $A_{nr\ell} = 0$ if $\ell > 3r$ or $\ell < r + 2$ and hence certainly if $\ell > 3i_0 - 6$ or $\ell < 3$. Thus, if $1 \le r \le i_0 - 2$, using (A4) and (A7),

$$\tilde{P}_{n,r}(z) = \frac{1}{r!}\frac{\partial^r f_n}{\partial u^r}(0,z) = \frac{1}{r!}\sum_{\ell=0}^{\infty} A_{nr\ell}\frac{z^\ell}{\ell!} = \frac{1}{r!}\sum_{\ell=3}^{3i_0-6} A_{nr\ell}\frac{z^\ell}{\ell!}.$$

Then, using $\tilde{P}_{n,0}(z) = 1$,

$$\begin{aligned}\sum_{r=0}^{i_0-2}\tilde{P}_{n,r}(z)u^r &= 1 + \sum_{r=1}^{i_0-2}\frac{1}{r!}\sum_{\ell=3}^{3i_0-6} A_{nr\ell}\frac{z^\ell}{\ell!}u^r \\ &= 1 + \sum_{\ell=3}^{3i_0-6}\left(\sum_{r=1}^{i_0-2} A_{nr\ell}\frac{u^r}{r!}\right)\frac{z^\ell}{\ell!}.\end{aligned}$$

Using (A10), we can continue with

$$\begin{aligned}&= 1 + \sum_{\ell=3}^{3i_0-6}\left(\frac{\partial^\ell f_n}{\partial z^\ell}(u,0) - \sum_{r=i_0-1}^{3i_0-6} A_{nr\ell}\frac{u^r}{r!}\right)\frac{z^\ell}{\ell!} \\ &= 1 + \sum_{\ell=3}^{3i_0-6}\frac{\partial^\ell f_n}{\partial z^\ell}(u,0)\frac{z^\ell}{\ell!} - \sum_{\ell=3}^{3i_0-6}\sum_{r=i_0-1}^{3i_0-6} A_{nr\ell}\frac{u^r}{r!}\frac{z^\ell}{\ell!}\end{aligned}$$

so that, taking $u = 1/\sqrt{n}$ and using (A9),

$$\sum_{r=0}^{i_0-2}\tilde{P}_{n,r}(z)n^{-r/2} = 1 + \sum_{\ell=3}^{3i_0-6}\alpha_{n,\ell}z^\ell - \sum_{\ell=3}^{3i_0-6}\sum_{r=i_0-1}^{3i_0-6} A_{nr\ell}\frac{1}{r!}\frac{z^\ell}{\ell!}n^{-r/2}.$$

It follows that for any integer $j \ge 0$

$$\begin{aligned}&\frac{d^j}{dx^j}\sum_{r=0}^{i_0-2} n^{-r/2}\tilde{P}_{n,r}\left(-\frac{d}{dx}\right)\Phi_{0,V_n}(x) \\ &= \frac{d^j}{dx^j}\left(\Phi_{0,V_n}(x) + \sum_{\ell=3}^{3i_0-6}\alpha_{n,\ell}\left(-\frac{d}{dx}\right)^\ell\Phi_{0,V_n}(x)\right) + I_n(x),\end{aligned}$$

where

$$I_n(x) = -\sum_{\ell=3}^{3i_0-6}\sum_{r=i_0-1}^{3i_0-6} n^{-r/2}\frac{A_{nr\ell}}{r!\ell!}\left(-\frac{d}{dx}\right)^{\ell+j}\Phi_{0,V_n}(x) = O(n^{\frac{1-i_0}{2}})$$

uniformly in x, because from claim (ii) above the $A_{nr\ell}$ are bounded as functions of n, $\left(d^{\ell+j}/dx^{\ell+j}\right)\Phi_{0,V_n}(x)$ is bounded as a function of x and n in view of (A2), and $n^{-r/2} \le n^{(1-i_0)/2}$ when $r \ge i_0 - 1$. The statement of the proposition follows from (A8). □

Appendix A.2. On the Power Series of the Exponential of a Power Series

To prove (39), we use the following lemma.

Lemma A3. *Suppose that for $n \ge 1$, $j \ge 3$, $|b_{n,j}| \le k_j n^{(2-j)/2}$ for some real number k_j. If*

$$\sum_{j=0}^{\infty} a_{n,j}s^j = \exp\left(\sum_{j=3}^{\infty} b_{n,j}s^j\right),$$

then $a_{n,0} = 1$, $a_{n,j} = 0$ for $1 \le j \le 2$, and for every integer $j \ge 0$, there exists a constant K_j such that $|a_{n,j}| \le K_j n^{-j/6}$.

Proof. The fact that $a_{n,0} = 1$, $a_{n,j} = 0$ for $1 \le j \le 2$ is clear. Suppose now that $|a_{n,m}| \le K_m n^{-m/6}$ for $0 \le m \le j$, where $j \ge 2$. Then, by Lemma A1, for $j \ge 2$,

$$a_{n,j+1} = b_{n,j+1} + \sum_{m=3}^{j-2} \frac{m}{j+1} b_{n,m} a_{n,j+1-m}.$$

Thus,

$$\begin{aligned} |a_{n,j+1}| &\le k_{j+1} n^{(1-j)/2} + \sum_{m=3}^{j-2} \frac{m}{j+1} k_m n^{(2-m)/2} K_{j+1-m} n^{-(j+1-m)/6} \\ &\le \left[k_{j+1} + \sum_{m=3}^{j-2} \frac{m k_m K_{j+1-m}}{j+1} \right] n^{-(j+1)/6} \\ &= K_{j+1} n^{-(j+1)/6}. \end{aligned}$$

The lemma follows by induction on j. □

Appendix A.3. Properties of ψ

Recall that given an integer $i_0 \ge 3$, a sequence of real numbers $a = (a_0, a_1, \ldots)$ and a positive real number λ, ψ is defined by

$$\psi_{i_0,x,\lambda}(a) = \sum_{j=0}^{3i_0-6} (-1)^j a_j \frac{d^j}{dx^j} \Phi\left(\frac{x}{\sqrt{\lambda}}\right).$$

In the following lemma, we prove some properties of ψ, which we need.

First, there is some notation. For $a = (a_0, a_1, \ldots)$, we define $T^s a$ by $(T^s a)_j = a_{j-s}$ if $j \ge s$, $(T^s a)_j = 0$ if $0 \le j < s$.

Lemma A4. *The following properties of ψ hold:*

(i) If $b_j = \alpha a_j + \beta \bar{a}_j$ for $j = 0, \ldots, 3i_0 - 6$,

$$\psi_{i_0,x,\lambda}(b) = \alpha \psi_{i_0,x,\lambda}(a) + \beta \psi_{i_0,x,\lambda}(\bar{a}).$$

(ii) If $\sigma_n^{-1} > 0$ is bounded and $a_{n,j} = O(n^{-j/6})$ for $0 \le j \le 3i_0 - 6$, then for $s \ge 0$,

$$\sup_{x \in \mathbb{R}} \left| \frac{d^s}{dx^s} \psi_{i_0,x,\sigma_n^2}(a_n) - (-1)^s \psi_{i_0,x,\sigma_n^2}(T^s a_n) \right| = O\left(n^{\frac{s+1-i_0}{2}} \right).$$

(iii) If $\sigma_n^{-1} > 0$ is bounded and $a_{n,j} - \bar{a}_{n,j} = O(n^\alpha)$ for $0 \le j \le 3i_0 - 6$, where α is real, then

$$\sup_{x \in \mathbb{R}} \left| \psi_{i_0,x,\sigma_n^2}(a_n) - \psi_{i_0,x,\sigma_n^2}(\bar{a}_n) \right| = O(n^\alpha).$$

(iv) Suppose that

$$\sum_{j=0}^{\infty} a_{n,j} s^j = \exp\left(\sum_{j=2}^{\infty} b_{n,j} s^j \right), \quad \sum_{j=0}^{\infty} \bar{a}_{n,j} s^j = \exp\left(\sum_{j=2}^{\infty} \bar{b}_{n,j} s^j \right)$$

where for each $j \ge 2$

$$|b_{n,j}| + |\bar{b}_{n,j}| = O(1), \quad b_{n,j} - \bar{b}_{n,j} = O(n^{-(1+i_0)/2}).$$

Then, for each $j \geq 2$, $a_{n,j} - \bar{a}_{n,j} = O(n^{-(1+i_0)/2})$ and if, in addition, $\sigma_n^{-1} > 0$ is bounded, then

$$\sup_{x \in \mathbb{R}} \left| \psi_{i_0,x,\sigma_n^2}(a_n) - \psi_{i_0,x,\sigma_n^2}(\bar{a}_n) \right| = O(n^{-(1+i_0)/2}).$$

Proof. The proof is left as an exception for the reader. □

Appendix A.4. Replacing the Variance by the Sheppard-Corrected Variance in ψ

We use the proposition below to verify (43). To prove the proposition, we need a lemma.

Lemma A5. *For every integer $k \geq 0$ and any $\lambda > 0$,*

$$\frac{\partial^{2k}}{\partial x^{2k}} \Phi\left(\frac{x}{\sqrt{\lambda}}\right) = 2^k \frac{\partial^k}{\partial \lambda^k} \Phi\left(\frac{x}{\sqrt{\lambda}}\right) \tag{A11}$$

and hence for any sequence a

$$\frac{\partial^{2k}}{\partial x^{2k}} \psi_{i_0,x,\lambda}(a) = 2^k \frac{\partial^k}{\partial \lambda^k} \psi_{i_0,x,\lambda}(a). \tag{A12}$$

Proof. The proof of (A11) is a calculus exercise. (A12) follows from

$$\psi_{i_0,x,\lambda}(a) = \sum_{j=0}^{3i_0-6} (-1)^j a_j \frac{\partial^j}{dx^j} \Phi\left(\frac{x}{\sqrt{\lambda}}\right)$$

since then

$$\begin{aligned}
\frac{\partial^{2k}}{\partial x^{2k}} \psi_{i_0,x,\lambda}(a) &= \sum_{j=0}^{3i_0-6} (-1)^j a_j \frac{\partial^{j+2k}}{\partial x^{j+2k}} \Phi\left(\frac{x}{\sqrt{\lambda}}\right) \\
&= \sum_{j=0}^{3i_0-6} (-1)^j a_j \frac{\partial^j}{\partial x^j} 2^k \frac{\partial^k}{\partial \lambda^k} \Phi\left(\frac{x}{\sqrt{\lambda}}\right) \\
&= 2^k \frac{\partial^k}{\partial \lambda^k} \left(\sum_{j=0}^{3i_0-6} (-1)^j a_j \frac{\partial^j}{\partial x^j} \Phi\left(\frac{x}{\sqrt{\lambda}}\right) \right) \\
&= 2^k \frac{\partial^k}{\partial \lambda^k} \psi_{i_0,x,\lambda}(a).
\end{aligned}$$

□

Proposition A2. *Suppose that $a_n = (a_{n,0}, a_{n,1}, \ldots)$ is such that for each j, $a_{n,j} = O(n^{-j/6})$ and σ_n is a sequence of positive numbers such that $\sigma_n^2 \to V > 0$. Suppose also that $\bar{a}_n$ is a sequence determined by*

$$\sum_{j=0}^{\infty} \bar{a}_{n,j} s^j = e^{cs^2/n} \sum_{j=0}^{\infty} a_{n,j} s^j,$$

where c is a constant. Then, for $i_0 \geq 3$ and n being sufficiently large, such that $b_{n,2} := \sigma_n^2 + 2c/n > 0$,

$$\sup_{x \in \mathbb{R}} \left| \psi_{i_0,x,\sigma_n^2}(\bar{a}_n) - \psi_{i_0,x,b_{n,2}}(a_n) \right| = O(n^{(1-i_0)/2}).$$

Proof. Clearly

$$\sum_{j=0}^{\infty} \bar{a}_{n,j} s^j = \left(\sum_{k=0}^{\infty} \frac{c^k}{n^k k!} s^{2k} \right) \sum_{j=0}^{\infty} a_{n,j} s^j = \sum_{k=0}^{\infty} \frac{c^k}{n^k k!} \sum_{j=0}^{\infty} a_{n,j} s^{j+2k}$$
$$= \sum_{k=0}^{\infty} \frac{c^k}{n^k k!} \sum_{\ell=2k}^{\infty} a_{n,\ell-2k} s^{\ell} = \sum_{k=0}^{\infty} \frac{c^k}{n^k k!} \sum_{\ell=0}^{\infty} (T^{2k} a_n)_{\ell} s^{\ell}$$
$$= \sum_{\ell=0}^{\infty} \left[\sum_{k=0}^{\infty} \frac{c^k}{n^k k!} (T^{2k} a_n)_{\ell} \right] s^{\ell},$$

where $T^{2k} a_n$ is defined as in Appendix A.3. Hence

$$\bar{a}_n = \sum_{k=0}^{\infty} \frac{c^k}{n^k k!} T^{2k} a_n.$$

However, when $0 \le j \le 3i_0 - 6$,

$$\bar{a}_{n,j} = \sum_{k=0}^{N} \frac{c^k}{n^k k!} (T^{2k} a_n)_j$$

where $N = \text{floor}((3i_0 - 6)/2)$, since $(T^{2k} a_n)_j = a_{n,j-2k} = 0$ if $k > j/2$. Recall that $a_{n,j} = O(n^{-j/6})$. Hence, by (i) and (ii) in Lemma A4,

$$\psi_{i_0,x,\sigma_n^2}(\bar{a}_n) = \sum_{k=0}^{N} \frac{c^k}{n^k k!} \psi_{i_0,x,\sigma_n^2}(T^{2k} a_n)$$
$$= \sum_{k=0}^{N} \frac{c^k}{n^k k!} \left[\frac{\partial^{2k}}{\partial x^{2k}} \psi_{i_0,x,\sigma_n^2}(a_n) + O(n^{(2k+1-i_0)/2}) \right]$$

so that

$$\psi_{i_0,x,\sigma_n^2}(\bar{a}_n) = \sum_{k=0}^{N} \frac{c^k}{n^k k!} \frac{\partial^{2k}}{\partial x^{2k}} \psi_{i_0,x,\sigma_n^2}(a_n) + O(n^{(1-i_0)/2}). \tag{A13}$$

Next, by Taylor expansion, since $b_{n,2} = \sigma_n^2 + 2c/n$,

$$\psi_{i_0,x,b_{n,2}}(a_n) = \sum_{k=0}^{N} \frac{2^k c^k}{n^k k!} \frac{\partial^k}{\partial \lambda_n^k} \psi_{i_0,x,\lambda_n}(a_n)|_{\lambda_n = \sigma_n^2} + R_{N+1},$$

where, with θ_n between σ_n^2 and $b_{n,2}$,

$$R_{N+1} = \frac{2^{N+1} c^{N+1}}{n^{N+1}(N+1)!} \frac{\partial^{N+1}}{\partial \lambda_n^{N+1}} \psi_{i_0,x,\lambda_n}(a_n)|_{\lambda_n = \theta_n}$$

and, using Lemma A5,

$$R_{N+1} = \frac{c^{N+1}}{(N+1)!} \frac{\partial^{2(N+1)}}{\partial x^{2(N+1)}} \psi_{i_0,x,\theta_n}(a_n) n^{-(N+1)} = O(n^{-(N+1)})$$

uniformly with respect to x, since

$$\frac{\partial^{2(N+1)}}{\partial x^{2(N+1)}} \psi_{i_0,x,\theta_n}(a_n) = \sum_{j=0}^{3i_0-6} (-1)^j a_{n,j} \frac{\partial^{2(N+1)+j}}{\partial x^{2(N+1)+j}} \Phi(x/\sqrt{\theta_n})$$

is bounded as a function of n and x: this follows from Lemma A2 using (A2) with V_n replaced by θ_n which $\to V > 0$ as $n \to \infty$, and the fact that $a_{n,j}$ is bounded for each j. Hence,

$$\psi_{i_0,x,b_{n,2}}(a_n) = \sum_{k=0}^{N} \frac{2^k c^k}{n^k k!} \frac{\partial^k}{\partial \lambda_n^k} \psi_{i_0,x,\lambda_n}(a_n)|_{\lambda_n=\sigma_n^2} + O(n^{-(N+1)}). \tag{A14}$$

Now, using Lemma A5 again and also (A13),

$$\begin{aligned}\sum_{k=0}^{N} \frac{2^k c^k}{n^k k!} \frac{\partial^k}{\partial \lambda_n^k} \psi_{i_0,x,\lambda_n}(a_n)|_{\lambda_n=\sigma_n^2} &= \sum_{k=0}^{N} \frac{c^k}{n^k k!} \frac{\partial^{2k}}{\partial x^{2k}} \psi_{i_0,x,\sigma_n^2}(a_n) \\ &= \psi_{i_0,x,\sigma_n^2}(\bar{a}_n) + O(n^{(1-i_0)/2}),\end{aligned}$$

Combining this with (A14), we obtain

$$\begin{aligned}\psi_{i_0,x,\sigma_n^2}(\bar{a}_n) &= \psi_{i_0,x,b_{n,2}}(a_n) + O(n^{(1-i_0)/2}) + O(n^{-(N+1)}) \\ &= \psi_{i_0,x,b_{n,2}}(a_n) + O(n^{(1-i_0)/2})\end{aligned}$$

since $N+1 > \frac{3i_0-6}{2} > \frac{i_0-1}{2}$. All the O-terms are uniform in x, and therefore the result follows. □

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