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Some Stochastic Orders over an Interval with Applications

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Abstract: In this article, we study stochastic orders over an interval. Mainly, we focus on orders related to the Laplace transform. The results are then applied to obtain a bound for heavy-tailed distributions and are illustrated by some examples. We also indicate how these ordering relationships can be adapted to the classical risk model in order to derive a moment bound for ruin probability. Finally, we compare it with other existing bounds.

Keywords: stochastic orders; Laplace transform; heavy-tailed distribution; ruin probability; bound

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1. Introduction

It is often an extremely complicated task to carry out explicit calculations of some stochastic models' characteristics. In order to extract as much information as possible, many mathematical concepts have been developed. A convenient technique to study these situations is a comparison of the random variables (r.v.'s) and functions associated with each of them. In the literature, the theory of stochastic orders has been widely studied, since it is a useful tool in many different areas of probability theory, statistics, and reliability; for more details, see Shaked and Shanthikumar (2007).

As it is well known, Laplace transforms, as well as orders related to them, play an important role in engineering, economics, actuarial science, etc. In the literature on stochastic orders, many authors define orders related to the Laplace transform, e.g., Alzaid et al. (1991) considered preservation properties and applications of the stochastic order based on the Laplace transform of a r.v. Also, Shaked and Wong (1997) defined stochastic orders based on ratios of Laplace transforms. More recently, Li et al. (2009) introduced a new stochastic order based on ratios of differentiated Laplace transforms.

Laplace transform (LT) has several interpretations in many areas of finance, insurance, and reliability theory.

1. *Finance*: Buser (1986) showed that the cash flow distribution can be represented by its Laplace transform. In particular, the present value (PV) of a cash flow $C(t)$ for a given rate of discount r is
$$M(r) = \int_0^\infty e^{-rt} C(t) dt.$$

The PV is a useful tool to compare financial amounts, such as incomes, annuities, bonds, etc., that have stipulated structured payment schedules. Buser's argument introduces a change to the meaning of the variable r. We generally think of PV as a function of the discount rate, but in the LT, the variable r is allowed to take complex values. The real part of r can be described as the discount rate, while the imaginary part is interpreted as a frequency. Insurance applications of the Laplace transform in a PV context have appeared in several papers (see Goovaerts and De Schepper 1997; De Schepper et al. 1992). Here, we focus on the deterministic nature of the discount rate and its applications.

2. *Utility:* The usefulness of money may not be evaluated solely on a monetary scale. Thus, the usefulness of €x for an individual or a company is a function $u(x)$, the utility of €x. The expected utility hypothesis serves as a reference guide for decision makers with utility u, where the random future incomes modeled by the r.v.'s X and Y, i.e.,

$Eu(X) \leq Eu(Y)$, provided the expectations exist. In this framework, the Laplace transform order represents preferences of decision makers with a *negative exponential* utility function given by

$$u(x) = 1 - e^{hx}, \quad h < 0. \tag{1}$$

Here, $-h$ is the Arrow–Pratt measure of absolute risk aversion (see Goovaerts and Laeven 2008). Thus, a result $X \leq_{Lt} Y$ means that the income Y is preferred to the income X by all decision makers with a utility function of the form (1), for any constant risk aversion. For further research in this framework, we refer to Denuit (2001). Also, Goovaerts et al. (2004) gave some applications of the comparison of Laplace transforms in the context of risk measures.

3. *Life Insurance:* The topic of PV is central to actuarial science. Let (x) denote a person aged x, where $x \geq 0$. We denote his or her remaining lifetime of (x) by a continuous r.v. T_x considering that the death of (x) can occur at any age greater than x. We typically assume that the interest rate is constant and fixed. This is appropriate, for example, if the premiums for an insurance policy are invested in risk-free bonds, all yielding the same interest rate, so that the term structure is flat. The whole life insurance plan pays EUR 1 at death. Since the PV of a future payment depends on the payment date, the PV of the benefit payment is a function of the time of death, and is therefore modeled as a r.v. For our (x), the PV of a benefit of EUR 1 payable immediately on death is represented by a r.v. denoted as Z. This r.v. is defined as $Z = e^{-\delta T_x}$, where δ is known as the continuously compounded rate of interest. The expected present value (EPV) of the whole life insurance benefit payment with a sum insured of EUR 1 euro is $Ee^{-\delta T_x} \equiv \overline{A}_x$.

   As T_x has probability density function $f_x(t)$, we have

$$\overline{A}_x = \int_0^\infty e^{-\delta t} f_x(t) dt.$$

When remaining lifetime r.v.'s need to be compared, if T_x and T_y are two such remaining lifetimes with $E(e^{-tT_x}) \geq E(e^{-tT_y}), t \geq 0$, it is easily seen that this order of the Laplace transforms of r.v.'s T_x and T_y then means that the whole life premium relating to T_x is always higher than the whole life premium relating to T_y, whatever the interest rate is; T_y thus represents a "longer" life-length than T_x.

Consider now the case when the annuity is payable continuously at a rate of 1 per year as long as (x) survives. Then, the EPV is denoted by $\bar{a}_x$. The underlying r.v. is Y. We can directly write

$$EY = \bar{a}_x = \int_0^\infty e^{-\delta t} {}_t p_x dt,$$

where ${}_t p_x$ is the probability that (x) will attain age $x + t$. In this context, other representations can be found in Alzaid et al. (1991) and Belzunce et al. (2007).

4. *Reliability:* Let a device (or a system) have survival function $\overline{F}(t)$ and s be the discount rate. If it produces one unit per minute when functioning and the PV of one unit produced at time t is $1 \cdot e^{-st}$, then the EPV of total output produced during the life of the device is $\int_0^\infty e^{-st}\overline{F}(t)dt$, which is the Laplace transform of survival function $\overline{F}(t)$. Also, a different interpretation of $\int_0^\infty e^{-st}\overline{F}(t)dt$ may be seen as the EPV of the total maintenance cost of a device (or a system); see Shaked and Wong (1997).

The aim of this paper is to further study stochastic orders in the context of reliability theory and risk theory, as they can be an efficient tool to derive useful bounds for quantities of interest. In the actuarial literature, stochastic orders have been widely used in risk theory. Cheng and Pai (2003) studied the stop loss ordering of claim severities, and Denuit et al. (2005) suggested stochastic bounds on functions of dependent insurance risks. Moreover, Mitric and Trufin (2016) applied stochastic ordering tools in order to study some properties of a ruin measure related to the maximal aggregate loss, while Tsai (2006) and Tsai (2009) used stochastic order concepts to compare ruin probabilities for two surplus processes perturbed by diffusion. Also, Escudero and Ortega (2008) proposed Laplace transform comparison bounds and results for the insurer's aggregate claims assuming dependence among the random retention levels.

In this paper, we discuss some new orders relaxing some hypotheses of known stochastic orders. Tsai and Lu (2010) modified definitions of some orders defining some new stochastic orders over an interval $[a, b]$. In a similar manner, we introduce some stochastic orders over $[a, b]$ related to Laplace transform.

The paper is organized as follows: in Section 2, we recall some known definitions of stochastic orders over an interval, and we introduce some new stochastic orders over $[a, b]$ related to Laplace transforms. In Section 3, we obtain preservation results of some stochastic orders over an interval, and we give some interesting counterexamples. Finally, in Section 4, we give some applications in the context of reliability theory and risk theory and propose an upper bound for heavy-tailed distributions.

2. Preliminaries

When comparing r.v.'s with respect to various partial orders, the variables may not be ordered, especially if they belong to different parametric families of distributions. To study this ordering problem more generally, we relax some stochastic orders to ordering over the interval $[a, b]$, for some $a, b \geq 0$. In the following definition, we give the notion of some stochastic orders, which are defined by Tsai and Lu (2010).

First, if Z is a r.v. with an absolutely continuous d.f. $F_Z = 1 - \overline{F}_Z$, then the hazard (failure) rate of Z is defined as $r_Z(t) = f_Z(t)/\overline{F}_Z(t)$, and the mean residual lifetime function is defined as $m_Z(t) = E(Z - t|Z > t) = \int_t^\infty \overline{F}_Z(u)du/\overline{F}_Z(t)$. The r.v. Z (or F_Z) is said to be DFR (IFR) or decreasing (increasing) failure rate if $r_Z(t)$ is non-increasing (non-decreasing) in t. Also, the r.v. Z (or F_Z) is said to be IMRL (DMRL), or increasing (decreasing) mean residual lifetime if $m_Z(t)$ is non-decreasing (non-increasing) in t.

Definition 1. *Let X, Y be two non-negative r.v.'s with d.f. F_X, F_Y, respectively. Then X is said to be smaller than Y in the*

(i) stochastic dominance order over $[a, b]$ (denoted by $X \leq_{st[a,b]} Y$) if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for $a \leq t \leq b$;
(ii) hazard rate order over $[a, b]$ (denoted by $X \leq_{hr[a,b]} Y$) if $r_X(t) \geq r_Y(t)$ for $a \leq t \leq b$;
(iii) mean residual lifetime order over $[a, b]$ (denoted by $X \leq_{mrl[a,b]} Y$) if $m_X(t) \leq m_Y(t)$ for $a \leq t \leq b$;
(iv) the harmonic mean residual lifetime order over $[a, b]$ (denoted by $X \leq_{hmrl[a,b]} Y$) if $\{(1/t) \int_0^t 1/m_X(z)dz\}^{-1} \leq \{(1/t) \int_0^t 1/m_Y(z)dz\}^{-1}$ for $a \leq t \leq b$.

As pointed out in the introduction, the Laplace transform has many applications. When remaining lifetime T_x or cash flow $C(t)$ r.v.'s need to be compared, we use stochastic orders related to the Laplace transform. Of course, this assumption relies on computational convenience rather than realism, since it is extremely rare to have very large discount rates of a whole life premium or interest rates of an insurance policy. Therefore, it is interesting to focus on a specific interval of values of these rates defining some new stochastic orders related to the Laplace transform over $[a, b]$.

Next, we give four definitions of stochastic orders related to the Laplace transform between two r.v.'s X and Y, which are assumed non-negative and independent.

Definition 2. *Let X and Y have densities f_1 and f_2, respectively. If*

$$\int_0^\infty f_1(t)e^{-st}dt \geq \int_0^\infty f_2(t)e^{-st}dt, \quad s \in [a, b],$$

then X is smaller than Y in the Laplace transform order over $[a, b]$ (denoted by $X \leq_{Lt[a,b]} Y$).

For the rest of this paper, we will use the definitions of the Laplace–Stieltjes transform of d.f. F and survival function $\overline{F}$ of X, as in Shaked and Shanthikumar (2007),

$$\mathcal{L}_X(s) = \int_0^\infty e^{-sx} dF(x) \quad \text{and} \quad \mathcal{L}^*_X(s) = \int_0^\infty e^{-sx} \overline{F}(x) dx,$$

respectively. It is well known that the Laplace transforms of F and $\overline{F}$, respectively, are connected by the following relation

$$\mathcal{L}^*_X(s) = \frac{1 - \mathcal{L}_X(s)}{s}, \quad s \geq 0. \tag{2}$$

By (2) an equivalent relation in Definition 2 is the following:

$$\int_0^\infty \overline{F}_1(t) e^{-st} dt \leq \int_0^\infty \overline{F}_2(t) e^{-st} dt, \quad s \in [a, b]. \tag{3}$$

Furthermore, we give two stochastic orders based on ratios of the Laplace transforms of X and Y over $[a, b]$.

Definition 3. *Let X and Y with Laplace transforms $\mathcal{L}_X$ and $\mathcal{L}_Y$, respectively. If the ratio*

$$\frac{\mathcal{L}_Y(s)}{\mathcal{L}_X(s)} \quad \textit{is non-increasing in} \quad s \in [a, b],$$

then X is smaller than Y in the Laplace transform ratio order over $[a, b]$ (denoted by $X \leq_{Lt-r[a,b]} Y$).

Definition 4. *If the ratio*

$$\frac{\mathcal{L}^*_Y(s)}{\mathcal{L}^*_X(s)} \quad \textit{is non increasing in} \quad s \in [a, b],$$

then X is smaller than Y in the reverse Laplace transform ratio order over $[a, b]$ (denoted by $X \leq_{r-Lt-r[a,b]}$).

The Laplace transform ratio orders, through the characterizations of Definitions 3 and 4 have a variety of interpretations corresponding to models. Shaked and Wong (1997) gave an interesting interpretation in the context of reliability theory (see also Section 2).

By similar thinking of stochastic order based on the differentiated Laplace transform ratio order (see Li et al. 2009), we define another stochastic order based on derivatives of the Laplace transforms.

Definition 5. *Let X and Y have derivatives of the Laplace transforms $\mathcal{L}'_X(s)$ and $\mathcal{L}'_Y(s)$, respectively. Then X is said to be smaller than Y in differentiated Laplace transform ratio order over $[a, b]$ (denoted by $X \leq_{d-Lt-r[a,b]} Y$) if the ratio*

$$\frac{\mathcal{L}'_Y(s)}{\mathcal{L}'_X(s)},$$

is a non-increasing function in $s \in [a, b]$.

3. Stochastic Orders over an Interval $[a, b]$

Tsai and Lu (2010) proved that the order $\leq_{st[0,b)}$ is closed under convolution. The orders $\leq_{Lt[a,b]}$ and $\leq_{Lt-r[a,b]}$ are also closed under convolution, as we prove in the following theorems.

Theorem 1. *Let four continuous, non-negative, and independent r.v.'s X_i and Y_i have densities f_i and g_i, $i = 1, 2$, respectively. If $X_1 \leq_{Lt[a,b]} Y_1$ and $X_2 \leq_{Lt[a,b]} Y_2$ then*

$$X_1 + X_2 \leq_{Lt[a,b]} Y_1 + Y_2.$$

Proof. Firstly, we calculate Laplace transform $\mathcal{L}_{X_1+X_2}(s)$, associated with the convolution between F_1 and F_2 for $s \in [a, b]$. It is well known (see Feller 1971, p. 434) that the Laplace transform of convolution $F_{X_1+X_2}$ is given by

$$\mathcal{L}_{X_1+X_2}(s) = \mathcal{L}_{X_1}(s)\mathcal{L}_{X_2}(s), \quad s > 0. \tag{4}$$

By assumption that $X_1 \leq_{Lt[a,b]} Y_1$ and $X_2 \leq_{Lt[a,b]} Y_2$, or equivalently

$$\int_0^\infty f_1(t)e^{-st}dt \geq \int_0^\infty g_1(t)e^{-st}dt, \quad s \in [a, b],$$

and

$$\int_0^\infty f_2(t)e^{-st}dt \geq \int_0^\infty g_2(t)e^{-st}dt, \quad s \in [a, b],$$

and the equality (4), it follows that

$$\begin{aligned}\mathcal{L}_{X_1+X_2}(s) &= \int_0^\infty f_1(t)e^{-st}dt \int_0^\infty f_2(t)e^{-st}dt \\ &\geq \int_0^\infty g_1(t)e^{-st}dt \int_0^\infty g_2(t)e^{-st}dt \\ &= \mathcal{L}_{Y_1+Y_2}(s), \quad s \in [a, b].\end{aligned}$$

□

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed (i.i.d.) and non-negative r.v.'s (resp. $Y_1, Y_2, \ldots, Y_n$) . If $X_i \leq_{Lt-r[a,b]} Y_i$, $i = 1, 2, \ldots, n$, then*

$$\sum_{i=1}^n X_i \leq_{Lt-r[a,b]} \sum_{i=1}^n Y_i.$$

Proof. It is well known that the Laplace transform of the r.v. of the sum of n independent r.v. equals the product of individual Laplace transforms (see (4)). If, in addition, they are identically distributed, then it is easy to see that

$$\mathcal{L}_{X_1+X_2+\cdots+X_n}(s) = \prod_{i=1}^n \mathcal{L}_{X_i}(s), \quad s \in [a, b].$$

Now, let

$$\frac{\mathcal{L}_{Y_i}(s)}{\mathcal{L}_{X_i}(s)} \quad \searrow s \in (a, b), \quad i = 1, 2, \ldots, n,$$

and it follows immediately that

$$\frac{\mathcal{L}_{\sum_{i=1}^n Y_i}(s)}{\mathcal{L}_{\sum_{i=1}^n X_i}(s)}, \quad \searrow s \in (a, b), \quad i = 1, 2, \ldots, n.$$

□

It is well established that the order $\leq_{Lt}$ is closed under convolution (see Alzaid et al. 1991), as well as the orders $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$ (see Shaked and Wong 1997). Here, we prove preservation results of the orders $\leq_{Lt[0,s^*]}$, $\leq_{Lt-r[0,s^*]}$ and $\leq_{r-Lt-r[0,s^*]}$ with similar arguments.

Theorem 3. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two sequences of non-negative independent r.v.'s, and let M and N two positive integer-valued r.v.'s, where M and N are independent of the $(X_i)_{i\in\mathbb{N}}$ and the $(Y_i)_{i\in\mathbb{N}}$, respectively. Suppose that there exists a non-negative r.v. Z, such that $X_i \leq_{Lt[0,s^*]} Z \leq_{Lt[0,s^*]} Y_j$, for all i and j and for some $s^* > 0$. If $M \overset{d}{=} N$, then*

$$\sum_{j=1}^{M} X_j \leq_{Lt[0,s^*]} \sum_{j=1}^{N} Y_j.$$

Proof. Note that for all $0 < s < s^*$, we take

$$\begin{aligned}
E\left[exp\left(-s\sum_{j=1}^{M} X_j\right)\right] &= \sum_{n=1}^{\infty} Pr(M=n)\prod_{j=1}^{n} E\left[exp(-sX_j)\right] \\
&\geq \sum_{n=1}^{\infty} Pr(M=n)(E(exp\{-sZ\}))^n \\
&= \sum_{n=1}^{\infty} Pr(M=n)exp\{-n(-\log E(exp\{-sZ\}))\} \\
&= \sum_{n=1}^{\infty} Pr(N=n)exp\{-n(-\log E(exp\{-sZ\}))\} \\
&= \sum_{n=1}^{\infty} Pr(N=n)(E(exp\{-sZ\}))^n \\
&\geq \sum_{n=1}^{\infty} Pr(N=n)\prod_{j=1}^{n} E\left(exp\{-sY_j\}\right) \\
&= E\left(exp\{-s\sum_{j=1}^{N} Y_j\}\right),
\end{aligned}$$

where the first and the last equalities result from the independence of M and N of the $(X_i)_{i\in\mathbb{N}}$ and the $(Y_i)_{i\in\mathbb{N}}$, the first and the last inequalities result from $X_i \leq_{Lt[0,s^*]} Z \leq_{Lt[0,s^*]} Y_j$ for all i and j, and the middle equality follows from $M \overset{d}{=} N$. The assertion now follows. □

Tsai and Lu (2010) proved that the order $\leq_{st[0,s^*]}$ is closed under geometric compounding, and in particular, it is preserved for ruin probabilities. As a result of Theorem 3, we get the following corollary that shows the order $\leq_{Lt[0,s^*]}$ is closed under convolution.

Corollary 1. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ each be a sequence of i.i.d. non-negative r.v.'s, such that $X_i \leq_{Lt[0,s^*]} Y_i, i = 1, 2, \ldots$. Also, let M and N be two integer-valued positive r.v.'s, which are independent of the sequences $(X_i)_{i\in\mathbb{N}}$ and the $(Y_i)_{i\in\mathbb{N}}$, respectively, such that $M \overset{d}{=} N$. Then*

$$\sum_{j=1}^{M} X_j \leq_{Lt[0,s^*]} \sum_{j=1}^{N} Y_j.$$

Theorem 4. *Let $X_1, X_2, \ldots$ be i.i.d. non-negative r.v.'s, and let N_1 and N_2 be two positive integer-valued r.v.'s that are independent of the X_i's. Then*

$$N_1 \leq_{Lt-r[0,p^*]} (\leq_{r-Lt-r[0,p^*]}) N_2 \Rightarrow \sum_{i=1}^{N_1} X_i \leq_{Lt-r[0,s^*]} (\leq_{r-Lt-r[0,s^*]}) \sum_{i=1}^{N_2} X_i,$$

where $p^* = -\log(\mathcal{L}_X(s^*))$.

Proof. For $j = 1, 2$, we get

$$\begin{aligned}\mathcal{L}_{X_1+X_2+\cdots+X_{N_j}}(s) &= \sum_{i=1}^{\infty} Pr(N_j = i)\mathcal{L}_{X_1+X_2+\cdots+X_i}(s) \\ &= \sum_{i=1}^{\infty} Pr(N_j = i)\big(\mathcal{L}_{X_1}(s)\big)^i \\ &= \mathcal{L}_{N_j}\big(-\log \mathcal{L}_{X_1}(s)\big).\end{aligned}$$

The stated results now follow from the assumptions. □

The order $\leq_{d-Lt-r[a,b]}$ is not closed under convolution, as we see in the following example.

Example 1. *Let $X_1, X_2, \ldots$ be i.i.d. non-negative r.v.'s with the same distribution as X and let $Y_1, Y_2, \ldots$ i.i.d. non-negative r.v.'s with the same distribution as Y such that $EX = EY$. Kanellopoulos and Politis (2023) proved that if*

$$\frac{\mathcal{L}'_Y(s)}{\mathcal{L}'_X(s)} \quad \nearrow [0, s_0] \quad \textit{and} \quad \searrow [s_0, \infty],$$

then

$$\frac{\mathcal{L}'_{T_n}(s)}{\mathcal{L}'_{S_n}(s)} \nearrow [0, s_C] \quad \textit{and} \quad \searrow [s_C, \infty],$$

where $s_C < s_0$ and the sums $S_n = X_1 + X_2 + \cdots + X_n$, $T_n = Y_1 + Y_2 + \cdots + Y_n$.

In many cases, when we consider stochastic orders, it is interesting to study whether or not a stochastic order is preserved when taking equilibrium distributions. We recall that the d.f. F_e of a r.v. X^e with tail $\overline{F}_e(x) = \int_x^{\infty} \overline{F}(z)dz/EX$ is called the equilibrium distribution of F. Denuit (2001) showed that the Laplace transform order is closed when taking the equilibrium distribution. We prove that $\leq_{Lt[0,s^*]}$ is also closed in the equilibrium distribution.

Proposition 1. *Let two non-negative r.v.'s X and Y such that $EX = EY$. Then,*

$$X \leq_{Lt[0,s^*]} Y \Leftrightarrow Y^e \leq_{Lt[0,s^*]} X^e.$$

Proof. It is well known that

$$\mathcal{L}_X(s) = s \cdot \int_0^{\infty} e^{-st} F_X(t)dt, \quad 0 \leq s < s^*, \tag{5}$$

which yields

$$\mathcal{L}_X(s) = 1 - s \cdot \int_0^{\infty} e^{-st} \overline{F}_X(t)dt, \quad 0 < s < s^*. \tag{6}$$

Denuit (2001) showed that

$$\mathcal{L}_{X_e}(s) = \frac{1 - \mathcal{L}_X(s)}{s \cdot EX}, \quad s \geq 0. \tag{7}$$

By (5)–(7) the result follows immediately. □

Tsai and Lu (2010) show that $X \leq_{hmrl[0,s^*)} Y \Rightarrow X^e \leq_{st[0,s^*)} Y^e$ for $s^* > 0$. Willmot and Lin (2001) proved that $r_{X^e}(t) = 1/m_X(t)$, for $t \geq 0$. As a result, it follows that $X \leq_{mrl[0,s^*)} Y \Leftrightarrow X^e \leq_{hr[0,s^*)} Y^e$.

In a similar manner, we formulate the following two propositions.

Proposition 2. *Let X, Y be two non-negative r.v.'s. Then, it holds that*

$$X \leq_{Lt-r[0,s^*]} Y \Rightarrow X \leq_{Lt[0,s^*]} Y.$$

Proof. Let $X \leq_{Lt-r[0,s^*]} Y$. Then the ratio $\mathcal{L}_Y(s)/\mathcal{L}_X(s)$ is a non-increasing function in $s \in [0,s^*]$. If $s_1 < s_2$, for $s_1, s_2 \in [0,s^*]$, we have

$$\frac{\mathcal{L}_Y(s_1)}{\mathcal{L}_X(s_1)} \geq \frac{\mathcal{L}_Y(s_2)}{\mathcal{L}_X(s_2)} \overset{s_1=0}{\Rightarrow} 1 \geq \frac{\mathcal{L}_Y(s_2)}{\mathcal{L}_X(s_2)} \Rightarrow \mathcal{L}_X(s_2) \geq \mathcal{L}_Y(s_2).$$

Since $s_2 \in [0,s^*]$ is arbitrary, we get the result $X \leq_{Lt[0,s^*]} Y$. □

Proposition 3. *Let X, Y be two non-negative r.v.'s. Then*

$$X \leq_{hr[0,s^*)} Y \Rightarrow X \leq_{st[0,s^*)} Y.$$

Proof. By assumption we have $r_X(t) \geq r_Y(t), t \in (0,s^*)$. Integrating with respect to $t \in (0,s^*)$, we obtain

$$\int_0^t r_X(z)dz \geq \int_0^t r_Y(z)dz, \quad t \in (0,s^*),$$

so that

$$e^{-\int_0^t r_X(z)dz} \leq e^{-\int_0^t r_Y(z)dz}, \quad t \in (0,s^*),$$

or equivalently, $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for $t \in (0,s^*)$. □

It should be mentioned that the last proposition does not remain true for any interval (a,b). In the following example, we show that $X \leq_{hr[a,b]} Y \not\Rightarrow X \leq_{st[a,b]} Y$.

Example 2. *Let H be a r.v. with tail*

$$\overline{F}_H(t) = \frac{\left(\theta + \lambda + \theta\lambda \log\left(\frac{t}{x_0}\right)\right)}{\theta + \lambda}\left(\frac{t}{x_0}\right)^{-\theta}, \quad t \geq x_0 > 0, \tag{8}$$

and hazard rate

$$r_H(t) = \frac{\theta}{t}\left(1 - \frac{\lambda}{\theta + \lambda + \theta\lambda \log\left(\frac{t}{x_0}\right)}\right), \quad t \geq x_0 > 0,$$

where $\lambda \geq 0$ and $\theta > 0$. The r.v. H is called a mixture Pareto-loggamma distribution and is denoted as $\mathcal{MPLG}(\theta, \lambda, x_0)$. The Pareto (Type I) distribution with tail $\overline{F}_H(t) = (x_0/t)^{\theta}$, $t \geq x_0$ is obtained when $\lambda = 0$ in (8); for further details, see Bhati et al. (2019). Let also Z be an exponential r.v. with a mean of 1. For $\theta = 2.5, \lambda = 1.5$ and $x_0 = 1$, it easy to see (using MATHEMATICA) that $r_H(t) \geq r_Z(t)$ for $t \in (1, 1.9)$. However, it is not true that $\overline{F}_H(t) \leq \overline{F}_Z(t)$ for $t \in (1, 1.9)$. Equivalently,

$$H \leq_{hr[1,1.9]} Z \not\Rightarrow H \leq_{st[1,1.9]} Z.$$

It is well known that the $\leq_{st}$ order is stronger than $\leq_{Lt}$ order, but this is not valid for associated stochastic orders over $(0,s^*)$, in particular $\leq_{st[a,b]} \not\Rightarrow \leq_{Lt[a,b]}$, as we see in the next example.

Example 3. *Let X have a density function*

$$f_X(t) = e^{-2t} + 2te^{-2t}, \quad t \geq 0.$$

Also, let Y be a mixture of two exponentials with parameters 3 and 1, and respective weights $1/4$ and $3/4$, so that $EX = 3/4 < 5/6 = EY$. It is easy to see that

$$\overline{F}_X(t) \geq \overline{F}_Y(t), \quad 0 \leq t \leq 0.8316,$$

and,

$$\overline{F}_X(t) \leq \overline{F}_Y(t), \quad t > 0.8316.$$

However, if we solve the equation $\mathcal{L}_X(s) - \mathcal{L}_Y(s) = 0$, we see that it has a unique solution, such that

$$\int_0^\infty e^{-st}\overline{F}_X(t)dt \leq \int_0^\infty e^{-st}\overline{F}_Y(t)dt, \quad 0 \leq s \leq 0.7321.$$

and,

$$\int_0^\infty e^{-st}\overline{F}_X(t)dt \geq \int_0^\infty e^{-st}\overline{F}_Y(t)dt, \quad s > 0.7321.$$

In particular, it follows that $X \leq_{st[0,0.8316)} Y$ but also $X \leq_{Lt[0,0.7321]} Y$.

4. Applications

4.1. Upper Bound on DFR Reliability Function

Moment bounds on the reliability of a device (or a system) have been a topic of great interest in reliability theory. We derive smooth upper bounds for the survival function when it belongs to a common family (e.g., DFR) with a known moment by applying the results from the ordering over an interval between a DFR lifetime distribution and an exponential distribution. In the literature of reliability theory, consideration of computing bounds can be found in Barlow and Proschan (1981), Sengupta (1994), and Sengupta and Das (2016).

Let X be a non-negative r.v. and Y be an exponential r.v., respectively, with $EX = \mu$ and $EY = 1/b$ where b is going to be determined. We also assume that $r_X(t)$ is non-increasing in $s \in [0, s^*]$ for some $s^* > 0$. If we consider the equation, for any fixed $s \in [0, s^*]$

$$\int_0^s r_X(t) - r_Y(t)dt = \int_0^s (r_X(t) - b)dt = 0$$

we get that

$$b = \frac{1}{s}\int_0^s r_X(t)dt \triangleq f_b(s), \quad s \in [0, s^*], \tag{9}$$

implying that $\int_0^z [r_X(t) - r_Y(t)]dt = \int_0^z [r_X(t) - f_b(s)]dt \geq 0$ for any $z \in [0, s]$, or $X \leq_{hr[0,s^*)} Y$. Now, we consider the r.v. Y is to be exponentially distributed with mean $1/f_b(\mu)$, where f_b is given by (9). Then, by Proposition 3 we have $X \leq_{st[0,s^*)} Y$, where $Y \sim Exp(f_b(\mu))$, or equivalently

$$\overline{F}_X(t) \leq e^{-f_b(\mu)t} := U_{\mathrm{LK}}(t), \quad t \in [0, \mu].$$

In other words, we derive an upper bound for the survival function of r.v. X that belongs to the DFR class (we recall that the DFR class is typically associated with a heavy tail, see Willmot and Lin 2001). Sengupta and Das (2016) obtained a bound for survival function of an r.v. X, when X is IMRL (see also Sengupta 1994). In particular, they proved that

$$\overline{F}(t) \leq K_1(t) = \begin{cases} e^{-\frac{t}{\mu}}, & t \leq \mu, \\ \frac{\mu}{t}e^{-1}, & t > \mu. \end{cases} \tag{10}$$

In the following examples, we illustrate that the bound $U_{\mathrm{LK}}(t)$ is better than bound $K_1(t)$ when X follows Pareto, Weibull, or a mixture of two exponential distributions.

Example 4. *Let $X \sim Pareto(\alpha, \lambda)$ with survival function*

$$\overline{F}_X(t) = \left(\frac{\lambda}{\lambda + t}\right)^{\alpha}, \quad x > 0, a > 1.$$

In this case, the function (9) is given by

$$f_b(t) = \frac{\alpha}{t} ln\left(\frac{\lambda + t}{\lambda}\right), \quad t \in [0, s^*].$$

For $t \leq \mu$, it is easy to see that the inequality $f_b(\mu) \geq \frac{1}{\mu}$ is equivalent to the inequality $\left(1 + \frac{1}{\alpha - 1}\right)^{\alpha} \geq e$, which is well-known result for $\alpha > 1$ (see Dörrie 1965, chp. 12). Then, $U_{LK}(t) \leq K_1(t)$ for $t \in [0, \mu]$.

Example 5. *Let $X \sim Weibull(\lambda, \alpha)$ with survival function*

$$\overline{F}_X(t) = e^{-(\lambda t)^{\alpha}}, \quad t \geq 0, 0 < \alpha < 1, \lambda > 0.$$

Then, we obtain

$$f_b(u) = \lambda^{\alpha} u^{\alpha - 1}, \quad 0 < \alpha < 1, \lambda \geq 0.$$

So, we get $f_b(\mu) \geq 1/\mu$, or equivalently,

$$(\lambda \mu)^{\alpha} \geq 1 \Leftrightarrow$$

$$\left(\lambda \frac{1}{\lambda} \Gamma\left(1 + \frac{1}{\alpha}\right)\right)^{\alpha} \geq 1.$$

For $a < 1$, we have that $\Gamma(1 + \frac{1}{\alpha}) > \Gamma(2) = 1$ and it follows immediately that $U_{LK}(t) \leq K_1(t)$ for $t \in [0, \mu]$.

Example 6. *Let X follow a mixture of two exponentials distributions with density function*

$$f_X(t) = a_1 e^{-b_1 t} + a_2 e^{-b_2 t},$$

where $a_1 + a_2 = 1$. In this case, $f_b(\mu) \geq 1/\mu$ is equivalent to the inequality

$$\frac{1}{\mu} \int_0^{\mu} \frac{a_1 b_1 e^{-b_1 t} + a_2 b_2 e^{-b_2 t}}{a_1 e^{-b_1 t} + a_2 e^{-b_2 t}} dt \geq \frac{1}{\mu}.$$

Keeping in mind that $\mu = \frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2}$, after some straightforward calculations it follows that

$$ln\left(\frac{1}{a_1 e^{-b_1 \mu} + a_2 e^{-b_2 \mu}}\right) \geq 1,$$

or,

$$1 \geq e(a_1 e^{-b_1 \mu} + a_2 e^{-b_2 \mu}).$$

Therefore, we get that

$$1 \geq a_1 e^{a_2 \frac{b_2 - b_1}{b_2}} + a_2 e^{a_1 \frac{b_1 - b_2}{b_1}}.$$

Without loss of generality we assume that $b_2 > b_1$, so we get

$$a_1 e^{a_2 \frac{b_2 - b_1}{b_2}} + a_2 e^{a_1 \frac{b_1 - b_2}{b_1}} \leq a_1 e^{a_2} + a_2 \leq a_1 + a_2 = 1,$$

and the result follows.

Following the above, three numerical examples are given for illustration when the distributions are Pareto, Weibull, and a mixture of two exponentials, respectively; Our proposed bound is compared with an existing bound (see (10)). In particular, in Table 1 we consider three cases, when $X \sim Pareto(4, 3)$, $X \sim Weibul(2, 1/3)$ and X has a survival function $\overline{F}(x) = (1/3)e^{-2x} + (2/3)e^{-x/5}$, respectively.

Table 1. Upper bounds for DFR distribution functions.

t	$\overline{F}(t)$	$K_1(t)$	$U_{LK}(t)$
	(**a**) Pareto (4, 3)		
0.05	0.936	0.951	0.944
0.1	0.877	0.904	0.891
0.2	0.772	0.818	0.794
0.5	0.539	0.606	0.562
0.33	0.448	0.513	0.464
0.75	0.409	0.472	0.421
1	0.316	0.367	0.316
	(**b**) Mixture of two exponentials		
0.05	0.961	0.985	0.984
0.1	0.926	0.971	0.968
0.2	0.863	0.944	0.938
0.5	0.725	0.866	0.854
1	0.590	0.751	0.729
3.2	0.352	0.402	0.364
3.5	0.331	0.367	0.331
	(**c**) Weibull (2, 1/3)		
1/10	0.5572	0.9672	0.9412
1/5	0.4786	0.9355	0.8859
1/2	0.3678	0.8464	0.7387
1	0.2836	0.7165	0.5456
1.5	0.2363	0.6065	0.4031
2	0.2044	0.5134	0.2977
2.5	0.1808	0.4345	0.2199
2.9	0.1658	0.3803	0.1726
3	0.1624	0.3678	0.1624

Remark 1. *Heavy-tailed distributions play a significant role in modeling insurance loss data and financial returns. While infrequent in occurrence, these losses are the ones that have the most important impact on an insurer's operations and could potentially result in the company facing bankruptcy. Under such conditions, heavy-tailed distributions have demonstrated their suitability in modeling this type of data (see McNeil 1997 and Resnick 1997). In particular, distributions such as Weibull and gamma are suitable for small losses, whereas Pareto and Lomax are suitable for large losses. Actuaries frequently use these distributions for modeling insurance losses, but calculations with such distributions in actuarial models may sometimes be a challenging or even impossible task. In order to overcome these difficulties, they study methods that lead to bounds, asymptotic formulas, or approximations of these distributions. In Table 1 we see a robust bound for three known heavy-tailed distributions. Moreover, in the following section, we provide another numerical example (see Example 7) to handle the difficulties of calculating a closed form for the ruin probability in situations where claim sizes or ladder heights follow heavy-tailed distributions (see Rolski et al. 1999).*

4.2. Classical Risk Model

Explicit expressions of the ruin probability for the classical risk model are often hard or impossible to evaluate. The theory of stochastic orders over an interval can be an efficient method to obtain a smooth upper bound for the heavy-tailed distribution of claim size and the ruin probability for the classical risk model. Many authors have utilized the theory of stochastic orders in risk models. Tsai and Lu (2010) applied some stochastic orders (see Definition 1) to obtain bounds for ruin probabilities for the surplus process perturbed by diffusion.

Consider the following surplus process:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0, \tag{11}$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the premium rate, and the Y_i's represent the sizes of claims. We assume that the sizes of these claims $Y_1, Y_2, \ldots$ are i.i.d. positive r.v.'s with a distribution P concentrated on $(0, \infty)$ and they arrive at an insurer according to a Poisson process $\{N(t) : t \geq 0\}$ with intensity λ. We further assume that the claims are independent of the claim-arrival process.

Let $\psi(u)$ denote the probability of ruin with initial capital u, i.e., $\psi(u) = Pr(U(t) < 0$ for some $t > 0)$. Let F be the equilibrium distribution associated with P, and it has density $\overline{P}(t)/\mu$, where μ is the first moment of the claim-size distribution. We write ϕ for $(1+\theta)^{-1}$, where $\theta = (c - \lambda\mu)/\lambda\mu$ is the premium loading factor associated with the surplus process. We assume that is positive, which ensures in particular that $\psi(u) < 1$ for all u. Then $0 < \phi < 1$, and ϕF is a defective probability distribution. The Pollaczek–Kinchine formula (see Rolski et al. 1999, Theorem 5.3.4) gives

$$\psi(u) = \sum_{n=1}^{\infty} (1-\phi)\phi^n \overline{F^{*n}}(u), \quad u > 0, \tag{12}$$

where $F^{*n}(u)$ is the $n-$fold Lebesque–Stieltjes convolution power of F and $\overline{F^{*n}}(u) = 1 - F^{*n}(u)$. The probability of ruin is given by the tail of a particular compound geometric distribution (see Willmot and Lin 2001, chp. 7). In the general case, P may not be available analytically, and so the corresponding probability of ruin. It is worth mentioning that, for heavy-tailed claims, getting explicit expressions for the ruin probability is a very difficult task. Next, we give an example deriving bound for the ruin probability $\psi(u)$ when the claim size distribution follows Pareto $(3, 1)$ and Pareto $(3, 2)$, respectively.

Example 7. *Suppose that the claim size distribution is Pareto with tail*

$$\overline{P}(x) = \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}. \tag{13}$$

The mean claim size is then $EY = \lambda/(\alpha - 1)$*. We assume that the relative security loading is* $\theta = 1/10$*, which implies that* $\phi = 10/11$*. In Table 2, we compare the performance of the bound derived for the ruin* $\psi(u)$ *in Section 4.1 against other bounds obtained in the literature for two cases of (13). In particular, for* $\alpha = 3$ *and* $\lambda = 1$*, the bound* U_{CP} *is the upper bound for the ruin probability given in Theorem 4.5 of Chadjiconstantinidis and Politis (2005), while the bounds* $U_{PP1}(u)$ *and* $U_{PP2}(u)$ *are those obtained from Theorem 4.1 of Psarrakos and Politis (2008) using* $U(u) = U_{CP}(u)$ *as their initial bound. We see that for all values of u, our upper bound performs better than the upper bound* $U_{CP}(u)$*, and for some values near the mean, it performs better than the upper bounds* $U_{PP_1}(u)$ *and* $U_{PP_2}(u)$*. Similarly, in the second case, when claim size distribution follows Pareto(3, 2), we notice that our bound has better performance of* $U_{CP}(u)$ *for all u and it performs better than* $U_{PP_1}(u)$ *and* $U_{PP_2}(u)$ *for some values near to the mean. We point out that an advantage of our bound in comparison with bounds* $U_{PP_1}(u)$ *and* $U_{PP_2}(u)$ *is that our method does not require an initial bound.*

Table 2. Upper bounds for $\psi(u)$.

u	$U_{\mathrm{LK}}(u)$	$U_{PP1}(u)$	$U_{PP2}(u)$	$U_{CP}(u)$
		(**a**) Pareto(3, 1)		
0.2	0.8865	0.8808	0.8806	0.8888
0.25	0.8809	0.8750	0.8747	0.8844
0.5	0.8536	0.8509	0.8493	0.8635
0.75	0.8271	0.8322	0.8283	0.8437
0.9	0.8116	0.8225	0.8170	0.8322
1	0.8014	0.8166	0.8099	0.8247
		(**b**) Pareto (3, 2)		
0.5	0.8809	0.8750	0.8747	0.8844
1	0.8536	0.8509	0.8493	0.8635
1.25	0.8402	0.8411	0.8384	0.8535
1.5	0.8271	0.8322	0.8283	0.8437
1.75	0.8142	0.8241	0.8189	0.8341
2	0.8014	0.8166	0.8099	0.8247

Finally, we give an example using Laplace transform order over an interval in the context of the classical risk model.

Example 8. *Let X and Y as Example 3 be the ladder heights of two different surplus processes (see Equation (12)). Suppose the relative security loading is the same for both portfolios,* $\theta = 0.1$*. Then, with a simple program in MATHEMATICA, we see that*

$$\psi_X(u) \geq \psi_Y(u), \quad 0 \leq u \leq 1.5336,$$

and

$$\int_0^\infty e^{-st}\psi_X(t)dt \geq \int_0^\infty e^{-st}\psi_X(t)dt, \quad 0 \leq s \leq 0.7321.$$

With other words the stochastic order $\leq_{Lt[0,s^*]}$ *is closed in compound geometric distributions.*

5. Conclusions

In this paper, we have defined some new stochastic orders over an interval related to the Laplace transform. The motivation comes from Tsai and Lu (2010), who provided bounds for the ruin probability perturbed by diffusion using stochastic comparisons over an interval between claim size random variables. It is often more realistic to compare financial quantities (for instance discount rate) for an interval. We proposed a bound for heavy-tailed distributions using stochastic orders over an interval and illustrate the results giving numerical examples of well-known heavy-tailed distributions. We also provided a way to derive bounds for the ruin probability in the context of classical risk model.

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References

Alzaid, Abdulhamid, Jee Soo Kim, and Frank Proschan. 1991. Laplace ordering and its applications. *Journal of Applied Probability* 28: 116–30. [CrossRef]

Barlow, Richard E., and Frank Proschan. 1981. *Statistical Theory of Reliability and Life Testing: Probability Models.* Silver Spring: To Begin With.

Belzunce, Félix, Eva-María Ortega, and José Ruiz. 2007. On non-monotonic ageing properties from the Laplace transform, with actuarial applications. *Insurance: Mathematics and Economics* 40: 1–14. [CrossRef]

Bhati, Deepesh, Enrique Calderin-Ojeda, and Mareeswaran Meenakshi. 2019. A new heavy tailed class of distributions which includes Pareto. *Risks* 7: 99. [CrossRef]

Buser, Stephen A. 1986. Laplace transforms as present value rules: A note. *The Journal of Finance* 41: 243–47. [CrossRef]

Chadjiconstantinidis, Stathis, and Konstadinos Politis. 2005. Non-exponential bounds for stop-loss premiums and ruin probabilities. *Scandinavian Actuarial Journal* 5: 335–57. [CrossRef]

Cheng, Yu, and Jeffrey S. Pai. 2003. On the *n*th stop-loss transform order of ruin probability. *Insurance: Mathematics and Economics* 32: 51–60. [CrossRef]

De Schepper, Ann, Marc J. Goovaerts, and Freddy Delbaen. 1992. The Laplace transform of annuities certain with exponential time distribution. *Insurance: Mathematics and Economics* 11: 291–94. [CrossRef]

Denuit, Michel. 2001. Laplace transform ordering of actuarial quantities. *Insurance: Mathematics and Economics* 29: 83–102. [CrossRef]

Denuit, Michel, Jan Dhaene, Marc J. Goovaerts, and Rob Kaas. 2005. *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. New York: Wiley.

Dörrie, Heinrich. 1965. *100 Great Problems of Elementary Mathematics.* New York: Dover Publications, INC.

Escudero, Laureano F., and Eva-María Ortega. 2008. Actuarial comparisons for aggregate claims with randomly right-truncated claims. *Insurance: Mathematics and Economics* 43: 255–62. [CrossRef]

Feller, William. 1971. *An Introduction to Probability Theory and Its Application*. New York: Wiley, vol. II.

Goovaerts, Marc J., and Ann De Schepper. 1997. IBNR reserves under stochastic interest rates. *Insurance: Mathematics and Economics* 21: 225–44. [CrossRef]

Goovaerts, Marc J., and Roger J.A. Laeven. 2008. Actuarial risk measures for financial derivative pricing. *Insurance: Mathematics and Economics* 42: 540–47. [CrossRef]

Goovaerts, Marc J., Rob Kaas, Roger J. A. Laeven, and Qihe Tang. 2004. A comonotonic image of independence for additive risk measures. *Insurance: Mathematics and Economics* 35: 581–94. [CrossRef]

Kanellopoulos, Lazaros, and Konstadinos Politis. 2023. Some Results for Stochastic Orders and Aging Properties Related to the Laplace Transform. *Submitted for publication*.

Li, Xiaohu, Xiaoliang Ling, and Ping Li. 2009. A new stochastic order based upon Laplace transform with applications. *Journal of Statistical Planning and Inference* 139: 2624–30. [CrossRef]

McNeil, Alexander J. 1997. Estimating the tails of loss severity distributions using extreme value theory. *Astin Bulletin* 27: 117–37. [CrossRef]

Mitric, Ilie-Radu, and Julien Trufin. 2016. On a risk measure inspired from the ruin probability and the expected deficit at ruin. *Scandinavian Actuarial Journal* 2016: 932–51. [CrossRef]

Psarrakos, Georgios, and Konstadinos Politis. 2008. Tail bounds for the joint distribution of the surplus prior to and at ruin. *Insurance: Mathematics and Economics* 42: 163–76. [CrossRef]

Resnick, Sidney I. 1997. Discussion of the Danish data on large fire insurance losses. *Astin Bulletin* 27: 139–51. [CrossRef]

Rolski, Tomasz, Hanspeter Schmidli, Volker Schmidt, and Jozef Teugels. 1999. *Stochastic Processes for Insurance and Finance.* Chichester: Wiley.

Sengupta, Debasis. 1994. Another look at the moment bounds on reliability. *Journal of Applied Probability* 31: 777–87. [CrossRef]

Sengupta, Debasis, and Sudipta Das. 2016. Sharp bounds on DMRL and IMRL classes of life distributions with specified mean. *Statistics and Probability Letters* 119: 101–7. [CrossRef]

Shaked, Moshe, and J. George Santhikumar. 2007. *Stochastic Orders.* Springer Series in Statistics. Berlin/Heidelberg: Springer.

Shaked, Moshe, and Tityik Wong. 1997. Stochastic orders based on ratios of Laplace transforms. *Journal of Applied Probability* 34: 404–19. [CrossRef]

Tsai, Cary Chi-Liang. 2006. On the stop loss transform and order for the surplus process perturbed by diffusion. *Insurance: Mathematics and Economics* 39: 151–70. [CrossRef]

Tsai, Cary Chi-Liang. 2009. On the ordering of ruin probabilities for the surplus process perturbed by diffusion. *Scandinavian Actuarial Journal* 2009: 187–204. [CrossRef]

Tsai, Cary Chi-Liang, and Yi Lu. 2010. An effective method for constructing bounds for ruin probabilities for the surplus process perturbed by diffusion. *Scandinavian Actuarial Journal* 3: 200–20. [CrossRef]

Willmot, Gordon E., and X. Sheldon Lin. 2001. *Lundberg Approximations for Compound Distributions with Insurance Applications*. New York: Springer.