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Bidual Representation of Expectiles

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Abstract: Downside risk measures play a very interesting role in risk management problems. In particular, the value at risk (VaR) and the conditional value at risk (CVaR) have become very important instruments to address problems such as risk optimization, capital requirements, portfolio selection, pricing and hedging issues, risk transference, risk sharing, etc. In contrast, expectile risk measures are not as widely used, even though they are both coherent and elicitable. This paper addresses the bidual representation of expectiles in order to prove further important properties of these risk measures. Indeed, the bidual representation of expectiles enables us to estimate and optimize them by linear programming methods, deal with optimization problems involving expectile-linked constraints, relate expectiles with VaR and CVaR by means of both equalities and inequalities, give VaR and CVaR hyperbolic upper bounds beyond the level of confidence, and analyze whether co-monotonic additivity holds for expectiles. Illustrative applications are presented.

Keywords: VaR and CVaR; expectile; dual and bidual representations; risk optimization; risk bounds and equalities

JEL Classification: G21; C22; C61; C02

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1. Introduction

Downside risk measures have been used in actuarial science, mathematical finance, and more general risk management problems. The value at risk (VaR) and the conditional value at risk (CVaR) are probably the most famous downside risk measures, since they are very intuitive and easy to interpret in practice. Expectile risk measures are much less commonly used, perhaps because their practical interpretation is not so obvious. This lower interest in expectiles is observed even in the Basel III (banking) and Solvency II (insurance) regulatory systems. Nevertheless, expectile risk measures reflect very interesting properties. They are coherent, in the sense of Artzner et al. (1999), and elicitable (Bellini et al. 2014). In particular, their elicitability has important implications in backtesting (Bellini and Di Bernardino 2017) and other important applications (Embrechts et al. 2021). Both coherence (which fails for VaR) and elicitability (which fails for CVaR) may justify the use of expectiles. Moreover, Zou (2014), Bellini and Di Bernardino (2017), and Tadese and Drapeau (2020), among others, have shown close relationships between expectiles and CVaR, making it easier to interpret expectiles as downside risks.

Dual representation is another cornerstone in downside risk (Artzner et al. 1999; Föllmer and Schied 2002, etc.). Indeed, dual representations have played a critical role in estimating, managing, and optimizing downside risks in practice. Since dual representation frequently implies that a downside risk measure is the optimal value of a linear optimization problem, it makes sense to study the dual optimization problem of the dual representation, that is, the bidual representation of a downside risk measure. That is exactly the main objective of this paper, with a special focus on expectiles. In other words, the leitmotif of this paper is the use of bidual representation and several properties of the duality theory of linear programming as a unified methodology to address important issues affecting expectiles.

The paper outline is as follows. Section 2 begins by synthesizing the most important ideas related to VaR, CVaR, and expectiles. Their usual definitions and properties are presented despite the fact that they are not new; however, the main purpose is to facilitate the reading of the paper. This approach allows us to introduce the new Theorem 5 and its proof in a natural manner. Theorem 5 is one of the most important results of this paper, since it provides us with the bidual representation of the expectile and a new way to relate expectiles and linear programming methods. Theorem 5 leads to Theorems 6 and 7, which are the most important results of Section 3. Indeed, Theorem 6 shows that the optimization of expectiles can be reduced to a linear (convex) problem if the constraints are linear (convex). Analogously, Theorem 7 shows that optimization problems involving expectile-linked constraints can be linearized as well. Needless to say, the linearization of risk optimization problems has been a very important question in the risk analysis literature (Rockafellar and Uryasev 2000; Konno et al. 2005; Balbás and Charron 2019, etc.).

Important relationships between CVaR and expectiles have been addressed in Delbaen (2013), Bellini and Di Bernardino (2017), and Tadese and Drapeau (2020), to name a few. Section 4 is devoted to showing that these relationships may be also addressed by means of the bidual representation given in Theorem 5. In particular, by computing the expectation and the expectile of an arbitrary random gain, you will have an upper bound of the CVaR given by a simple hyperbolic function of the CVaR confidence level. Every confidence level may be involved in this simple formula. The other important finding of this section is Corollary 6, since it allows us to characterize whether co-monotonic additivity holds for expectiles. In general, the co-monotonic additivity of a downside risk measure is required in many practical applications (Dhaene et al. 2002; Bellini et al. 2021; Balbás et al. 2022, etc.), but it may fail for expectiles (Delbaen 2013).

The relationships in Section 4 are inequalities, and an obvious question is whether they can be improved. This problem is addressed in Section 5, where it is shown that they often become equalities, i.e., they cannot be improved (Theorems 10 and 11 and Corollary 9). Moreover, when they are equalities, they provide new ways to estimate different risk measures in practical applications, with special focus on the value at risk. It is known that topics related to the practical estimation of risks are very important in real-world applications (Buch et al. 2023; Dacorogna 2023, etc.).

Section 6 is devoted to illustrating all the above findings in a very popular actuarial example, namely, the (maybe optimal) combination of reinsurance contracts and financial investments. To the best of our knowledge, this is the first paper that simultaneously involves reinsurance contracts, financial markets, VaR, CVaR, and expectiles. A very general approach would significantly increase the paper length and therefore is beyond the scope of this work, so several simplifications are incorporated. In any case, the simplifications are sufficient to illustrate the main results of the previous sections, which are the major purpose. In fact, the bidual representation of expectiles enables us to estimate and optimize them by means of linear programming methods, relate them with VaR and CVaR by means of both equalities and inequalities, give VaR and CVaR upper bounds beyond the level of confidence, and analyze whether co-monotonic additivity holds for expectiles. Section 7 presents both a general discussion about contributions/limitations and the main conclusion.

2. Dual and Bidual Representations

As indicated in the introduction, this section is devoted to synthesizing several important and well-known properties of VaR, CVaR, and expectiles. They are given without proofs, but their presence will facilitate the reading of the paper. Furthermore, a new representation theorem for expectiles is given (Theorem 5), which will be vital in future sections.

2.1. VaR and CVaR

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of states of nature Ω, the σ-algebra $\mathcal{F}$ reflecting the information available at a future date T, and the probability measure $\mathbb{P}$. Denote by $\mathbb{E}(y)$ the mathematical expectation of every $\mathbb{R}$-valued random variable y defined on Ω. Unless the opposite is indicated, $\mathbb{E}(y)$ exists and is finite for every random variable in this paper. In other words, every random variable belongs to the classical space L^1 (or $L^1(\Omega, \mathcal{F}, \mathbb{P})$, if necessary).

Fix $\mu^* \in (0,1)$. If $F_y(x) = \mathbb{P}(y \leq x)$ is the cumulative distribution function of the random variable y,[1] then the value at risk of y with the level of confidence $1 - \mu^* \in (0,1)$ is given by

$$VaR_{1-\mu^*}(y) := -Inf\{x \in \mathbb{R};\ F_y(x) > \mu^*\} \in \mathbb{R},$$

and the conditional value at risk of y with the same confidence level is given by

$$CVaR_{1-\mu^*}(y) := \frac{1}{\mu^*} \int_0^{\mu^*} VaR_{1-\beta}(y) d\beta \in \mathbb{R}.$$

Previous papers provided us with representation theorems for both VaR and $CVaR$.

Theorem 1 (VaR bidual representation, Balbás et al. 2017)**.** *$VaR_{1-\mu^*}(y)$ is the optimal value of the bounded and solvable problem*

$$Min\ \lambda \begin{cases} y = \lambda_m - \lambda_M - \lambda \\ \mathbb{E}(z^*) = 1 \\ z^* \leq 1/\mu^* \\ \lambda_m z^* = \lambda_M (1/\mu^* - z^*) = 0 \\ z^*,\ \lambda_m,\ \lambda_M \geq 0, \end{cases} \tag{1}$$

with $(\lambda, z^*, \lambda_m, \lambda_M) \in \mathbb{R} \times \left(L^1\right)^3$ being the decision variable.

Theorem 1 provides us with a bidual representation of VaR. There exists a dual representation as well (Koenker 2005), but it is not needed in this paper. Theorem 1 was first proved in Balbás et al. (2017), and later Balbás and Charron (2019) showed that similar methods may allow us to prove extensions and/or slight modifications of Theorem 1. A particular case is Corollary 2 below.

Corollary 1. *If there exists $\varepsilon > 0$ such that $F_y(x) < \mu^*$ for $-VaR_{1-\mu^*}(y) - \varepsilon < x < -VaR_{1-\mu^*}(y)$, then $(\lambda, \lambda_m, \lambda_M)$ remains constant for every (1)-feasible $(\lambda, z^*, \lambda_m, \lambda_M)$, and $\lambda = VaR_{1-\mu^*}(y)$ holds.*

Remark 1. *In particular, $(\lambda, \lambda_m, \lambda_M)$ is unique if there exists $\varepsilon > 0$ such that F_y is strictly increasing in the interval $\left(-VaR_{1-\mu^*}(y) - \varepsilon, -VaR_{1-\mu^*}(y)\right)$. The existence of such an ε obviously holds if there exist $-\infty \leq u < v \leq +\infty$ such that $F_y : (u,v) \to \mathbb{R}$ is strictly increasing, $F_y(x) = 0$ for $x < u$, and $F_y(x) = 1$ for $x > v$.*

Theorem 2 (CVaR dual representation, Rockafellar et al. 2006)**.** *$CVaR_{1-\mu^*}(y)$ is the optimal value of the bounded and solvable problem*

$$Max\ \ -\mathbb{E}(yz^*) \begin{cases} \mathbb{E}(z^*) = 1 \\ 0 \leq z^* \leq 1/\mu^*, \end{cases} \tag{2}$$

where $z^* \in L^1$ is the decision variable.

Theorem 3 (CVaR bidual representation, Balbás et al. 2021)**.** *$CVaR_{1-\mu^*}(y)$ is the optimal value of the bounded and solvable problem*

$$Min\ \lambda + \mathbb{E}(\lambda_M)/\mu^* \begin{cases} y = \lambda_m - \lambda_M - \lambda \\ \lambda_m,\ \lambda_M \geq 0, \end{cases} \tag{3}$$

with $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times \left(L^1\right)^2$ being the decision variable. Moreover, if $z^* \in L^1$ is (2)-feasible and if $(\lambda, \lambda_m, \lambda_M)$ is (3)-feasible, then they solve the corresponding problem if and only if the complementary slackness conditions

$$\lambda_m z^* = \lambda_M \left(\frac{1}{\mu^*} - z^* \right) = 0 \tag{4}$$

hold.

Corollary 2. *Take $(\lambda, z^*, \lambda_m, \lambda_M) \in \mathbb{R} \times \left(L^1\right)^3$. Then, $(\lambda, z^*, \lambda_m, \lambda_M)$ is (1)-feasible if and only if z^* solves (2) and $(\lambda, \lambda_m, \lambda_M)$ solves (3), in which case the equalities*

$$CVaR_{1-\mu^*}(y) = -\mathbb{E}(yz^*) = \lambda + \mathbb{E}(\lambda_M)/\mu^*$$

hold.

Corollary 3. $\left(VaR_{1-\mu^*}(y), \left(y + VaR_{1-\mu^*}(y)\right)^+, \left(y + VaR_{1-\mu^*}(y)\right)^- \right)$ *is a solution to (3).*[2]

Proof. $\left(VaR_{1-\mu^*}(y), \left(y + VaR_{1-\mu^*}(y)\right)^+, \left(y + VaR_{1-\mu^*}(y)\right)^- \right)$ is obviously (3)-feasible. Take $(\lambda, z^*, \lambda_m, \lambda_M)$, solving (1). $\lambda = VaR_{1-\mu^*}(y)$ is obvious, and the constraints of (1) lead to

$$y + VaR_{1-\mu^*}(y) = \lambda_m - \lambda_M. \tag{5}$$

If $\lambda_M > \left(y + VaR_{1-\mu^*}(y)\right)^-$, then the objective of (3) at $(\lambda, \lambda_m, \lambda_M)$ would be strictly higher than it is at

$$\left(VaR_{1-\mu^*}(y), \left(y + VaR_{1-\mu^*}(y)\right)^+, \left(y + VaR_{1-\mu^*}(y)\right)^- \right), \tag{6}$$

which cannot hold because $(\lambda, \lambda_m, \lambda_M)$ solves (3). Thus,

$$\lambda_M = \left(y + VaR_{1-\mu^*}(y)\right)^-,$$

and therefore

$$\lambda_m = \left(y + VaR_{1-\mu^*}(y)\right)^+$$

due to (5). □

2.2. Expectiles

Fix $\mu \in (0, 1/2)$. There exits a unique solution to the equation

$$\mu \mathbb{E}\left((y - x)^+ \right) = (1 - \mu) \mathbb{E}\left((x - y)^+ \right), \tag{7}$$

where $x \in \mathbb{R}$ is the unknown (Bellini and Di Bernardino 2017). This solution is denoted by $E_\mu(y)$ and is said to be the expectile of y at level μ.[3] The expectile risk measure at level μ is defined by $\mathcal{E}_\mu(y) := -E_\mu(y)$. Bearing in mind the equality $(y - x)^+ + x = y + (x - y)^+$, it is easy to see that $x \in \mathbb{R}$ solves (7) if and only if x solves

$$\mathbb{E}(y) = x + \frac{1 - 2\mu}{1 - \mu} \mathbb{E}\left((y - x)^+ \right), \tag{8}$$

so (8) also characterizes the expectile risk measure. Furthermore, several authors have shown that $\mathcal{E}_\mu$ is a continuous, coherent (in the sense of Artzner et al. (1999)), expectation-bounded (in the sense of Rockafellar et al. (2006)), and law-invariant risk measure (Ziegel 2016). In other words, $\mathcal{E}_\mu$ is continuous, sub-additive ($\mathcal{E}_\mu(y_1 + y_2) \leq \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$), positively homogeneous ($\mathcal{E}_\mu(\lambda y) = \lambda \mathcal{E}_\mu(y)$ if $\lambda \geq 0$), translation invariant ($\mathcal{E}_\mu(y + \lambda) = \mathcal{E}_\mu(y) - \lambda$ if $\lambda \in \mathbb{R}$), decreasing ($\mathcal{E}_\mu(y_1) \leq \mathcal{E}_\mu(y_2)$ if $y_1 \geq y_2$), mean dominating ($\mathcal{E}_\mu(y) \geq -\mathbb{E}(y)$) and law invariant ($\mathcal{E}_\mu(y_1) = \mathcal{E}_\mu(y_2)$ if $F_{y_1} = F_{y_2}$). Notice that there are some redundancies in the latter sentence. Firstly, positively homogeneous and decreasing imply continuous. Secondly, coherent and law invariant imply expectation bounded (mean dominating). Anyway, we have preferred to give an exhaustive list of properties. Recall that $CVaR_{1-\mu^*}$ also satisties all the properties above, whereas $VaR_{1-\mu^*}$ is neither continuous, nor sub-additive, nor mean dominating.

Theorem 4 (Dual representation of expectiles, Delbaen 2013)**.** *$\mathcal{E}_\mu(y)$ is the optimal value of the bounded and solvable problem*

$$Max \ -\mathbb{E}(yz) \begin{cases} \mathbb{E}(z) = 1 \\ \xi \leq z \leq \xi \dfrac{1-\mu}{\mu}, \end{cases} \tag{9}$$

where $(\xi, z) \in \mathbb{R} \times L^1$ is the decision variable.

Proposition 1. *(a) If (ξ, z) is (9)-feasible, then $\mu/(1-\mu) \leq \xi \leq 1$. (b) $(\xi = 1, z)$ is (9)-feasible if and only if $z = 1$. (c) $(\xi = \mu/(1-\mu), z)$ is (9)-feasible if and only if $z = 1$.*

Proof. (a) Taking expectations in the constraints of (9), one has $\xi \leq \mathbb{E}(z) = 1 \leq \xi \dfrac{1-\mu}{\mu}$.

(b) It is obvious that $(\xi, z) = (1, 1)$ is (9)-feasible. Conversely, if $(\xi = 1, z)$ is (9)-feasible, then the constraints of this problem lead to $z \geq \xi = 1$ and $\mathbb{E}(z) = 1$, so $z = 1$.

(c) It is obvious that $(\xi, z) = (\mu/(1-\mu), 1)$ is (9)-feasible. Conversely, if

$$(\xi = \mu/(1-\mu), z)$$

is (9)-feasible, then the constraints of this problem lead to $1 = \xi \dfrac{1-\mu}{\mu} \geq z$ and $\mathbb{E}(z) = 1$, so $z = 1$. □

Theorem 5 (Bidual representation of expectiles)**.** *$\mathcal{E}_\mu(y)$ is the optimal value of the bounded and solvable problem*

$$Min \ \lambda \begin{cases} y = \lambda_m - \lambda_M - \lambda \\ \mathbb{E}(\lambda_m) = \dfrac{1-\mu}{\mu} \mathbb{E}(\lambda_M) \\ \lambda_m, \ \lambda_M \geq 0, \end{cases} \tag{10}$$

with $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times (L^1)^2$ being the decision variable. Furthermore, if (ξ, z) is (10)-feasible and $(\lambda, \lambda_m, \lambda_M)$ is (2)-feasible, then they solve the related problem if and only if the complementary slackness conditions

$$\lambda_m(z - \xi) = \lambda_M \left(\xi \frac{1-\mu}{\mu} - z \right) = 0 \tag{11}$$

hold.

Proof. Following Anderson and Nash (1987), (10) is the dual problem of the linear problem (9) if one proves the constraint $(\lambda_m, \lambda_M) \in (L^1)^2$. Nevertheless, this constraint may be proved with the method used in Balbás et al. (2021) to relate (2) and (3). Since (9) and (10) may be infinite-dimensional problems, the so-called duality gap between them could arise, but it may be proved that this duality gap vanishes by using a method similar to that in Balbás et al. (2021) for (2) and (3). The absence of a duality gap guarantees that (11) are necessary and sufficient optimality conditions (Anderson and Nash 1987). □

Corollary 4. *The assertions below are equivalent:* (a) $y = \mathbb{E}(y)$. (b) *There exists* $z \in L^1$ *such that* $(\xi = 1, z)$ *solves (9).* (c) *There exists* $z \in L^1$ *such that* $(\xi = \mu/(1-\mu), z)$ *solves (9). Furthermore, under the affirmative case,* (d), (e), *and* (f) *below hold:* (d) *One can take* $z = 1$ *in* (b) *and* (c). (e) $(\xi, z = 1)$ *solves (9) for every* $\xi \in [\mu/(1-\mu), 1]$, *and* $\mathcal{E}_\mu(y) = -\mathbb{E}(y)$. (f) *Every (9)-feasible* (ξ, z) *solves (9).*

Proof. $(a) \Rightarrow (b)$ Suppose that (a) holds, and take $(\xi, z) = (1, 1)$. It is evident that (ξ, z) is (9)-feasible. Take $(\lambda, \lambda_m, \lambda_M) = (-\mathbb{E}(y), 0, 0)$, and it is evident that $(\lambda, \lambda_m, \lambda_M)$ is (10)-feasible. Moreover, (11) trivially holds.

$(a) \Rightarrow (c)$ Suppose that (a) holds, and take $(\xi, z) = (\mu/(1-\mu), 1)$. It is evident that (ξ, z) is (9)-feasible. Take $(\lambda, \lambda_m, \lambda_M) = (-\mathbb{E}(y), 0, 0)$, and it is evident that $(\lambda, \lambda_m, \lambda_M)$ is (10)-feasible. Moreover, (11) trivially holds.

$(b) \Rightarrow (a)$ Suppose that $(1, \tilde{z})$ solves (9). Proposition 1 implies that $\tilde{z} = 1$. $E_\mu(y) = \mathbb{E}(y\tilde{z}) = \mathbb{E}(y)$ and (8) imply that

$$\mathbb{E}(y) = \mathbb{E}(y) + \frac{1-2\mu}{1-\mu}\mathbb{E}\Big((y - \mathbb{E}(y))^+\Big),$$

i.e., $\mathbb{E}\Big((y - \mathbb{E}(y))^+\Big) = 0$ and therefore $y \geq \mathbb{E}(y)$, which is equivalent to (a).

$(c) \Rightarrow (a)$ Suppose that $(\mu/(1-\mu), \tilde{z})$ solves (9). Proposition 1 implies that $\tilde{z} = 1$. The rest of the proof is similar to that of $(b) \Rightarrow (a)$.

(d) Obvious consequence of Proposition 1.

(e) $\mathcal{E}_\mu(y) = -\mathbb{E}(y)$ holds because $\mathcal{E}_\mu$ is coherent, and every coherent risk measure satisfies this equality Artzner et al. (1999). If $\xi \in [\mu/(1-\mu), 1]$, then it is evident that $(\xi, z = 1)$ is (9)-feasible. Moreover, the objective function of (9) at $(\xi, z = 1)$ satisfies that $-\mathbb{E}(yz) = -\mathbb{E}(y) = \mathcal{E}_\mu(y)$.

(f) As in (e), $-\mathbb{E}(yz) = -\mathbb{E}(y)\mathbb{E}(z) = -\mathbb{E}(y)$ because $\mathbb{E}(z) = 1$. □

3. Optimization Problems Involving Expectiles

Since Rockafellar and Uryasev (2000) proved that the CVaR minimization may often be addressed by means of linear programming methods, many authors have extended the analysis and dealt with other risk measures (Konno et al. 2005; Balbás and Charron 2019, for instance). Let us show that the optimization of expectiles may be also linearized. Accordingly, consider a functional $\alpha : L^1 \longrightarrow \mathbb{R}$, an arbitrary set X, a function $\beta : X \longrightarrow L^1$, and the optimization problem

$$Min\ \{\alpha(\beta(x)); x \in X\}. \tag{12}$$

Taking into account Theorems 3 and 5, the proof of Theorem 6 below becomes simple and therefore omitted.

Theorem 6. (a) *Consider* $\mu^* \in (0,1)$, $\alpha = CVaR_{1-\mu^*}$, *and Problem*

$$Min\ \lambda + \mathbb{E}(\lambda_M)/\mu^* \begin{cases} \beta(x) = \lambda_m - \lambda_M - \lambda \\ x \in X,\ \lambda_m,\ \lambda_M \geq 0, \end{cases} \tag{13}$$

with $(\lambda, x, \lambda_m, \lambda_M) \in \mathbb{R} \times X \times (L^1)^2$ *being the decision variable.* x *solves (12) if and only if there exists* $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times (L^1)^2$ *such that* $(\lambda, x, \lambda_m, \lambda_M)$ *solves (13), in which case* $CVaR_{1-\mu^*}(\beta(x)) = \lambda + \mathbb{E}(\lambda_M)/\mu^*$ *holds.* (b) *Consider* $\mu \in (0, 1/2)$, $\alpha = \mathcal{E}_\mu$, *and Prob-*

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$$Min\ \lambda \begin{cases} \beta(x) = \lambda_m - \lambda_M - \lambda \\ \mathbb{E}(\lambda_m) = \dfrac{1-\mu}{\mu}\mathbb{E}(\lambda_M) \\ x \in X,\ \lambda_m,\ \lambda_M \geq 0, \end{cases} \tag{14}$$

with $(\lambda, x, \lambda_m, \lambda_M) \in \mathbb{R} \times X \times \left(L^1\right)^2$ *being the decision variable.* x *solves (12) if and only if there exists* $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times \left(L^1\right)^2$ *such that* $(\lambda, x, \lambda_m, \lambda_M)$ *solves (14), in which case* $\mathcal{E}_\mu(\beta(x)) = \lambda$ *holds.*

Notice that both (13) and (14) may inherit several properties of the set X and the function β. In particular, if X is given by linear (convex) constraints and β is linear, then (13) and (14) become linear (convex) optimization problems. In other words, Theorem 6*b* may play a critical role in linearizing the minimization of expectiles, and Theorem 6*a* presents a way to linearize the CVaR minimization.

Expectile-linked constraints can be also linearized by the application of Theorem 5. Indeed, consider a subset $Y \subset L^1$, a real-valued function $f : Y \longrightarrow \mathbb{R}$, a real number $k \in \mathbb{R}$, and the optimization problem

$$Opt\ f(y) \begin{cases} \mathcal{E}_\mu(y) \leq k \\ Y \in Y, \end{cases} \tag{15}$$

where "*Opt*" applies for both "*Max*" or "*Min*". Then, one has the following.

Theorem 7. *Consider the optimization problem*

$$Opt\ f(y) \begin{cases} y = \lambda_m - \lambda_M - \lambda \\ \mathbb{E}(\lambda_m) = \dfrac{1-\mu}{\mu}\mathbb{E}(\lambda_M) \\ \lambda \leq k \\ y \in Y,\ \lambda_m,\ \lambda_M \geq 0, \end{cases} \tag{16}$$

with $(\lambda, y, \lambda_m, \lambda_M) \in \mathbb{R} \times Y \times \left(L^1\right)^2$ *being the decision variable.* y *solves (15) if and only if there exists* $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times \left(L^1\right)^2$ *such that* $(\lambda, y, \lambda_m, \lambda_M)$ *solves (16).*

Proof. Theorem 5 implies that $\mathcal{E}_\mu(y) \leq k$ holds if and only if there exists $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times \left(L^1\right)^2$ such that $(\lambda, y, \lambda_m, \lambda_M)$ is (16)-feasible. □

Problem (16) again inherits the properties of Y and f. In particular, if Y is given by linear (convex, concave) constraints and f is linear (convex, concave), then (16) is a linear (convex, concave) problem.

4. Linking *CVaR* and Expectiles

Several authors have pointed out the existence of inequalities involving VaR, CVaR, and expectiles (Delbaen 2013; Bellini and Di Bernardino 2017; Tadese and Drapeau 2020, to name a few). Let us show that this type of relationship may also be addressed by means of the bidual approach. Some of the inequalities below are quite similar to others proved in Tadese and Drapeau (2020). Nevertheless, the use of the bidual representation in Theorem 5 may simplify the proofs. Needless to say, the simplification of proofs may deserve the interest of many researchers (Herdegen and Munari 2023, for instance). Moreover, bidual representation will allow us to study the potential co-monotonic additivity of expectiles, as well as to verify in Section 5 whether the given inequalities may become exact equalities.

First of all, the expectile risk measure $\mathcal{E}_\mu$ will be the envelope of a family of continuous, coherent, expectation-bounded, law-invariant and co-monotonically additive risk measures.[4] In order to show that, let us fix the subsets A and B of $\mathbb{R}^2$ given by

$$\begin{cases} A := \left\{(\mu,\xi) \in \mathbb{R}^2; 0 < \mu < 1/2, \mu/(1-\mu) < \xi < 1\right\} \\ B := \left\{(\mu^*,\xi) \in \mathbb{R}^2; 0 < \mu^* < 1, 0 < \xi < 1\right\} = (0,1)^2, \end{cases} \tag{17}$$

and for $(\mu,\xi) \in A$ and $\mu^* \in (0,1)$, let us consider the sets

$$\begin{cases} C_{(\mu,\xi)} := \left\{z \in L^1; \mathbb{E}(z) = 1, \xi \le z \le \xi \dfrac{1-\mu}{\mu}\right\} \\ D_{\mu^*} := \left\{z^* \in L^1; \mathbb{E}(z^*) = 1, 0 \le z^* \le \dfrac{1}{\mu^*}\right\}. \end{cases} \tag{18}$$

Let us prove an instrumental lemma.

Lemma 1. (a) *The function*

$$A \ni (\mu,\xi) \to I(\mu,\xi) = (\mu^*,\xi) \in B$$

given by

$$\mu^* := \frac{(1-\xi)\mu}{\xi(1-2\mu)} \tag{19}$$

is well-defined and a one-to-one bijection whose inverse

$$B \ni (\mu^*,\xi) \to I^{-1}(\mu^*,\xi) = (\mu,\xi) \in A$$

is given by

$$\mu = \frac{\xi\mu^*}{1-\xi(1-2\mu^*)}. \tag{20}$$

(b) *Fix* $\mu \in (0,1/2)$. *If* μ^* *is given by (19), then the function*

$$(\mu/(1-\mu),1) \ni \xi \to \mu^* \in (0,1)$$

is well-defined and a one-to-one bijection whose inverse is given by

$$\xi = \frac{\mu}{\mu^* - 2\mu^*\mu + \mu}. \tag{21}$$

(c) *Fix* $(\mu,\xi) \in A$ *and* $I(\mu,\xi) = (\mu^*,\xi) \in B$. *The function*

$$C_{(\mu,\xi)} \ni z \to z^* = I_{(\mu,\xi)}(z) := \frac{z-\xi}{1-\xi} \in D_{\mu^*}$$

is well-defined and a one-to-one bijection whose inverse is given by

$$D_{\mu^*} \ni z^* \to z = I^{-1}_{(\mu,\xi)}(z^*) = \xi + (1-\xi)z^* \in C_{(\mu,\xi)}. \tag{22}$$

(d) *Fix* $(\mu,\xi) \in A$, $I(\mu,\xi) = (\mu^*,\xi) \in B$, $z \in C_{(\mu,\xi)}$ *and* $z^* = I_{(\mu,\xi)}(z) \in D_{\mu^*}$. *Then, out of a* $\mathbb{P}$*-null set,*

$$\left\{\omega \in \Omega;\ \xi\frac{1-\mu}{\mu} - z(\omega) = 0\right\} = \{\omega \in \Omega;\ 1/\mu^* - z^*(\omega) = 0\}.$$

Proof. In order to see that (19) is well-defined, one must show that $0 < \mu^* < 1$. The first inequality is evident, so let us see the second one. Since the one-to-one bijection

$$(0,1) \ni x \to \frac{1-x}{x} \in (0,+\infty)$$

is strictly decreasing, and $\mu/(1-\mu) < \xi$, one has that

$$\frac{1-\xi}{\xi} < \frac{1-\mu/(1-\mu)}{\mu/(1-\mu)},$$

and (19) leads to

$$\mu^* < \frac{1-\mu/(1-\mu)}{\mu/(1-\mu)} \frac{\mu}{1-2\mu} = 1.$$

Additionally, trivial manipulations of (19) lead to (20), so let us see that, for $(\mu^*, \xi) \in B = (0,1)^2$, (20) implies that $0 < \mu < 1/2$. The first inequality is obvious, so let us prove the second one. One has that

$$\frac{\xi}{2} < \frac{1}{2} \Rightarrow \xi\mu^* < \xi\mu^* + \frac{1}{2} - \frac{\xi}{2} \Rightarrow \xi\mu^* < \frac{1}{2}(1 - \xi(1-2\mu^*)),$$

and therefore $\mu < 1/2$ (see (20)). Lastly, it only remains to see that $\mu/(1-\mu) < \xi$ holds. Bearing in mind (20), it is equivalent to

$$\xi > \left(\frac{\xi\mu^*}{1-\xi(1-2\mu^*)}\right) \Big/ \left(1 - \frac{\xi\mu^*}{1-\xi(1-2\mu^*)}\right),$$

that is,

$$1 > \frac{\mu^*}{1-\xi+\xi\mu^*},$$

or

$$1 - \xi + \xi\mu^* > \mu^*.$$

This is equivalent to

$$1 - \xi > \mu^*(1-\xi),$$

which is obvious because $1 - \xi > 0$ and $0 < \mu^* < 1$.

(*b*) (21) trivially follows from (19), so it is sufficient to see that $\mu^* - 2\mu^*\mu + \mu > 0$. Since $0 < \mu < 1/2$ and $0 < \mu^* < 1$, one has that $\mu^* - 2\mu^*\mu + \mu > \mu^* - 2\mu^*\mu = \mu^*(1-2\mu) > 0$.

(*c*) $\mathbb{E}(z^*) = 1$ and $0 \leq z^*$ are evident, so let us see that $z^* \leq 1/\mu^*$ and therefore $z^* \in D_{\mu^*}$. Indeed, (19) shows that one must prove that

$$\frac{z-\xi}{1-\xi} \leq \frac{\xi(1-2\mu)}{(1-\xi)\mu},$$

i.e.,

$$z - \xi \leq \frac{\xi(1-2\mu)}{\mu} = \frac{\xi}{\mu} - 2\xi,$$

which is equivalent to

$$z \leq \frac{\xi}{\mu} - \xi = \xi\left(\frac{1}{\mu} - 1\right) = \xi\frac{1-\mu}{\mu},$$

which must hold because $z \in C_{(\mu,\xi)}$. Additionally, the equivalence

$$z^* = \frac{z-\xi}{1-\xi} \Longleftrightarrow z = \xi + (1-\xi)z^*$$

is obvious, so it only remains to prove that $z = \xi + (1-\xi)z^* \in C_{(\mu,\xi)}$ if $z^* \in D_{\mu^*}$. $\mathbb{E}(z) = 1$ and $0 \leq z$ are evident, so let us see that $z \leq \xi \dfrac{1-\mu}{\mu}$. One has that

$$z \leq \xi + (1-\xi)\frac{1}{\mu^*},$$

and (20) shows that

$$z \leq \xi + (1-\xi)\frac{\xi(1-2\mu)}{(1-\xi)\mu} = \xi + \frac{\xi(1-2\mu)}{\mu} = \frac{\xi}{\mu} - \xi = \xi\frac{1-\mu}{\mu}.$$

(d) Notice that

$$\frac{1}{\mu^*} - z^* = \frac{\xi(1-2\mu)}{(1-\xi)\mu} - \frac{z-\xi}{1-\xi} = \frac{\xi - \xi\mu - z\mu}{(1-\xi)\mu}$$

and

$$\frac{\xi(1-\mu)}{\mu} - z = \frac{\xi - \xi\mu - z\mu}{\mu},$$

so the given equivalence becomes evident. □

Theorem 8. *Consider $y \in L^1$, $(\mu, \xi) \in A$, and the optimization problems*

$$Max \ -\mathbb{E}(yz)\begin{cases} \mathbb{E}(z) = 1 \\ \xi \leq z \leq \xi\dfrac{1-\mu}{\mu}, \end{cases} \tag{23}$$

where $z \in L^1$ is the decision variable, and

$$Min \ \lambda + \xi\left(\frac{1-\mu}{\mu}\mathbb{E}(\lambda_M) - \mathbb{E}(\lambda_m)\right) + \begin{cases} y = \lambda_m - \lambda_M - \lambda \\ \lambda_m, \ \lambda_M \geq 0, \end{cases} \tag{24}$$

where $(\lambda, \lambda_m, \lambda_M) \in \mathbb{R} \times (L^1)^2$ is the decision variable. Equations (23) and (24) are bounded and solvable, and the optimal value of both problems coincide. If z is (23)-feasible and $(\lambda, \lambda_m, \lambda_M)$ is (24)-feasible, then they solve the corresponding problem if and only if

$$\lambda_m(z - \xi) = \lambda_M\left(\xi\frac{1-\mu}{\mu} - z\right) = 0 \tag{25}$$

hold.[5]

Proof. The (23)-feasible set is included in the space L^∞ of essentially bounded random variables. Moreover, this feasible set is $\sigma(L^\infty, L^1)$-compact owing to Alaoglu's theorem (Zeidler 1995). Since the objective function of (23) is $\sigma(L^\infty, L^1)$-continuous (Zeidler 1995), Weierstrass' theorem implies that (23) is bounded and solvable. Following Anderson and Nash (1987), (24) is the dual problem of (23), although one must show the constraint $(\lambda_m, \lambda_M) \in (L^1)^2$ because the dual space of L^∞ is larger than L^1. Nevertheless, this constraint may be proved with the method used in Balbás et al. (2021) to relate (2) and (3). Since (24) and (23) are infinite-dimensional problems, the so-called duality gap between them could arise. However, it may be proved that this duality gap vanishes by using a method similar to that in Balbás et al. (2021) for (2) and (3). The absence of a duality gap guarantees that (3) is bounded and solvable, and (25) reflects necessary and sufficient optimality conditions (Anderson and Nash 1987). □

Definition 1. *Consider $y \in L^1$ and $(\mu, \xi) \in A$. The sub-expectile $\mathcal{E}_{(\mu,\xi)}(y)$ of y will be given by the optimal value of (23) or (24).*

Remark 2. *Notice that $C_{(\mu,\xi)}$ and D_{μ^*} are the (23)-feasible and the (2)-feasible sets, respectively (see (17) and (18)). Notice also that the (3)-feasible and the (24)-feasible sets coincide.*

Theorem 9. *Consider $y \in L^1$, $(\mu, \xi) \in A$, $z \in C_{(\mu,\xi)}$, $z^* = I_{(\mu,\xi)}(z) \in D_{\mu^*}$, and a (3)-feasible $(\lambda, \lambda_m, \lambda_M)$. (a) z solves (23) if and only if z^* solves (2). (b) $(\lambda, \lambda_m, \lambda_M)$ solves (3) if and only if $(\lambda, \lambda_m, \lambda_M)$ solves (24). (c)*

$$\mathcal{E}_{(\mu,\xi)}(y) = -\xi \mathbb{E}(y) + (1-\xi) CVaR_{1-\mu^*}(y). \tag{26}$$

(d) $\mathcal{E}_{(\mu,\xi)}$ is a continuous, coherent, expectation-bounded, law-invariant, and co-monotonically additive risk measure.

Proof. (a) and (b) According to 8, z solves (23), and $(\lambda, \lambda_m, \lambda_M)$ solves (24) if and only if (25) holds. Lemmas 1c and 1d imply that the fulfillment of (25) is equivalent to the fulfillment of (4), which holds (Theorem 5) if and only if z^* solves (2) and $(\lambda, \lambda_m, \lambda_M)$ solves (3).

(c) Take a solution z for (23). (a) implies that z^*, given by (22), solves (2). Theorems 3 and 8 lead to $CVaR_{1-\mu^*}(y) = -\mathbb{E}(yz^*)$ and $\mathcal{E}_{(\mu,\xi)}(y) = -\mathsf{I} - 1.5pt\mathsf{E}(yz)$, respectively. Additionally, (22) and $\mathbb{E}(z) = 1$ (see (23)) lead to

$$\mathbb{E}(yz) = \xi \mathbb{E}(y) + (1-\xi)\mathbb{E}(yz^*).$$

(d) It trivially follows from (26) because $CVaR_{1-\mu^*}$ is continuous, coherent, expectation-bounded, law-invariant, and co-monotonically additive. □

Corollary 5. *Consider $\mu \in (0, 1/2)$.*

$$\mathcal{E}_\mu(y) = Max\{-\xi \mathbb{E}(y) + (1-\xi) CVaR_{1-\mu^*}(y); \xi \in (\mu/(1-\mu), 1)\}.$$

Proof. It trivially follows from Theorems 4 and 5, Proposition 1, Corollary 4, and (26). □

Remark 3. *Consider $\mu \in (0, 1/2)$. Theorem 9 and Corollary 5 imply that $\mathcal{E}_\mu$ is the maximum of a family of continuous, coherent, expectation-bounded, law-invariant, and co-monotonically additive risk measures. Nevertheless, something may be missed when taking the maximum. Indeed, $\mathcal{E}_\mu$ is not co-monotonically additive (Delbaen 2013). Theorem 9 also allows us to give necessary and sufficient conditions guaranteeing that the expectile "respects the co-monotonic additivity". This is important in many actuarial and/or financial applications, since co-monotonic additivity significantly simplifies many technical problems (Bellini et al. 2021; Balbás et al. 2022, among many others).*

Corollary 6. *Consider $\mu \in (0, 1/2)$ and suppose that the random variables y_1 and y_2 are co-monotonic. Consider (9) for $y = y_j$, $j = 1, 2$, and their solutions (ξ_1, z_1) and (ξ_2, z_2), respectively. Consider also (10) for $y = y_j$, $j = 1, 2$, and their solutions $\left(\lambda^{(1}, \lambda_m^{(1}, \lambda_M^{(1}\right)$ and $\left(\lambda^{(2}, \lambda_m^{(2}, \lambda_M^{(3}\right)$, respectively. (a) If $\xi_1 = \xi_2$, then $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$. (b) If $z_1 = z_2$, then $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$. (c) $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$ if and only if there exists (ξ, z), solving (9) for both $y = y_1$ and $y = y_2$. (d) $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$ if and only if*

$$\left(\lambda^{(1}, \lambda_m^{(1}, \lambda_M^{(1}\right) + \left(\lambda^{(2}, \lambda_m^{(2}, \lambda_M^{(3}\right)$$

solves (10) for $y = y_1 + y_2$.[6]

Proof. (a) Take $\xi = \xi_1 = \xi_2$. Equations (23) and (24) lead to $\mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2) = \mathcal{E}_{(\mu,\xi)}(y_1) + \mathcal{E}_{(\mu,\xi)}(y_2) = \mathcal{E}_{(\mu,\xi)}(y_1 + y_2) \leq \mathcal{E}_\mu(y_1 + y_2)$, and the opposite inequality holds because $\mathcal{E}_\mu$ is subadditive.

(b) Take $z = z_1 = z_2$. The (9)-feasible set does not depend on y, and the objective of (9) does not depend on ξ, so (ξ_1, z) is feasible for $y = y_2$, and the objective remains the same as it is for (ξ_2, z). In other words, (ξ_1, z) solves (9) for both $y = y_1$ and $y = y_2$, and the equality $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$ follows from (a).

(c) The existence of (ξ, z) and (a) (or (b)) trivially implies the equality $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$. Conversely, if $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$ holds, then take (ξ, z), solving (9) for $y = y_1 + y_2$. One has that

$$-\mathbb{E}(y_j z) \leq \mathcal{E}_\mu(y_1), \quad j = 1, 2. \tag{27}$$

Additionally,

$$\begin{cases} \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2) = \mathcal{E}_\mu(y_1 + y_2) = \\ -\mathbb{E}((y_1 + y_2)z) = -\mathbb{E}(y_1 z) + -\mathbb{E}(y_2 z). \end{cases} \tag{28}$$

Equations (27) and (28) trivially imply that $-\mathbb{E}(y_j z) = \mathcal{E}_\mu(y_j)$, $j = 1, 2$.

(d) $\left(\lambda^{(1}, \lambda_m^{(1}, \lambda_M^{(1}\right) + \left(\lambda^{(2}, \lambda_m^{(2}, \lambda_M^{(3}\right)$ is obviously (10)-feasible for $y = y_1 + y_2$. If this is a solution to the problem, then $\mathcal{E}_\mu(y_j) = \lambda^{(j}$ for $j = 1, 2$ leads to $\mathcal{E}_\mu(y_1 + y_2) = \lambda^{(1} + \lambda^{(2} = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2)$. Conversely, $\mathcal{E}_\mu(y_1 + y_2) = \mathcal{E}_\mu(y_1) + \mathcal{E}_\mu(y_2) = \lambda^{(1} + \lambda^{(2}$ implies that the minimum $\mathcal{E}_\mu(y_1 + y_2)$ of (10) for $y = y_1 + y_2$ is attained at $\left(\lambda^{(1}, \lambda_m^{(1}, \lambda_M^{(1}\right) + \left(\lambda^{(2}, \lambda_m^{(2}, \lambda_M^{(3}\right)$. $\square$

Corollary 7. *Consider $(\mu^*, \xi) \in B = (0,1)^2$ and take $(\mu, \xi) = I^{-1}(\mu^*, \xi) \in A$. Then,*

$$CVaR_{1-\mu^*}(y) \leq \frac{\mathcal{E}_\mu(y) + \xi \mathbb{E}(y)}{1 - \xi}, \tag{29}$$

and the equality holds if and only if there exists $z \in L^1$ such that (ξ, z) solves (9).

Proof. Equation (29) trivially follows from Lemma 1 and Corollary 5. Additionally, the existence of z implies that $\mathcal{E}_\mu(y) = \mathcal{E}_{(\mu,\xi)}(y)$ (recall Definition 1), and (26) leads to the equality in (29). Conversely, if (29) becomes an equality, then (26) leads to $\mathcal{E}_\mu(y) = \mathcal{E}_{(\mu,\xi)}(y)$, so every solution z for (23), whose existence is guaranteed by Theorem 8, satisfies that (ξ, z) solves (9). $\square$

Corollary 8.

$$CVaR_{1-\mu^*}(y) \leq \mathcal{E}_\mu(y) + \frac{\mu}{\mu^*(1 - 2\mu)} \left(\mathcal{E}_\mu(y) + \mathbb{E}(y)\right) \tag{30}$$

holds for every $y \in L^1$, every $\mu^* \in (0,1)$, and every $\mu \in (0, 1/2)$.

Proof. Fix $\mu^* \in (0,1)$ and $\mu \in (0, 1/2)$. Lemma 1b implies that ξ, given by (21), satisfies that $I(\mu, \xi) = (\mu^*, \xi)$. Thus,

$$CVaR_{1-\mu^*}(y) \leq \frac{(\mu^* - 2\mu^*\mu + \mu)\mathcal{E}_\mu(y) + \mu \mathbb{E}(y)}{\mu^* - 2\mu^*\mu} \tag{31}$$

trivially follows from (21) and (29), and (30) trivially follows from (31). $\square$

Remark 4. *Given an arbitrary $\mu \in (0, 1/2)$, (30) indicates that $CVaR_{1-\mu^*}(y)$ is bounded from above by a simple hyperbolic function of μ^*, namely,*

$$CVaR_{1-\mu^*}(y) \leq \eta_1 + \frac{\eta_2}{\mu^*} \tag{32}$$

holds for every $\mu^* \in (0,1)$ *if*

$$\begin{cases} \eta_1 = \mathcal{E}_\mu(y) \\ \eta_2 = \dfrac{\mu\left(\mathcal{E}_\mu(y) + \mathbb{E}(y)\right)}{(1-2\mu)} = \nu\left(\mathcal{E}_\mu(y) + \mathbb{E}(y)\right) \end{cases} \tag{33}$$

where $\nu = \mu/(1-2\mu)$. *Table 1 below provides us with the value of* ν *for several values of* μ. ν *has been rounded to the second decimal place.*

Table 1. Coefficients of the CVaR hyperbolic upper bound.

μ	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
ν	0.01	0.06	0.13	0.21	0.33	0.50	0.75	1.17	2.00	4.50

Obviously, the coefficients η_1 *and* η_2 *of (32) depend on* y *(see (33)), but the value of* ν *and* y *are independent. Although* ν *grows,* μ *does as well, and (32) may become better because* $\mathcal{E}_\mu(y)$ *becomes lower. Actually, (32) will yield a finite collection of hyperbolic upper bounds of* $CVaR_{1-\mu^*}(y)$ *if a finite collection of values for* μ *are considered.*

5. Linking VaR and Expectiles

Expressions such as (29) and (30) establish several relationships between the conditional value at risk and the expectile. Obviously, since $VaR_{1-\mu^*}(y) \le CVaR_{1-\mu^*}(y)$, one has that

$$VaR_{1-\mu^*}(y) \le \frac{\mathcal{E}_\mu(y) + \xi\mathbb{E}(y)}{1-\xi} \tag{34}$$

for $y \in L^1$ and $(\mu,\xi) = I^{-1}(\mu^*,\xi)$, and

$$VaR_{1-\mu^*}(y) \le \mathcal{E}_\mu(y) + \frac{\mu}{\mu^*(1-2\mu)}\left(\mathcal{E}_\mu(y) + \mathbb{E}(y)\right) \tag{35}$$

for $y \in L^1$, $\mu^* \in (0,1)$, and $\mu \in (0,1/2)$. Let us deal with the value at risk in order to show that some of the given inequalities may become equalities. In particular, the equality $\mathcal{E}_\mu(y) = VaR_{1-\mu^*}(y)$ may frequently hold.[7]

Theorem 10. *Consider* $y \in L^1$ *and* $\mu^* \in (0,1)$. *Suppose that there exists* $(\mu,\xi,z) \in A \times L^1$ *such that* $(\mu^*,\xi) = I(\mu,\xi)$, *and* (ξ,z) *solves (9). Consider a solution* $(\lambda,\lambda_m,\lambda_M)$ *to (10). Then,* $z^* = I_{(\mu,\xi)}(z)$ *solves (2),* $(\lambda,\lambda_m,\lambda_M)$ *solves (3),*

$$VaR_{1-\mu^*}(y) \le \mathcal{E}_\mu(y) \le CVaR_{1-\mu^*}(y), \tag{36}$$

and (29) and (30) are fulfilled as equalities. Furthermore,

$$\begin{cases} \mathbb{P}(y = \mathbb{E}(y)) = 1 \Longrightarrow \begin{cases} VaR_{1-\mu^*}(y) = \mathcal{E}_\mu(y) = \\ = CVaR_{1-\mu^*}(y) = -\mathbb{E}(y) \end{cases} \\ \mathbb{P}(y = \mathbb{E}(y)) < 1 \Longrightarrow \mathcal{E}_\mu(y) < CVaR_{1-\mu^*}(y). \end{cases} \tag{37}$$

Lastly, if there exists $\varepsilon > 0$ *such that* $F_y(x) < \mu^*$ *for* $-VaR_{1-\mu^*}(y) - \varepsilon < x < -VaR_{1-\mu^*}(y)$, *then* $\mathcal{E}_\mu(y) = VaR_{1-\mu^*}(y)$ *and*

$$CVaR_{1-\mu^*}(y) = VaR_{1-\mu^*}(y) + \frac{\mu}{\mu^*(1-2\mu)}\left(VaR_{1-\mu^*}(y) + \mathbb{E}(y)\right). \tag{38}$$

Proof. z obviously solves (23), and therefore Theorem 9 implies that z^* solves (2). Equations (9) and (11) in Theorem 5 imply that $(\lambda,\lambda_m,\lambda_M)$ is (24)-feasible and that (25) holds for (ξ,z)

and $(\lambda, \lambda_m, \lambda_M)$; therefore, Theorem 8 implies that $(\lambda, \lambda_m, \lambda_M)$ solves (24), and Theorem 9 implies that $(\lambda, \lambda_m, \lambda_M)$ solves (3).

Since $(\lambda, \lambda_m, \lambda_M)$ solves both (3) and (10), one has that $CVaR_{1-\mu^*}(y) = \lambda + \mathbb{E}(\lambda_M)/\mu^* \geq \lambda = \mathcal{E}_\mu(y)$ (Theorems 3 and 5). Additionally, Theorem 3 implies the fulfillment of (4), so $(\lambda, z^*, \lambda_m, \lambda_M)$ is (1)-feasible, and Theorem 1 leads to $\lambda \geq VaR_{1-\mu^*}(y)$, i.e., $\mathcal{E}_\mu(y) \geq VaR_{1-\mu^*}(y)$. In other words, (36) holds.

In order to see that (29) becomes an equality, notice that it is sufficient to see that $\mathcal{E}_\mu(y) = \mathcal{E}_{(\mu,\xi)}(y)$ because (26) applies. Since $(\lambda, \lambda_m, \lambda_M)$ solves (24) and (10), it is (10)-feasible, that is, $\mathbb{E}(\lambda_m) = [(1-\mu)/\mu]\mathbb{E}(\lambda_M)$ holds, and one has that

$$\mathcal{E}_{(\mu,\xi)}(y) = \lambda - \xi\left(\mathbb{E}(\lambda_m) - \frac{1-\mu}{\mu}\mathbb{E}(\lambda_M)\right) = \lambda = \mathcal{E}_\mu(y).$$

In order to see that (30) becomes an equality, notice that $(\mu^*, \xi) = I(\mu, \xi)$ trivially implies the fulfillment of (21), and therefore it is enough to proceed as in the proof of Corollary 8.

Suppose that $\mathbb{P}(y = \mathbb{E}(y)) < 1$. If $\mathcal{E}_\mu(y) = CVaR_{1-\mu^*}(y)$, then (Theorems 2 and 4) $\mathbb{E}(yz) = \mathbb{E}(yz^*) = \xi\mathbb{E}(y) + (1-\xi)\mathrm{I} - 1.5pt\mathrm{E}(yz)$, that is, $\mathrm{I} - 1.5pt\mathrm{E}(y) = \mathbb{E}(yz) = \mathbb{E}(yz^*)$ and therefore $CVaR_{1-\mu^*}(y) = -\mathbb{E}(y)$. Consequently, $\mathbb{P}(y = \mathbb{E}(y)) = 1$ (Rockafellar et al. 2006), which is a contradiction.

Lastly, if $\varepsilon > 0$ exists, then $\lambda = VaR_{1-\mu^*}(y)$ (Corollary 1), and the equality $\lambda = \mathcal{E}_\mu(y)$ was already proved. Furthermore, (38) becomes trivial because (30) becomes an equality. □

Example 1. *(Counter-example) According to Theorem 10, given $y \in L^1$ and $\mu^* \in (0,1)$, the existence of (μ, ξ, z) fulfilling the required conditions generates the spread (36) containing $\mathcal{E}_\mu(y)$; implies that (29) and (30) cannot be improved because they become equalities; and also implies that, under weak conditions, $CVaR_{1-\mu^*}(y)$ can be computed from $VaR_{1-\mu^*}(y)$, $\mathbb{E}(y)$, and (38). Hence, it is a natural problem to analyze the existence of such a (μ, ξ, z). In general, this existence does not hold. Indeed, suppose that $\mathbb{P}(y \geq 0) = 1$ and $1 > \mathbb{P}(y > 0) > 0$. Take $\mu^* = \mathbb{P}(y = 0)$, $z^* = \chi_{y=0}/\mathbb{P}(y = 0)$ where $\chi_{y=0}$ represents the usual indicator of the set $y = 0$, $\lambda_m = y$, $\lambda_M = 0$, and $\lambda = 0$. It is clear that z^* is (2)-feasible, $(\lambda, \lambda_m, \lambda_M)$ is (3)-feasible, and $(\lambda, z^*, \lambda_m, \lambda_M)$ is (1)-feasible. Consequently, $(\lambda, \lambda_m, \lambda_M)$ solves (3), and $CVaR_{1-\mu^*}(y) = 0$ (Corollary 6). Suppose now that y is selected in such a manner that $\mathbb{P}(y < 1/\mu^*) = 1$. If there were a second (1)-feasible element $(\lambda', z', \lambda'_m, \lambda'_M)$ with $\lambda' < 0$, then $\lambda'_M = 0$ should hold ((4) and Corollary 2), and therefore $CVaR_{1-\mu^*}(y) = \lambda' < 0$, which is a contradiction. Such a $(\lambda', z', \lambda'_m, \lambda'_M)$ cannot exist, and therefore $VaR_{1-\mu^*}(y) = 0$ (Theorem 1). The fulfillment of (36) would lead to $\mathcal{E}_\mu(y) = 0$, and (8) would lead to*

$$\mathbb{E}(y) = \frac{1-2\mu}{1-\mu}\mathbb{E}(y),$$

which cannot hold because $\mathbb{E}(y) > 0$ and $\mu > 0$.

Remark 5. *If $\mathbb{P}(y = \mathbb{E}(y)) = 1$, then*

$$VaR_{1-\mu^*}(y) = \mathcal{E}_\mu(y) = CVaR_{1-\mu^*}(y) = -\mathbb{E}(y),$$

so the fulfillment of (29) and (30) as equalities and the fulfillment of (36) are evident. Hence, the open problem presented in Example 1 may be addressed for $\mathbb{P}(y = \mathbb{E}(y)) < 1$.

Theorem 11. *Consider $y \in L^1$ with $\mathbb{P}(y = \mathbb{E}(y)) < 1$ and $\mu^* \in (0,1)$. There exists $(\mu, \xi, z) \in A \times L^1$ such that $(\mu^*, \xi) = I(\mu, \xi)$, and (ξ, z) solves (9) if and only if there exists a solution $(\lambda, \lambda_m, \lambda_M)$ to (3) such that*

$$\mathbb{E}(\lambda_m) > \mathbb{E}(\lambda_M) > 0. \tag{39}$$

Furthermore, if (39) holds, then one can take

$$\mu = \frac{\mathbb{E}(\lambda_M)}{\mathbb{E}(\lambda_m) + \mathbb{E}(\lambda_M)}, \tag{40}$$

and ξ given by (21).

Proof. Suppose that $(\mu, \xi, z) \in A \times L^1$ exists. Theorem 10 shows that the solution $(\lambda, \lambda_m, \lambda_M)$ to (10) also solves (3). Moreover, the second constraint of (10) leads to either $\mathsf{I} - 1.5pt\mathsf{E}(\lambda_m) = \mathsf{I} - 1.5pt\mathsf{E}(\lambda_M) = 0$ or $\mathbb{E}(\lambda_m)/\mathbb{E}(\lambda_M) > 1$. Nevertheless, $\mathbb{E}(\lambda_m) = \mathbb{E}(\lambda_M) = 0$, and the third constraint of (10) would imply that $\lambda_m = \lambda_M = 0$; therefore, the first constraint would imply $y = \mathbb{E}(y)$, against the assumptions.

Conversely, suppose that (39) holds for some solution $(\lambda, \lambda_m, \lambda_M)$ to (3). Take μ as in (40). Equation (39) implies that $0 < \mu < 1/2$. $\mathbb{E}(\lambda_m) = ((1-\mu)/\mu)\mathbb{E}(\lambda_M)$ trivially follows from (40), so the constraints of (3) show that $(\lambda, \lambda_m, \lambda_M)$ is (10)-feasible. Moreover, Lemma 1b) implies that $(\mu^*, \xi) = I(\mu, \xi)$ if ξ is given by (21). Take a solution z^* to (2), and consider $z = I^{-1}_{(\mu,\xi)}(z^*)$. Theorem 9 implies that z solves (23) and $(\lambda, \lambda_m, \lambda_M)$ solves (24), and Theorem 8 implies that (25) holds; that is, (11) holds and therefore (ξ, z) solves (9) because it is (9)-feasible and $(\lambda, \lambda_m, \lambda_M)$ is (10)-feasible (Theorems 4 and 5). □

Remark 6. *If (39) holds, then (40) shows that μ decreases as $\mathbb{E}(\lambda_m)/\mathbb{E}(\lambda_M)$ increases. Actually, μ tends to 0 as $\mathbb{E}(\lambda_m)/\mathbb{E}(\lambda_M)$ tends to infinity, and μ tends to $1/2$ as $\mathbb{E}(\lambda_m)/\mathbb{E}(\lambda_M)$ tends to 1. Since the lower the parameter μ, the higher the risk $\mathcal{E}_\mu(y)$ (Bellini and Di Bernardino 2017), a large (low) ratio $\mathbb{E}(\lambda_m)/\mathbb{E}(\lambda_M)$ implies a "low (large) risk aversion" when comparing $\mathcal{E}_\mu(y)$ with $VaR_{1-\mu^*}(y)$ and $CVaR_{1-\mu^*}(y)$.*

Corollary 9. *Consider $y \in L^1$ with $\mathbb{P}(y = \mathbb{E}(y)) < 1$ and $\mu^* \in (0,1)$. If*

$$\mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^+\Big) > \mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^-\Big) > 0, \tag{41}$$

then there exists $(\mu, \xi, z) \in A \times L^1$ such that $(\mu^*, \xi) = I(\mu, \xi)$, and (ξ, z) solves (9). Furthermore, one can take

$$\mu = \frac{\mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^-\Big)}{\mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^+\Big) + \mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^-\Big)}, \tag{42}$$

and ξ given by (21). Moreover, if there exists $\varepsilon > 0$ such that $F_y(x) < \mu^*$ for $-VaR_{1-\mu^*}(y) - \varepsilon < x < -VaR_{1-\mu^*}(y)$, then (μ, ξ, z) exists if and only if (41) holds, in which case $VaR_{1-\mu^*}(y) = \mathcal{E}_\mu(y) < CVaR_{1-\mu^*}(y)$.[8]

Proof. If (41) holds, then the given implications are evident consequences of Corollary 3 and Theorem 11. Moreover, the existence of ε, Remark 1, and Corollary 3 show that (6) is the unique solution to (3), and therefore $VaR_{1-\mu^*}(y) = \mathcal{E}_\mu(y)$. Additionally, (41) is implied by (40), and $\mathcal{E}_\mu(y) < CVaR_{1-\mu^*}(y)$ coincides with expression (37). □

Corollary 10. *Consider $y \in L^1$ with $\mathbb{P}(y = \mathbb{E}(y)) < 1$ and $\mu^* \in (0,1)$, and suppose that there exists $(\mu, \xi, z) \in A \times L^1$ such that $(\mu^*, \xi) = I(\mu, \xi)$, and (ξ, z) solves (9). Suppose that there exist $-\infty \le u < v \le +\infty$ such that $F_y : (u,v) \to \mathbb{R}$ is strictly increasing, $F_y(x) = 0$ for $x < u$, and $F_y(x) = 1$ for $x > v$. Then, given $\tilde{\mu}^* \in (0,1)$ such that $\tilde{\mu}^* < \mu^*$ and $VaR_{1-\tilde{\mu}^*}(y) < CVaR_{1-\tilde{\mu}^*}(y)$, there exists $(\tilde{\mu}, \tilde{\xi}, \tilde{z}) \in A \times L^1$ such that $(\tilde{\mu}^*, \tilde{\xi}) = I(\tilde{\mu}, \tilde{\xi})$, and $(\tilde{\xi}, \tilde{z})$ solves (9).*

Proof. The existence of (u, v) implies the existence of $\varepsilon > 0$ satisfying the assumptions of the latter corollary (Remark 1). Therefore, (41) holds. Additionally, $VaR_{1-\tilde{\mu}^*}(y) \geq VaR_{1-\mu^*}(y)$ implies that

$$\begin{cases} \mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^+\Big) \leq \mathbb{E}\Big(\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^+\Big) \\ \mathbb{E}\Big(\big(y + VaR_{1-\mu^*}(y)\big)^-\Big) \geq \mathbb{E}\Big(\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^-\Big), \end{cases}$$

and therefore

$$\mathbb{E}\Big(\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^+\Big) > \mathbb{E}\Big(\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^-\Big).$$

If $\mathbb{E}\Big(\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^-\Big) = 0$, then $\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^- = 0$ (notice that $\big(y + VaR_{1-\tilde{\mu}^*}(y)\big)^- \geq 0$), which leads to

$$y + VaR_{1-\tilde{\mu}^*}(y) \geq 0. \tag{43}$$

Replace μ^* with $\tilde{\mu}^*$ in (3). Equation (43) implies that $\big(VaR_{1-\tilde{\mu}^*}(y), y + VaR_{1-\tilde{\mu}^*}(y), 0\big)$ satisfies the restrictions. Hence, bearing in mind the objective function of (3), $CVaR_{1-\tilde{\mu}^*}(y) \leq VaR_{1-\tilde{\mu}^*}(y)$, in contradiction with the assumptions. □

Remark 7. *Suppose that there exist* $-\infty \leq u < v \leq +\infty$ *such that* $F_y : (u, v) \to \mathbb{R}$ *is strictly increasing,* $F_y(x) = 0$ *for* $x < u$*, and* $F_y(x) = 1$ *for* $x > v$*. According to the latter corollary, if, for a given* $\mu^* \in (0, 1)$*, there exists* $\mu \in (0, 1/2)$ *such that (30) cannot be improved, then for every* $\tilde{\mu}^* < \mu^*$ *with* $CVaR_{1-\tilde{\mu}^*}(y) > VaR_{1-\tilde{\mu}^*}(y)$*, there exists* $\tilde{\mu} \in (0, 1/2)$ *such that (30) cannot be improved.*

6. Illustrative Example

6.1. Combining Actuarial and Financial Risks

Let us illustrate the ideas of Sections 2–5 by dealing with an important actuarial problem, that is, the optimal combination of reinsurance contracts and financial instruments. The particular problems are the selection of the optimal reinsurance, which arises if one imposes that the selected financial strategy must equal zero, and the portfolio choice problem, which arises when there is no actuarial risk involved. Both the optimal reinsurance problem and the portfolio selection one have been addressed by dealing with downside risk measures,[9] and a recent line of research integrates both problems into a single one, which is our first focus in this section, namely, the optimal combination. Though there are several interesting perspectives, we will focus on that of Balbás et al. (2023), since it is very general and properly fits the illustrative objective of this section.[10]

Suppose that $\mathcal{Y} \geq 0$ reflects the global indemnification to be paid by a direct insurer within the time interval $[0, T]$. There is a reinsurance market, and $\mathcal{Y}$ can be divided according to $\mathcal{Y} = \mathcal{Y}_c + \mathcal{Y}_r$, where $\mathcal{Y}_c$ represents the ceded risk, and $\mathcal{Y}_r$ represents the retained one. In order to guarantee that $\mathcal{Y}$, $\mathcal{Y}_c$, and $\mathcal{Y}_r$ are co-monotonic, a typical requirement to prevent the moral hazard, let us deal with the marginal retained indemnification rather than the retained indemnification itself. Accordingly, consider the interval $(0, +\infty)$, its Borel σ-algebra $\mathcal{B}$, and the Lebesgue measure $\mathbb{L}$. For every $\mathbb{R}$-valued, essentially bounded measurable function $x \in L^\infty((0, \infty), \mathcal{B}, \mathbb{L})$ such that

$$0 \leq x \leq 1, \tag{44}$$

the retained indemnification will be given by

$$\mathcal{Y}_r(\omega) = J_{\mathcal{Y}}(x)(\omega) = \int_0^{\mathcal{Y}(\omega)} x(s)ds \tag{45}$$

for $\omega \in \Omega$. If $\mathcal{Y}$ has a finite expectation, then, according to Balbás et al. (2023), (44) and (45) lead to a random variable $\mathcal{Y}_r$ with a finite expectation such that $\mathcal{Y}$, $\mathcal{Y}_r$, and $\mathcal{Y}_c = \mathcal{Y} - \mathcal{Y}_r$ are co-monotonic. There is also a financial market, and the insurer may focus on an international stock index whose stochastic behavior is conducted by a geometric Brownian motion (GBM). Hence, the evolution $\{S_t; 0 \leq t \leq T\}$ of the index quotation is given by $dS_t = S_t((r^* - \gamma)dt + \sigma dB_t)$, where r^* is the index drift, γ is the index dividend yield, and σ is the index volatility. There are future contracts whose underlying asset is the index above. If r_0 denotes the riskless rate, it is known that the future quotation $\{F_t; 0 \leq t \leq T\}$ is another GBM and evolves according to $dF_t = F_t(rdt + \sigma dB_t)$, where $r = r^* - r_0$ is the index excess return. Since the Black–Scholes–Merton (BSM) model is complete, given $\delta \in L^\infty((0,\infty), \mathcal{B}, \mathbb{L})$, the European-style derivative security

$$J_{F_T}(\delta)(\omega) = \int_0^{F_T(\omega)} \delta(s) ds \tag{46}$$

for $\omega \in \Omega$ may be replicated by means of a self-financing stochastic strategy combining the future contract and the riskless security. It has been shown by Balbás et al. (2023) that δ is the usual delta-Greek (sensitivity, or first-order mathematical derivative) at T of the derivative $J_{F_T}(\delta)$ with respect to F_T. If the insurer selects the marginal retained indemnity x and the financial Greek δ, then its random wealth at T will become

$$y = P - J_{\mathcal{Y}}(x) + J_{F_T}(\delta) - \Pi(J_{\mathcal{Y}}(x)), \tag{47}$$

where P is the global premium paid by insureds, the random variable $\mathcal{Y}_r = J_{\mathcal{Y}}(x)$ is given by (45), $\Pi(J_{\mathcal{Y}}(x))$ is the reinsurance price, and the random pay-off $J_{F_T}(\delta)$ is given by (46). The insurer problem may be the risk minimization under a minimum expected value R of y and a maximum Greek $\Delta \in L^\infty((0,\infty), \mathcal{B}, \mathbb{L})$,[11] where the risk is going to be measured by means of $\mathcal{E}_\mu$ for some $\mu \in (0, 1/2)$. Thus, bearing in mind that $\mathcal{E}_\mu$ is translation-invariant, the insurer problem becomes

$$\begin{cases} Min \begin{cases} \mathcal{E}_\mu\big(J_{F_T}(\delta) - J_{\mathcal{Y}}(x)\big) \\ +\Pi(J_{\mathcal{Y}}(x)) - P \end{cases} \\ \mathbb{E}\big(J_{F_T}(\delta) - J_{\mathcal{Y}}(x)\big) - \Pi(J_{\mathcal{Y}}(x)) \geq R_0 \\ 0 \leq x \leq 1,\ \delta \leq \Delta \end{cases} \tag{48}$$

where $(x, \delta) \in L^\infty((0,\infty), \mathcal{B}, \mathbb{L})^2$ is the decision variable and $R_0 = R - P$. Theorem 6 enables us to transform (48) into the equivalent problem

$$\begin{cases} Min\ \lambda \\ \begin{cases} P - J_{\mathcal{Y}}(x) + J_{F_T}(\delta) - \Pi(J_{\mathcal{Y}}(x)) \\ = \lambda_m - \lambda_M - \lambda \end{cases} \\ \mathbb{E}\big(J_{F_T}(\delta) - J_{\mathcal{Y}}(x)\big) - \Pi(J_{\mathcal{Y}}(x)) \geq R_0 \\ \mathbb{E}(\lambda_m) = \dfrac{1-\mu}{\mu} \mathbb{E}(\lambda_M) \\ 0 \leq x \leq 1,\ \delta \leq \Delta,\ \lambda \in \mathbb{R},\ \lambda_m,\ \lambda_M \geq 0, \end{cases} \tag{49}$$

with $(x, \delta, \lambda, \lambda_m, \lambda_M) \in L^\infty((0,\infty), \mathcal{B}, \mathbb{L})^2 \times \mathbb{R} \times \big(L^1(\Omega, \mathcal{F}, \mathbb{P})\big)^2$ being the decision variable (see (14)). The premium principle Π is frequently convex (Pichler 2014), and therefore (49) is a convex problem. Furthermore, (49) is linear if the reinsurer premium principle Π is linear too, and in particular, under the expected value premium principle

$$\Pi(J_{\mathcal{Y}}(x)) = (1 + K)\mathbb{E}(\mathcal{Y} - J_{\mathcal{Y}}(x)),$$

$K \geq 0$ being the reinsurer loading rate.

Equations (48) and (49) present an illustrative example showing that Theorems 5 and 6 may be useful in practice in order to address the minimization of the expectile risk measure by linear programming methods. As already mentioned, this is just an illustrative section, and a complete solution to (49) is beyond the scope and would significantly increase the paper length. Moreover, Balbás et al. (2023) have presented an exhaustive methodology to solve (49), which does not need to be repeated here. Nevertheless, if $(x, \delta, \lambda, \lambda_m, \lambda_M)$ solves (49) and y is given by (47), then (30) and (32) yield upper bounds for the insurer CVaR, applying to every confidence level.

6.2. Numerical Experiment

In order to illustrate how the results of Sections 4 and 5 may apply, let us select a couple (not necessarily (49)-feasible and/or (49)-optimal) (x, δ). For instance,

$$x(s) = \begin{cases} 1, & s < 10 \\ 0, & otherwise \end{cases}$$

and $\delta = 5$ for every $s > 0$. Evidently, $J_{\mathcal{Y}}(x) = Min\{\mathcal{Y}, 1\}$; that is, the selected (x, δ) implies the purchase of 5 futures plus a stop-loss reinsurance whose deductible equals 10. Suppose that $\mathcal{Y}$ and F_T are independent log-normal distributions,[12] consider the notation of Section 6.1, and take

$$\begin{pmatrix} F_0, & r, & \sigma, & P, & K, & \mathbb{E}(\mathcal{Y}), & Variance(\mathcal{Y}) \\ 1, & 10\%, & 15\%, & 11.5, & 5\%, & 10, & 15 \end{pmatrix}.$$

Also consider the insurer's final wealth

$$y = P + J_{F_T}(\delta) - J_{\mathcal{Y}}(x) - (1+K)\mathbb{E}(\mathcal{Y} - J_{\mathcal{Y}}(x)).$$

Then, (41) and (42) enable us to verify that (30) and (32) cannot be improved, in the sense that, for every $\mu^* < 0.48$, there exists $\mu \in (0, 1/2)$ such that they are satisfied as equalities. Table 2 below shows a selected sample for μ^*. In all cases, μ and $\mathcal{E}_\mu(y)$ have been rounded,[13] and the equality $VaR_{1-\mu^*}(y) = \mathcal{E}_\mu(y)$ is implied by Theorem 6 or Corollary 9.

Since $\mathbb{E}(y) = 0.07$, the obtained upper bounds become

$$\begin{cases} CVaR_{1-\mu^*}(y) \le 6.40 + \dfrac{3.30 \times 10^{-5}}{\mu^*} \\ CVaR_{1-\mu^*}(y) \le 5.49 + \dfrac{0.002}{\mu^*} \\ CVaR_{1-\mu^*}(y) \le 4.5 + \dfrac{0.05}{\mu^*} \\ CVaR_{1-\mu^*}(y) \le 3.89 + \dfrac{0.13}{\mu^*} \\ CVaR_{1-\mu^*}(y) \le 0.43 + \dfrac{1.31}{\mu^*} \end{cases}$$

for every $\mu^* \in (0, 1)$, with equality in those cases presented in Table 2.

Table 2. Main Hyperbolic upper bounds become equalities.

$1 - \mu^*$	99.98%	99%	90%	80%	55%
μ	5.1×10^{-6}	4.2×10^{-4}	0.01	0.03	0.42
$\mathcal{E}_\mu(y) = VaR_{1-\mu^*}(y)$	6.40	5.49	4.50	3.89	0.43

To sum up, Sections 6.1 and 6.2 have illustrated that the minimization of actuarial/financial risks given by expectiles may be frequently linearized, that this minimization may permit us to control other downside risks beyond any parameter/confidence level, and that the inequalities connecting VaR and expectiles or CVaR and expectiles may often become equalities. Moreover, to the best of our knowledge, this is the first study combining reinsurance contracts, financial markets, and expectiles.

7. Discussion and Conclusions

7.1. Discussion

As already indicated, expectiles are much less used in practice than VaR or CVaR, and this lower use is also reflected in the regulatory and supervisory systems. However, expectiles have very important analytical properties, and for this reason they have deserved the attention of many researchers. In particular, their coherence, elicitability, and relationships with CVaR have been extensively studied. This paper has presented a theoretical study based on the relationships between the dual representation and the bidual representation of expectiles. In this sense, the approach seems to be new, since the main instrument of analysis is the duality theory of linear programming. This methodology enables us to integrate under the same prism different problems affecting expectiles. Indeed, the methodology has allowed us to recover important inequalities relating CVaR and expectiles, but further issues have been addressed, including relationships between VaR and expectiles, potential improvements to the CVaR-linked inequalities, the potential co-monotonic additivity of expectiles, and the linearization of (actuarial, financial, or risk management) optimization problems involving risks in both the objective function and the constraints. This suggests that bidual representation and the duality theory of linear programming could also be a powerful tool for dealing with potential problems that may arise in the future. Additionally, since both the dual and the bidual representation of expectiles may lead to infinite-dimensional linear optimization problems, perhaps the most important practical limitation of this linear programming-linked approach is the lack of universal algorithms valid for every infinite-dimensional optimization problem.

7.2. Conclusions

The bidual representation of expectiles may be a powerful instrument to address important properties of these coherent and elicitable downside risk measures. In particular, this representation leads to new estimation and optimization methods by means of linear programming, new ways to analyze whether the co-monotonic additivity holds for expectiles, further relationships involving VaR, CVaR, and expectiles, and hyperbolic upper bounds of VaR and CVaR applying to every confidence level. Some theoretical findings have been illustrated in classical actuarial problems.

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Notes

1 Similar notation will apply in similar situations.

2 Recall the usual notations $x^+ := Max\{x, 0\}$ and $x^- := Max\{-x, 0\}$ for every $x \in \mathbb{R}$.

3 The initial definition of expectile was introduced in Newey and Powell (1987), where it was defined for a random variable y with finite expectation and variance as the unique minimizer x of $\mu\mathbb{E}\left(\left((y-x)^+\right)^2\right)+(1-\mu)\mathbb{E}\left(\left((x-y)^+\right)^2\right)$. If y has finite expectation and variance, both definitions are equivalent.

4 Recall that the random variables $U_1, U_2, .., U_n$ are said to be co-monotonic if their joint distribution is given by the Fréchet–Hoeffding copula

$$c(u_1, u_2, .., u_n) = Min\{u_1, u_2, .., u_n\}$$

for $0 \le u_i \le 1$, $i = 1, 2, .., n$ (Dhaene et al. 2002). Recall also that a risk measure α is said to be co-monotonically additive if $\alpha(y_1 + y_2) = \alpha(y_1) + \alpha(y_2)$ holds when y_1 and y_2 are co-monotonic. In particular, both $VaR_{1-\mu^*}$ and $CVaR_{1-\mu^*}$ are co-monotonically additive, but, according to Delbaen (2013), $\mathcal{E}_\mu$ is not.

5 Notice the analogy between (11) and (25).

6 With a similar proof this corollary is easily extended if there are more than two involved co-monotonic risks.

7 CVaR is coherent, and therefore its analytical properties are better than they are for VaR. Nevertheless, for some specific applications of risk measurement, some authors have pointed out that VaR may present some advantages with respect to CVaR (Koch-Medina et al. 2017, among others).

8 Obviously, if $u \in L^1$ one has that

$$u = u^+ - u^- \Longrightarrow u^+ = u^- + u \Longrightarrow \mathbb{E}(u^+) = \mathbb{E}(u^-) + \mathbb{E}(u).$$

Thus, taking $u = y + VaR_{1-\mu^*}(y)$, (42) is equivalent to

$$\mu = \frac{\mathbb{E}\left(\left(y + VaR_{1-\mu^*}(y)\right)^-\right)}{\mathbb{E}(y) + VaR_{1-\mu^*}(y) + 2\mathbb{E}\left(\left(y + VaR_{1-\mu^*}(y)\right)^-\right)}.$$

9 Cheung et al. (2019), Xie et al. (2023), and Avanzi et al. (2023), among others, are recent papers involving downside risk measures in the optimal reinsurance problem. Furthermore, Xie et al. (2023) also deal with expectiles. Similarly, Stoyanov et al. (2007), Lejeune and Shen (2016), and Strub et al. (2019) are papers dealing with downside risk measures and optimal financial strategies.

10 The optimal reinsurance–portfolio combination is not a unique optimization problem involving both actuarial and financial ideas. Many other interesting problems might be presented (Goovaerts and Laeven 2008).

11 If the existence of Δ is not imposed, then (48) becomes unbounded (Balbás et al. 2023).

12 Recall that F_T must be log-normal because we are dealing with the BSM pricing model.

13 They have been rounded to the second decimal place when the obtained rounded value is strictly higher than zero.

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