

Article

A Functional-Analytic Framework for Sensitivity Analysis in Actuarial Valuation

Emmanuel Van Yeboah 

Department of Mathematics, College of Science and Engineering, Texas Christian University,
Fort Worth, TX 76129, USA; emmanuelyeboah17@augustana.edu

Abstract

Actuarial valuation of long-duration insurance liabilities depends on assumptions regarding mortality, lapse behavior, and discount rates, making it important to understand how valuation outcomes respond to changes in these assumptions. Existing approaches primarily rely on stress testing, scenario analysis, and numerical sensitivity analysis, which provide limited theoretical insight into the stability and sensitivity of the valuation process. This paper formulates actuarial valuation as a nonlinear operator acting on a space of actuarial assumptions. We establish well-posedness, Lipschitz continuity, Fréchet differentiability, and second-order sensitivity expressions that characterize the response of valuation outcomes to simultaneous perturbations in mortality, lapse, and discount assumptions. The theoretical results are illustrated through a numerical example using mortality data from the Social Security Administration Actuarial Life Table, where the first- and second-order approximations closely match the exact perturbed valuations under the assumptions considered. The framework developed in this paper provides a new way to study how actuarial valuations respond to changes in mortality, lapse, and discount assumptions.

Keywords: model risk; actuarial valuation; assumption sensitivity; functional analysis; Fréchet differentiability; nonlinear operators

1. Introduction

Actuarial valuation is fundamental to the measurement of insurance liabilities and underpins premium determination, reserve estimation, product pricing, and financial and regulatory reporting ([Actuarial Standards Board 2017](#); [Bowers et al. 1997](#)). The valuation of long-duration insurance products depends critically on assumptions regarding mortality, policyholder lapse behavior, and discount rates, all of which are uncertain and subject to periodic revision. Because valuation outcomes are sensitive to these assumptions, understanding how changes in model inputs affect liability estimates is important for financial reporting, solvency oversight, and actuarial decision-making.

In practice, insurers commonly assess the impact of assumption uncertainty through stress testing, scenario analysis, and numerical sensitivity analysis. These approaches are valuable for model validation and risk management. However, they are primarily numerical and provide limited theoretical insight into how valuation outcomes respond to changes in actuarial assumptions. Existing work on actuarial model risk and assumption uncertainty has largely focused on these approaches ([International Actuarial Association 2013](#); [Pitacco 2016](#)).

In contrast, relatively little attention has been given to studying actuarial valuation from an operator-theoretic perspective. Rather than relying solely on numerical sensitivity



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analysis, the present paper studies actuarial valuation analytically by formulating the valuation process V as a nonlinear operator acting on an infinite-dimensional parameter space Θ consisting of mortality intensities, lapse rates, and discount functions.

While numerical sensitivity analysis examines the effect of selected changes in actuarial assumptions, the proposed framework characterizes the stability and sensitivity of the valuation process itself. Using tools from functional analysis, we establish Lipschitz continuity with respect to integrated (L^1) perturbations, prove Fréchet differentiability, derive second-order sensitivity expressions that capture interactions among actuarial assumptions, and illustrate these results with a numerical example.

From a mathematical perspective, the contribution is not the development of new functional-analytic techniques, but rather the formulation of actuarial valuation as a nonlinear operator whose stability, Fréchet differentiability, and higher-order sensitivity can be studied analytically. This provides theoretical properties of the valuation process that are not directly available from traditional stress testing or scenario analysis while also providing a natural framework for future extensions to stochastic assumptions, dependent decrement models, and other actuarial valuation settings.

2. Literature Review

Actuarial valuation depends on assumptions that are uncertain and subject to revision over time. As these assumptions change, valuation results are affected, making sensitivity analysis an important part of actuarial practice. Existing approaches primarily rely on stress testing, scenario analysis, and numerical sensitivity analysis to evaluate the impact of assumption changes on valuation results. Although these developments have improved the modeling of uncertainty, they remain primarily concerned with the effects of assumption changes rather than the analytical properties of the valuation process itself.

A complementary line of research has focused on the application of mathematical techniques to insurance and actuarial problems. For example, stochastic order methods have been used to compare insurance risks and study the effects of policy limits and deductibles on actuarial quantities (Bouhadjar et al. 2015, 2016). In quantitative finance, sensitivity analysis developed through PDE-based valuation models beginning with the Black–Scholes framework (Black and Scholes 1973) and later insurance applications (Brennan and Schwartz 1976). These methods introduced analytical and numerical techniques for studying valuation sensitivity, but their primary focus has been pricing and hedging rather than the mathematical properties of actuarial valuation operators. Functional analysis provides a natural framework for studying continuity, differentiability, and stability of mappings between function spaces (Kreyszig 1978). Although these ideas have been successfully applied in mathematical finance (Bayazit and Nolder 2013), actuarial valuation continues to rely predominantly on finite-dimensional parameter shocks and scenario-based methods (DeLong et al. 2019; Dhaene et al. 2017).

Taken together, the existing literature illustrates the growing interaction between modern mathematical analysis and actuarial modeling. However, relatively little attention has been given to studying actuarial valuation itself as a nonlinear operator with analytically tractable stability and sensitivity properties. In this paper, actuarial valuation is formulated as a nonlinear operator, providing a framework to study these properties.

3. Deterministic Valuation as a Nonlinear Operator

3.1. Valuation Operator and Parameter Space

We begin by formalizing actuarial valuation as a nonlinear operator acting on an infinite-dimensional space of model assumptions.

Definition 1 (Parameter Space). Let $T > 0$ be a fixed time horizon. Define

$$\Theta := \left\{ (\lambda, \ell, r) \in L^\infty([0, T])^3 : \lambda(t), \ell(t), r(t) \geq 0 \text{ a.e. on } [0, T] \right\}.$$

For $\theta = (\lambda, \ell, r) \in \Theta$, define

$$\|\theta\|_\Theta := \|\lambda\|_\infty + \|\ell\|_\infty + \|r\|_\infty.$$

Remark 1. Here, λ , ℓ , and r denote mortality intensity, lapse intensity, and discount rate functions, respectively.

The admissible parameter set Θ is a closed convex cone in $L^\infty([0, T])^3$, equipped with the metric induced by $\|\cdot\|_\Theta$. Since $L^\infty([0, T])^3$ is a Banach space and Θ is closed, Θ is complete as a metric subspace (Kreyszig 1978).

Definition 2 (Cash Flow Structure). Let $c \in L^\infty([0, T])$ denote a deterministic liability cash-flow function with

$$c(t) \geq 0 \quad \text{a.e. on } [0, T].$$

Definition 3 (Accumulation Function). For $\theta = (\lambda, \ell, r) \in \Theta$, define

$$A_\theta(t) := \int_0^t (r(s) + \ell(s) + \lambda(s)) ds, \quad 0 \leq t \leq T.$$

Definition 4 (Valuation Operator). The valuation operator $V : \Theta \rightarrow \mathbb{R}$ is defined by

$$V(\theta) := \int_0^T c(t) e^{-A_\theta(t)} dt.$$

Equivalently,

$$V(\lambda, \ell, r) = \int_0^T c(t) \exp\left(-\int_0^t (r(s) + \ell(s) + \lambda(s)) ds\right) dt.$$

Remark 2. The operator V represents the expected present value of deterministic future liability cash flows under independent decrement assumptions (Dhaene et al. 2017).

Remarks on the Assumptions

The boundedness, non-negativity, and independent decrement assumptions are used to establish the analytical properties of the valuation operator. These assumptions may not hold in more realistic settings where key model components vary over time or interact with one another. Extending the analysis to weaker regularity assumptions or models with dependent decrements is beyond the scope of the present paper but remains a natural direction for future research. Despite these limitations, these assumptions provide a useful mathematical foundation for understanding the stability and sensitivity of actuarial valuation.

The results obtained under this framework can help inform the development and analysis of more realistic valuation models that account for time-varying assumptions encountered in practice. In this sense, the present framework provides a reference point for understanding how additional sources of uncertainty may affect the valuation of realistic insurance portfolios.

3.2. Well-Definedness of the Valuation Operator

We first verify that the valuation operator is well-defined on Θ .

Theorem 1. For every $\theta \in \Theta$, the valuation $V(\theta)$ is well-defined, finite, and nonnegative.

Proof. Let $\theta = (\lambda, \ell, r) \in \Theta$. Since

$$r, \ell, \lambda \geq 0 \quad \text{a.e. on } [0, T],$$

we have

$$0 \leq e^{-A_\theta(t)} \leq 1$$

for all $t \in [0, T]$.

Because $c \in L^\infty([0, T])$ and $c(t) \geq 0$ a.e.,

$$0 \leq c(t)e^{-A_\theta(t)} \leq \|c\|_\infty.$$

Hence,

$$0 \leq V(\theta) \leq \int_0^T \|c\|_\infty dt = T\|c\|_\infty < \infty.$$

□

Remark 3. The preceding result requires only boundedness and non-negativity assumptions on the model parameters and cash-flow function.

3.3. Lipschitz Stability of the Valuation Operator

We now study the stability of the valuation operator under perturbations of the model parameters.

Theorem 2 (Lipschitz Stability). Let $c \in L^\infty([0, T])$ with $c(t) \geq 0$ a.e. on $[0, T]$. Then for any

$$\theta_i = (\lambda_i, \ell_i, r_i) \in \Theta, \quad i = 1, 2,$$

the valuation operator satisfies

$$|V(\theta_1) - V(\theta_2)| \leq T\|c\|_\infty(\|r_1 - r_2\|_{L^1} + \|\ell_1 - \ell_2\|_{L^1} + \|\lambda_1 - \lambda_2\|_{L^1}).$$

Consequently, V is Lipschitz continuous on Θ with respect to the L^1 perturbation metric

$$d(\theta_1, \theta_2) := \|r_1 - r_2\|_{L^1} + \|\ell_1 - \ell_2\|_{L^1} + \|\lambda_1 - \lambda_2\|_{L^1}.$$

Proof. Let $\theta_i = (\lambda_i, \ell_i, r_i) \in \Theta, i = 1, 2$ and define

$$A_i(t) := \int_0^t (r_i(s) + \ell_i(s) + \lambda_i(s)) ds.$$

Then

$$V(\theta_i) = \int_0^T c(t)e^{-A_i(t)} dt.$$

Hence,

$$|V(\theta_1) - V(\theta_2)| \leq \int_0^T c(t) \left| e^{-A_1(t)} - e^{-A_2(t)} \right| dt.$$

Since $x \mapsto e^{-x}$ is Lipschitz on $[0, \infty)$ with constant 1,

$$\left| e^{-A_1(t)} - e^{-A_2(t)} \right| \leq |A_1(t) - A_2(t)|.$$

Moreover,

$$|A_1(t) - A_2(t)| \leq \|r_1 - r_2\|_{L^1} + \|\ell_1 - \ell_2\|_{L^1} + \|\lambda_1 - \lambda_2\|_{L^1}.$$

Therefore,

$$|V(\theta_1) - V(\theta_2)| \leq T\|c\|_\infty(\|r_1 - r_2\|_{L^1} + \|\ell_1 - \ell_2\|_{L^1} + \|\lambda_1 - \lambda_2\|_{L^1}).$$

□

Remark 4. The Lipschitz estimate shows that valuation sensitivity is quantitatively controlled by L^1 perturbations in the model parameters. The factor T indicates that sensitivity increases with the contract horizon.

3.4. Fréchet Differentiability of the Valuation Operator

We now study first-order sensitivity of the valuation operator with respect to perturbations in the model parameters.

Let

$$\theta = (\lambda, \ell, r) \in \Theta$$

and let

$$h = (h_\lambda, h_\ell, h_r) \in L^1([0, T])^3$$

be a perturbation direction. For sufficiently small $\varepsilon \in \mathbb{R}$, define

$$\theta + \varepsilon h = (\lambda + \varepsilon h_\lambda, \ell + \varepsilon h_\ell, r + \varepsilon h_r).$$

We study the asymptotic behavior of

$$V(\theta + \varepsilon h) - V(\theta)$$

as $\varepsilon \rightarrow 0$, which naturally leads to the notion of Fréchet differentiability.

Remark 5. Up to this point, the valuation operator has been studied on the admissible parameter space

$$\Theta \subset L^\infty([0, T])^3.$$

Fréchet differentiability, however, is naturally formulated on open subsets of normed vector spaces (Kreyszig 1978). Since Θ includes non-negativity constraints, it is not open in $L^1([0, T])^3$. For the differentiability analysis, we therefore consider the valuation operator on the ambient Banach space.

$$X := L^1([0, T])^3,$$

equipped with the norm

$$\|h\|_X = \|h_\lambda\|_{L^1} + \|h_\ell\|_{L^1} + \|h_r\|_{L^1},$$

Theorem 3 (Fréchet Differentiability). Let $\theta \in X$. Then the valuation operator

$$V : X \rightarrow \mathbb{R}$$

is Fréchet differentiable at θ . Moreover, the derivative

$$DV(\theta) : X \rightarrow \mathbb{R}$$

is the bounded linear functional given by

$$DV(\theta)[h] = - \int_0^T c(t) e^{-A_\theta(t)} \left(\int_0^t (h_r(s) + h_\ell(s) + h_\lambda(s)) ds \right) dt,$$

for every

$$h = (h_\lambda, h_\ell, h_r) \in X.$$

Proof. Fix $\theta = (\lambda, \ell, r) \in X$ and let

$$h = (h_\lambda, h_\ell, h_r) \in X.$$

Define

$$A_\theta(t) := \int_0^t (r(s) + \ell(s) + \lambda(s)) ds$$

and

$$B_h(t) := \int_0^t (h_r(s) + h_\ell(s) + h_\lambda(s)) ds.$$

Then

$$V(\theta + h) = \int_0^T c(t) e^{-A_\theta(t)} e^{-B_h(t)} dt.$$

Hence,

$$V(\theta + h) - V(\theta) = \int_0^T c(t) e^{-A_\theta(t)} (e^{-B_h(t)} - 1) dt.$$

Using

$$e^{-x} - 1 = -x + \rho(x), \quad |\rho(x)| \leq Cx^2$$

for $|x|$ sufficiently small, we obtain

$$V(\theta + h) - V(\theta) = - \int_0^T c(t) e^{-A_\theta(t)} B_h(t) dt + R(h),$$

where

$$R(h) = \int_0^T c(t) e^{-A_\theta(t)} \rho(B_h(t)) dt.$$

It follows from the preceding expansion that

$$V(\theta + h) - V(\theta) = DV(\theta)[h] + R(h),$$

where $DV(\theta)$ is the linear functional stated in the theorem. This map is linear in h . We now show that it is bounded. Since

$$|A_\theta(t)| \leq \int_0^t (|r(s)| + |\ell(s)| + |\lambda(s)|) ds \leq \|\theta\|_X,$$

we have

$$|e^{-A_\theta(t)}| \leq e^{\|\theta\|_X}.$$

Moreover,

$$|B_h(t)| \leq \|h_r\|_{L^1} + \|h_\ell\|_{L^1} + \|h_\lambda\|_{L^1} = \|h\|_X.$$

Therefore,

$$|DV(\theta)[h]| \leq T \|c\|_\infty e^{\|\theta\|_X} \|h\|_X.$$

Thus, $DV(\theta)$ is bounded.

Since

$$|B_h(t)|^2 \leq \|h\|_X^2,$$

we have

$$\int_0^T |B_h(t)|^2 dt \leq T \|h\|_X^2.$$

Using

$$|e^{-A_\theta(t)}| \leq e^{\|\theta\|_X},$$

Now combining these estimates

$$|R(h)| \leq CT \|c\|_\infty e^{\|\theta\|_X} \|h\|_X^2.$$

Therefore,

$$\frac{|R(h)|}{\|h\|_X} \leq CT \|c\|_\infty e^{\|\theta\|_X} \|h\|_X,$$

which converges to 0 as $\|h\|_X \rightarrow 0$.

Thus,

$$\frac{|V(\theta + h) - V(\theta) - DV(\theta)[h]|}{\|h\|_X} \rightarrow 0 \quad \text{as } \|h\|_X \rightarrow 0.$$

This proves that V is Fréchet differentiable at θ . $\square$

Remark 6 (Actuarial Interpretation). *The Fréchet derivative provides a first-order approximation of valuation changes under simultaneous perturbations in mortality, lapse, and discount assumptions. For admissible nonnegative assumptions, the derivative depends on the cumulative effect of perturbations up to time t , where the contribution of each perturbation is weighted by the survival-discount factor*

$$e^{-A_\theta(t)}.$$

Consequently, assumption changes occurring earlier in the contract horizon may have amplified effects on long-term liability values.

3.5. Second-Order Sensitivity

We now establish existence of the second Fréchet derivative of the valuation operator and characterize its structure.

Theorem 4 (Second-Order Fréchet Differentiability). *Let $\theta \in X$. Then the valuation operator*

$$V : X \rightarrow \mathbb{R}$$

is twice Fréchet differentiable at θ . For any

$$h = (h_\lambda, h_\ell, h_r) \in X$$

and

$$k = (k_\lambda, k_\ell, k_r) \in X,$$

the second derivative is given by

$$D^2V(\theta)[h, k] = \int_0^T c(t) e^{-A_\theta(t)} \left(\int_0^t (h_r(s) + h_\ell(s) + h_\lambda(s)) ds \right) \left(\int_0^t (k_r(s) + k_\ell(s) + k_\lambda(s)) ds \right) dt.$$

Proof. From the first derivative formula, for $h \in X$,

$$DV(\theta)[h] = - \int_0^T c(t) e^{-A_\theta(t)} B_h(t) dt$$

Now let $k \in X$. Since

$$A_{\theta+k}(t) = A_{\theta}(t) + B_k(t),$$

we have

$$DV(\theta+k)[h] = - \int_0^T c(t) e^{-A_{\theta}(t)} e^{-B_k(t)} B_h(t) dt.$$

Therefore,

$$DV(\theta+k)[h] - DV(\theta)[h] = - \int_0^T c(t) e^{-A_{\theta}(t)} (e^{-B_k(t)} - 1) B_h(t) dt.$$

Using

$$e^{-x} - 1 = -x + \rho(x), \quad |\rho(x)| \leq Cx^2$$

for $|x|$ sufficiently small, we obtain

$$DV(\theta+k)[h] - DV(\theta)[h] = \int_0^T c(t) e^{-A_{\theta}(t)} B_h(t) B_k(t) dt + R_2(h, k)$$

where

$$R_2(h, k) = - \int_0^T c(t) e^{-A_{\theta}(t)} \rho(B_k(t)) B_h(t) dt.$$

The map $D^2V(\theta)[h, k]$ is bilinear. Moreover,

$$|B_h(t)| \leq \|h\|_X, \quad |B_k(t)| \leq \|k\|_X.$$

Hence,

$$|D^2V(\theta)[h, k]| \leq T \|c\|_{\infty} e^{\|\theta\|_X} \|h\|_X \|k\|_X,$$

so $D^2V(\theta)$ is bounded.

Finally, using the estimate established in the proof of Theorem 3.4,

$$|R_2(h, k)| \leq C \|c\|_{\infty} e^{\|\theta\|_X} \int_0^T |B_k(t)|^2 |B_h(t)| dt \leq CT \|c\|_{\infty} e^{\|\theta\|_X} \|k\|_X^2 \|h\|_X.$$

Thus,

$$\frac{|R_2(h, k)|}{\|k\|_X \|h\|_X} \leq CT \|c\|_{\infty} e^{\|\theta\|_X} \|k\|_X \rightarrow 0 \quad \text{as } \|k\|_X \rightarrow 0.$$

□

This completes the proof that V is twice Fréchet differentiable at θ with second Fréchet derivative $D^2V(\theta)$.

Remark 7 (Functional-Analytic Interpretation). *The second Fréchet derivative $D^2V(\theta)$ admits the following functional-analytic properties.*

- **Symmetry.** *The second derivative is symmetric, i.e.,*

$$D^2V(\theta)[h, k] = D^2V(\theta)[k, h],$$

which follows immediately from the product structure of the integrand. The second-order sensitivity is independent of the order in which perturbations are applied.

- **Bounded Bilinearity.** *The map*

$$D^2V(\theta) : X \times X \rightarrow \mathbb{R}$$

is a bounded bilinear form. As a result, the second-order approximation varies continuously with the perturbation directions.

- **Positive Semi-definiteness.** Since

$$D^2V(\theta)[h, h] \geq 0$$

for every $h \in X$, the valuation operator is locally convex. Thus, the second derivative characterizes the local curvature of the valuation operator.

Remark 8 (Actuarial Interpretation). The second Fréchet derivative captures nonlinear interaction effects between simultaneous perturbations in mortality, lapse, and discount assumptions. In particular, the quadratic structure of

$$D^2V(\theta)[h, k]$$

reflects how combined assumption shocks interact within the valuation process.

3.6. Convexity and Actuarial Interpretation

For any perturbation direction

$$h \in X,$$

the second derivative satisfies

$$D^2V(\theta)[h, h] \geq 0.$$

Hence, the valuation operator is convex, but generally not strictly convex, with respect to perturbations in mortality, lapse, and discount assumptions because it depends only on their combined intensity

$$r(t) + \ell(t) + \lambda(t).$$

Convexity implies that simultaneous assumption shocks may generate valuation responses beyond those predicted by first-order sensitivity alone. In particular, persistent perturbations over long horizons may produce amplified cumulative effects even when instantaneous shocks are small.

When perturbations are restricted to individual components, the second derivative describes curvature with respect to mortality, lapse, or discount assumptions separately. More generally, combined perturbations produce nonlinear interaction effects through the quadratic structure of

$$D^2V(\theta)[h, k],$$

which are not captured by the first-order approximation alone. Accordingly, the second-order sensitivity may provide additional insight when assessing reserve adequacy, capital requirements, and stress testing scenarios in which multiple model assumptions are varied simultaneously.

4. Numerical Illustration

4.1. Data and Baseline Assumptions

We now illustrate the first- and second-order sensitivity formulas using a simple deterministic liability model. The purpose of this numerical example is to illustrate deterministic local sensitivity under the assumptions of the model rather than to quantify statistical uncertainty. We use age-specific mortality rates from the Social Security Administration actuarial life table (Social Security Administration 2024). For illustration, we consider ages 40–70 and convert annual death probabilities into continuous-time mortality intensities using

$$\lambda_x = -\log(1 - q_x).$$

Here, q_x denotes the annual mortality probability from the published life table, whereas $\ell(t)$ denotes the lapse intensity introduced in Section 3.

This conversion is standard in continuous-time actuarial modeling and allows the mortality assumptions to be incorporated into the valuation operator framework.

For each starting age, the corresponding mortality intensity is assumed to remain constant over the 30-year valuation horizon. This keeps the numerical illustration simple. In practice, mortality varies with attained age, and incorporating age-varying mortality is left for future work.

We assume a constant lapse intensity

$$\ell(t) = 0.03,$$

representing a baseline annual policy surrender rate, and a constant discount rate

$$r(t) = 0.04,$$

representing a fixed interest rate environment, and a valuation horizon of

$$T = 30$$

years. We further normalize the cash flow by setting

$$c(t) = 1,$$

so that the effect of changes in the actuarial assumptions can be examined without changes in the cash-flow pattern. These simplifying assumptions isolate the effects of parameter changes. In practice, lapse rates, discount rates, and cash flows vary over time.

The baseline valuation is given by

$$V(\theta) = \int_0^T \exp\left(-\int_0^t (r(s) + \ell(s) + \lambda(s)) ds\right) dt.$$

4.2. Mortality Perturbation

To specify the mortality component $\lambda(t)$, we use selected age-specific death probabilities q_x from the Social Security Administration actuarial life table and convert them into continuous-time mortality intensities using

$$\lambda_x = -\log(1 - q_x).$$

As expected, increasing mortality probabilities across age groups correspond to lower life expectancy. We consider a proportional mortality perturbation of the form

$$h_\lambda(t) = 0.1\lambda(t)$$

corresponding to a 10% increase in mortality intensity.

We compare the resulting perturbed valuation with its first- and second-order approximations derived from the Fréchet sensitivity formulas.

The male mortality intensities presented in Table 1 were used to compute the valuation results reported in Table 2. These calculations were carried out using a spreadsheet developed for the numerical illustration. The mortality intensities were obtained from the published life table using the conversion $\lambda_x = -\log(1 - q_x)$ after which the baseline and perturbed valuations were evaluated under the specified lapse, discount, and cash-flow

assumptions. The first- and second-order approximations were then computed directly from the Fréchet derivative formulas derived in Sections 3.4 and 3.5.

Table 1. Selected entries from the Social Security Administration actuarial life table (2022) (Social Security Administration 2024).

Age	Male				Female			
	q_x	Lives	Life Exp.	λ_x	q_x	Lives	Life Exp.	λ_x
40	0.003353	94,339	37.67	0.003359	0.001803	97,129	41.86	0.000784
45	0.004175	92,625	33.32	0.004184	0.002352	96,156	37.25	0.001023
50	0.005666	90,468	29.05	0.005682	0.003407	94,847	32.73	0.001482
55	0.008348	87,520	24.94	0.008383	0.005040	92,982	28.34	0.002194
60	0.012458	83,277	21.08	0.012536	0.007658	90,217	24.12	0.003339
65	0.017897	77,402	17.48	0.018059	0.011018	86,231	20.12	0.004812
70	0.024828	69,838	14.09	0.025141	0.016025	80,893	16.27	0.007016

Table 2. Baseline and perturbed valuation results under a 10% mortality-intensity shock.

Age	q_x	λ_x	$V(\theta)$	$V(\theta + h)$	First-Order	Second-Order
40	0.003353	0.003359	12.122376	12.082190	12.082082	12.082190
45	0.004175	0.004184	12.024035	11.974662	11.974498	11.974663
50	0.005666	0.005682	11.848714	11.783308	11.783014	11.783309
55	0.008348	0.008383	11.542968	11.450676	11.450070	11.450680
60	0.012458	0.012536	11.097287	10.968289	10.967041	10.968299
65	0.017897	0.018059	10.547067	10.376918	10.374595	10.376945
70	0.024828	0.025141	9.905261	9.693101	9.689174	9.693162

Remark 9. The numerical results show close agreement between the exact perturbed valuation and its first- and second-order approximations. The second-order approximation is consistently more accurate, reflecting the nonlinear sensitivity structure established in the theoretical analysis. As mortality intensity increases across age groups, the valuation decreases due to higher cumulative decrement effects. The results illustrate how Fréchet sensitivity formulas can provide accurate local approximations for deterministic actuarial valuation under parameter perturbations.

5. Conclusions

In this paper, we studied actuarial valuation using a functional-analytic framework. The valuation process was formulated as an operator depending on mortality, lapse, and discount assumptions. We showed that the valuation operator is stable under perturbations of the model parameters and established first- and second-order Fréchet differentiability. These results provide explicit formulas for approximating valuation changes under assumption shocks.

The numerical illustration showed close agreement between the exact perturbed valuation and the corresponding first- and second-order approximations. In particular, the second-order approximation captured effects that are not visible from first-order sensitivity alone. Although the framework is deterministic, it allows the effects of changes in model assumptions to be studied independently of additional uncertainty arising from stochastic mortality, interest rates, or policyholder behavior. This framework provides a clearer understanding of the stability and sensitivity of the valuation process and serves as a foundation for future extensions.

Possible extensions of this work include stochastic mortality models, stochastic interest rates, dynamic policyholder behavior, path-dependent liabilities, and valuation problems involving stochastic differential equations or partial differential equations.

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Data Availability Statement: The mortality data used in this study are publicly available from the U.S. Social Security Administration Actuarial Life Table (<https://www.ssa.gov/oact/STATS/table4c6.html>, accessed on 1 June 2026). The valuation results were generated using these publicly available data together with the analytical methods developed in the paper. No additional datasets were created or analyzed.

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