

Article

Heat Transfer Analysis of MHD Flow in a Porous Tube Under Local Thermal Nonequilibrium Conditions Using the Keller-Box Method

Spoorthi Kadikol Math ¹, Nagaraj N. Katagi ^{1,*}, Ashwini Bhat ^{1,*}, Manjunath Shettar ¹ and Rajashekhar V. Choudhari ²

¹ Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal 576104, India

² Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Manipal 576104, India

* Correspondence: nn.katagi@manipal.edu (N.N.K.); ashwini.bhat@manipal.edu (A.B.)

Abstract

The present study investigates heat transfer characteristics in the thermally developing region of a porous tube under the local thermal nonequilibrium (LTNE) model. The influence of magnetohydrodynamic (MHD) flow on an electrically conducting fluid flowing through a porous medium under a transverse magnetic field is examined. Under the LTNE framework, two separate energy equations are employed to describe the temperature fields of the fluid and solid phases. The coupled governing equations are solved numerically using the Keller-box method. The results indicate that increasing the interphase heat transfer parameter strengthens thermal coupling between the fluid and solid phases, thereby reducing temperature differences and promoting local thermal equilibrium. In contrast, an increase in the Prandtl number reduces thermal diffusion, leading to larger temperature gradients and greater disparity between the two phases. Furthermore, the magnetic field suppresses both the velocity and temperature distributions through the Lorentz force. An increase in permeability reduces the velocity profiles due to the combined effects of the MHD and Prandtl numbers while increasing the temperature profiles. Increasing the interphase heat transfer rate drives the system from the LTNE to the LTE phase. The study confirms that LTNE effects play a significant role in thermal transfer processes in porous media and are relevant for various industrial heat transfer applications.

Keywords: local thermal nonequilibrium (LTNE); magnetohydrodynamics (MHD); porous medium; Keller-box method; interphase heat transfer; thermally developing flow; two-temperature model

Academic Editor: Wolfgang Linert

Received: 7 May 2026

Revised: 10 June 2026

Accepted: 12 June 2026

Published: 25 June 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

The Local Thermal Equilibrium (LTE) assumption is valid only when the interphase heat transfer is sufficiently strong and the thermal conductivities of both phases are comparable. However, in many practical porous media applications, especially those involving high-conductivity solids such as metal foams or rapid heating, the temperatures of the fluid and solid phases differ significantly. This limitation led to the development of the Local Thermal Nonequilibrium (LTNE) model, which employs separate energy equations for the fluid and solid phases to accurately capture the interphase temperature difference.

One of the earliest contributions in this direction was presented by Kuznetsov [1], who analyzed transient forced convection in a porous concentric annulus under LTNE conditions. His results revealed that the temperature difference between the phases propagates as localized thermal waves originating at the inlet. Minkowycz et al. [2] subsequently investigated deviations from LTE under transient heating conditions and introduced the Sparrow number as a quantitative measure of thermal nonequilibrium. Their analysis demonstrated that parameters such as pore size, interfacial heat transfer coefficient, heating rate, and characteristic layer thickness significantly influence the onset and magnitude of LTNE effects.

Further developments were carried out by Nield [3], who studied forced convection in porous channels, including conjugate conduction in bounding slabs. Their analytical solutions indicated that LTNE intensity and heat transfer characteristics depend strongly on porosity, the conductivity ratio, and interphase heat exchange. Quintard and Whitaker [4] established closure-based criteria for selecting LTE and LTNE models. Their analysis showed that strong conductivity contrasts and weak interfacial heat exchange necessitate LTNE formulations. Nield [5] also emphasized that structured porous media require careful LTNE modelling because pore geometry influences transport mechanisms.

He and Jin [6] incorporated LTNE effects into thermo-poroelasticity frameworks and reported that nonequilibrium temperatures substantially modify stress fields. Using weighted-temperature averaging, Biot theory, and Laplace transforms, they obtained solutions to the cylindrical hole problem and derived the corresponding constitutive relations. Their findings indicated that LTNE strongly affects the development of pore pressure and thermal stresses, particularly during early transient stages. Yang [7] obtained exact LTNE solutions for fully developed convection in an annulus and showed that LTE assumptions may fail under typical engineering conditions, especially in metal–foam–air systems. In the same year, Kayhani et al. [8] extended LTNE modelling to porous media subjected to sudden heating, demonstrating that thermal shocks and internal heat generation significantly promote nonequilibrium behavior. Li et al. [9] derived an analytical solution for the Nusselt number in thermally developing forced convection in a porous circular tube using an LTNE formulation with asymmetric inlet temperature conditions. Their results revealed a strong dependence of heat transfer on the Peclet number, Biot number, conductivity ratio, and inlet asymmetry. The proposed solution also provides a valuable benchmark for validating LTNE-based numerical simulations. In addition, Shashi Prabha et al. [10] and Keshavarzian and Sayehvand [11] applied LTNE models to natural convection over cylinders and wedge-type surfaces, respectively, and reported that thermal gradients are highly sensitive to the conductivity ratio and interphase heat transfer coefficients.

Shashi Prabha et al. [12] examined mixed convection over a moving wedge and showed that LTNE conditions lead to dual solution branches, with the stability of the upper branch confirmed through perturbation analysis. Additionally, Shashi Prabha et al. [13] extended this analysis to three-dimensional stagnation boundary layers and reported that LTNE intensity depends on governing parameters and results in boundary-layer thinning. Tayebi [14] analyzed entropy generation under LTNE conditions in a non-Darcy cavity and introduced a parameter to compare thermal irreversibility contributions from the fluid and solid phases. The same author also incorporated LTNE effects in nanofluid-saturated elliptical annuli, showing that nonequilibrium intensifies when the interphase heat transfer coefficient decreases, the Darcy number increases, or the porous medium conductivity ratio decreases. Shyabal et al. [15] conducted a stability analysis of Darcy–Brinkman convection with non-uniform basic temperature profiles. They demonstrated that LTNE alters the critical Rayleigh number, whereas different base-temperature gradients yield qualitatively symmetric responses. In addition, Sivanandam and Alqurashi [16]

studied the influences of activation energy, radiation, and cross-diffusion on rotating nanofluid flow in a porous medium. They demonstrated that heat transfer rates decrease with increasing inertia, rotation, and permeable parameters. Further, Kiran et al. [17] studied conjugate convective thermal transfer in annular porous geometry. From this work, they showed that the conductivity ratio and wall thickness had an important role in controlling thermal transfer through the annular space.

More recently, LTNE-based magnetohydrodynamic (MHD) flow in porous media has received considerable attention due to its relevance in nuclear reactors and aerospace systems. Sivasankaran et al. [18] investigated the effects of chemical reaction on double diffusive MHD squeezing copper water nanofluid flow between parallel plates. Here, they showed that the concentration field decreases for positive values of Cr and enhances for negative values of Cr . Siddabasappa et al. [19] explored LTNE effects in an MHD porous channel with induced magnetic fields. They found that the Darcy number significantly governs momentum transfer, magnetic induction, and temperature differences between the fluid and solid phases. Entropy generation studies have also highlighted the significance of LTNE modeling. Additionally, Sivandam et al. [20] studied MHD double diffusive convective squeezing parallel ternary nanofluid flow over parallel plates, and they included activation energy and viscous dissipation. The findings showed that increasing activation energy enhances concentration distribution and reduces the Sherwood number. Kefene et al. [21] investigated the magnetic effects on unsteady ferrofluid flow in a heated microchannel with permeable walls. Using a finite difference scheme with Runge–Kutta–Fehlberg integration, they found that the magnetic field and nanoparticle volume fraction reduce velocity but enhance temperature, while suction improves heat transfer rates. Gupta et al. [22] reported that couple-stress effects and magnetic forces substantially influence LTNE temperature distributions, particularly when interphase heat transfer is weak. Kumar et al. [23] investigated MHD mixed convection in porous channels and emphasized the dominant roles of the porous medium parameters and the Hartmann number in shaping LTNE temperature fields. In the same year, Goli et al. [24] studied MHD mixed convection of developing slip flow in a vertical porous microchannel under LTNE conditions. Using the finite volume method with the algorithm, they analyzed the effects of rarefaction, magnetic field strength, porous medium properties, and fluid–solid thermal interaction on velocity and temperature distributions, as well as on the Nusselt number. Khan et al. [25] investigated entropy generation in radiatively coupled stress–MHD nanofluid flow through a permeable channel. Their results indicate that fractional effects significantly influence irreversibility and heat transfer behavior, thereby enhancing energy efficiency. Kiran et al. [26] investigated magnetohydrodynamic fluid flow through a porous medium under LTNE conditions by using the Bernoulli wavelet method. Magnetic fields suppress fluid flow and heat transfer, while inertial effects enhance flow and skin friction. Higher Prandtl number, interphase heat transfer, and thermal property ratios improve phase coupling and solid-phase heat transfer. Çiçek et al. [27] investigated the flow and heat transfer behavior of a dusty hybrid nanofluid over a stretchable surface embedded in a porous medium, accounting for the LTNE approach, thermophoresis, and gravity-buoyancy effects. Ahmed et al. [28] studied thermal performance in star-shaped thermal energy storage (TES) units filled with nano-enhanced phase change materials (NEPCMs). Higher Rayleigh numbers strengthen buoyancy effects, while rotation enhances mixing and increases flow activity by 44.7% under external heating.

In the present work, two-dimensional fluid flow in a porous tube subjected to a transverse magnetic field is examined under LTNE conditions. The fluid- and porous-solid-phase temperatures are treated as distinct, and the governing equations are solved numerically using the Keller-box method. Although LTNE and MHD heat transfer in porous media have been extensively investigated, studies on thermally developing axisymmetric

flow in porous cylindrical tubes under simultaneous LTNE and magnetic-field effects remain limited. Furthermore, detailed analyses of separate fluid- and solid-phase temperature fields using the Keller-box method are scarce. Therefore, the present study focuses on analyzing the influence of magnetic parameter, permeability resistance, interphase heat transfer, and thermal conductivity parameters on the momentum and thermal transport characteristics of the fluid and solid phases under LTNE conditions.

2. Mathematical Formulation

The present study considers the steady, laminar, incompressible, and axisymmetric flow of an electrically conducting, viscous fluid through a porous cylindrical tube saturated with a homogeneous, isotropic porous medium, as illustrated in Figure 1. Here, v_w represents the uniform transverse velocity at the porous bounding surface. A uniform transverse magnetic field of strength B_0 is applied perpendicular to the flow direction. The induced magnetic field is neglected under the assumption of a small magnetic Reynolds number. The porous medium is modeled using the Darcy–Brinkman formulation to account for viscous diffusion and permeability resistance effects.

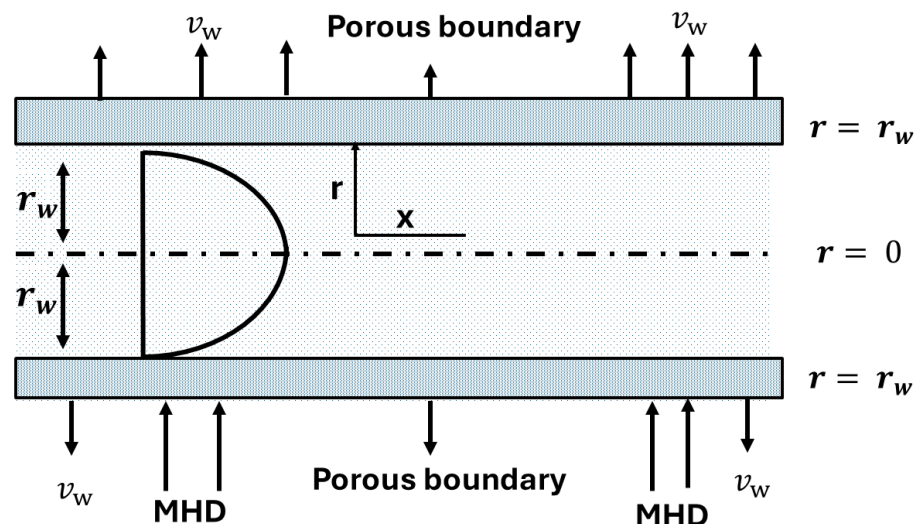


Figure 1. Physical model.

LTNE conditions are assumed between the fluid and solid phases of the porous medium; therefore, separate energy equations are employed for the fluid and solid phases. Thermophysical properties are assumed constant, while viscous dissipation, Joule heating, and thermal radiation effects are neglected.

Under the above assumptions, the governing equations for continuity, momentum, and heat transfer in cylindrical coordinates (x, r) are given as follows.

$$\nabla \cdot \vec{q} = 0 \quad (1)$$

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{q} - \frac{\mu}{\rho K} \vec{q}_D - (J \times B) \quad (2)$$

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T \right) = k \nabla^2 T \quad (3)$$

where ρ —fluid density; μ —dynamic viscosity; K —porous medium permeability; C_p —isobaric specific heat; $\vec{q}_D$ —Darcy velocity vector; p —pressure; T —temperature; and k —thermal conductivity. The Dupuit–Forchheimer relationship [29] is employed to balance

these values because the intrinsic $\vec{q}$ and the Darcy velocity $\vec{q}_D$ diverge in a homogeneous porous medium.

$$\vec{q}_D = \epsilon \vec{q} \quad (4)$$

The porous medium's porosity is denoted by ϵ . The majority of naturally occurring materials have porosity in the range [0, 0.6), while the range for artificial materials is [0.6, 1). According to the model, a heated, porous solid medium is traversed by the entering cold fluid. The temperature T_1 of the hot impermeable surface is situated in the freestream of the ambient temperature T_2 . The heat transfer Equation (3) for the LTNE phase is divided into

$$(\rho C_p)_f \left(\frac{\partial T_f}{\partial t} + \vec{q} \cdot \nabla T_f \right) = k_f \epsilon \nabla^2 T_f + h(T_s - T_f) \quad (5)$$

$$(\rho C_p)_s (1 - \epsilon) \left(\frac{\partial T_s}{\partial t} \right) = (1 - \epsilon) k_s \nabla^2 T_s + h(T_f - T_s) \quad (6)$$

In this context, k_f and k_s represent the thermal conductivity of the fluid and solid components of the porous medium, respectively, [30,31]. The interfacial heat transfer coefficient (h) acts as a critical parameter, $h = h_{sf} a_v$, where h_{sf} is the volumetric interfacial heat transfer coefficient and a_v is the specific surface of the porous medium. Higher values of h signify an LTE regime, wherein the fluid and solid phases of the porous medium attain identical local temperatures; for analytical convenience, h is subsequently expressed in non-dimensional form as H [32].

The multiplication of Equation (2) by ϵ^{-2} yields the following continuity, momentum, and energy equations.

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial(rv)}{\partial r} = 0 \quad (7)$$

$$\frac{1}{\epsilon^2} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v_{eff} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial x^2} \right) - \frac{v}{K} u - \frac{\sigma B_0^2}{\rho} u \quad (8)$$

$$\frac{1}{\epsilon^2} \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial r} + v_{eff} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{\partial^2 v}{\partial r^2} \right) - \frac{v}{K} v \quad (9)$$

$$u \frac{\partial T_f}{\partial x} + v \frac{\partial T_f}{\partial r} = \frac{k_f \epsilon}{\rho C_p} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_f}{\partial r} \right) + \frac{\partial^2 T_f}{\partial x^2} \right) + \frac{h}{\rho C_p} (T_s - T_f) \quad (10)$$

$$0 = (1 - \epsilon) k_s \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_s}{\partial r} \right) + \frac{\partial^2 T_s}{\partial x^2} \right) - h(T_s - T_f) \quad (11)$$

The boundary conditions employed in the present study are adopted from Bhat et al. [33].

$$u(x, r_w) = -\left(\frac{\sqrt{k}}{\alpha}\right) \left(\frac{\partial u}{\partial r}\right); \quad \left(\frac{\partial u}{\partial r}\right)_{r=0} = 0; \quad v(x, 0) = 0; \quad v(x, r_w) = v_w; \quad \frac{\partial T_s}{\partial r} = \frac{\partial T_f}{\partial r} = 0 \text{ at } r = 0; \quad T_f = T_s \rightarrow T_w \text{ at } r = r_w \quad (12)$$

where v is the kinematic viscosity, and v_{eff} is the effective kinematic viscosity. By introducing a similarity transformation ψ , the governing PDE equations are reduced to non-linear ODEs, where $\eta = \left(\frac{r}{r_w}\right)^2$. To simplify the governing partial differential equations, similarity transformations are introduced such that the continuity equation is satisfied identically. The flow variables are transformed into dimensionless forms using the characteristic tube radius, the wall suction velocity, and the temperature difference between the wall and the mainstream fluid.

$$\psi(\eta) = \frac{r_w}{2} (r_w \bar{u}_0 - 2v_w x) f(\eta) \quad (13)$$

Using the stream function formulation, the velocity components are obtained as

$$u(x, \eta) = \left(\bar{u}_0 - \frac{2v_w x}{r_w} \right) f'(\eta); \quad v(\eta) = \frac{v_w}{\sqrt{\eta}} f(\eta) \quad (14)$$

The dimensionless temperature functions for the fluid and solid phases are defined as

$$T_f = T \theta(\eta) + T_2; \quad T_s = T \Phi(\eta) + T_2 \quad (15)$$

where $T = T_w - T_2$; the reference and mainstream temperatures are T_w and T_2 .

Substituting the above similarity transformations into the governing momentum and energy equations yields the following coupled nonlinear ordinary differential equations.

$$\frac{1}{\varepsilon^2} (-f'(\eta) f''(\eta) + f(\eta) f'''(\eta)) = \left(\frac{2\Lambda}{Rey} \right) (\eta f''''(\eta) + 2f'''(\eta)) - \left(\frac{\Omega}{2} \right) f''(\eta) - \left(\frac{M}{2} \right) f''(\eta) \quad (16)$$

$$\left(\frac{Rey Pr}{\varepsilon} \right) f(\eta) \theta'(\eta) = 2\eta \theta''(\eta) + \theta'(\eta) + H(\Phi - \theta) \quad (17)$$

$$2\eta \Phi''(\eta) + \Phi'(\eta) - HY(\Phi - \theta) = 0 \quad (18)$$

Subjected to the boundary conditions,

$$\sqrt{\eta} f''(\eta) \rightarrow 0, \text{ as } \eta \rightarrow 0; f'(1) = -2Sf''(1); f(0) = 0; f(1) = 1$$

$$\theta'(0) = 0; \theta(1) = 1; \Phi'(0) = 0; \Phi(1) = 1; \quad (19)$$

The transformed boundary conditions are consistent with the adopted similarity transformation in cylindrical coordinates. The condition involving $\sqrt{\eta}$ ensures finite and physically bounded velocity gradients near the axisymmetric region.

The Nusselt number and skin friction coefficient are given as

$$Nu_f = -\theta'(1), Nu_s = -\Phi'(1) \quad (20)$$

$$C_f = -f''(1) \quad (21)$$

where $\left(\frac{v_{eff}}{v} \right) = \Lambda$ is the ratio of effective viscosity to fluid kinematic viscosity, $\Omega = \left(\frac{vr_w}{v_w K} \right)$ is the permeability resistance parameter, $Pr = \left(\frac{\rho c_p v}{k_f} \right)$ is the Prandtl number, $H = \left(\frac{hr_w^2}{k_f \varepsilon} \right)$ is the microscopic inter-phase rate of heat transfer, $Y = \left(\frac{k_f \varepsilon}{(1-\varepsilon)k_s} \right)$ is porosity-scaled conductivity, and $M = \left(\frac{\sigma B_0^2 r_w}{v_w \mu} \right)$ is the magnetic parameter [34].

Here, Λ represents the ratio of effective viscosity to fluid kinematic viscosity. Consequently, Λ characterizes the additional momentum resistance introduced by the porous matrix. For $\Lambda = 1$, the effective viscosity equals the fluid viscosity, whereas larger values of Λ indicate stronger viscous diffusion effects within the porous structure. Therefore, Λ plays an important role in wall shear stress and momentum transport in porous media flows.

3. Methodology

The coupled nonlinear ODEs are solved using the implicit finite difference Keller-box method. Cebeci and Bradshaw [35] provided a detailed explanation of this approach in the context of boundary-layer problems. The solution procedure initially applies finite difference approximations and solves the equations using a box scheme. The process begins by expressing the system as a set of first-order equations by initiating new variables.

$$\mathbf{u} = \mathbf{f}', \mathbf{v} = \mathbf{f}'', \mathbf{f}''' = \mathbf{w}, \theta = \mathbf{th}, \theta' = \mathbf{m}, \Phi = \mathbf{ph}, \Phi' = \mathbf{g} \quad (22)$$

Using Equation (22), Equations (16)–(18) are rewritten as

$$\frac{1}{\varepsilon^2} [\mathbf{u} \mathbf{v} + \mathbf{f} \mathbf{w}] - \left(\frac{2\Lambda}{Re\gamma} \right) [\eta \mathbf{w}' + 2\mathbf{w}] + \left(\frac{\Omega}{2} \right) \mathbf{v} + \left(\frac{M}{2} \right) \mathbf{v} = \mathbf{0} \quad (23)$$

$$(2\eta \mathbf{m}' + \mathbf{m}) + H(\mathbf{ph} - \mathbf{th}) - \left(\frac{Re\gamma Pr}{\varepsilon} \right) \mathbf{f} \mathbf{m} = \mathbf{0} \quad (24)$$

$$(2\eta \mathbf{g}' + \mathbf{g}) - H\gamma(\mathbf{ph} - \mathbf{th}) = \mathbf{0} \quad (25)$$

And the boundary conditions are written as

$$\begin{aligned} \sqrt{\eta} \mathbf{v}(0) &= \mathbf{0}; \quad \mathbf{u}(1) = -2S\mathbf{v}(1); \quad \mathbf{f}(0) = \mathbf{0}; \quad \mathbf{f}(1) = \mathbf{1}; \\ \mathbf{m}(0) &= \mathbf{0}; \quad \mathbf{th}(1) = \mathbf{1}; \quad \mathbf{g}(0) = \mathbf{0}; \quad \mathbf{ph}(1) = \mathbf{1}; \end{aligned} \quad (26)$$

The central difference scheme is employed to convert the first-order differential equations into a system of difference equations.

$$\begin{aligned} f_{j-1/2} &= \frac{f_j + f_{j-1}}{2} \\ \left(\frac{df}{d\eta} \right)_{j-1/2} &= \frac{f_j - f_{j-1}}{\Delta \eta} \end{aligned} \quad (27)$$

The system of equations above consists of nonlinear algebraic equations that are difficult to solve if we employ the finite difference approach in the same way. Consequently, we linearize them by incorporating

$$\mathbf{a}^{(k+1)} = \mathbf{a}^k + \delta \mathbf{a}^k \quad (28)$$

where $\mathbf{a} = [\mathbf{f}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{th}, \mathbf{m}, \mathbf{ph}, \mathbf{g}]^T$. We obtain a system of linear algebraic equations in $\delta \mathbf{a}^k$ by substituting the above linearized equation into the finite difference equation and ignoring any products of $\delta \mathbf{a}$. The following equations are further obtained:

$$(\mathbf{a}_1)_j \delta \mathbf{w}_j + (\mathbf{a}_2)_j \delta \mathbf{w}_{j-1} + (\mathbf{a}_3)_j \delta \mathbf{v}_j + (\mathbf{a}_4)_j \delta \mathbf{v}_{j-1} + (\mathbf{a}_5)_j \delta \mathbf{u}_j + (\mathbf{a}_6)_j \delta \mathbf{u}_{j-1} + (\mathbf{a}_7)_j \delta \mathbf{f}_j + (\mathbf{a}_8)_j \delta \mathbf{f}_{j-1} = \mathbf{0} \quad (29)$$

$$(\mathbf{b}_1)_j \delta \mathbf{m}_j + (\mathbf{b}_2)_j \delta \mathbf{m}_{j-1} + (\mathbf{b}_3)_j \delta \mathbf{ph}_j + (\mathbf{b}_4)_j \delta \mathbf{ph}_{j-1} + (\mathbf{b}_5)_j \delta \mathbf{th}_j + (\mathbf{b}_6)_j \delta \mathbf{th}_{j-1} + (\mathbf{b}_7)_j \delta \mathbf{f}_j + (\mathbf{b}_8)_j \delta \mathbf{f}_{j-1} = \mathbf{0} \quad (30)$$

$$(\mathbf{c}_1)_j \delta \mathbf{g}_j + (\mathbf{c}_2)_j \delta \mathbf{g}_{j-1} + (\mathbf{c}_3)_j \delta \mathbf{ph}_j + (\mathbf{c}_4)_j \delta \mathbf{ph}_{j-1} + (\mathbf{c}_5)_j \delta \mathbf{th}_j + (\mathbf{c}_6)_j \delta \mathbf{th}_{j-1} = \mathbf{0} \quad (31)$$

where

$$(\mathbf{a}_1)_j = \left[\left(\frac{2\Lambda}{Re\gamma} \right) (\eta + 2d_j) + \frac{d_j (f_j + f_{j-1})}{\varepsilon^2} \right] \quad (32)$$

$$(\mathbf{a}_2)_j = \left[- \left(\frac{2\Lambda}{Re\gamma} \right) (\eta + 2d_j) + \frac{d_j (f_j + f_{j-1})}{\varepsilon^2} \right] \quad (33)$$

$$(\mathbf{a}_3)_j = \left[\left(\left(\frac{\Omega}{2} \right) + \left(\frac{M}{2} \right) \right) d_j - \frac{d_j (u_j + u_{j-1})}{\varepsilon^2} \right] \quad (34)$$

$$(\mathbf{a}_4)_j = (\mathbf{a}_3)_j \quad (35)$$

$$(\mathbf{a}_5)_j = \left[- \frac{d_j (v_j + v_{j-1})}{\varepsilon^2} \right] \quad (36)$$

$$(\mathbf{a}_6)_j = (\mathbf{a}_5)_j \quad (37)$$

$$(a_7)_j = \left[\frac{d_j (w_j + w_{j-1})}{\varepsilon^2} \right] \quad (38)$$

$$(a_8)_j = (a_7)_j \quad (39)$$

$$(b_1)_j = \left[2\eta - d_j - \left(\frac{Rey Pr}{\varepsilon} \right) d_j \frac{(f_j + f_{j-1})}{2} \right] \quad (40)$$

$$(b_2)_j = \left[-2\eta - d_j - \left(\frac{Rey Pr}{\varepsilon} \right) d_j \frac{(f_j + f_{j-1})}{2} \right] \quad (41)$$

$$(b_3)_j = [-Hd_j] \quad (42)$$

$$(b_4)_j = (b_3)_j \quad (43)$$

$$(b_5)_j = [Hd_j] \quad (44)$$

$$(b_6)_j = (b_5)_j \quad (45)$$

$$(b_7)_j = \left[- \left(\frac{Rey Pr}{\varepsilon} \right) d_j \right] \quad (46)$$

$$(b_8)_j = (b_7)_j \quad (47)$$

$$(c_1)_j = [2\eta + d_j] \quad (48)$$

$$(c_2)_j = [-2\eta + d_j] \quad (49)$$

$$(c_3)_j = [-Hyd_j] \quad (50)$$

$$(c_4)_j = (c_3)_j \quad (51)$$

$$(c_5)_j = [Hyd_j] \quad (52)$$

$$(c_6)_j = (c_5)_j \quad (53)$$

Construct a vector matrix representation of the linearized equations. To solve the linear system in triangular form, employ the decomposition method.

$$A\delta = r \quad (54)$$

$$[A] = \begin{bmatrix} [A_1][C_1] & \dots & \dots \\ [B_2][A_2][C_2] & \ddots & \vdots \\ \vdots & \dots & [B_{J-1}][A_{J-1}][C_{J-1}] \\ \vdots & \dots & [B_J][A_J] \end{bmatrix}, [\delta] = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ \vdots \\ [\delta_{J-1}] \\ [\delta_J] \end{bmatrix}, [r] = \begin{bmatrix} [r_1] \\ [r_2] \\ \vdots \\ [r_{J-1}] \\ [r_J] \end{bmatrix} \quad (55)$$

where, in the above equation, the elements are defined by

$$A_j = \begin{bmatrix} 0 & 0 & 0 & -d_j & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -d_j & 0 & 0 & 0 \\ -d_j & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -d_j & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -d_j & 0 \\ (a_2)_j & 0 & 0 & (a_6)_j & (a_3)_j & 0 & 0 & (a_7)_j \\ 0 & (b_6)_j & (b_4)_j & 0 & 0 & (b_1)_j & 0 & (b_7)_j \\ 0 & (c_6)_j & (c_4)_j & 0 & 0 & 0 & (c_1)_j & 0 \end{bmatrix}, 1 \leq j \leq J-1$$

$$C_j = \begin{bmatrix} 0 & 0 & 0 & -d_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -d_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (a_1)_j & 0 & 0 & (a_5)_j & 0 & 0 & 0 & 0 \\ 0 & (b_5)_j & (b_3)_j & 0 & 0 & 0 & 0 & 0 \\ 0 & (c_5)_j & (c_3)_j & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -d_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_j & 0 \\ 0 & 0 & 0 & 0 & (a_4)_j & 0 & 0 & (a_8)_j \\ 0 & 0 & 0 & 0 & 0 & (b_2)_j & 0 & (b_8)_j \\ 0 & 0 & 0 & 0 & 0 & 0 & (c_2)_j & 0 \end{bmatrix}, 2 \leq j \leq J$$

$$A_J = \begin{bmatrix} 0 & 0 & 0 & -d_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -d_j & 0 & 0 & 0 \\ -d_j & 0 & 0 & 0 & 1 & 0 & 0 & -d_j \\ 0 & -1 & 0 & 0 & 0 & -d_j & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -d_j & 0 \\ (a_2)_J & 0 & 0 & (a_6)_J & (a_3)_J & 0 & 0 & (a_1)_J \\ 0 & (b_6)_J & (b_4)_J & 0 & 0 & (b_1)_J & 0 & 0 \\ 0 & (c_6)_J & (c_4)_J & 0 & 0 & 0 & (c_1)_J & 0 \end{bmatrix}$$

$$[\delta_j] = [\delta w_{j-1}, \delta th_{j-1}, \delta ph_{j-1}, \delta u_{j-1}, \delta v_j, \delta m_j, \delta g_j, \delta f_j], 1 \leq i \leq i-1$$

and

$$[\delta_J] = [\delta w_{j-1}, \delta th_{j-1}, \delta ph_{j-1}, \delta u_{j-1}, \delta v_j, \delta m_j, \delta g_j, \delta w_j]^T$$

$$[r_j] = [(r_1)_j, (r_2)_j, (r_3)_j, (r_4)_j, (r_5)_j, (r_6)_j, (r_7)_j, (r_8)_j]^T, 1 \leq j \leq J$$

The LU decomposition method is used to solve the block tridiagonal matrix, i.e.,

$$A = LU \quad (56)$$

where L and U are the lower and upper triangular matrices.

$$L = \begin{bmatrix} [\alpha_1] & \dots & \dots & \dots & \dots & \dots \\ \dots & [\alpha_2] & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & [B_J] & [\alpha_J] \end{bmatrix}$$

$$U = \begin{bmatrix} [I] & [\Gamma_1] & \dots & \dots & \dots & \dots \\ \dots & [I] & [\Gamma_1] & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & [I] & [\Gamma_{j-1}] \\ \dots & \dots & \dots & \dots & \dots & [I] \end{bmatrix}$$

In this case, the unit matrix is denoted by $[I]$, and the elements of the 8×8 matrices $[\alpha_j]$ and $[\Gamma_j]$ are defined as:

$$[\alpha_1] = [A_1]$$

$$[C_1] = [A_1][\Gamma_1]$$

$$[\alpha_1] = [A_j] - [B_j][\Gamma_{j-1}], j = 2, 3, \dots, J$$

$$[C_j] = [\alpha_j][\Gamma_j], j = 2, 3, \dots, J - 1$$

Equation LU is substituted in Equation (56) in (54), and we have

$$LU\delta = r \quad (57)$$

Define $u\delta = w$. We have

$$Lw = r \quad (58)$$

where

$$w = [(w_1), (w_2), (w_3), (w_4), (w_5), (w_6), (w_7), (w_8)]^T \quad (59)$$

and the $[w_j]$ are 8×1 column matrices. The elements w can be solved as

$$[r_1] = [\alpha_j][w_j]$$

$$[\alpha_1][w_j] = [r_j] - [B_j][w_{j-1}]$$

These solutions for δ can be used to find the $(i + 1)^{th}$ iteration. The Keller-box algorithm is implemented in MATLAB R2022a using a uniform step size of **0.001** and a convergence tolerance of 10^{-6} .

4. Results and Discussion

In this work, two-dimensional fluid flow in a porous tube under a transverse magnetic field is investigated. We focus on heat transfer between a fluid and a porous surface (solid matrix), where the two phases are not in thermal equilibrium; therefore, two separate energy equations are employed to describe the temperature distributions of the fluid and solid phases. The temperature difference between the fluid and solid phases can quantify the degree of LTNE in the porous medium. To characterize the strength of thermal nonequilibrium, the LTNE intensity is defined as

$$I_{LTNE} = |\theta(\eta) - \Phi(\eta)|$$

where $\theta(\eta)$ and $\Phi(\eta)$ denote the dimensionless fluid-phase and solid-phase temperatures, respectively. The LTNE intensity provides a direct measure of the thermal nonequilibrium between the two phases. Larger values of I_{LTNE} indicate stronger thermal nonequilibrium and weaker interphase thermal coupling, whereas $I_{LTNE} \rightarrow 0$ corresponds to LTE, where both phases attain identical temperatures. Throughout the present study, variations in I_{LTNE} are used to assess the influence of governing parameters on the thermal interaction between the fluid and solid phases.

The problem is solved numerically using the Keller-box technique, and the effects of various governing parameters on the velocity and temperature distributions are discussed below.

Figure 2 illustrates the influence of the magnetic parameter M on the velocity and temperature fields. Figure 2a shows that the axial velocity profile $f'(\eta)$ decreases significantly with increasing values of M . The interaction between the applied magnetic field and the electrically conducting fluid generates an electromagnetic resistive force, commonly known as the Lorentz force, which acts in the opposite direction to the flow. This resistive force suppresses fluid motion and reduces momentum transport throughout the boundary layer. The suppression of fluid motion also increases the dominance of viscous effects relative to inertial transport within the flow field.

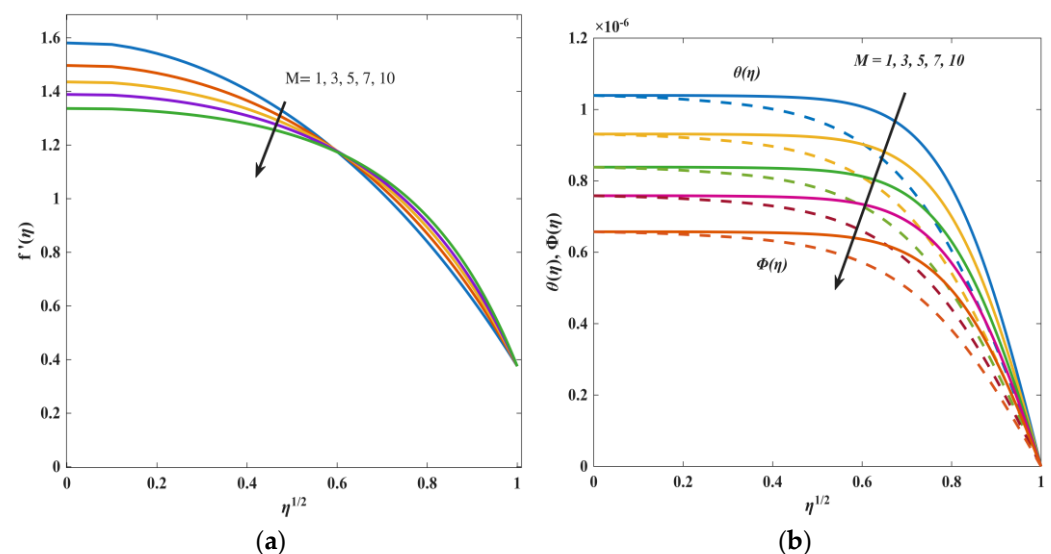


Figure 2. Effect of magnetic parameter M on (a) velocity profiles and (b) fluid/solid temperature profiles.

Figure 2b presents the corresponding variations in the fluid-phase and solid-phase temperature distributions, $\theta(\eta)$ and $\Phi(\eta)$, for different values of M . Increasing M reduces the temperature profiles in both phases throughout the thermal boundary layer. The reduction in fluid velocity weakens convective thermal transport and limits the movement of thermal energy away from the heated wall region. Thus, thermal penetration inside the porous medium decreases and the thermal boundary layer becomes thinner. In addition, the electromagnetic resistance alters the balance between convective and conductive heat transfer mechanisms within the LTNE system. Since the fluid phase loses its ability to transport thermal energy effectively in stronger magnetic fields, the interphase thermal interaction between the fluid and solid phases also weakens. As a result, both the momentum and thermal transport characteristics are strongly influenced by the magnetic field, demonstrating the coupled interaction between electromagnetic effects and heat transfer behavior inside the porous medium. Since viscous dissipation and Joule heating are neglected in the present model, the magnetic field primarily suppresses convective transport through the Lorentz force.

Figure 3 illustrates the influence of the permeability resistance parameter Ω on the velocity and temperature distributions within the porous medium. As shown in Figure 3a, the axial velocity profile $f'(\eta)$ decreases with increasing values of Ω . Additionally, larger values of Ω correspond to lower permeability of the porous medium, thereby increasing the resistance to fluid motion. The enhanced porous drag suppresses momentum transport and weakens fluid acceleration within the boundary layer. Subsequently, the

axial velocity decreases progressively with increasing permeability resistance, indicating stronger momentum-damping effects inside the porous structure.

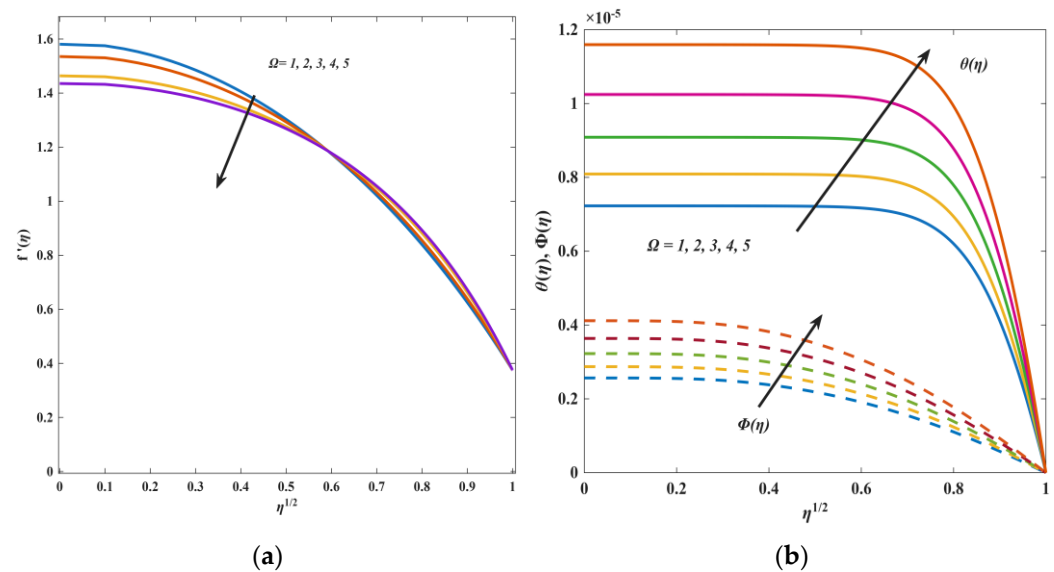


Figure 3. Effect of permeability resistance parameter Ω on (a) velocity profiles and (b) temperature profiles at $H = 0.01$, $\gamma = 80$, $Pr = 0.71$, and $M = 3$.

Figure 3b presents the corresponding variations in the fluid-phase and solid-phase temperature profiles, $\theta(\eta)$ and $\Phi(\eta)$, for different values of Ω . Increasing Ω enhances the temperature distributions in both phases. The reduction in fluid motion weakens convective heat transport and promotes thermal energy accumulation near the wall region, thereby increasing the thermal boundary layer thickness. Under LTNE conditions, finite interphase heat transfer between the fluid and solid phases prevents instantaneous thermal equilibrium, leading to distinct temperature distributions in the two phases. The increase in permeability resistance also enhances the thermal boundary layer thickness due to reduced convective heat transport within the porous medium.

Figure 4 illustrates the effects of the porosity-scaled conductivity parameter γ and the interphase heat transfer parameter H on the fluid-phase and solid-phase temperature distributions under LTNE conditions. Figure 4a shows the variation of the temperature profiles $\theta(\eta)$ and $\Phi(\eta)$ for different values of γ . It is observed that increasing γ modifies the thermal interaction between the fluid and solid phases within the porous medium. Larger γ values enhance the effective thermal conductivity of the porous structure and strengthen heat diffusion through the medium. Subsequently, thermal energy is transferred more efficiently between the two phases, reducing the temperature difference between the fluid and solid regions. The fluid-phase temperature decreases slightly due to enhanced thermal redistribution, whereas the solid-phase temperature exhibits a marginal increase because of improved heat conduction through the porous matrix. The reduced temperature difference indicates weaker LTNE effects. Thus, increasing γ promotes a transition toward thermal equilibrium behavior inside the porous medium.

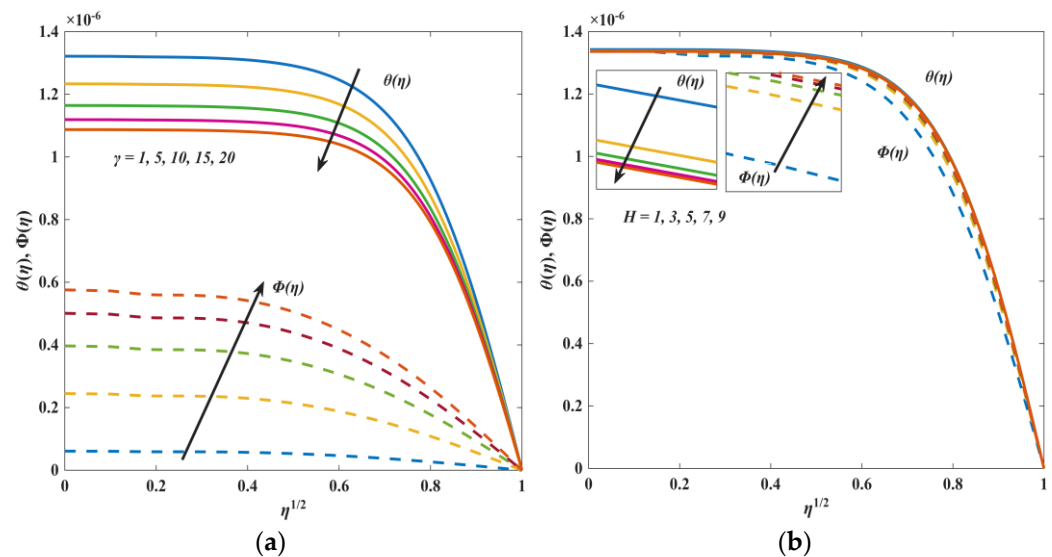


Figure 4. Variation of fluid- and solid-phase temperature profiles for (a) different values of γ ; (b) different values of H at $\gamma = 80$, $H = 0.01$, $S = 0.1$, and $\varepsilon = 0.971$.

Figure 4b presents the influence of the interphase heat transfer parameter H on the temperature distributions for a fixed value of γ . The parameter H governs the intensity of thermal energy exchange between the fluid and solid phases and therefore plays a significant role in LTNE heat transport behavior. At lower values of H , a noticeable temperature difference exists between $\theta(\eta)$ and $\Phi(\eta)$, indicating weak thermal coupling and strong LTNE behavior. As H increases, the interphase heat exchange becomes stronger, allowing thermal energy to be transferred more rapidly between the two phases. Therefore, the temperature difference between the fluid and solid phases decreases progressively. For larger H values, both temperature profiles converge, indicating that the system gradually approaches local thermal equilibrium. This behavior confirms that stronger interphase heat transfer suppresses LTNE effects and enhances thermal synchronization between the fluid and porous matrix. Increasing γ and H reduces the thermal disparity between the two phases and promotes stronger thermal coupling across the boundary layer.

The decrease in $|\theta(\eta) - \Phi(\eta)|$ with increasing H shows that the fluid- and solid-phase temperatures gradually become closer, indicating a transition toward thermal equilibrium within the porous medium.

Figure 5 illustrates the influence of the Prandtl number Pr on the fluid-phase and solid-phase temperature distributions under LTNE conditions. The Prandtl number represents the ratio of momentum diffusivity to thermal diffusivity and therefore controls the rate of thermal energy diffusion within the porous medium. As shown in Figure 5a, increasing Pr values enhance the solid-phase temperature profile, $\Phi(\eta)$. Larger Prandtl numbers correspond to lower thermal diffusivity, which weakens heat diffusion through the porous structure and reduces the rate of thermal energy dissipation from the wall region. Consequently, thermal energy is retained for a longer duration within the solid matrix, resulting in higher solid-phase temperatures throughout the domain.

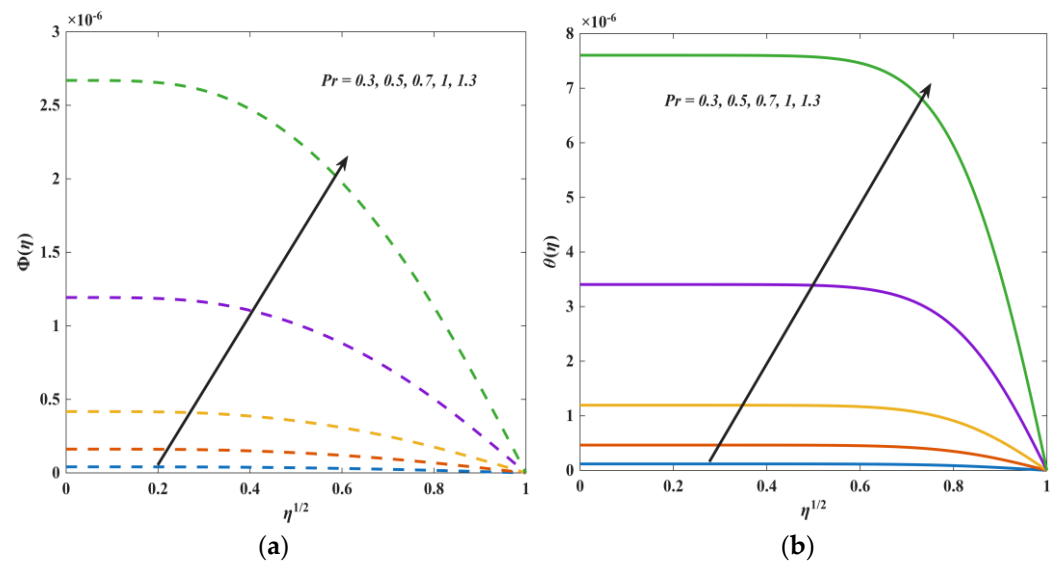


Figure 5. Effect of Prandtl number Pr (a) $\Phi(\eta)$ and (b) $\theta(\eta)$ at $H = 0.01$, $\gamma = 80$, $\varepsilon = 0.971$, $\Omega = 1$ and $M = 3$.

A similar trend is observed in Figure 5b for the fluid-phase temperature profile $\theta(\eta)$. Higher Pr values reduce thermal diffusivity and restrict heat diffusion away from the heated surface. As a result, thermal energy accumulates within the boundary layer, raising the fluid's temperature. Unlike classical forced-convection systems, the present LTNE porous medium formulation accounts for separate thermal transport in the fluid and solid phases, coupled through finite interphase heat exchange. Under such conditions, the reduction in thermal diffusivity becomes more dominant than convective thermal transport, thereby promoting thermal energy retention inside the porous structure. Larger Prandtl numbers further compress the thermal boundary layer and reduce thermal penetration into the porous medium due to weaker thermal diffusion.

Figure 6 illustrates the combined effects of the interphase heat transfer parameter H and the thermal conductivity ratio γ on the fluid-phase and solid-phase temperature distributions under LTNE conditions. Figure 6a presents the temperature profiles corresponding to $H = 0.04$ and $\gamma = 1000$. The large value of γ enhances thermal conduction within the porous medium and promotes stronger redistribution of thermal energy between the fluid and solid phases. Although the interphase heat transfer parameter remains moderate, the enhanced conductive transport produces more uniform temperature distributions across the boundary layer region. Consequently, the temperature difference between $\theta(\eta)$ and $\Phi(\eta)$ decreases, indicating weaker LTNE behavior. The enhanced thermal conduction, therefore, strengthens thermal coupling between the two phases and promotes the gradual establishment of thermal equilibrium within the porous structure.

Figure 6b illustrates the temperature profiles for $H = 0.05$ and $\gamma = 500$. Increasing the interphase heat transfer parameter intensifies the thermal interaction between the fluid and solid phases. The stronger interfacial heat exchange enhances energy transfer across the two phases and further reduces the temperature difference between $\theta(\eta)$ and $\Phi(\eta)$. Although the conductivity ratio γ is smaller compared with Figure 6a, it remains sufficiently large to maintain effective conductive heat transport within the porous medium. As a result, both temperature profiles remain closely aligned throughout the flow region, indicating a strong tendency toward local thermal equilibrium. These results show that increasing H and γ promotes LTE conditions.

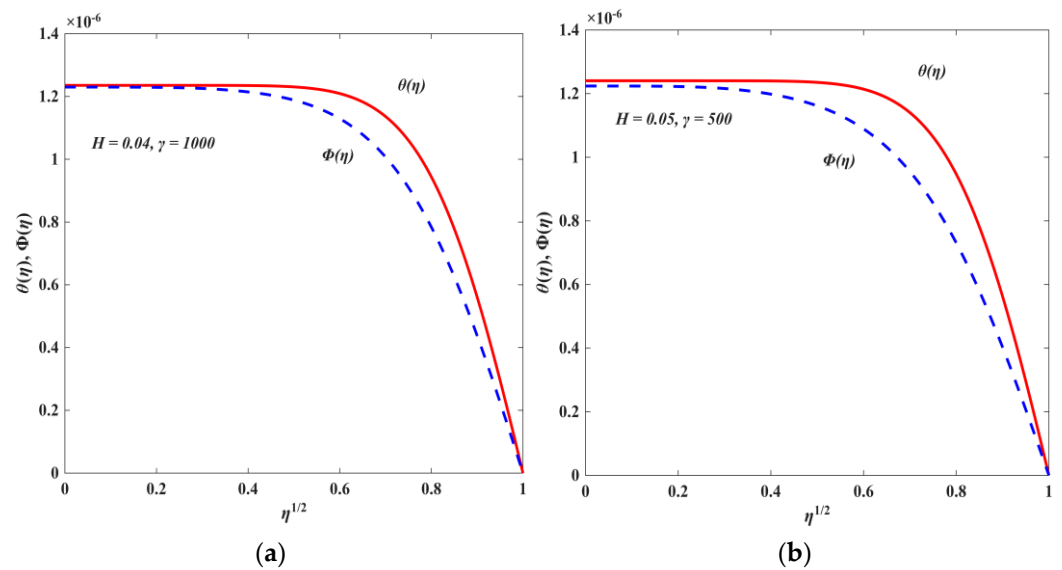


Figure 6. Temperature profiles for (a) $H = 0.04$ and $\gamma = 1000$ and (b) $H = 0.05$ and $\gamma = 500$ at $M = 3$, $\varepsilon = 0.971$, $Pr = 0.71$, $\Omega = 1$, and $S = 0.1$.

Figure 7 presents the variation of the wall temperature gradients $\theta'(1)$ and $\Phi'(1)$ with the interphase heat transfer parameter H for different values of the Prandtl number Pr and permeability resistance parameter Ω . Figure 7a illustrates the influence of the Prandtl number on the wall temperature gradients. Increasing Pr increases both wall temperature gradients. Since larger Prandtl numbers correspond to lower thermal diffusivity, thermal energy diffusion within the boundary layer becomes weaker. Accordingly, heat transfer is confined to a thinner thermal region near the wall, leading to steeper surface-temperature gradients. The enhanced wall gradients also indicate stronger thermal resistance within the boundary layer at higher Prandtl numbers.

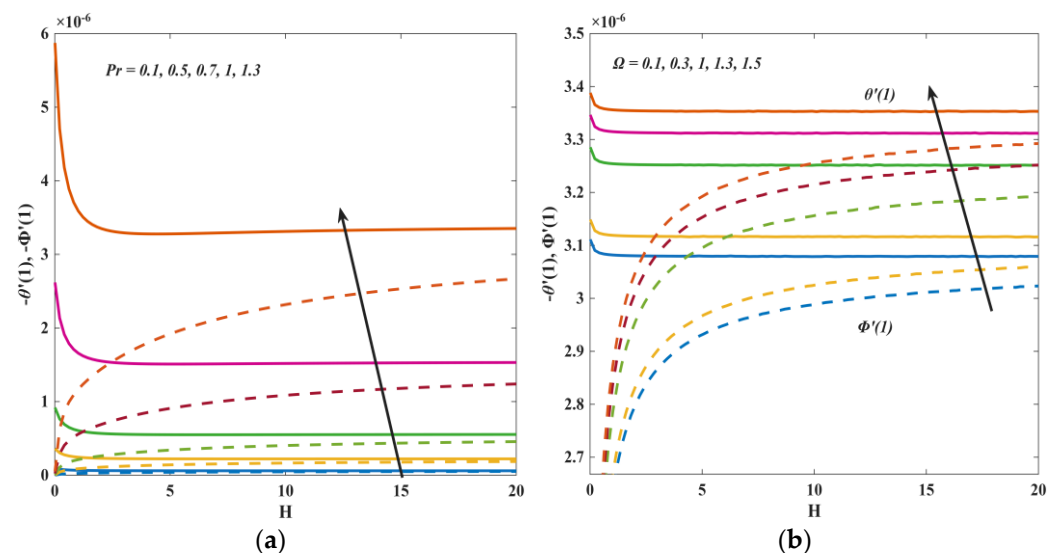


Figure 7. Variation of wall temperature gradients $\theta'(1)$ and $\Phi'(1)$ with H for different (a) Pr and (b) Ω at $\varepsilon = 0.971$, $\gamma = 5$, and $M = 3$.

Figure 7b illustrates the influence of the permeability resistance parameter Ω on the wall temperature gradients $\theta'(1)$ and $\Phi'(1)$. Increasing Ω enhances the magnitudes of both gradients throughout the considered range of H . Larger values of Ω correspond to lower permeability of the porous medium, which suppresses fluid motion and weakens convective heat transport inside the flow domain. As convection diminishes, heat transfer

near the wall becomes increasingly dominated by conduction, leading to steeper temperature variations near the surface. Consequently, both wall temperature gradients increase with increasing permeability resistance. The solid-phase temperature gradient $\Phi'(1)$ exhibits a more pronounced increase with H before approaching an asymptotic behavior, whereas the fluid-phase gradient $\theta'(1)$ shows relatively smaller variation. Furthermore, a finite difference between $\theta'(1)$ and $\Phi'(1)$ persists even at larger Ω values, indicating that complete thermal equilibrium is not achieved and LTNE effects remain significant within the porous medium.

Table 1 presents the grid-independence study for the skin-friction coefficient, $-f''(1)$, obtained using different mesh resolutions. As the mesh is refined from 50 to 801 grid points, the computed values gradually converge toward a nearly constant solution. Simultaneously, the relative error decreases significantly, indicating improved numerical accuracy with mesh refinement. The variation between solutions obtained with 401 and 801 mesh points is extremely small, demonstrating that the numerical solution is essentially independent of the computational grid. Since further refinement beyond $\Delta y = 0.001$ produces only negligible changes in $-f''(1)$, the 801-point mesh is adopted for all subsequent computations to ensure both accuracy and computational efficiency.

Table 1. Grid-independence analysis of the Keller-box solutions for $-f''(1)$ using different mesh resolutions.

Δy	Mesh Points	$-f''(1)$	Error
0.02	50	1.48215	-
0.01	101	1.48702	0.328
0.005	201	1.48801	0.067
0.0025	401	1.48826	0.017
0.001	801	1.48832	0.004

Table 2 compares the skin-friction coefficient, $-f''(1)$, obtained using the present Keller-box method with the corresponding values computed using the BVP4C solver for different step sizes and error tolerances. The results obtained from both numerical methods are in excellent agreement, with percentage differences of approximately 0.1% in all cases. This close agreement demonstrates the accuracy and reliability of the Keller-box implementation for solving the present LTNE-MHD porous flow problem.

Table 2. Validation of the Keller-box method against MATLAB BVP4C for the skin-friction coefficient, when $M = 3$, $\Omega = 1$, $\gamma = 100$, $Rey = 1$, and $H = 0.01$.

Δy	Err	$-f''(1)$ (BVP4C)	$-f''(1)$ (Keller-Box)	Error %
0.01	10^{-15}	2.19024	2.18813	0.0963
0.01	10^{-7}	2.19024	2.18813	0.0963
0.001	10^{-15}	2.16173	2.15973	0.0965
0.001	10^{-7}	2.16173	2.15973	0.0965

Table 3 compares the present Keller-box solutions for $f''(1)$ with the results reported by Bhat et al. [33] for various Reynolds numbers and slip parameters S . Excellent agreement is observed for all cases considered, with only negligible differences between the two sets of results. The close correspondence validates the accuracy and reliability of the present numerical implementation of the Keller-box method.

Table 3. Comparison of the present results for $f''(1)$ with those of Bhat et al. [33] for different Reynolds numbers and slip parameters S .

Rey	$S = 0$		$S = 0.1$		$S = 0.5$	
	Bhat et al. [33]	Present Study	Bhat et al. [33]	Present Study	Bhat et al. [33]	Present Study
1	−1.90254	−1.905135	−1.37926	−1.378776	−0.656025	−0.652931
3	−1.54942	−1.550111	−1.20575	−1.203415	−0.620434	−0.621123
4	−1.12690	−1.123690	−1.00827	−1.007358	−0.585166	−0.584757
−1	−2.07282	−2.071919	−1.46600	−1.464941	−0.674872	−0.678836
−20	−2.41038	−2.410281	−1.65778	−1.651688	−0.719378	−0.718537

Table 4 shows the effect of the interphase heat transfer parameter H on the fluid-phase wall temperature gradient, solid-phase wall temperature gradient, and LTNE intensity. For larger values of H , the fluid-phase and solid-phase wall temperature gradients become progressively closer, indicating stronger thermal interaction between the two phases. The reduction in LTNE intensity demonstrates that the temperature difference between the fluid and solid phases weakens significantly as the interfacial heat exchange becomes stronger. Consequently, the porous medium gradually approaches local thermal equilibrium conditions.

Table 4. Effect of H on wall temperature gradients and LTNE intensity.

H	$-\theta'(1)$	$-\Phi'(1)$	LTNE Intensity
1	0.94488	0.99949	0.0546
2	0.94469	0.98352	0.0388
3	0.9446	0.97633	0.0317
10	0.94442	0.96175	0.0173
20	0.94436	0.95658	0.0122
50	0.9443	0.95201	0.007
100	0.94427	0.94972	0.0054

Table 5 illustrates the influence of the magnetic parameter M on the skin friction coefficient, fluid-phase temperature gradient, and solid-phase temperature gradient. Increasing M reduces all three quantities throughout the flow domain. The applied magnetic field generates a Lorentz force that opposes the fluid motion and suppresses momentum transport within the porous medium. So, the velocity gradient near the wall decreases, thereby reducing wall shear stress. The suppression of fluid motion also weakens convective heat transfer within the thermal boundary layer, thereby reducing the rate of heat transfer between the fluid phase and the boundary layer. A similar decrease in the solid-phase temperature gradient is observed due to interphase thermal coupling under LTNE conditions. However, the magnetic field's influence on the solid phase remains comparatively weak because heat transfer within the porous matrix is primarily by conduction.

Table 5. $-f''(1)$, $-\theta'(1)$, and $-\Phi'(1)$ for different M with $H = 0.01$, $Pr = 0.7$, $\varepsilon = 0.971$, $R = 1$, and $\Omega = 1$.

M	$-f''(1)$	$-\theta'(1)$	$-\Phi'(1)$
1	1.5982	0.94964	1.1707
2	1.5559	0.94857	1.1706
3	1.5114	0.94744	1.1705
4	1.4645	0.94623	1.1703
5	1.4151	0.94494	1.1702

The wall-temperature gradients across the fluid and solid phases reflect the local Nusselt number behavior under LTNE conditions. In addition, LTNE intensity is evaluated through the temperature difference between the fluid and solid phases to characterize the degree of thermal nonequilibrium within the porous medium.

Table 6 presents the influence of the permeability resistance parameter Ω on the skin friction coefficient, fluid-phase temperature gradient, and solid-phase temperature gradient under LTNE conditions. Larger values of Ω correspond to lower permeability of the porous medium, thereby enhancing resistance to fluid motion within the porous structure. The enhanced porous drag weakens momentum transport and reduces the velocity gradient near the wall, thereby lowering the wall shear stress. In contrast, restricting fluid motion suppresses convective transport and promotes thermal energy accumulation near the heated surface. Consequently, heat transfer within the thermal boundary layer becomes increasingly dominated by conduction. This behavior slightly strengthens the fluid-phase temperature gradient and enhances thermal retention within the porous medium. The solid-phase temperature gradient exhibits only marginal variation because thermal transport in the porous matrix is primarily governed by conduction and inter-phase thermal interaction rather than by direct fluid convection.

Table 6. $-f''(1)$, $-\theta'(1)$, and $-\Phi'(1)$ for different Ω with $H = 0.01$, $Pr = 0.7$, $\varepsilon = 0.971$, $R = 1$, and $M = 3$.

Ω	$-f''(1)$	$-\theta'(1)$	$-\Phi'(1)$
0.5	1.4883	0.9480	1.1705
1	1.4645	0.9485	1.1706
2	1.4151	0.9496	1.1701
3	1.3628	0.9506	1.1708
5	1.2485	0.9525	1.171

Table 7 demonstrates the influence of the Reynolds number R on the skin friction coefficient, fluid-phase temperature gradient, and solid-phase temperature gradient under LTNE conditions. As the Reynolds number increases, inertial transport within the porous tube becomes increasingly dominant over viscous diffusion. This weakens momentum diffusion near the wall and reduces the hydrodynamic resistance in the boundary layer, leading to lower wall shear stress. The stronger axial transport associated with larger Reynolds numbers also reduces the thermal residence time of the fluid near the heated surface. Consequently, thermal energy is converted more rapidly in the flow direction, limiting thermal penetration into the boundary layer. This behavior weakens the fluid-phase wall-temperature gradient and reduces the associated heat transfer rate. A similar trend is observed in the solid-phase temperature gradient, as reduced interphase thermal interaction weakens heat exchange between the fluid and the porous matrix.

Table 7. $-f''(1)$, $-\theta'(1)$, and $-\Phi'(1)$ for different R with $H = 0.01$, $Pr = 0.7$, $\varepsilon = 0.971$, $\Omega = 1$, and $M = 3$.

R	$-f''(1)$	$-\theta'(1)$	$-\Phi'(1)$
0.5	2.4341	1.065	1.1796
1	1.9518	0.9485	1.1706
2	1.6834	0.7370	1.1521
5	1.5177	0.5534	1.1331
10	1.4623	0.4506	1.0922

5. Conclusions

This study presents a numerical investigation of MHD flow and heat transfer in a porous tube under LTNE conditions. The governing equations are transformed using similarity variables and solved using the Keller-box method to analyze the influence of key physical parameters on velocity and temperature distributions of both fluid and solid phases.

The major findings of the study are summarized as follows:

- The applied magnetic field (M) significantly suppresses the velocity field due to the resistive Lorentz force. This suppression weakens convective heat transfer, leading to lower temperature profiles in both the fluid and solid phases.
- An increase in the permeability resistance parameter (Ω) decreases the velocity due to enhanced resistance of the porous medium. Consequently, convective heat transfer is weakened, leading to higher temperature distributions in both phases due to thermal energy accumulation near the wall.
- The porosity-scaled conductivity parameter (γ) and the microscopic interphase heat transfer parameter (H) strongly influence thermal interaction between the fluid and solid phases. Higher values of these parameters enhance interphase heat exchange, significantly reducing the temperature difference and driving the system toward LTE.
- Increasing the magnetic parameter, permeability resistance, and Reynolds number suppresses fluid motion and weakens wall gradients, resulting in lower skin friction and reduced fluid and solid Nusselt numbers.
- Increasing the Prandtl number decreases thermal diffusivity, resulting in thinner thermal boundary layers and elevated temperature profiles in both phases.
- Numerical findings obtained using the Keller-box method show remarkable agreement with the existing literature.

Author Contributions: Conceptualization, S.K.M., N.N.K. and A.B.; methodology, S.K.M., N.N.K. and A.B.; software, S.K.M.; validation, N.N.K., A.B. and M.S.; formal analysis, S.K.M., N.N.K. and A.B.; investigation, S.K.M., N.N.K. and R.V.C.; resources, N.N.K. and R.V.C.; data curation, S.K.M. and A.B.; writing—original draft preparation, S.K.M. and M.S.; writing—review and editing, A.B. and M.S.; visualization, N.N.K. and R.V.C.; supervision, N.N.K. and A.B.; project administration, N.N.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data that support the findings of this study are available from the corresponding authors [N.N.K. & A.B.] upon reasonable request.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

ρ	Fluid density
p	Pressure
μ	Dynamic viscosity
K	Porous medium permeability
$\vec{q}_D$	Darcy velocity vector
C_p	Isobaric specific heat
T	Temperature

k_f, k_s	Thermal conductivity of the fluid and solid
h	Interfacial heat transfer coefficient
ν	Kinematic viscosity
ν_{eff}	Effective kinematic viscosity.
T_f, T_s	Temperature of the fluid and solid phases
Nu_f, Nu_s	Nusselt number for fluid and solid phase
C_f	Skin friction
Λ	Ratio of effective viscosity to fluid kinematic viscosity
$\Omega = \frac{1}{Da Rey}$	Permeability resistance parameter
Pr	Prandtl number
H	Microscopic interphase rate of heat transfer
γ	Porosity-scaled conductivity
$M = \frac{Ha^2}{Rey}$	Magnetic parameter
u, v	Velocity along the x and r direction
x, r	Polar coordinates
$f'(\eta)$	Axial velocity profile
θ, Φ	Temperature profiles for the fluid and solid phases
η	Dimensionless radial coordinate
Rey	Reynolds number
S	Velocity slip parameter
ε^2	Porosity of the porous medium
r_w	Radius of the tube
v_w	Transverse velocity at porous bounding surface
T_1 and T_2	Reference and mainstream temperature
Da	Darcy number
Ha	Hartmann number

References

1. Kuznetsov, A.V. Analysis of a non-thermal equilibrium fluid flow in a concentric tube annulus filled with a porous medium. *Int. Commun. Heat Mass Transf.* **1996**, *23*, 929–938. [https://doi.org/10.1016/0735-1933\(96\)00075-9](https://doi.org/10.1016/0735-1933(96)00075-9).
2. Minkowycz, W.J.; Haji-Sheikh, A.; Vafai, K. On departure from local thermal equilibrium in porous media due to a rapidly changing heat source: The Sparrow number. *Int. J. Heat Mass Transf.* **1999**, *42*, 3373–3385. [https://doi.org/10.1016/S0017-9310\(99\)00043-5](https://doi.org/10.1016/S0017-9310(99)00043-5).
3. Nield, D.A.; Kuznetsov, A.V. Local thermal nonequilibrium effects in forced convection in a porous medium channel: A conjugate problem. *Int. J. Heat Mass Transf.* **1999**, *42*, 3245–3252. [https://doi.org/10.1016/S0017-9310\(98\)00386-X](https://doi.org/10.1016/S0017-9310(98)00386-X).
4. Quintard, M.; Whitaker, S. One- and Two-Equation Models for Transient Diffusion Processes in Two-Phase Systems. In *Advances in Heat Transfer*; Elsevier: Amsterdam, The Netherlands, 1993; Volume 23, pp. 369–464. [https://doi.org/10.1016/S0065-2717\(08\)70009-1](https://doi.org/10.1016/S0065-2717(08)70009-1).
5. Nield, D.A. note on the modeling of local thermal nonequilibrium in a structured porous medium. *Int. J. Heat Mass Transf.* **2002**, *45*, 4367–4368. [https://doi.org/10.1016/S0017-9310\(02\)00138-2](https://doi.org/10.1016/S0017-9310(02)00138-2).
6. He, L.-W.; Jin, Z.-H. A Local Thermal Nonequilibrium Poroelastic Theory for Fluid Saturated Porous Media. *J. Therm. Stress.* **2010**, *33*, 799–813. <https://doi.org/10.1080/01495739.2010.482358>.
7. Yang, C. Thermal Nonequilibrium Forced Convective Flow in an Annulus Filled with a Porous Medium. *Open Transp. Phenom. J.* **2011**, *3*, 31–39. <https://doi.org/10.2174/1877729501103010031>.
8. Kayhani, M.H.; Abbasi, A.O.; Sadi, M. Study of Local thermal nonequilibrium in porous medium due to temperature sudden change and heat generation. *Mechanics* **2011**, *17*, 57–63. <https://doi.org/10.5755/j01.mech.17.1.204>.
9. Li, P.; Zhang, J.; Wang, K.; Xu, Z. Heat transfer characteristics of thermally developing forced convection in a porous circular tube with asymmetric entrance temperature under LTNE condition. *Appl. Therm. Eng.* **2019**, *154*, 326–331. <https://doi.org/10.1016/j.applthermaleng.2019.03.109>.
10. Prabha, G.S.S.; Bharathi, M.C.; Kudenatti, R.B. Heat transfer through mixed convection boundary layer in a porous medium: LTNE analysis. *Appl. Therm. Eng.* **2020**, *179*, 115705. <https://doi.org/10.1016/j.applthermaleng.2020.115705>.

11. Keshavarzian, B.; Sayehvand, H. The effect of the flow and solid matrix parameters on the Nusselt number for free convection over horizontal cylinder by considering the boundary layer and local thermal nonequilibrium model. *J. Therm. Anal. Calorim.* **2023**, *148*, 8087–8096. <https://doi.org/10.1007/s10973-022-11801-x>.
12. Gogate, S.S.P.; Bharathi, M.C.; Kudenatti, R.B. Linear stability on the local thermal nonequilibrium model of mixed convection boundary layer flow over a moving wedge in a porous medium: Viscous dissipation and radiation effects. *J. Heat Transf.* **2021**, *143*, 041802. <https://doi.org/10.1115/1.4049514>.
13. Gogate, S.P.; Bharathi, M.C.; Misbah, N.E.; Kudenatti, R.B. A computational study of three-dimensional laminar boundary layer flow and forced convective heat transfer in a porous medium. *Arch. Appl. Mech.* **2023**, *93*, 551–569. <https://doi.org/10.1007/s00419-022-02285-0>.
14. Tayebi, T. Analysis of the local non-equilibria on the heat transfer and entropy generation during thermal natural convection in a non-Darcy porous medium. *Int. Commun. Heat Mass Transf.* **2022**, *135*, 106133. <https://doi.org/10.1016/j.icheatmasstransfer.2022.106133>.
15. Shyabal, S.; Hanumagowda, B.N.; Ravisha, M.; Mamatha, A.L.; Shivaraju, N.; Soumya, D.O.; Alkarni, S.; Shah, N.A. The Onset of Darcy–Brinkman Convection in a Porous Layer with Mutual Impact of Thermal Nonequilibrium and Non-Uniform Temperature Gradients. *Symmetry* **2023**, *15*, 1695. <https://doi.org/10.3390/sym15091695>.
16. Sivanandam, S.; Alqurashi, T.J. Activation energy and cross-diffusion effects on 3D rotating nanofluid flow in a Darcy–Forchheimer porous medium with radiation and convective heating. *Open Phys.* **2025**, *23*, 20250186. <https://doi.org/10.1515/phys-2025-0186>.
17. Kiran, S.; Sankar, M.; Sivasankaran, S. Numerical study on conjugate convective thermal transport in an annular porous geometry. *Kuwait J. Sci.* **2022**, *49*, 10789. <https://doi.org/10.48129/kjs.10789>.
18. Sivasankaran, S.; Chandrapushpam, T.; Bhuvaneswari, M.; Karthikeyan, S.; Alzahrani, A.K. Effect of chemical reaction on double diffusive MHD squeezing copper water nanofluid flow between parallel plates. *J. Mol. Liq.* **2022**, *368*, 120768. <https://doi.org/10.1016/j.molliq.2022.120768>.
19. Siddabasappa, C.; Kalpana, G.; Babitha, A. numerical study on heat transfer and MHD flow of a Newtonian through a porous channel—Local thermal nonequilibrium model. *Heat Transf.* **2023**, *52*, 949–967. <https://doi.org/10.1002/htj.22724>.
20. Sivanandam, S. MHD double diffusive convective squeezing ternary nano fluid flow between parallel plates with activation energy and viscous dissipation. *Int. J. Numer. Methods Heat Fluid Flow* **2024**, *34*, 4409–4431. <https://doi.org/10.1108/HFF-05-2024-0365>.
21. Kefene, M.; Makinde, D.; Enyadene, L. Magnetic Effects of Ferrofluid Flow in a Heated Microchannel with Permeable Walls. *Lat. Am. Appl. Res. Int. J.* **2022**, *52*, 247–257. <https://doi.org/10.52292/j.laar.2022.666>.
22. Gupta, N.; Bhargavi, D.; Makinde, O.D. Heat transfer in a MHD couple-stress fluid in a channel filled with porous material: A computational analysis. *Int. Commun. Heat Mass Transf.* **2024**, *155*, 107586. <https://doi.org/10.1016/j.icheatmasstransfer.2024.107586>.
23. Kumar, A.; Saini, A.; Negi, A.S. A LTNE approach on the study of transverse hydromagnetic on mixed convection in a channel. *J. Phys. Conf. Ser.* **2024**, *2844*, 012015. <https://doi.org/10.1088/1742-6596/2844/1/012015>.
24. Goli, A.; Zahmatkesh, I.; Saleh, S.R.; Alavi, S.M.A. MHD Mixed Convection of Developing Slip Flow in a Vertical Porous Microchannel Under Local Thermal Non–Equilibrium Conditions. *Nanoscale Microscale Thermophys. Eng.* **2024**, *28*, 28–45. <https://doi.org/10.1080/15567265.2023.2257748>.
25. Khan, Z.H.; Makinde, O.D.; Usman, M.; Ahmad, R.; Khan, W.A. Fractional order analysis of radiating couple stress MHD nanofluid flow in a permeable wall channel. *J. Taibah Univ. Sci.* **2025**, *19*, 2485396. <https://doi.org/10.1080/16583655.2025.2485396>.
26. Kumar, D.L.K.; Nagappanavar, S.N.; Raghunatha, K.R.; Sangamesh. Numerical analysis of steady MHD flow in a saturated porous medium under local thermal nonequilibrium conditions using the Bernoulli wavelet method. *Arab J. Basic Appl. Sci.* **2025**, *32*, 455–472. <https://doi.org/10.1080/25765299.2025.2572183>.
27. Çiçek, O.; Erdem, D.; Baytaş, A.C. Numerical Analysis of Dusty Hybrid Nanofluid Flow in a Porous Medium Under Ltne Conditions In The Impacts Of Thermophoresis And Gravity-Buoyancy Forces. *J. Porous Media* **2026**, *29*, 91–121. <https://doi.org/10.1615/JPorMedia.2025056114>.
28. Ahmed, S.E.; Raizha, Z.; Hafed, Z.; Alhazmi, M. Optimization of heat transfer in star-shaped thermal energy storage systems with NEPCM, and LTNE porous media. *Open Phys.* **2026**, *24*, 20250276. <https://doi.org/10.1515/phys-2025-0276>.
29. Nield, D.A.; Bejan, A. *Convection in Porous Media*; Springer International Publishing: Cham, Switzerland, 2017. <https://doi.org/10.1007/978-3-319-49562-0>.

30. Straughan, B. Porous convection with local thermal nonequilibrium temperatures and with Cattaneo effects in the solid. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2013**, *469*, 20130187. <https://doi.org/10.1098/rspa.2013.0187>.
31. Vafai, K.; Amiri, A. Non-Darcian Effects in Confined Forced Convective Flows. In *Transport Phenomena in Porous Media*; Elsevier: Amsterdam, The Netherlands, 1998; pp. 313–329. <https://doi.org/10.1016/B978-008042843-7/50013-1>.
32. Rees, D.; Pop, I. LOCAL Thermal Nonequilibrium in Porous Medium Convection. In *Transport Phenomena in Porous Media III*; Elsevier: Amsterdam, The Netherlands, 2005; pp. 147–173. <https://doi.org/10.1016/B978-008044490-1/50010-7>.
33. Bhat, A.; Katagi, N.N.; Bujurke, N.M. Analysis of laminar flow in a porous pipe with slip velocity. *J. Mech. Eng. Res. Dev.* **2017**, *40*, 526–536. Available online: <https://www.jmerd.org/current/vol40no4/335.html> (accessed on 6 May 2026).
34. Tangsali, P.R.; Katagi, N.N.; Kudenatti, R.B.; Bhat, A.; Shettar, M. Analysis of local thermal nonequilibrium effects in three-dimensional flow over a stretching sheet embedded in a porous medium. *Arab J. Basic Appl. Sci.* **2026**, *33*, 48–59. <https://doi.org/10.1080/25765299.2026.2617729>.
35. Cebeci, T.; Bradshaw, P. Finite-Difference Solution of Boundary-Layer Equations. In *Physical and Computational Aspects of Convective Heat Transfer*; Springer: Berlin/Heidelberg, Germany, 1984; pp. 385–428. https://doi.org/10.1007/978-3-662-02411-9_13.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.