

## Article

# Flexible Bivariate Generalized Shifted Inverse Trinomial Distributions for Over- and Under-Dispersed Count Data

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## Abstract

Modeling bivariate count data with complex dispersion and dependence structures remains a significant challenge in statistical data analysis. This article introduces two new bivariate count distributions derived from the generalized shifted inverse trinomial distribution. The proposed models, denoted by BGIT-I and BGIT-II, are constructed using convolution and trivariate reduction methods. They provide flexible joint frameworks for modeling correlated count data while accommodating different marginal dispersion patterns. BGIT-I allows negative, near-zero, and positive dependence, whereas BGIT-II induces non-negative dependence through a common component. The proposed models have simple, tractable probability generating functions, which facilitate the derivation of probabilistic properties and motivate a probability-generating-function-based estimation approach alongside maximum-likelihood estimation. The finite-sample performance of the estimators is further examined through a Monte Carlo simulation study. The practical utility of the proposed models is illustrated using two real bivariate count data sets involving shunter accidents and patient counts in critical care and emergency room settings. The results show that the proposed BGIT models provide competitive alternatives for modeling bivariate count data with different dispersion and dependence characteristics.

**Keywords:** convolution; trivariate reduction; probability-generating function-based estimators; empirical modeling; safety-critical occupations; healthcare

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## 1. Introduction

Flexible bivariate discrete distributions are important in statistical modeling because they allow the joint behavior of two count variables to be described. Such distributions are useful in many fields, including epidemiology, finance, and the social sciences, where two discrete outcomes may be associated. For example, in epidemiology, modeling disease counts across different regions can help describe disease spread and control, while in finance, joint modeling of default and bankruptcy counts can support risk assessment.

An important feature of count data is dispersion. Depending on the relationship between the mean and variance, count data may be under-, equi-, or over-dispersed. Hence, a flexible model capable of handling different dispersion patterns is highly useful. In the univariate case, the Conway–Maxwell–Poisson distribution is a popular model for handling different dispersion levels [1,2,3,4,5]. Recent developments in this area can also be found in [6,7]. In the bivariate case, however, relatively few distributions are available that can simultaneously accommodate different marginal dispersion patterns and model dependence between the two count variables.

Numerous bivariate discrete distributions have been defined and studied using different construction methods; see, for example, refs. [8–10] introduced a bivariate generalized Poisson distribution using trivariate reduction, but the model permits only positive correlation between  $X_1$  and  $X_2$ . The bivariate negative binomial distribution defined using a copula function in [11] is suitable mainly for over-dispersed data. More recently, ref. [12] constructed bivariate Conway–Maxwell–Poisson distributions based on the Sarmanov family, allowing a range of correlation structures. A bivariate distribution with binomial convolution marginals was also considered in [13], generalizing the bivariate binomial distribution and its variants. Although these models provide useful frameworks for bivariate count data, some are restricted to positive dependence, some are mainly designed for over-dispersed counts, and relatively few provide a unified construction that combines flexible marginal dispersion with different admissible dependence structures. This motivates the development of alternative bivariate discrete models such as the proposed BGIT models.

This paper builds on the work of [14], which studied a special case of the generalized shifted inverse trinomial distribution, denoted by  $\text{GIT}_{3,1}$ . The  $\text{GIT}_{3,1}$  distribution can accommodate over-dispersion when  $p_3 > p_2$ , under-dispersion ( $p_3 < p_2$ ) and equi-dispersion ( $p_3 = p_2$ ). Its probability mass function (pmf) [15] is given by

$$f_n(x) = \sum_{i=0}^{\min(n,x)} \frac{n}{n+x-i} \binom{n+x-i}{n-i, i, x-i} p_1^{n-i} p_2^i p_3^{x-i} \quad (1)$$

for  $x = 0, 1, 2, \dots$ ,  $n \in \mathbb{Z}^+$  and  $p_i \geq 0$  for  $i = 1, 2, 3$ ;  $\sum_{i=1}^3 p_i = 1$ . Equation (1) may also be expressed as

$$f_n(x) = \binom{n+x-1}{x} p_1^n p_3^x {}_2F_1\left(-n, -x; -n-x+1; -\frac{p_2}{p_1 p_3}\right)$$

in terms of the Gauss hypergeometric function. The probability generating function (pgf) is

$$\phi(t) = \left(\frac{p_1+p_2t}{1-p_3t}\right)^n = \left(\frac{p_1+p_2t}{p_1+p_2}\right)^n \left(\frac{1-p_3}{1-p_3t}\right)^n, \quad 1-p_3 = p_1 + p_2$$

It reduces to a binomial pgf if  $p_3 = 0$ , negative binomial pgf if  $p_2 = 0$ , and shifted negative binomial (shifted  $n$  steps to the right) pgf if  $p_1 = 0$ . As highlighted by [14],  $\text{GIT}_{3,1}$  exhibits several interesting features: it can handle over-, under-, and equi-dispersed data; it remains valid across parameter ranges; and it possesses a straightforward pgf. Interestingly, when the index of dispersion is 1, it reduces to a non-Poisson distribution.

In this study, we construct two bivariate extensions of the  $\text{GIT}_{3,1}$  distribution, denoted by BGIT-I and BGIT-II. The first construction is based on convolution and yields

BGIT-I, which accommodates negative, near-zero, and positive dependence. The second construction is based on trivariate reduction and yields BGIT-II, which induces non-negative dependence via a common component. The main contribution of this paper is therefore the development of two complementary bivariate  $\text{GIT}_{3,1}$ -based models that retain the dispersion flexibility and simple pgf structure of the univariate  $\text{GIT}_{3,1}$  distribution, while extending it to the bivariate setting. The proposed models can accommodate under-, equi-, and over-dispersed marginal behavior and produce a range of joint shapes through their positive integer parameters.

The remainder of the paper is organized as follows. Section 2 introduces the proposed BGIT-I and BGIT-II distributions. Section 3 presents their main probabilistic properties, including marginal distributions, moments, covariance, and correlation. Section 4 illustrates the shape characteristics of the proposed distributions. Section 5 discusses parameter estimation using maximum likelihood and pgf-based methods. Section 6 presents a Monte Carlo simulation study to evaluate the finite-sample performance of the estimators. Section 7 applies the proposed models to two real bivariate count data sets and compares their performance with existing bivariate count models, including the bivariate Conway–Maxwell–Poisson models of [12]. Section 8 concludes the paper.

## 2. Formulation of Bivariate Distributions

Two formulations of the bivariate distributions are considered:

- (a) BGIT-I, constructed through the convolution of bivariate binomial and bivariate negative binomial distributions;
- (b) BGIT-II, constructed using the method of trivariate reduction [9].

### 2.1. Type I Bivariate $\text{GIT}_{3,1}$ (BGIT-I) Using the Convolution Method

The probability generating function (pgf) of  $\text{GIT}_{3,1}$  distribution can be written as

$$\phi(t) = \left( \frac{p_1 + p_2 t}{1 - p_3 t} \right)^n = \left( \frac{p_1}{p_1 + p_2} + \frac{p_2 t}{p_1 + p_2} \right)^n \left( \frac{1 - p_3}{1 - p_3 t} \right)^n. \quad (2)$$

If  $p_3 = 0$ , it reduces to the pgf of the binomial distribution, and when  $p_2 = 0$ , it is the negative binomial pgf. Thus, the  $\text{GIT}_{3,1}$  distribution is a convolution of the binomial  $B\left(n, \frac{p_1}{p_1 + p_2}\right)$  and negative binomial  $\text{NB}(n, p_3)$  distributions. In a similar manner, we define a BGIT-I as the mixture of bivariate binomial and bivariate negative binomial distributions, where its joint pgf is given as

$$\phi(t_1, t_2) = (p_3 + p_1 t_1 + p_2 t_2)^{n_1} \left( \frac{q_3}{1 - q_1 t_1 - q_2 t_2} \right)^{n_2} \quad (3)$$

where  $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$  and  $n_1, n_2$  are both positive integers.

Equation (3) can be interpreted as the product of two probability generating functions. The first factor corresponds to a bivariate binomial component, while the second factor corresponds to a bivariate negative binomial component. Hence, the BGIT-I random vector can be viewed as the sum of two independent bivariate components. Let  $(Y_1, Y_2)$  denote the bivariate binomial component and let  $(U, V)$  denote the bivariate negative binomial component. Then  $X_1 = Y_1 + U$ , and  $X_2 = Y_2 + V$ .

For fixed observed values  $X_1$  and  $X_2$ , suppose that the bivariate negative binomial component contributes  $U$  to  $X_1$  and  $V$  to  $X_2$ . Then, the bivariate binomial component must contribute  $X_1 - U$  to  $X_1$  and  $X_2 - V$  to  $X_2$ . Since different values of  $u$  and  $v$  can lead to the same observed pair  $(X_1, X_2)$ , the joint pmf is obtained by summing over all admissible values of  $U$  and  $V$ . Thus,

$$f(x_1, x_2) = q_3^{n_2} p_1^{x_1} p_2^{x_2} p_3^{n_1 - x_1 - x_2} \times \sum_{u=0}^{x_1} \sum_{v=0}^{x_2} \frac{n_1!}{(x_1 - u)!(x_2 - v)!(n_1 - x_1 - x_2 + u + v)!} \frac{(u + v + n_2 - 1)!}{u!v!(n_2 - 1)!} \left( \frac{q_1}{p_1} \right)^u \left( \frac{q_2}{p_2} \right)^v p_3^{u+v}. \quad (4)$$

The summation is taken over  $U(0, 1, \dots, x_1)$  and  $V(0, 1, \dots, x_2)$  subject to the condition  $n_1 - x_1 - x_2 + u + v \geq 0$ . Terms that do not satisfy this condition are treated as zero.

## 2.2. Type II Bivariate GIT<sub>3,1</sub> (BGIT-II) by Trivariate Reduction Method

This subsection introduces the second type of BGIT, which extends the flexibility and utility of the univariate GIT<sub>3,1</sub> to a bivariate form through the method of trivariate reduction. We have  $X_1 = (Y_1 + Y_2)$  and  $X_2 = (Y_1 + Y_3)$  where  $Y_1, Y_2$ , and  $Y_3$  are independent GIT<sub>3,1</sub> random variables. Now, let  $\phi_1, \phi_2$ , and  $\phi_3$  be the pgf of  $Y_1, Y_2$ , and  $Y_3$ , respectively, that is

$$\phi_1 = \left(\frac{p_1+p_2t}{1-p_3t}\right)^{n_1}; \phi_2 = \left(\frac{q_1+q_2t}{1-q_3t}\right)^{n_2}; \phi_3 = \left(\frac{r_1+r_2t}{1-r_3t}\right)^{n_3}.$$

We define the joint pgf of  $(X_1, X_2)$  as follows:

$$\phi(t_1, t_2) = \left(\frac{p_1+p_2(t_1t_2)}{1-p_3(t_1t_2)}\right)^{n_1} \left(\frac{q_1+q_2t_1}{1-q_3t_1}\right)^{n_2} \left(\frac{r_1+r_2t_2}{1-r_3t_2}\right)^{n_3}. \quad (5)$$

Here,  $n_1, n_2$  and  $n_3$  denote positive integers and  $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = r_1 + r_2 + r_3 = 1$ .

From Equation (5), the joint pmf of BGIT-II is derived as

$$f(x_1, x_2) = p_1^{n_1} q_1^{n_2} r_1^{n_3} \sum_{k=0}^{\min(x_1, x_2)} (A_1(k) \times A_2(x_1 - k) \times A_3(x_2 - k)) \quad (6)$$

where  $n_1, n_2, n_3$  are positive integers,  $p_1 + p_2 + p_3 = 1$ ,  $q_1 + q_2 + q_3 = 1$ ,  $r_1 + r_2 + r_3 = 1$ ,

$$A_1(k) = \binom{n_1 + k - 1}{k} \times {}_2F_1\left(-n_1, -k; -n_1 - k + 1; -\frac{p_2}{p_1 p_3}\right) p_3^k,$$

$$A_2(x_1 - k) = \binom{n_2 + x_1 - k - 1}{x_1 - k} \times {}_2F_1\left(-n_2, -(x_1 - k); -n_2 - (x_1 - k) + 1; -\frac{q_2}{q_1 q_3}\right) q_3^{x_1 - k},$$

$$A_3(x_2 - k) = \binom{n_3 + x_2 - k - 1}{x_2 - k} \times {}_2F_1\left(-n_3, -(x_2 - k); -n_3 - (x_2 - k) + 1; -\frac{r_2}{r_1 r_3}\right) r_3^{x_2 - k}.$$

Here,  $\binom{a}{b}$  denotes the binomial coefficient, and  ${}_2F_1(a, b; c; z)$  denotes the Gaussian hypergeometric function.

## 3. Properties of the Bivariate Distributions

This section examines the properties of the BGIT. We will analyze both BGIT-I and BGIT-II, focusing on key properties such as the marginal pgfs of  $X_1$  and  $X_2$ , their marginal means and variances, as well as the covariances and correlations among the variables.

### 3.1. Properties of BGIT-I

In this subsection, we discuss the main properties of the BGIT-I distribution, including its marginal distributions, marginal moments, index of dispersion, covariance, and correlation.

#### 3.1.1. Marginal Distributions of BGIT-I

The marginal pgf of  $X_i$  for  $i = 1, 2$  is given by

$$\phi_{X_i}(t) = (1 - p_i + p_i t)^{n_1} \left(\frac{q_3}{q_3 + q_i(1 - t)}\right)^{n_2}.$$

This shows that each marginal distribution is obtained by combining a binomial component with a negative binomial component.

#### 3.1.2. Marginal Means and Variances of BGIT-I

For  $i = 1, 2$ , the marginal mean is

$$E(X_i) = \frac{n_2 q_i}{q_3} + n_1 p_i,$$

and the marginal mean is

$$\text{Var}(X_i) = n_1 p_i (1 - p_i) + \frac{n_2 q_i (q_i + q_3)}{q_3^2}.$$

To investigate the marginal dispersion behavior of BGIT-I, we evaluated the index of dispersion under the parameter configurations listed in Table 1. The index of dispersion is defined as

$$ID(X_i) = \frac{\text{Var}(X_i)}{E(X_i)}.$$

The results in Table 2 show that the BGIT-I distribution can accommodate under-dispersed, approximately equi-dispersed, and over-dispersed marginal distributions. In general, as the ratio between  $(n_1/n_2)$  increases, the marginal index of dispersion decreases for both  $X_1$  and  $X_2$ . Table 2 reports the results for  $X_1$ ; the same qualitative pattern was also observed for  $X_2$ .

**Table 1.** Parameter configurations.

|        | $p_1$ | $p_2$ | $p_3$ | $q_1$ | $q_2$ | $q_3$ |
|--------|-------|-------|-------|-------|-------|-------|
| Case 1 | 0.1   | 0.1   | 0.8   | 0.1   | 0.1   | 0.8   |
| Case 2 | 0.8   | 0.1   | 0.1   | 0.8   | 0.1   | 0.1   |
| Case 3 | 0.1   | 0.8   | 0.1   | 0.1   | 0.8   | 0.1   |

**Table 2.** Marginal index of dispersion for  $X_1$ .

|           |      |      |      |
|-----------|------|------|------|
| $n_1$     | 10   | 10   | 10   |
| $n_2$     | 50   | 6    | 1    |
| $n_1/n_2$ | 0.2  | 1.7  | 10   |
| Case 1    | 1.09 | 1.00 | 0.93 |
| Case 2    | 8.83 | 7.74 | 4.60 |
| Case 3    | 1.98 | 1.84 | 1.45 |

### 3.1.3. Covariance and Correlation of BGIT-I

Using the component structure of BGIT-I, the covariance between  $X_1$  and  $X_2$  is given by

$$\text{Cov}(X_1, X_2) = \frac{n_2 q_1 q_2}{q_3^2} - n_1 p_1 p_2.$$

The covariance consists of two competing components. The term  $\frac{n_2 q_1 q_2}{q_3^2}$  arises from the bivariate negative binomial component and is non-negative, while the term  $-n_1 p_1 p_2$  arises from the bivariate binomial component and is negative. Hence, the overall covariance may be positive, negative, or close to zero, depending on the relative magnitudes of these two terms.

The corresponding correlation coefficient is

$$\rho(X_1, X_2) = \frac{n_2 q_1 q_2 - n_1 p_1 p_2 q_3^2}{\sqrt{((n_2 q_1 (1 - q_2) + n_1 p_1 (1 - p_1) q_3^2)(n_2 q_2 (1 - q_1) + n_1 p_2 (1 - p_2) q_3^2))}}.$$

To show that the correlation coefficient is admissible, it is sufficient to show that the squared covariance is smaller than the product of the two variances, that is

$$\{\text{Cov}(X_1, X_2)\}^2 < \text{Var}(X_1) \text{Var}(X_2).$$

Equivalently, define  $D = \text{Var}(X_1)\text{Var}(X_2) - \{\text{Cov}(X_1, X_2)\}^2$ . If  $D > 0$ , then  $\frac{\{\text{Cov}(X_1, X_2)\}^2}{\text{Var}(X_1)\text{Var}(X_2)} < 1$ , which implies that  $\rho^2(X_1, X_2) < 1$ . Hence,  $-1 < \rho(X_1, X_2) < 1$ . It remains to show that  $D > 0$  for the BGIT-I distribution. Let  $A = \frac{n_2}{q_3^2}$ ,  $B = n_1$ , we have

$$\text{Var}(X_1) = Aq_1(1 - q_2) + Bp_1(1 - p_1),$$

$$\text{Var}(X_2) = Aq_2(1 - q_1) + Bp_2(1 - p_2),$$

$$\text{and } \text{Cov}(X_1, X_2) = Aq_1q_2 - Bp_1p_2.$$

After simplification,

$$D = A^2q_1q_2q_3 + B^2p_1p_2p_3 + AB\{q_1(1 - q_2)p_2(1 - p_2) + q_2(1 - q_1)p_1(1 - p_1) + 2q_1q_2p_1p_2\}.$$

Under the usual interior parameter conditions,  $A > 0, B > 0, p_i > 0, q_i > 0$ , for  $i = 1, 2, 3$ . Therefore, every term on the right-hand side is positive or non-negative, and at least one term is strictly positive. Hence,  $D > 0$ . This implies that  $\rho^2(X_1, X_2) < 1$  and therefore  $-1 < \rho(X_1, X_2) < 1$ . Thus, the correlation coefficient of the BGIT-I distribution is well defined and lies within the admissible interval.

### 3.2. Properties of BGIT-II

In this subsection, we discuss the main properties of BGIT-II, including its marginal distributions, marginal means and variances, dispersion behavior, covariance, and correlation.

#### 3.2.1. Marginal Distributions of BGIT-II

The marginal pgfs of  $X_1$  and  $X_2$  are respectively given by

$$\phi_{X_1}(t) = \left(\frac{p_1 + p_2 t}{1 - p_3 t}\right)^{n_1} \left(\frac{q_1 + q_2 t}{1 - q_3 t}\right)^{n_2} \text{ and } \phi_{X_2}(t) = \left(\frac{p_1 + p_2 t}{1 - p_3 t}\right)^{n_1} \left(\frac{r_1 + r_2 t}{1 - r_3 t}\right)^{n_3}.$$

The first factor is common to both marginal pgfs and represents the shared component of  $X_1$  and  $X_2$ , while the remaining factors are margin specific.

#### 3.2.2. Marginal Means and Variances of BGIT-II

The marginal means of  $X_1$  and  $X_2$  are given by

$$E(X_1) = n_1 \left(\frac{1 - p_1}{1 - p_3}\right) + n_2 \left(\frac{1 - q_1}{1 - q_3}\right) \text{ and } E(X_2) = n_1 \left(\frac{1 - p_1}{1 - p_3}\right) + n_3 \left(\frac{1 - r_1}{1 - r_3}\right).$$

The corresponding marginal variances are

$$\text{Var}(X_1) = n_1 \frac{(1 - p_1)(1 - p_2)}{(1 - p_3)^2} + n_2 \frac{(1 - q_1)(1 - q_2)}{(1 - q_3)^2} \text{ and } \text{Var}(X_2) = n_1 \frac{(1 - p_1)(1 - p_2)}{(1 - p_3)^2} + n_3 \frac{(1 - r_1)(1 - r_2)}{(1 - r_3)^2}.$$

These expressions show that each marginal distribution is formed by the convolution of a common component and a margin-specific component.

To illustrate the marginal dispersion behavior, consider the symmetric special case where  $n_1 = n_2 = n_3$ ,  $p_1 = q_1 = r_1$ ,  $p_2 = q_2 = r_2$ , and  $p_3 = q_3 = r_3$ . Under this setting, the two marginal pgfs are identical and reduce to

$$\phi_{X_1}(t) = \phi_{X_2}(t) = \left(\frac{p_1 + p_2 t}{1 - p_3 t}\right)^{2n}.$$

This pgf can be factorized as

$$\left(\frac{p_1 + p_2 t}{1 - p_3 t}\right)^{2n} = \left[\frac{p_1}{p_1 + p_2} + \left(\frac{p_2}{p_1 + p_2}\right)t\right]^{2n} \times \left(\frac{1 - p_3}{1 - p_3 t}\right)^{2n}.$$

Hence, each marginal distribution can be viewed as the convolution of a binomial distribution with parameters  $2n$  and  $\frac{p_2}{p_1 + p_2}$ , and a negative binomial distribution with

size parameter  $\frac{p_1}{p_1+p_2}$ . This special case shows that the marginals may be approximately equi-dispersed, over-dispersed, or under-dispersed depending on the relative magnitudes of  $p_2$  and  $p_3$ . In particular, the marginals are approximately equi-dispersed when  $p_2 = p_3$ , over-dispersed when  $p_3 > p_2$ , and under-dispersed when  $p_3 < p_2$ .

### 3.2.3. Covariance and Correlation of BGIT-II

The covariance of  $X_1$  and  $X_2$  are given by

$$\text{Cov}(X_1, X_2) = \frac{n_1(1-p_1)(1-p_2)}{(1-p_3)^2}.$$

Under the usual interior parameter conditions ( $n_1 > 0, (1-p_1) > 0, (1-p_2) > 0$ , and  $(1-p_3) > 0$ ), this covariance is strictly positive. Hence, BGIT-II is suitable for modeling non-negative dependence.

The corresponding correlation coefficient is

$$\rho(X_1, X_2) = \frac{n_1(1-p_1)(1-p_2)(q_1+q_2)(r_1+r_2)}{\sqrt{(n_1(1-p_1)(1-p_2)(q_1+q_2)^2 + n_2(1-q_1)(1-q_2)(p_1+p_2)^2) \times (n_1(1-p_1)(1-p_2)(r_1+r_2)^2 + n_3(1-r_1)(1-r_2)(p_1+p_2)^2)}}.$$

In the explicit expression for  $\rho(X_1, X_2)$ , all factors in the numerator are positive under the corresponding non-degenerate parameter restrictions. Hence, BGIT-II has a strictly positive correlation in the interior of the parameter space. On the boundary of the parameter space, the dependence may reduce to zero, so the model induces non-negative dependence more generally.

## 4. Characteristics of Distributions

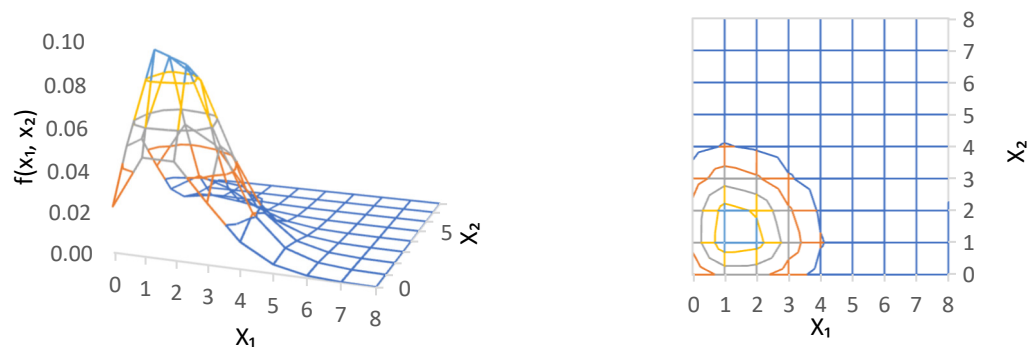
To examine the shapes of the proposed distributions, selected joint pmf plots are presented for BGIT-I and BGIT-II under different parameter configurations. Let  $f(x_1, x_2)$  denote the joint pmf. For graphical illustration, the pmf is evaluated over a finite grid, where  $x_1 = 0, 1, 2, \dots, r$  and  $x_2 = 0, 1, 2, \dots, s$ .

### 4.1. Characteristics of BGIT-I

To investigate the effect of the parameters  $n_1$  and  $n_2$  on the shape of the BGIT-I distribution, we fix  $p_1 = p_2 = q_1 = q_2 = 0.1$ . Hence,  $p_3 = q_3 = 0.8$ . In Figure 1,  $n_1$  is set to  $r + s$ , while  $n_2 = 1$ . In Figure 2, the value of  $n_1$  is increased, while  $n_2$  is kept fixed. In Figure 3,  $n_2$  is increased while  $n_1$  is kept at the same value as in Figure 2.

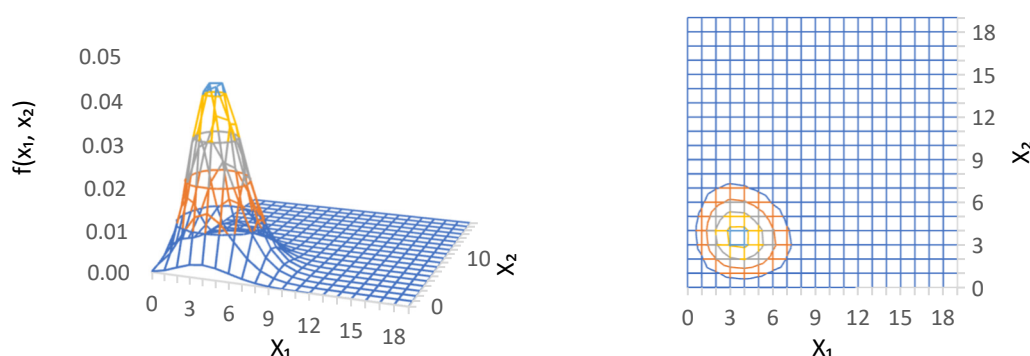
Figures 1 and 2 show that increasing  $n_1$  shifts the local mode away from the origin  $(0, 0)$  and reduces the height of the peak. This indicates that a larger  $n_1$  produces a distribution with greater spread and a mode located farther from the origin. Comparing Figures 2 and 3, increasing  $n_2$  further flattens the local mode and spreads the probability mass over a wider range of values.

Overall, the graphical results show that smaller values of  $n_1$  and  $n_2$  tend to produce distributions with probability mass concentrated near the origin and a relatively sharp local mode. Increasing  $n_1$  shifts the mode away from the origin and lowers the peak, while increasing both  $n_1$  and  $n_2$  produces a flatter and more dispersed distribution. When both parameters are large, the distribution tends to be more bell-shaped, with the main probability mass farther from the origin.



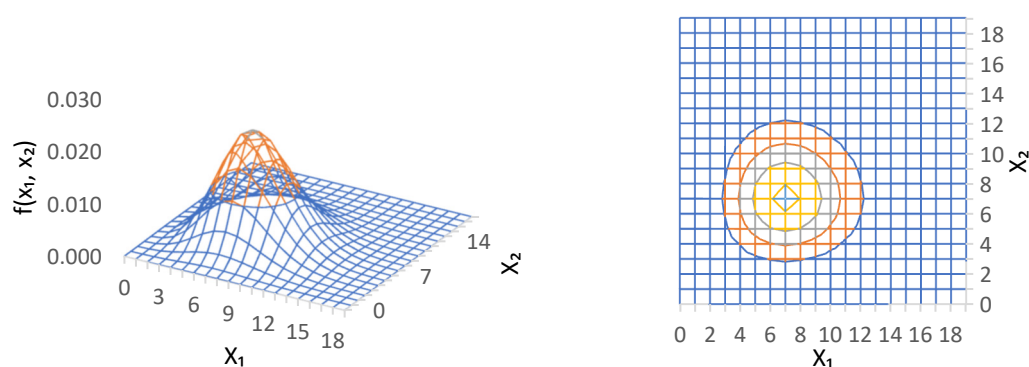
A local mode at (1, 1).

**Figure 1.** Joint pmf surface and corresponding contour plot of the BGIT-I distribution for  $p_1=0.1$ ,  $p_2=0.1$ ,  $q_1=0.1$ ,  $q_2=0.1$ ,  $n_1=16$ ,  $n_2=1$ . The colored curves represent selected equal-pmf contour levels.



A local mode at (4, 4).

**Figure 2.** Joint pmf surface and corresponding contour plot of the BGIT-I distribution for  $p_1=0.1$ ,  $p_2=0.1$ ,  $q_1=0.1$ ,  $q_2=0.1$ ,  $n_1=38$ ,  $n_2=1$ . The colored curves represent selected equal-pmf contour levels.



Local mode is at (7, 7).

**Figure 3.** Joint pmf surface and corresponding contour plot of the BGIT-I distribution for  $p_1=0.1$ ,  $p_2=0.1$ ,  $q_1=0.1$ ,  $q_2=0.1$ ,  $n_1=38$ ,  $n_2=30$ . The colored curves represent selected equal-pmf contour levels.

#### 4.2. Characteristics of BGIT-II

Figures 4–10 illustrate the behavior of BGIT-II under different combinations of the positive integer parameters  $n_1$ ,  $n_2$ , and  $n_3$ , as summarized in Table 3. These parameters control the contributions of the common component and the two margin-specific compo-

nents. By varying  $n_1$ ,  $n_2$ , and  $n_3$ , changes in the location, height, and spread of the joint pmf can be observed.

**Table 3.** Parameter configurations for BGIT-II analysis.

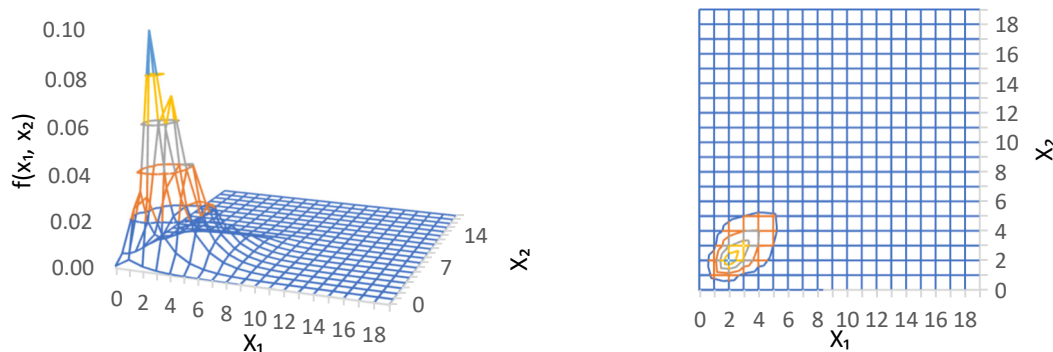
| Figure | Figure 4 | Figure 5 | Figure 6 | Figure 7 | Figure 8 | Figure 9 | Figure 10 |
|--------|----------|----------|----------|----------|----------|----------|-----------|
| $n_1$  | 1        | 5        | 1        | 1        | 1        | 5        | 5         |
| $n_2$  | 1        | 1        | 5        | 1        | 5        | 5        | 1         |
| $n_3$  | 1        | 1        | 1        | 5        | 5        | 1        | 5         |

In Figure 4, where  $n_1 = n_2 = n_3$ , the joint pmf has a sharp local mode located near the origin, at (2, 2). When  $n_1$  is increased while  $n_2$  and  $n_3$  are kept fixed, as shown in Figure 5, the local mode shifts to (9, 9). This indicates that increasing  $n_1$ , which corresponds to the common component, shifts the probability mass jointly in both the  $X_1$  and  $X_2$  directions. The peak height also decreases, and the contours become more elongated.

Figure 6 illustrates the effect of increasing  $n_2$  while keeping  $n_1$  and  $n_3$  fixed at 1. Since  $n_2$  controls the component specific to  $X_1$ , the probability mass shifts mainly toward larger  $X_1$  values, with the local mode located at (9, 2). In contrast, Figure 7 shows the effect of increasing  $n_3$  while keeping  $n_1$  and  $n_2$  fixed at 1. Since  $n_3$  controls the component specific to  $X_2$ , the probability mass shifts mainly toward larger  $X_2$  values, with the local mode located at (2, 9). Thus, Figures 6 and 7 show analogous margin-specific effects, with the roles of  $X_1$  and  $X_2$  interchanged.

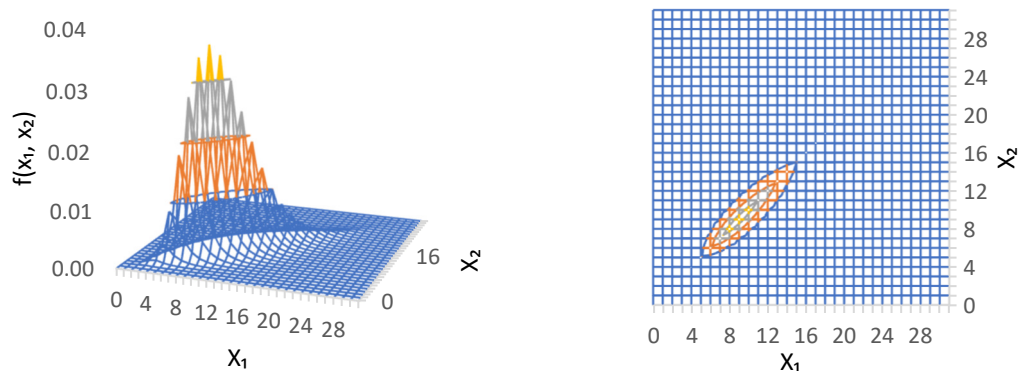
Figures 8–10 show the effect of increasing two of the three integer parameters while keeping the remaining parameter fixed. In Figure 8, both margin-specific parameters  $n_2$  and  $n_3$  are increased, giving a local mode at (9, 9). Compared with Figure 5, the peak is lower, and the contours are broader, indicating greater marginal spread in both directions. In Figure 9, both  $n_1$  and  $n_2$  are increased, resulting in a local mode at (15, 9). This reflects the combined effect of the common component and the  $X_1$ -specific component. Similarly, in Figure 10, both  $n_1$  and  $n_3$  are increased, resulting in a local mode at (9, 15), reflecting the combined effect of the common component and the  $X_2$ -specific component.

Overall, the graphical results demonstrate the flexibility of BGIT-II in producing a range of joint probability shapes. Smaller values of  $n_1$ ,  $n_2$ , and  $n_3$  tend to produce a sharper local mode near the origin. Increasing  $n_1$  shifts the probability mass jointly for  $X_1$  and  $X_2$ , while increasing  $n_2$  or  $n_3$  mainly affects the marginal spread of  $X_1$  or  $X_2$ , respectively. Hence, the integer parameters provide useful control over the location, spread, and shape of the joint distribution.



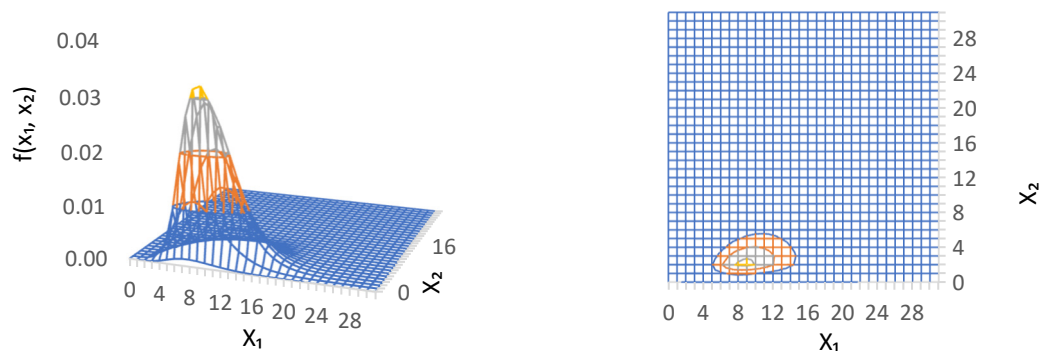
A local mode at (2, 2).

**Figure 4.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = n_2 = n_3 = 1$ . The coloured curves represent selected equal-pmf contour levels.



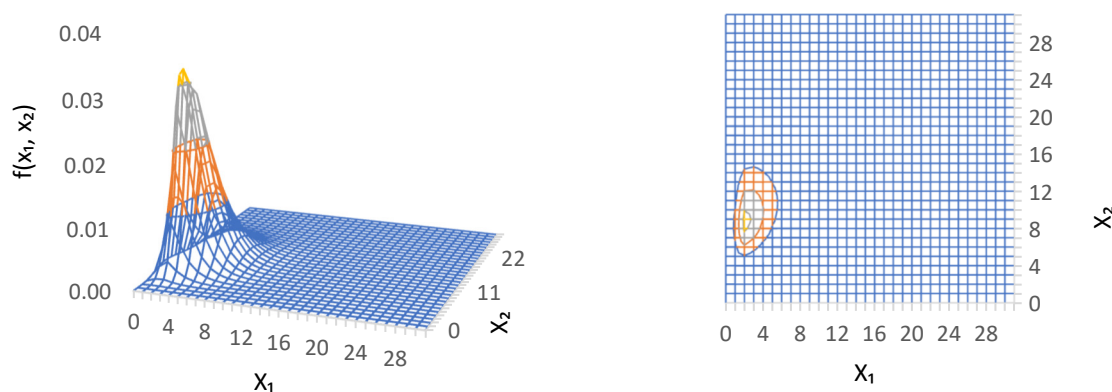
A local mode at (9, 9).

**Figure 5.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = 5, n_2 = n_3 = 1$ . The colored curves represent selected equal-pmf contour levels.



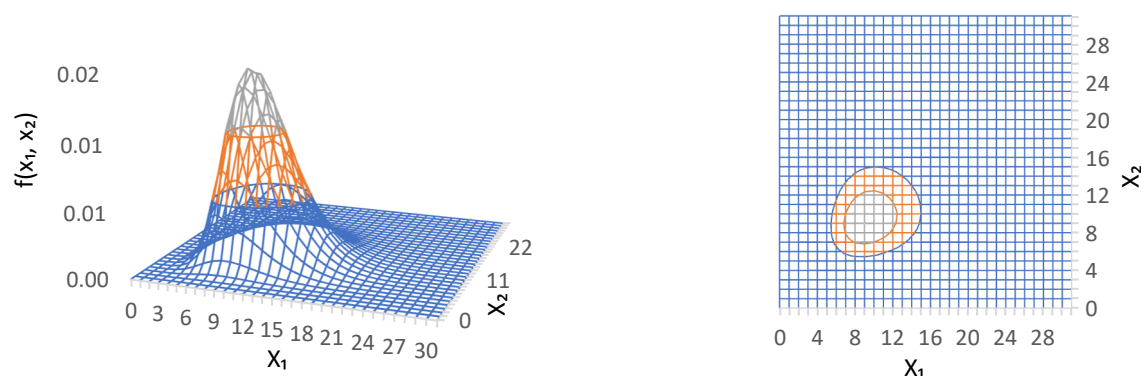
Local mode at (9, 2).

**Figure 6.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = 1, n_2 = 5, n_3 = 1$ . The colored curves represent selected equal-pmf contour levels.



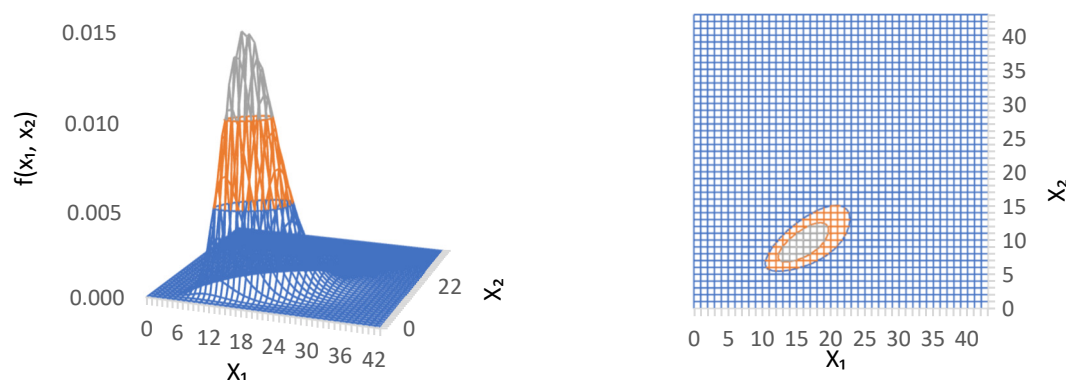
Local mode at (2, 9).

**Figure 7.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = 1, n_2 = 1, n_3 = 5$ . The colored curves represent selected equal-pmf contour levels.



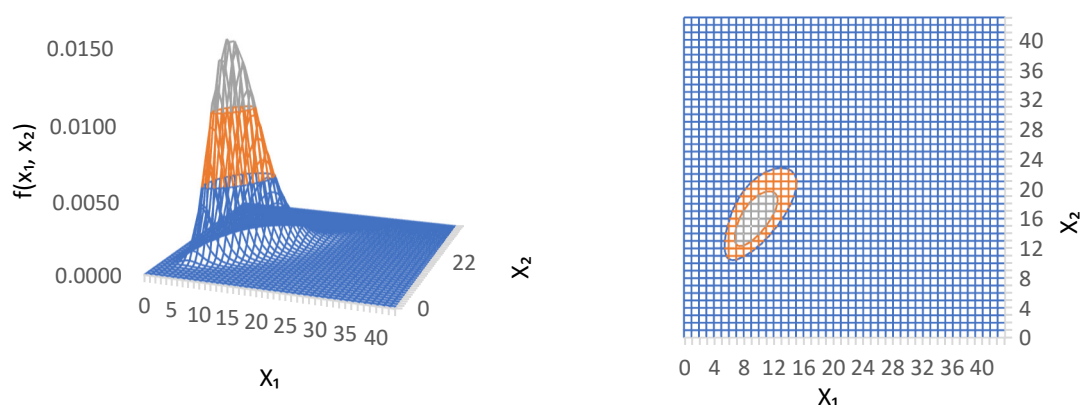
Local mode at (9, 9).

**Figure 8.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = 1, n_2 = 5, n_3 = 5$ . The colored curves represent selected equal-pmf contour levels.



Local modes at (15, 9).

**Figure 9.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = 5, n_2 = 5, n_3 = 1$ . The colored curves represent selected equal-pmf contour levels.



Local modes at (9, 15)

**Figure 10.** Joint pmf surface and corresponding contour plot of the BGIT-II distribution for  $p_1 = 0.1, p_2 = 0.4, q_1 = 0.1, q_2 = 0.4, r_1 = 0.1, r_2 = 0.4, n_1 = 5, n_2 = 1, n_3 = 5$ . The colored curves represent selected equal-pmf contour levels.

## 5. Parameter Estimation

This section discusses parameter estimation for the proposed BGIT-I and BGIT-II distributions. We consider two estimation approaches: maximum-likelihood estimation (MLE) and pgf-based estimation. Both methods involve continuous probability parameters and unknown positive integer parameters. For BGIT-I, the integer parameters are  $n_1$  and  $n_2$ , while for BGIT-II, they are  $n_1$ ,  $n_2$ , and  $n_3$ .

Let  $\theta$  denote the vector of continuous probability parameters. For BGIT-I,  $(\theta = (p_1, p_2, p_3, q_1, q_2, q_3))$ , where  $(p_1 + p_2 + p_3 = 1)$  and  $(q_1 + q_2 + q_3 = 1)$ . For BGIT-II,  $(\theta = (p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3))$ , where each triplet sums to one. In the numerical implementation, these simplex constraints were imposed during optimization, with a small lower bound  $\varepsilon$  used to keep the probability parameters in the interior of the parameter space and to avoid numerical boundary problems.

For a fixed set of integer parameters, the continuous parameters were estimated by constrained numerical optimization using the Sequential Least Squares Programming (SLSQP) algorithm implemented in scipy. Optimize running under Python version 3.13.9 (Anaconda distribution) in Spyder version 6.1.0. The convergence tolerance was set to  $1.0 \times 10^{-9}$ . Since the objective functions are generally non-linear and may be multi-modal, convergence to a global optimum cannot be guaranteed by a local optimizer alone. Therefore, the optimization results were checked using the objective function values, convergence status, and parameter admissibility. To reduce sensitivity to initialization, the optimization was repeated from different admissible starting values, including balanced starting values such as  $(1/3, 1/3, 1/3)$  and alternative interior points favoring different components. The solution giving the best objective value was retained.

A further challenge is that the integer parameters are unknown. Since these parameters are discrete, they cannot be optimized directly using standard gradient-based continuous optimization. We therefore treated their selection as a discrete profiling step over a finite candidate set of positive integer values. For each candidate integer vector, the continuous probability parameters were first estimated using either MLE or the pgf-based criterion. The fitted model was then evaluated using a chi-square statistic based on grouped frequencies, where adjacent categories were combined when necessary to avoid small expected frequencies. The candidate integer vector that produced the smallest chi-square value was selected.

This chi-square criterion was used only for selecting the integer parameters. Specifically, for each candidate integer vector, the continuous probability parameters were first estimated, and the corresponding grouped chi-square statistic was then used to identify the integer vector giving the smallest discrepancy between the observed and fitted frequencies. After the integer values and continuous parameters had been selected, the final goodness-of-fit assessment was conducted using the grouped goodness-of-fit tables reported in the empirical sections.

Likelihood-based criteria such as AIC and BIC were reported for models fitted by MLE, but they are not directly applicable to the pgf-based estimator because the pgf-based method does not maximize the likelihood. Therefore, the chi-square criterion provides a common and consistent selection rule for the integer parameters under both MLE and pgf-based estimation.

The computational cost of this procedure increases with the size of the candidate set for the integer parameters. For BGIT-I, the search is over pairs  $(n_1, n_2)$ , while for BGIT-II it is over triples  $(n_1, n_2, n_3)$ . Thus, BGIT-II is computationally more intensive. Nevertheless, the integer search is feasible because it is performed over a finite candidate set, and each candidate requires only a constrained optimization over the continuous probability parameters.

### 5.1. Maximum-Likelihood Estimation

MLE is used as a standard likelihood-based method for estimating the parameters of the proposed bivariate distributions. For fixed-integer parameters, the continuous probability parameters are estimated by maximizing the log-likelihood function subject to the required probability constraints.

Let  $\pi_{ij}$  denote the observed frequency in the  $(i, j)$  cell for  $i = 0, 1, 2, \dots, r$  and  $j = 0, 1, 2, \dots, s$ . Let  $f_{\theta}(i, j)$  denote the joint pmf evaluated at  $(i, j)$  where  $\theta$  is the vector of unknown parameters. The log-likelihood function is

$$\ln L = \sum_{i=0}^r \sum_{j=0}^s \pi_{ij} \ln f_{\theta}(i, j).$$

For each fixed candidate value of the integer parameters, the negative log-likelihood was minimized numerically subject to the simplex constraints on the probability parameters.

Under standard regularity conditions, including identifiability, an interior true parameter value, differentiability of the log-likelihood, and non-singularity of the Fisher information matrix, the MLE is consistent and asymptotically normal. It is also asymptotically efficient among regular estimators. In finite samples, however, the likelihood surface may contain local optima, so the numerical checks described above were used to assess the reliability of the obtained solution.

### 5.2. The Probability Generating Function-Based Estimator

The pgf-based estimator provides a computationally attractive alternative to MLE. The proposed distributions have tractable pgfs, whereas their pmfs involve finite summations and, in the case of BGIT-II, hypergeometric functions. Therefore, estimation based on the pgf reduces the computational burden of repeated pmf evaluations.

The pgf-based approach was considered by [16] in the univariate context as a minimum Hellinger-type and squared distance statistic. Ref. [17] proposed a pgf-based Jeffrey's divergence and ref. [18] examined a weighted pgf-based analog of corrected Kullback–Leibler divergence for parameter estimation. Ref. [19] examined pgf-based estimation for the bivariate extensions. Ref. [20] considered the alternating minimization (AM) algorithm to analyze these pgf-based parameter estimation methods. We will adopt this approach, where the pgf-based estimator is given as

$$T_2 = \int_0^1 \int_0^1 (\phi_N(t_1, t_2) - \phi_{\theta}(t_1, t_2))^2 dt_1 dt_2$$

where  $\phi_{\theta}(t_1, t_2)$  is the theoretical pgf and  $\phi_N(t_1, t_2)$  is the empirical pgf of the distribution given by  $\phi_N(t_1, t_2) = \sum_{i,j} \frac{n_{ij}}{n} t_1^i t_2^j$ .

Thus, for fixed-integer parameters, the pgf-based estimator is obtained by minimizing  $T_2$  subject to the same probability constraints used in the MLE approach. The estimator can be interpreted as a minimum-distance, or M-estimation, procedure based on the discrepancy between the empirical pgf and the theoretical pgf. Under suitable regularity conditions, such as identifiability, compactness of the parameter space, continuity of the objective function, and uniqueness of the minimizer, minimum-distance, and M-estimators are generally consistent and asymptotically normal; see, for example, refs. [21,22]. For pgf-based estimation of count distributions, ref. [23] provides related asymptotic results for estimators obtained by minimizing distances between the empirical pgf and the model pgf. Therefore, the pgf-based estimator considered here is expected to have similar large-sample behavior when the corresponding regularity and identifiability conditions hold. Compared with MLE, the pgf-based estimator may be less statistically efficient, but it offers a useful computational alternative for the proposed BGIT models.

## 6. Monte Carlo Simulation Study

A Monte Carlo simulation study was conducted to evaluate the finite-sample performance of the proposed estimation procedures. The study considers the BGIT-I and BGIT-II models under different marginal dispersion behaviors and dependence structures. Three sample sizes were considered, namely  $N = 100, 500$ , and  $1000$ , representing small, moderate, and large samples, respectively. For each setting and sample size, 1000 Monte Carlo replications were generated.

For notational convenience, the probability parameters are written as  $p = p_1, p_2, p_3$ ,  $q = q_1, q_2, q_3$ , and  $r = r_1, r_2, r_3$ . For BGIT-I, the model involves  $p$  and  $q$ , while BGIT-II involves  $p, q$ , and  $r$ . The simulation settings are summarized in Table 4.

**Table 4.** Monte Carlo simulation design for BGIT-I and BGIT-II.

| Model   | Setting     | Marginal Behaviour           | Dependence Regime | Integer Parameter (s)                   | $p, q, r$                                                                        |
|---------|-------------|------------------------------|-------------------|-----------------------------------------|----------------------------------------------------------------------------------|
| BGIT-I  | Setting I   | Under-dispersed              | Negative          | $n_1 = 12,$<br>$n_2 = 1$                | $p = (0.20, 0.20, 0.60)$<br>$q = (0.03, 0.03, 0.94)$                             |
| BGIT-I  | Setting II  | Approximately equi-dispersed | Near-zero         | $n_1 = 17,$<br>$n_2 = 5$                | $p = (0.20, 0.12, 0.68)$<br>$q = (0.22, 0.11, 0.67)$                             |
| BGIT-I  | Setting III | Over-dispersed               | Positive          | $n_1 = 14,$<br>$n_2 = 6$                | $p = (0.07, 0.16, 0.77)$<br>$q = (0.15, 0.475, 0.375)$                           |
| BGIT-II | Setting I   | Under-dispersed              | Weak positive     | $n_1 = 1,$<br>$n_2 = 15,$<br>$n_3 = 15$ | $p = (0.90, 0.08, 0.02)$<br>$q = (0.75, 0.20, 0.05)$<br>$r = (0.75, 0.20, 0.05)$ |
| BGIT-II | Setting II  | Approximately equi-dispersed | Moderate positive | $n_1 = 4,$<br>$n_2 = 10,$<br>$n_3 = 10$ | $p = (0.75, 0.15, 0.10)$<br>$q = (0.60, 0.20, 0.20)$<br>$r = (0.60, 0.20, 0.20)$ |
| BGIT-II | Setting III | Over-dispersed               | Strong positive   | $n_1 = 12,$<br>$n_2 = 6,$<br>$n_3 = 6$  | $p = (0.25, 0.20, 0.55)$<br>$q = (0.20, 0.25, 0.55)$<br>$r = (0.20, 0.25, 0.55)$ |

For each simulated sample, the probability parameters were estimated using the MLE and pgf-based estimation procedures described in Section 5. The same parameter constraints were imposed in both estimation methods to ensure that each probability vector remained valid. The integer parameters were set according to the simulation design in Table 4, and the reported performance measures focus on the estimation of the probability parameters.

The performance of the estimators was evaluated using the Monte Carlo mean, bias, empirical standard error, root-mean-square error, convergence rate, and average computation time. For a parameter  $\theta$ , the bias was computed as the difference between the Monte Carlo mean of the estimates and the true value. The empirical standard error was computed as the standard deviation of the estimates across successful Monte Carlo replications, while the RMSE combines both bias and variability. The convergence rate was computed as the proportion of successful replications out of 1000 attempted replications. The simulation results are reported separately for each parameter.

Table 5(a)–(d) report the parameter-specific Monte Carlo results for the MLE and pgf-based estimators. For BGIT-I, both estimators show satisfactory finite-sample performance across the three dispersion and dependence settings. In Setting I, where  $q_1$  and  $q_2$  are close to zero, the  $q$ -parameters are more difficult to estimate than the  $p$ -parameters. This is reflected in the larger empirical SE and RMSE values for  $q_1, q_2$ , and  $q_3$ , particularly when  $N = 100$ . Nevertheless, the RMSE values decrease as the sample size increases, indicating improved estimation accuracy.

**Table 5.** (a) Monte Carlo results for BGIT-I using MLE. (b) Monte Carlo results for BGIT-I using pgf-based estimator. (c) Monte Carlo results for BGIT-II using MLE. (d) Monte Carlo results for BGIT-II using pgf-based estimator.

| Setting     | N    | Parameter | True Value | Mean   | Bias    | Empirical SE | RMSE   |
|-------------|------|-----------|------------|--------|---------|--------------|--------|
| (a)         |      |           |            |        |         |              |        |
| Setting I   | 100  | $p_1$     | 0.2000     | 0.1952 | −0.0048 | 0.0173       | 0.0169 |
|             |      | $p_2$     | 0.2000     | 0.1951 | −0.0049 | 0.0173       | 0.0177 |
|             |      | $p_3$     | 0.6000     | 0.6097 | 0.0097  | 0.0224       | 0.0239 |
|             |      | $q_1$     | 0.0300     | 0.0630 | 0.0330  | 0.0943       | 0.1001 |
|             |      | $q_2$     | 0.0300     | 0.0630 | 0.0330  | 0.0990       | 0.1042 |
|             |      | $q_3$     | 0.9400     | 0.8739 | −0.0661 | 0.1249       | 0.1414 |
|             | 500  | $p_1$     | 0.2000     | 0.1983 | −0.0017 | 0.0100       | 0.0085 |
|             |      | $p_2$     | 0.2000     | 0.1984 | −0.0016 | 0.0100       | 0.0085 |
|             |      | $p_3$     | 0.6000     | 0.6033 | 0.0033  | 0.0100       | 0.0114 |
|             |      | $q_1$     | 0.0300     | 0.0453 | 0.0153  | 0.0640       | 0.0661 |
|             |      | $q_2$     | 0.0300     | 0.0412 | 0.0112  | 0.0608       | 0.0617 |
|             |      | $q_3$     | 0.9400     | 0.9135 | −0.0265 | 0.0794       | 0.0837 |
|             | 1000 | $p_1$     | 0.2000     | 0.1985 | −0.0015 | 0.0000       | 0.0067 |
|             |      | $p_2$     | 0.2000     | 0.1991 | −0.0009 | 0.0000       | 0.0061 |
|             |      | $p_3$     | 0.6000     | 0.6025 | 0.0025  | 0.0100       | 0.0085 |
|             |      | $q_1$     | 0.0300     | 0.0423 | 0.0123  | 0.0539       | 0.0549 |
|             |      | $q_2$     | 0.0300     | 0.0377 | 0.0077  | 0.0510       | 0.0515 |
|             |      | $q_3$     | 0.9400     | 0.9199 | −0.0201 | 0.0663       | 0.0695 |
| Setting II  | 100  | $p_1$     | 0.2000     | 0.2017 | 0.0017  | 0.0374       | 0.0377 |
|             |      | $p_2$     | 0.1200     | 0.1209 | 0.0009  | 0.0300       | 0.0294 |
|             |      | $p_3$     | 0.6800     | 0.6773 | −0.0027 | 0.0412       | 0.0412 |
|             |      | $q_1$     | 0.2200     | 0.2135 | −0.0065 | 0.0728       | 0.0733 |
|             |      | $q_2$     | 0.1100     | 0.1068 | −0.0032 | 0.0616       | 0.0620 |
|             |      | $q_3$     | 0.6700     | 0.6797 | 0.0097  | 0.0663       | 0.0668 |
|             | 500  | $p_1$     | 0.2000     | 0.2000 | −0.0000 | 0.0173       | 0.0162 |
|             |      | $p_2$     | 0.1200     | 0.1198 | −0.0002 | 0.0141       | 0.0142 |
|             |      | $p_3$     | 0.6800     | 0.6802 | 0.0002  | 0.0173       | 0.0175 |
|             |      | $q_1$     | 0.2200     | 0.2195 | −0.0005 | 0.0316       | 0.0320 |
|             |      | $q_2$     | 0.1100     | 0.1096 | −0.0004 | 0.0300       | 0.0305 |
|             |      | $q_3$     | 0.6700     | 0.6709 | 0.0009  | 0.0265       | 0.0269 |
|             | 1000 | $p_1$     | 0.2000     | 0.2004 | 0.0004  | 0.0100       | 0.0112 |
|             |      | $p_2$     | 0.1200     | 0.1204 | 0.0004  | 0.0100       | 0.0098 |
|             |      | $p_3$     | 0.6800     | 0.6793 | −0.0007 | 0.0141       | 0.0124 |
|             |      | $q_1$     | 0.2200     | 0.2195 | −0.0005 | 0.0224       | 0.0217 |
|             |      | $q_2$     | 0.1100     | 0.1086 | −0.0014 | 0.0200       | 0.0206 |
|             |      | $q_3$     | 0.6700     | 0.6719 | 0.0019  | 0.0200       | 0.0191 |
| Setting III | 100  | $p_1$     | 0.0700     | 0.0724 | 0.0024  | 0.0374       | 0.0369 |
|             |      | $p_2$     | 0.1600     | 0.1650 | 0.0050  | 0.0656       | 0.0659 |
|             |      | $p_3$     | 0.7700     | 0.7626 | −0.0074 | 0.0755       | 0.0758 |
|             |      | $q_1$     | 0.1500     | 0.1484 | −0.0016 | 0.0300       | 0.0307 |
|             |      | $q_2$     | 0.4750     | 0.4715 | −0.0035 | 0.0361       | 0.0368 |
|             |      | $q_3$     | 0.3750     | 0.3801 | 0.0051  | 0.0283       | 0.0284 |
|             | 500  | $p_1$     | 0.0700     | 0.0703 | 0.0003  | 0.0173       | 0.0169 |
|             |      | $p_2$     | 0.1600     | 0.1613 | 0.0013  | 0.0300       | 0.0295 |
|             |      | $p_3$     | 0.7700     | 0.7685 | −0.0015 | 0.0332       | 0.0337 |
|             |      | $q_1$     | 0.1500     | 0.1501 | 0.0001  | 0.0141       | 0.0136 |

|             |      |       |        |        |         |        |        |
|-------------|------|-------|--------|--------|---------|--------|--------|
|             |      | $q_2$ | 0.4750 | 0.4739 | −0.0011 | 0.0173 | 0.0160 |
|             |      | $q_3$ | 0.3750 | 0.3760 | 0.0010  | 0.0100 | 0.0122 |
|             | 1000 | $p_1$ | 0.0700 | 0.0702 | 0.0002  | 0.0100 | 0.0118 |
|             |      | $p_2$ | 0.1600 | 0.1601 | 0.0001  | 0.0200 | 0.0211 |
|             |      | $p_3$ | 0.7700 | 0.7697 | −0.0003 | 0.0245 | 0.0242 |
|             |      | $q_1$ | 0.1500 | 0.1497 | −0.0003 | 0.0100 | 0.0096 |
|             |      | $q_2$ | 0.4750 | 0.4749 | −0.0001 | 0.0100 | 0.0114 |
|             |      | $q_3$ | 0.3750 | 0.3754 | 0.0004  | 0.0100 | 0.0090 |
| <b>(b)</b>  |      |       |        |        |         |        |        |
| Setting I   | 100  | $p_1$ | 0.2000 | 0.1911 | −0.0089 | 0.0265 | 0.0278 |
|             |      | $p_2$ | 0.2000 | 0.1910 | −0.0090 | 0.0265 | 0.0286 |
|             |      | $p_3$ | 0.6000 | 0.6178 | 0.0178  | 0.0361 | 0.0398 |
|             |      | $q_1$ | 0.0300 | 0.1009 | 0.0709  | 0.1533 | 0.1690 |
|             |      | $q_2$ | 0.0300 | 0.1019 | 0.0719  | 0.1536 | 0.1697 |
|             |      | $q_3$ | 0.9400 | 0.7972 | −0.1428 | 0.2005 | 0.2461 |
|             | 500  | $p_1$ | 0.2000 | 0.1954 | −0.0046 | 0.0141 | 0.0158 |
|             |      | $p_2$ | 0.2000 | 0.1959 | −0.0041 | 0.0141 | 0.0149 |
|             |      | $p_3$ | 0.6000 | 0.6087 | 0.0087  | 0.0200 | 0.0214 |
|             |      | $q_1$ | 0.0300 | 0.0743 | 0.0443  | 0.1145 | 0.1227 |
|             |      | $q_2$ | 0.0300 | 0.0680 | 0.0380  | 0.1068 | 0.1134 |
|             |      | $q_3$ | 0.9400 | 0.8578 | −0.0822 | 0.1435 | 0.1652 |
|             | 1000 | $p_1$ | 0.2000 | 0.1961 | −0.0039 | 0.0100 | 0.0123 |
|             |      | $p_2$ | 0.2000 | 0.1962 | −0.0038 | 0.0100 | 0.0121 |
|             |      | $p_3$ | 0.6000 | 0.6077 | 0.0077  | 0.0141 | 0.0170 |
|             |      | $q_1$ | 0.0300 | 0.0665 | 0.0365  | 0.0954 | 0.1023 |
|             |      | $q_2$ | 0.0300 | 0.0677 | 0.0377  | 0.0954 | 0.1027 |
|             |      | $q_3$ | 0.9400 | 0.8658 | −0.0742 | 0.1212 | 0.1420 |
| Setting II  | 100  | $p_1$ | 0.2000 | 0.2083 | 0.0083  | 0.0640 | 0.0648 |
|             |      | $p_2$ | 0.1200 | 0.1204 | 0.0004  | 0.0469 | 0.0464 |
|             |      | $p_3$ | 0.6800 | 0.6712 | −0.0088 | 0.0671 | 0.0679 |
|             |      | $q_1$ | 0.2200 | 0.1887 | −0.0313 | 0.1300 | 0.1338 |
|             |      | $q_2$ | 0.1100 | 0.1100 | 0.0000  | 0.1034 | 0.1034 |
|             |      | $q_3$ | 0.6700 | 0.7013 | 0.0313  | 0.1183 | 0.1224 |
|             | 500  | $p_1$ | 0.2000 | 0.2006 | 0.0006  | 0.0316 | 0.0309 |
|             |      | $p_2$ | 0.1200 | 0.1211 | 0.0011  | 0.0265 | 0.0263 |
|             |      | $p_3$ | 0.6800 | 0.6783 | −0.0017 | 0.0300 | 0.0306 |
|             |      | $q_1$ | 0.2200 | 0.2166 | −0.0034 | 0.0671 | 0.0673 |
|             |      | $q_2$ | 0.1100 | 0.1068 | −0.0032 | 0.0608 | 0.0609 |
|             |      | $q_3$ | 0.6700 | 0.6766 | 0.0066  | 0.0520 | 0.0523 |
|             | 1000 | $p_1$ | 0.2000 | 0.2012 | 0.0012  | 0.0224 | 0.0216 |
|             |      | $p_2$ | 0.1200 | 0.1210 | 0.0010  | 0.0200 | 0.0188 |
|             |      | $p_3$ | 0.6800 | 0.6777 | −0.0023 | 0.0224 | 0.0217 |
|             |      | $q_1$ | 0.2200 | 0.2171 | −0.0029 | 0.0469 | 0.0468 |
|             |      | $q_2$ | 0.1100 | 0.1072 | −0.0028 | 0.0436 | 0.0433 |
|             |      | $q_3$ | 0.6700 | 0.6758 | 0.0058  | 0.0361 | 0.0367 |
| Setting III | 100  | $p_1$ | 0.0700 | 0.0792 | 0.0092  | 0.0566 | 0.0574 |
|             |      | $p_2$ | 0.1600 | 0.1683 | 0.0083  | 0.0849 | 0.0850 |
|             |      | $p_3$ | 0.7700 | 0.7524 | −0.0176 | 0.0933 | 0.0950 |
|             |      | $q_1$ | 0.1500 | 0.1427 | −0.0073 | 0.0520 | 0.0527 |
|             |      | $q_2$ | 0.4750 | 0.4716 | −0.0034 | 0.0557 | 0.0558 |
|             |      | $q_3$ | 0.3750 | 0.3857 | 0.0107  | 0.0361 | 0.0382 |
|             | 500  | $p_1$ | 0.0700 | 0.0712 | 0.0012  | 0.0300 | 0.0296 |
|             |      | $p_2$ | 0.1600 | 0.1618 | 0.0018  | 0.0436 | 0.0438 |

|            |      |       |        |        |         |        |        |
|------------|------|-------|--------|--------|---------|--------|--------|
|            |      | $p_3$ | 0.7700 | 0.7670 | −0.0030 | 0.0447 | 0.0444 |
|            |      | $q_1$ | 0.1500 | 0.1495 | −0.0005 | 0.0265 | 0.0272 |
|            |      | $q_2$ | 0.4750 | 0.4736 | −0.0014 | 0.0300 | 0.0303 |
|            |      | $q_3$ | 0.3750 | 0.3770 | 0.0020  | 0.0173 | 0.0172 |
|            | 1000 | $p_1$ | 0.0700 | 0.0711 | 0.0011  | 0.0200 | 0.0206 |
|            |      | $p_2$ | 0.1600 | 0.1605 | 0.0005  | 0.0300 | 0.0306 |
|            |      | $p_3$ | 0.7700 | 0.7684 | −0.0016 | 0.0316 | 0.0324 |
|            |      | $q_1$ | 0.1500 | 0.1491 | −0.0009 | 0.0173 | 0.0185 |
|            |      | $q_2$ | 0.4750 | 0.4747 | −0.0003 | 0.0200 | 0.0207 |
|            |      | $q_3$ | 0.3750 | 0.3761 | 0.0011  | 0.0141 | 0.0126 |
| (c)        |      |       |        |        |         |        |        |
| Setting I  | 100  | $p_1$ | 0.9000 | 0.7198 | −0.1802 | 0.3197 | 0.3668 |
|            |      | $p_2$ | 0.0800 | 0.2052 | 0.1252  | 0.2823 | 0.3086 |
|            |      | $p_3$ | 0.0200 | 0.0750 | 0.0550  | 0.1020 | 0.1157 |
|            |      | $q_1$ | 0.7500 | 0.7660 | 0.0160  | 0.0316 | 0.0349 |
|            |      | $q_2$ | 0.2000 | 0.1792 | −0.0208 | 0.0707 | 0.0739 |
|            |      | $q_3$ | 0.0500 | 0.0548 | 0.0048  | 0.0500 | 0.0498 |
|            |      | $r_1$ | 0.7500 | 0.7655 | 0.0155  | 0.0300 | 0.0339 |
|            |      | $r_2$ | 0.2000 | 0.1790 | −0.0210 | 0.0686 | 0.0714 |
|            |      | $r_3$ | 0.0500 | 0.0555 | 0.0055  | 0.0480 | 0.0487 |
|            | 500  | $p_1$ | 0.9000 | 0.8338 | −0.0662 | 0.2037 | 0.2142 |
|            |      | $p_2$ | 0.0800 | 0.1238 | 0.0438  | 0.1972 | 0.2019 |
|            |      | $p_3$ | 0.0200 | 0.0424 | 0.0224  | 0.0616 | 0.0653 |
|            |      | $q_1$ | 0.7500 | 0.7549 | 0.0049  | 0.0173 | 0.0178 |
|            |      | $q_2$ | 0.2000 | 0.1941 | −0.0059 | 0.0387 | 0.0394 |
|            |      | $q_3$ | 0.0500 | 0.0510 | 0.0010  | 0.0265 | 0.0268 |
|            |      | $r_1$ | 0.7500 | 0.7550 | 0.0050  | 0.0173 | 0.0178 |
|            |      | $r_2$ | 0.2000 | 0.1943 | −0.0057 | 0.0374 | 0.0383 |
|            |      | $r_3$ | 0.0500 | 0.0506 | 0.0006  | 0.0265 | 0.0262 |
|            | 1000 | $p_1$ | 0.9000 | 0.8569 | −0.0431 | 0.1612 | 0.1668 |
|            |      | $p_2$ | 0.0800 | 0.1060 | 0.0260  | 0.1622 | 0.1642 |
|            |      | $p_3$ | 0.0200 | 0.0371 | 0.0171  | 0.0520 | 0.0546 |
|            |      | $q_1$ | 0.7500 | 0.7526 | 0.0026  | 0.0141 | 0.0130 |
|            |      | $q_2$ | 0.2000 | 0.1985 | −0.0015 | 0.0283 | 0.0277 |
|            |      | $q_3$ | 0.0500 | 0.0489 | −0.0011 | 0.0173 | 0.0187 |
|            |      | $r_1$ | 0.7500 | 0.7532 | 0.0032  | 0.0141 | 0.0127 |
|            |      | $r_2$ | 0.2000 | 0.1963 | −0.0037 | 0.0265 | 0.0271 |
|            |      | $r_3$ | 0.0500 | 0.0505 | 0.0005  | 0.0173 | 0.0186 |
| Setting II | 100  | $p_1$ | 0.7500 | 0.7112 | −0.0388 | 0.1752 | 0.1793 |
|            |      | $p_2$ | 0.1500 | 0.1807 | 0.0307  | 0.2007 | 0.2029 |
|            |      | $p_3$ | 0.1000 | 0.1081 | 0.0081  | 0.0889 | 0.0893 |
|            |      | $q_1$ | 0.6000 | 0.6135 | 0.0135  | 0.0693 | 0.0708 |
|            |      | $q_2$ | 0.2000 | 0.1890 | −0.0110 | 0.1109 | 0.1112 |
|            |      | $q_3$ | 0.2000 | 0.1975 | −0.0025 | 0.0592 | 0.0594 |
|            |      | $r_1$ | 0.6000 | 0.6140 | 0.0140  | 0.0693 | 0.0704 |
|            |      | $r_2$ | 0.2000 | 0.1865 | −0.0135 | 0.1086 | 0.1096 |
|            |      | $r_3$ | 0.2000 | 0.1995 | −0.0005 | 0.0583 | 0.0581 |
|            | 500  | $p_1$ | 0.7500 | 0.7255 | −0.0245 | 0.1145 | 0.1169 |
|            |      | $p_2$ | 0.1500 | 0.1713 | 0.0213  | 0.1584 | 0.1599 |
|            |      | $p_3$ | 0.1000 | 0.1032 | 0.0032  | 0.0686 | 0.0684 |
|            |      | $q_1$ | 0.6000 | 0.6095 | 0.0095  | 0.0424 | 0.0438 |
|            |      | $q_2$ | 0.2000 | 0.1869 | −0.0131 | 0.0640 | 0.0654 |
|            |      | $q_3$ | 0.2000 | 0.2036 | 0.0036  | 0.0283 | 0.0283 |

|             |      |       |        |        |         |        |        |
|-------------|------|-------|--------|--------|---------|--------|--------|
| Setting III | 1000 | $r_1$ | 0.6000 | 0.6098 | 0.0098  | 0.0436 | 0.0441 |
|             |      | $r_2$ | 0.2000 | 0.1860 | −0.0140 | 0.0640 | 0.0657 |
|             |      | $r_3$ | 0.2000 | 0.2042 | 0.0042  | 0.0283 | 0.0289 |
|             |      | $p_1$ | 0.7500 | 0.7375 | −0.0125 | 0.0837 | 0.0847 |
|             |      | $p_2$ | 0.1500 | 0.1586 | 0.0086  | 0.1285 | 0.1288 |
|             |      | $p_3$ | 0.1000 | 0.1038 | 0.0038  | 0.0592 | 0.0596 |
|             |      | $q_1$ | 0.6000 | 0.6046 | 0.0046  | 0.0316 | 0.0312 |
|             |      | $q_2$ | 0.2000 | 0.1937 | −0.0063 | 0.0458 | 0.0458 |
|             |      | $q_3$ | 0.2000 | 0.2017 | 0.0017  | 0.0200 | 0.0196 |
|             | 100  | $r_1$ | 0.6000 | 0.6052 | 0.0052  | 0.0316 | 0.0315 |
|             |      | $r_2$ | 0.2000 | 0.1921 | −0.0079 | 0.0458 | 0.0466 |
|             |      | $r_3$ | 0.2000 | 0.2027 | 0.0027  | 0.0200 | 0.0204 |
|             |      | $p_1$ | 0.2500 | 0.2469 | −0.0031 | 0.0917 | 0.0916 |
|             |      | $p_2$ | 0.2000 | 0.2043 | 0.0043  | 0.1122 | 0.1124 |
|             |      | $p_3$ | 0.5500 | 0.5488 | −0.0012 | 0.0332 | 0.0324 |
|             |      | $q_1$ | 0.2000 | 0.2033 | 0.0033  | 0.1490 | 0.1488 |
|             |      | $q_2$ | 0.2500 | 0.2476 | −0.0024 | 0.1676 | 0.1677 |
|             |      | $q_3$ | 0.5500 | 0.5491 | −0.0009 | 0.0469 | 0.0468 |
|             | 500  | $r_1$ | 0.2000 | 0.1998 | −0.0002 | 0.1428 | 0.1429 |
|             |      | $r_2$ | 0.2500 | 0.2533 | 0.0033  | 0.1572 | 0.1572 |
|             |      | $r_3$ | 0.5500 | 0.5469 | −0.0031 | 0.0447 | 0.0450 |
|             |      | $p_1$ | 0.2500 | 0.2632 | 0.0132  | 0.0781 | 0.0789 |
|             |      | $p_2$ | 0.2000 | 0.1848 | −0.0152 | 0.0843 | 0.0856 |
|             |      | $p_3$ | 0.5500 | 0.5520 | 0.0020  | 0.0141 | 0.0150 |
|             |      | $q_1$ | 0.2000 | 0.1926 | −0.0074 | 0.1393 | 0.1395 |
|             |      | $q_2$ | 0.2500 | 0.2508 | 0.0008  | 0.1414 | 0.1413 |
|             |      | $q_3$ | 0.5500 | 0.5566 | 0.0066  | 0.0200 | 0.0214 |
| Setting I   | 1000 | $r_1$ | 0.2000 | 0.1888 | −0.0112 | 0.1414 | 0.1417 |
|             |      | $r_2$ | 0.2500 | 0.2563 | 0.0063  | 0.1439 | 0.1440 |
|             |      | $r_3$ | 0.5500 | 0.5548 | 0.0048  | 0.0200 | 0.0208 |
|             |      | $p_1$ | 0.2500 | 0.2604 | 0.0104  | 0.0721 | 0.0728 |
|             |      | $p_2$ | 0.2000 | 0.1873 | −0.0127 | 0.0768 | 0.0780 |
|             |      | $p_3$ | 0.5500 | 0.5523 | 0.0023  | 0.0100 | 0.0109 |
|             |      | $q_1$ | 0.2000 | 0.1969 | −0.0031 | 0.1300 | 0.1300 |
|             |      | $q_2$ | 0.2500 | 0.2479 | −0.0021 | 0.1308 | 0.1309 |
|             |      | $q_3$ | 0.5500 | 0.5552 | 0.0052  | 0.0141 | 0.0155 |
|             | 500  | $r_1$ | 0.2000 | 0.1956 | −0.0044 | 0.1308 | 0.1307 |
|             |      | $r_2$ | 0.2500 | 0.2494 | −0.0006 | 0.1315 | 0.1314 |
|             |      | $r_3$ | 0.5500 | 0.5550 | 0.0050  | 0.0141 | 0.0151 |
|             |      | (d)   |        |        |         |        |        |
|             | 100  | $p_1$ | 0.9000 | 0.7768 | −0.1232 | 0.3303 | 0.3524 |
|             |      | $p_2$ | 0.0800 | 0.1558 | 0.0758  | 0.2676 | 0.2781 |
|             |      | $p_3$ | 0.0200 | 0.0674 | 0.0474  | 0.1304 | 0.1388 |
|             |      | $q_1$ | 0.7500 | 0.7639 | 0.0139  | 0.0374 | 0.0394 |
|             |      | $q_2$ | 0.2000 | 0.1679 | −0.0321 | 0.0995 | 0.1045 |
|             |      | $q_3$ | 0.0500 | 0.0682 | 0.0182  | 0.0714 | 0.0740 |
|             |      | $r_1$ | 0.7500 | 0.7637 | 0.0137  | 0.0361 | 0.0392 |
|             |      | $r_2$ | 0.2000 | 0.1719 | −0.0281 | 0.0975 | 0.1015 |
|             |      | $r_3$ | 0.0500 | 0.0645 | 0.0145  | 0.0700 | 0.0718 |
|             | 500  | $p_1$ | 0.9000 | 0.8204 | −0.0796 | 0.2412 | 0.2539 |
|             |      | $p_2$ | 0.0800 | 0.1504 | 0.0704  | 0.2263 | 0.2368 |
|             |      | $p_3$ | 0.0200 | 0.0292 | 0.0092  | 0.0592 | 0.0598 |
|             |      | $q_1$ | 0.7500 | 0.7567 | 0.0067  | 0.0224 | 0.0236 |

|             |      |       |        |        |         |        |        |
|-------------|------|-------|--------|--------|---------|--------|--------|
| Setting II  | 1000 | $q_2$ | 0.2000 | 0.1865 | −0.0135 | 0.0648 | 0.0663 |
|             |      | $q_3$ | 0.0500 | 0.0568 | 0.0068  | 0.0469 | 0.0475 |
|             |      | $r_1$ | 0.7500 | 0.7570 | 0.0070  | 0.0224 | 0.0231 |
|             |      | $r_2$ | 0.2000 | 0.1849 | −0.0151 | 0.0640 | 0.0658 |
|             |      | $r_3$ | 0.0500 | 0.0580 | 0.0080  | 0.0469 | 0.0481 |
|             |      | $p_1$ | 0.9000 | 0.8515 | −0.0485 | 0.1871 | 0.1932 |
|             |      | $p_2$ | 0.0800 | 0.1249 | 0.0449  | 0.1836 | 0.1890 |
|             |      | $p_3$ | 0.0200 | 0.0236 | 0.0036  | 0.0469 | 0.0475 |
|             |      | $q_1$ | 0.7500 | 0.7537 | 0.0037  | 0.0141 | 0.0161 |
|             |      | $q_2$ | 0.2000 | 0.1943 | −0.0057 | 0.0458 | 0.0466 |
|             |      | $q_3$ | 0.0500 | 0.0521 | 0.0021  | 0.0346 | 0.0351 |
|             |      | $r_1$ | 0.7500 | 0.7536 | 0.0036  | 0.0173 | 0.0164 |
|             | 100  | $r_2$ | 0.2000 | 0.1941 | −0.0059 | 0.0458 | 0.0467 |
|             |      | $r_3$ | 0.0500 | 0.0523 | 0.0023  | 0.0346 | 0.0346 |
|             |      | $p_1$ | 0.7500 | 0.7792 | 0.0292  | 0.1732 | 0.1755 |
|             |      | $p_2$ | 0.1500 | 0.0795 | −0.0705 | 0.1697 | 0.1837 |
|             |      | $p_3$ | 0.1000 | 0.1413 | 0.0413  | 0.1030 | 0.1107 |
|             |      | $q_1$ | 0.6000 | 0.5922 | −0.0078 | 0.0742 | 0.0742 |
|             |      | $q_2$ | 0.2000 | 0.2220 | 0.0220  | 0.1360 | 0.1379 |
|             |      | $q_3$ | 0.2000 | 0.1858 | −0.0142 | 0.0866 | 0.0878 |
|             |      | $r_1$ | 0.6000 | 0.5942 | −0.0058 | 0.0735 | 0.0734 |
|             |      | $r_2$ | 0.2000 | 0.2155 | 0.0155  | 0.1393 | 0.1402 |
|             |      | $r_3$ | 0.2000 | 0.1903 | −0.0097 | 0.0917 | 0.0921 |
|             | 500  | $p_1$ | 0.7500 | 0.7610 | 0.0110  | 0.1082 | 0.1086 |
|             |      | $p_2$ | 0.1500 | 0.1052 | −0.0448 | 0.1463 | 0.1531 |
|             |      | $p_3$ | 0.1000 | 0.1338 | 0.0338  | 0.0693 | 0.0771 |
|             |      | $q_1$ | 0.6000 | 0.5973 | −0.0027 | 0.0447 | 0.0443 |
|             |      | $q_2$ | 0.2000 | 0.2053 | 0.0053  | 0.0819 | 0.0821 |
|             |      | $q_3$ | 0.2000 | 0.1973 | −0.0027 | 0.0469 | 0.0475 |
|             |      | $r_1$ | 0.6000 | 0.5984 | −0.0016 | 0.0458 | 0.0455 |
|             |      | $r_2$ | 0.2000 | 0.2024 | 0.0024  | 0.0843 | 0.0843 |
|             |      | $r_3$ | 0.2000 | 0.1991 | −0.0009 | 0.0469 | 0.0474 |
|             | 1000 | $p_1$ | 0.7500 | 0.7455 | −0.0045 | 0.1049 | 0.1048 |
|             |      | $p_2$ | 0.1500 | 0.1351 | −0.0149 | 0.1603 | 0.1609 |
|             |      | $p_3$ | 0.1000 | 0.1194 | 0.0194  | 0.0728 | 0.0755 |
|             |      | $q_1$ | 0.6000 | 0.6029 | 0.0029  | 0.0412 | 0.0409 |
|             |      | $q_2$ | 0.2000 | 0.1938 | −0.0062 | 0.0714 | 0.0717 |
|             |      | $q_3$ | 0.2000 | 0.2033 | 0.0033  | 0.0374 | 0.0369 |
|             |      | $r_1$ | 0.6000 | 0.6021 | 0.0021  | 0.0400 | 0.0406 |
|             |      | $r_2$ | 0.2000 | 0.1973 | −0.0027 | 0.0707 | 0.0709 |
|             |      | $r_3$ | 0.2000 | 0.2006 | 0.0006  | 0.0361 | 0.0367 |
| Setting III | 100  | $p_1$ | 0.2500 | 0.2673 | 0.0173  | 0.1020 | 0.1034 |
|             |      | $p_2$ | 0.2000 | 0.1887 | −0.0113 | 0.1257 | 0.1262 |
|             |      | $p_3$ | 0.5500 | 0.5439 | −0.0061 | 0.0480 | 0.0481 |
|             |      | $q_1$ | 0.2000 | 0.1620 | −0.0380 | 0.1530 | 0.1574 |
|             |      | $q_2$ | 0.2500 | 0.3020 | 0.0520  | 0.2030 | 0.2095 |
|             |      | $q_3$ | 0.5500 | 0.5360 | −0.0140 | 0.1122 | 0.1129 |
|             |      | $r_1$ | 0.2000 | 0.1599 | −0.0401 | 0.1526 | 0.1578 |
|             |      | $r_2$ | 0.2500 | 0.3046 | 0.0546  | 0.2037 | 0.2108 |
|             |      | $r_3$ | 0.5500 | 0.5355 | −0.0145 | 0.1118 | 0.1129 |
|             | 500  | $p_1$ | 0.2500 | 0.2844 | 0.0344  | 0.0748 | 0.0824 |
|             |      | $p_2$ | 0.2000 | 0.1637 | −0.0363 | 0.0781 | 0.0861 |
|             |      | $p_3$ | 0.5500 | 0.5519 | 0.0019  | 0.0200 | 0.0210 |

|      |       |        |        |         |        |        |
|------|-------|--------|--------|---------|--------|--------|
|      | $q_1$ | 0.2000 | 0.1476 | −0.0524 | 0.1237 | 0.1344 |
|      | $q_2$ | 0.2500 | 0.3041 | 0.0541  | 0.1292 | 0.1399 |
|      | $q_3$ | 0.5500 | 0.5483 | −0.0017 | 0.0520 | 0.0524 |
|      | $r_1$ | 0.2000 | 0.1477 | −0.0523 | 0.1241 | 0.1345 |
|      | $r_2$ | 0.2500 | 0.3040 | 0.0540  | 0.1281 | 0.1389 |
|      | $r_3$ | 0.5500 | 0.5483 | −0.0017 | 0.0520 | 0.0518 |
| 1000 | $p_1$ | 0.2500 | 0.2813 | 0.0313  | 0.0678 | 0.0750 |
|      | $p_2$ | 0.2000 | 0.1670 | −0.0330 | 0.0693 | 0.0767 |
|      | $p_3$ | 0.5500 | 0.5517 | 0.0017  | 0.0141 | 0.0150 |
|      | $q_1$ | 0.2000 | 0.1537 | −0.0463 | 0.1136 | 0.1227 |
|      | $q_2$ | 0.2500 | 0.2953 | 0.0453  | 0.1118 | 0.1207 |
|      | $q_3$ | 0.5500 | 0.5510 | 0.0010  | 0.0374 | 0.0367 |
|      | $r_1$ | 0.2000 | 0.1547 | −0.0453 | 0.1153 | 0.1240 |
|      | $r_2$ | 0.2500 | 0.2930 | 0.0430  | 0.1136 | 0.1215 |
|      | $r_3$ | 0.5500 | 0.5523 | 0.0023  | 0.0346 | 0.0350 |

Note. Mean denotes the Monte Carlo mean estimate. Empirical SE denotes the standard deviation of the estimates across successful replications. RMSE denotes root-mean-square error.

For BGIT-I Settings II and III, the estimation results are more stable. The biases are generally small, and the empirical SE and RMSE values decrease as  $N$  increases for most parameters. The MLE tends to produce lower RMSE values than the pgf-based estimator, consistent with the usual efficiency advantage of likelihood-based estimation under correct model specification. However, the pgf-based estimator remains competitive and provides reasonable parameter estimates, especially in Settings II and III.

For BGIT-II, the MLE generally yields lower RMSE than the pgf-based estimator. In Setting I, the parameters in the  $p$ -vector are the most difficult to estimate, especially for  $N = 100$ . This is expected because the true value ( $p_3 = 0.02$ ) is close to the boundary of the parameter space. In contrast, the  $q$ - and  $r$ -parameters are estimated more accurately, with smaller empirical SE and RMSE values. As the sample size increases, the RMSE values decrease for most parameters, showing improved estimation accuracy.

In BGIT-II Setting II, both estimators provide reasonably stable estimates. The  $q$ - and  $r$ -parameters are estimated with relatively small bias, while the  $p$ -parameters show larger variability, especially under the pgf-based estimator. Although the RMSE values generally decrease with increasing  $N$ , the decrease is not strictly monotone for every parameter. This is not unexpected in finite-sample simulation studies, particularly for a constrained multi-parameter model.

BGIT-II Setting III, which corresponds to over-dispersed margins with strong positive dependence, is the most challenging case. For MLE, the RMSE values generally decrease as  $N$  increases, although some biases are not strictly monotone. For the pgf-based estimator, several parameters show noticeable bias, particularly the  $q$ - and  $r$ -parameters. Nevertheless, the RMSE values still tend to decrease with increasing sample size, suggesting improved overall estimation accuracy among successful replications.

Table 6 summarizes the convergence rates and average computation times. The MLE achieved a convergence rate of 1.000 in all simulation settings. The pgf-based estimator also achieved convergence rates close to 1.000, except for BGIT-II Setting III, where the convergence rates were 0.987, 0.948, and 0.931 for  $N = 100, 500$ , and 1000, respectively. This confirms that the strong positive dependence case is computationally more challenging for the pgf-based estimator.

The computation time results show a clear computational advantage for the pgf-based estimator. Averaged over the three settings and three sample sizes, the pgf-based estimator reduces the average computation time from 7.1633 s to 1.5997 s for BGIT-I, corresponding to approximately 77.7% savings. For BGIT-II, the average computation time

decreases from 61.8043 s to 4.0399 s, corresponding to a reduction of approximately 93.5%. This reduction is particularly important for BGIT-II, where likelihood evaluation is more computationally demanding.

Overall, the Monte Carlo results show that the proposed estimators have satisfactory finite-sample performance under the simulated settings. MLE generally provides more accurate estimates, while the pgf-based estimator offers a computationally attractive alternative given the simpler pgf forms of the proposed BGIT models. The simulation study focuses on correctly specified BGIT models; robustness under model misspecification is left for future research.

**Table 6.** Convergence rates and average computation times.

| Model   | Setting     | N    | Estimator | Convergence Rate | Mean Time (s) |
|---------|-------------|------|-----------|------------------|---------------|
| BGIT-I  | Setting I   | 100  | MLE       | 1.000            | 11.8686       |
|         |             |      | PGF       | 1.000            | 6.0672        |
|         |             | 500  | MLE       | 1.000            | 19.2840       |
|         |             |      | PGF       | 0.998            | 1.0946        |
|         |             | 1000 | MLE       | 1.000            | 4.9542        |
|         |             |      | PGF       | 1.000            | 0.8644        |
|         | Setting II  | 100  | MLE       | 1.000            | 2.4379        |
|         |             |      | PGF       | 0.998            | 0.9946        |
|         |             | 500  | MLE       | 1.000            | 3.4907        |
|         |             |      | PGF       | 0.999            | 0.7961        |
|         |             | 1000 | MLE       | 1.000            | 4.1630        |
|         |             |      | PGF       | 1.000            | 0.7295        |
|         | Setting III | 100  | MLE       | 1.000            | 3.3261        |
|         |             |      | PGF       | 1.000            | 0.7435        |
|         |             | 500  | MLE       | 1.000            | 6.6970        |
|         |             |      | PGF       | 0.999            | 2.5126        |
|         |             | 1000 | MLE       | 1.000            | 8.2478        |
|         |             |      | PGF       | 1.000            | 0.5948        |
| BGIT-II | Setting I   | 100  | MLE       | 1.000            | 56.0637       |
|         |             |      | PGF       | 1.000            | 2.0201        |
|         |             | 500  | MLE       | 1.000            | 26.1095       |
|         |             |      | PGF       | 1.000            | 2.5083        |
|         |             | 1000 | MLE       | 1.000            | 52.0884       |
|         |             |      | PGF       | 1.000            | 2.0790        |
|         | Setting II  | 100  | MLE       | 1.000            | 18.1172       |
|         |             |      | PGF       | 1.000            | 2.3452        |
|         |             | 500  | MLE       | 1.000            | 27.4181       |
|         |             |      | PGF       | 1.000            | 5.3188        |
|         |             | 1000 | MLE       | 1.000            | 42.6128       |
|         |             |      | PGF       | 1.000            | 6.0158        |
|         | Setting III | 100  | MLE       | 1.000            | 64.3508       |
|         |             |      | PGF       | 0.987            | 7.1815        |
|         |             | 500  | MLE       | 1.000            | 135.2297      |
|         |             |      | PGF       | 0.948            | 4.6405        |
|         |             | 1000 | MLE       | 1.000            | 134.2486      |
|         |             |      | PGF       | 0.931            | 4.2501        |

Note. The convergence rate is computed as the proportion of successful replications out of 1000 Monte Carlo replications. Mean time refers to the average computation time per replication, in seconds.

## 7. Numerical Examples

To demonstrate the practical application of the proposed models, two real-life bivariate count data sets were considered. The first data set records the number of accidents sustained by 122 experienced shunters over two successive periods [24], and has previously been analyzed by [12,25]. This data set exhibits over-dispersion and has a sample correlation of 0.26. The second data set records the number of patients in two sections of a critical care and emergency room at San Agustin Hospital in Linares, Spain [26]. This data set is under-dispersed and has a sample correlation of  $-0.009$ .

For both data sets, the proposed BGIT-I and BGIT-II models were fitted using MLE and the pgf-based estimation method. The unknown positive integer parameters were selected using the minimum chi-square procedure described in Section 5, where the integer vector producing the smallest chi-square value was retained. For comparison, we also include the bivariate Conway–Maxwell Poisson models with given marginals and correlation, denoted by BCMPI and BCMPII, as proposed by [12]. For the first data set, the bivariate negative binomial model is included as an additional comparison.

To address sparse cells in the goodness-of-fit assessment, the frequency tables were grouped before computing the final chi-square goodness-of-fit statistics. Cells in the tail region were pooled so that the expected frequencies were sufficiently large for the chi-square approximation to be meaningful. The degrees of freedom are reported explicitly for each fitted model and were computed as

$df = \text{number of grouped cells} - 1 - \text{number of estimated free probability parameters}$ .

For BGIT-I, the number of estimated free probability parameters is four, while for BGIT-II, it is six. The selected integer parameters are reported separately.

To quantify the uncertainty in the proposed BGIT parameter estimates, a parametric bootstrap with 1000 replications was performed for each fitted BGIT model. The selected integer parameters were kept fixed at their fitted values, and the continuous probability parameters were re-estimated for each bootstrap sample. The bootstrap standard errors and 95% bootstrap confidence intervals are reported separately for each data set.

### 7.1. Shunter Accident Data

For the shunter accident data, the observed frequency table was grouped to ensure that the expected frequencies were adequate for the chi-square goodness-of-fit assessment. Table 7 reports the grouped observed frequencies together with the fitted frequencies from the competing models.

**Table 7.** Grouped observed and fitted frequencies for the shunter accident data.

| Model     | $X_2 = 0$ | $X_2 = 1$ | $X_2 = 2$ | $X_2 \geq 3$ | Total |
|-----------|-----------|-----------|-----------|--------------|-------|
| $X_1 = 0$ | 21        | 18        | 8         | 3            | 50    |
| BNB       | 21.90     | 16.67     | 7.98      | 4.52         | 51.07 |
| BCMPI     | 18.08     | 16.60     | 10.02     | 7.78         | 52.48 |
| BCMPII    | 21.09     | 15.44     | 8.63      | 6.48         | 51.64 |
| BGIT-I    | 20.45     | 18.28     | 8.89      | 4.86         | 52.47 |
| BGIT-II   | 23.09     | 17.24     | 7.07      | 4.92         | 52.31 |
| BGIT-I *  | 20.77     | 17.82     | 8.50      | 4.73         | 51.81 |
| BGIT-II * | 21.62     | 16.78     | 7.94      | 4.99         | 51.33 |
| $X_1 = 1$ | 13        | 14        | 10        | 6            | 43    |
| BNB       | 12.52     | 13.18     | 8.06      | 5.98         | 39.74 |
| BCMPI     | 12.98     | 11.95     | 7.38      | 5.87         | 38.18 |
| BCMPII    | 11.56     | 12.43     | 8.12      | 6.55         | 38.66 |
| BGIT-I    | 13.25     | 13.16     | 7.47      | 5.46         | 39.34 |

|              |       |       |      |      |       |
|--------------|-------|-------|------|------|-------|
| BGIT-II      | 10.79 | 14.43 | 8.06 | 5.61 | 38.89 |
| BGIT-I *     | 13.17 | 12.81 | 7.29 | 5.56 | 38.83 |
| BGIT-II *    | 11.61 | 14.78 | 8.74 | 5.95 | 41.08 |
| $X_1 \geq 2$ | 6     | 7     | 8    | 8    | 29    |
| BNB          | 6.23  | 9.05  | 7.37 | 8.52 | 31.17 |
| BCMPI        | 10.15 | 9.44  | 6.21 | 5.55 | 31.35 |
| BCMPII       | 7.93  | 10.42 | 7.17 | 6.17 | 31.69 |
| BGIT-I       | 6.85  | 8.62  | 6.53 | 8.18 | 30.19 |
| BGIT-II      | 6.44  | 9.57  | 7.55 | 7.24 | 30.80 |
| BGIT-I *     | 6.83  | 8.67  | 6.73 | 9.12 | 31.36 |
| BGIT-II *    | 6.38  | 9.76  | 7.72 | 7.04 | 30.90 |

\* pgf-based estimator, and others are fitted by MLE.

Table 8 summarizes the estimated parameters and goodness-of-fit results for the shunter accident data. Among the existing bivariate count models considered, the bivariate negative binomial model gives the smallest chi-square statistic, with  $\chi^2 = 1.75$  and  $p$ -value = 0.97, and also gives the smallest AIC and BIC values. This indicates that the bivariate negative binomial model provides the most favorable fit for this data set.

Nevertheless, the proposed BGIT models also provide good and competitive fits. Among the proposed models, BGIT-I performs particularly well. The MLE-based BGIT-I gives  $\chi^2 = 2.53$  and  $p$ -value = 0.92, with AIC = 696.10 and BIC = 712.92. The pgf-based BGIT-I\* gives a very similar goodness-of-fit result, with  $\chi^2 = 2.62$  and  $p$ -value = 0.92. These results show that BGIT-I can provide a close fit to the positively correlated over-dispersed shunter accident data. Therefore, although the bivariate negative binomial model gives the best overall fit, BGIT-I remains a competitive alternative among the models considered.

**Table 8.** Summary statistics for Table 7.

| Model  | Estimated Parameters                                                                                                                                   | $\chi^2$ | df | $p$ -Value | Log-likelihood Value | AIC    | BIC    |
|--------|--------------------------------------------------------------------------------------------------------------------------------------------------------|----------|----|------------|----------------------|--------|--------|
| BNB    | $\hat{p}_1 = 0.20,$<br>$\hat{p}_2 = 0.15,$<br>$\hat{p}_3 = 0.01,$<br>$\hat{v} = 3.88.$<br>$\hat{\lambda}_1 = 0.92,$<br>$\hat{v}_1 = 0.57,$             | 1.75     | 7  | 0.97       | −341.61              | 691.22 | 702.43 |
| BCMPI  | $\hat{\lambda}_2 = 0.73,$<br>$\hat{v}_2 = 0.53,$<br>$\hat{\alpha} = 0.58,$<br>$\hat{\beta} = 1.00$<br>$\hat{\lambda}_1 = 0.94,$<br>$\hat{v}_1 = 0.59,$ | 5.43     | 5  | 0.37       | −343.50              | 699.00 | 715.83 |
| BCMPII | $\hat{\lambda}_2 = 0.75,$<br>$\hat{v}_2 = 0.56,$<br>$\hat{\beta} = 1.00$<br>$\hat{p}_1 = 0.03$<br>$\hat{p}_2 = 0.04$                                   | 9.14     | 6  | 0.17       | −345.55              | 701.10 | 715.12 |
| BGIT-I | $\hat{p}_3 = 0.93$<br>$\hat{q}_1 = 0.26$<br>$\hat{q}_2 = 0.31$<br>$\hat{q}_3 = 0.43$                                                                   | 2.53     | 7  | 0.92       | −342.05              | 696.10 | 712.92 |

|          |                                   |      |   |      |         |        |        |
|----------|-----------------------------------|------|---|------|---------|--------|--------|
| BGIT-II  | Best $n_1 = 13$                   |      |   |      |         |        |        |
|          | Best $n_2 = 1$                    |      |   |      |         |        |        |
|          | $\hat{p}_1 = 0.87$                |      |   |      |         |        |        |
|          | $\hat{p}_2 = 0.02$                |      |   |      |         |        |        |
|          | $\hat{p}_3 = 0.11$                |      |   |      |         |        |        |
|          | $\hat{q}_1 = 0.57$                |      |   |      |         |        |        |
|          | $\hat{q}_2 = 0.05$                |      |   |      |         |        |        |
|          | $\hat{q}_3 = 0.37$                | 2.88 | 5 | 0.72 | -341.80 | 701.61 | 726.84 |
|          | $\hat{r}_1 = 0.44$                |      |   |      |         |        |        |
|          | $\hat{r}_2 = 0.15$                |      |   |      |         |        |        |
|          | $\hat{r}_3 = 0.41$                |      |   |      |         |        |        |
|          | Best $n_1 = 2$                    |      |   |      |         |        |        |
|          | Best $n_2 = 1$                    |      |   |      |         |        |        |
| BGIT-I*  | Best $n_3 = 1$                    |      |   |      |         |        |        |
|          | $\hat{p}_1 = 0.03$                |      |   |      |         |        |        |
|          | $\hat{p}_2 = 0.04$                |      |   |      |         |        |        |
|          | $\hat{p}_3 = 0.93$                |      |   |      |         |        |        |
|          | $\hat{q}_1 = 0.27$                | 2.62 | 7 | 0.92 | -       | -      | -      |
|          | $\hat{q}_2 = 0.32$                |      |   |      |         |        |        |
|          | $\hat{q}_3 = 0.41$                |      |   |      |         |        |        |
|          | Best $n_1 = 13$                   |      |   |      |         |        |        |
|          | Best $n_2 = 1$                    |      |   |      |         |        |        |
|          | $\hat{p}_1 = 0.91$                |      |   |      |         |        |        |
|          | $\hat{p}_2 = 5.38 \times 10^{-7}$ |      |   |      |         |        |        |
|          | $\hat{p}_3 = 0.09$                |      |   |      |         |        |        |
|          | $\hat{q}_1 = 0.54$                |      |   |      |         |        |        |
| BGIT-II* | $\hat{q}_2 = 0.13$                |      |   |      |         |        |        |
|          | $\hat{q}_3 = 0.32$                | 2.20 | 5 | 0.82 | -       | -      | -      |
|          | $\hat{r}_1 = 0.42$                |      |   |      |         |        |        |
|          | $\hat{r}_2 = 0.16$                |      |   |      |         |        |        |
|          | $\hat{r}_3 = 0.41$                |      |   |      |         |        |        |
|          | Best $n_1 = 3$                    |      |   |      |         |        |        |
|          | Best $n_2 = 1$                    |      |   |      |         |        |        |
|          | Best $n_3 = 1$                    |      |   |      |         |        |        |

Note. BGIT-I\* and BGIT-II\* denote the pgf-based estimates, while the other models are fitted by maximum-likelihood estimation. The degrees of freedom are computed after grouping the observed frequency table. Likelihood-based quantities such as log-likelihood, AIC, and BIC are not reported for pgf-based estimators because they are not obtained by maximizing a likelihood.

Table 9 reports the parametric bootstrap standard errors and 95% bootstrap confidence intervals for the proposed BGIT models fitted to the shunter accident data. For BGIT-I, the bootstrap standard errors are relatively small to moderate, and the bootstrap confidence intervals are reasonably narrow. This indicates that the probability parameters are estimated with acceptable precision. The pgf-based BGIT-I\* estimates show a similar pattern, although the bootstrap standard errors for the  $q$ -parameters are slightly larger than those obtained from MLE.

**Table 9.** Parametric bootstrap standard errors and confidence intervals for the shunter accident data.

| Model  | Parameter | Estimate | Bootstrap SE | 95% Bootstrap CI |
|--------|-----------|----------|--------------|------------------|
| BGIT-I | $p_1$     | 0.03     | 0.0110       | (0.0100, 0.0520) |
|        | $p_2$     | 0.04     | 0.0134       | (0.0184, 0.0698) |
|        | $p_3$     | 0.93     | 0.0161       | (0.8966, 0.9582) |

|          |       |                       |        |                                              |
|----------|-------|-----------------------|--------|----------------------------------------------|
|          | $q_1$ | 0.26                  | 0.0578 | (0.1314, 0.3635)                             |
|          | $q_2$ | 0.31                  | 0.0623 | (0.1849, 0.4102)                             |
|          | $q_3$ | 0.43                  | 0.0476 | (0.3563, 0.5395)                             |
| BGIT-I*  | $p_1$ | 0.03                  | 0.0137 | (0.0100, 0.0591)                             |
|          | $p_2$ | 0.04                  | 0.0168 | (0.0103, 0.0774)                             |
|          | $p_3$ | 0.93                  | 0.0207 | (0.8889, 0.9694)                             |
|          | $q_1$ | 0.27                  | 0.0773 | (0.0807, 0.3960)                             |
|          | $q_2$ | 0.32                  | 0.0873 | (0.1000, 0.4652)                             |
|          | $q_3$ | 0.41                  | 0.0659 | (0.3232, 0.5711)                             |
| BGIT-II  | $p_1$ | 0.87                  | 0.0416 | (0.7842, 0.9458)                             |
|          | $p_2$ | 0.01                  | 0.0613 | $(5.42 \times 10^{-7}, 0.1885)$              |
|          | $p_3$ | 0.11                  | 0.0571 | $(5.08 \times 10^{-6}, 0.1839)$              |
|          | $q_1$ | 0.57                  | 0.0671 | (0.4267, 0.6780)                             |
|          | $q_2$ | 0.05                  | 0.0862 | $(2.12 \times 10^{-6}, 0.2936)$              |
|          | $q_3$ | 0.37                  | 0.0611 | (0.1933, 0.4307)                             |
|          | $r_1$ | 0.44                  | 0.0623 | (0.3007, 0.5424)                             |
|          | $r_2$ | 0.15                  | 0.0818 | (0.0126, 0.3297)                             |
|          | $r_3$ | 0.41                  | 0.0517 | (0.2883, 0.4921)                             |
| BGIT-II* | $p_1$ | 0.91                  | 0.0345 | (0.8525, 0.9886)                             |
|          | $p_2$ | $5.38 \times 10^{-7}$ | 0.0082 | $(8.87 \times 10^{-8}, 2.00 \times 10^{-6})$ |
|          | $p_3$ | 0.09                  | 0.0346 | (0.0103, 0.1463)                             |
|          | $q_1$ | 0.54                  | 0.0741 | (0.4014, 0.6809)                             |
|          | $q_2$ | 0.13                  | 0.0879 | $(2.15 \times 10^{-6}, 0.3121)$              |
|          | $q_3$ | 0.32                  | 0.0700 | (0.1676, 0.4325)                             |
|          | $r_1$ | 0.42                  | 0.0681 | (0.2995, 0.5648)                             |
|          | $r_2$ | 0.16                  | 0.0828 | $(2.97 \times 10^{-6}, 0.3305)$              |
|          | $r_3$ | 0.41                  | 0.0586 | (0.2839, 0.5108)                             |

For BGIT-II and BGIT-II\*, the bootstrap results also show reasonable stability for most parameters. However, some small probability estimates, particularly  $p_2$ , have wider relative uncertainty. This is expected when fitted probabilities are close to the lower boundary of the parameter space. The remaining parameters have moderate bootstrap standard errors and confidence intervals that are consistent with the fitted estimates. Overall, the bootstrap results support the numerical stability of the proposed BGIT fits for the shunter accident data, with BGIT-I showing the more stable parameter estimates among the proposed models.

## 7.2. Critical Care and Emergency Room Data

For the critical care and emergency room data, the observed frequency table was grouped to ensure that the expected frequencies were adequate for the chi-square goodness-of-fit assessment. Table 10 reports the grouped observed frequencies together with the fitted frequencies from the competing models.

**Table 10.** Grouped observed and fitted frequencies for critical care and emergency room data.

|              | $X_2 \leq 1$ | $X_2 = 2$ | $X_2 = 3$ | $X_2 \geq 4$ |
|--------------|--------------|-----------|-----------|--------------|
| $X_1 \leq 1$ | 1            | 4         | 6         | 4            |
| BCMPI        | 1.31         | 3.76      | 4.63      | 4.44         |
| BCMPII       | 1.20         | 3.74      | 4.70      | 4.50         |
| BGIT-I       | 1.72         | 3.69      | 5.60      | 12.91        |
| BGIT-II      | 1.78         | 3.77      | 4.98      | 5.41         |
| BGIT-I*      | 1.23         | 2.86      | 4.69      | 12.50        |
| BGIT-II*     | 1.66         | 3.61      | 4.88      | 5.47         |

|              |       |       |       |       |
|--------------|-------|-------|-------|-------|
| $X_1 = 2$    | 3     | 10    | 13    | 14    |
| BCMPI        | 3.72  | 10.64 | 13.04 | 12.41 |
| BCMPII       | 3.70  | 10.67 | 13.06 | 12.38 |
| BGIT-I       | 3.71  | 6.92  | 9.17  | 15.40 |
| BGIT-II      | 4.00  | 8.49  | 11.23 | 12.20 |
| BGIT-I*      | 2.89  | 5.88  | 8.42  | 16.13 |
| BGIT-II*     | 3.67  | 7.99  | 10.81 | 12.11 |
| $X_1 = 3$    | 5     | 12    | 11    | 17    |
| BCMPI        | 4.62  | 13.13 | 16.02 | 15.17 |
| BCMPII       | 4.68  | 13.15 | 15.99 | 15.11 |
| BGIT-I       | 5.68  | 9.23  | 10.71 | 13.60 |
| BGIT-II      | 5.40  | 11.46 | 15.15 | 16.47 |
| BGIT-I*      | 4.77  | 8.48  | 10.63 | 15.22 |
| BGIT-II*     | 5.04  | 10.97 | 14.83 | 16.63 |
| $X_1 \geq 4$ | 6     | 15    | 15    | 16    |
| BCMPI        | 4.56  | 12.87 | 15.62 | 16.05 |
| BCMPII       | 4.61  | 12.85 | 15.57 | 16.07 |
| BGIT-I       | 13.27 | 15.68 | 13.74 | 10.96 |
| BGIT-II      | 5.76  | 12.21 | 16.14 | 17.54 |
| BGIT-I*      | 12.93 | 16.49 | 15.42 | 13.47 |
| BGIT-II*     | 5.77  | 12.56 | 16.97 | 19.03 |

Table 11 summarizes the estimated parameters and goodness-of-fit results for the critical care and emergency room data. Among the existing bivariate count models considered, BCMPI and BCMPII give the smallest chi-square statistics, with  $\chi^2 = 3.32$  and  $\chi^2 = 3.24$ , respectively, and large  $p$ -values. These results indicate that BCMPI-type models provide the best fits for this data set.

Among the proposed BGIT models, BGIT-II clearly provides the better fit. The MLE-based BGIT-II gives  $\chi^2 = 4.04$  and  $p$ -value = 0.91, while the pgf-based BGIT-II\* gives  $\chi^2 = 4.44$  and  $p$ -value = 0.88. In contrast, BGIT-I and BGIT-I\* give larger chi-square statistics and borderline  $p$ -values. These results show that BGIT-II provides a good, competitive fit to the under-dispersed data set with near-zero dependence. Therefore, although BCMPI and BCMPII give the best overall fits, BGIT-II remains a useful alternative and performs substantially better than BGIT-I for this data set.

**Table 11.** Summary statistics for Table 10.

| Model  | Estimated Parameters                                                                                                                                          | $\chi^2$ | df | $p$ -Value | Loglikelihood Value | AIC     | BIC     |
|--------|---------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|----|------------|---------------------|---------|---------|
| BCMPI  | $\hat{\lambda}_1 = 14.50,$<br>$\hat{\nu}_1 = 2.25,$<br>$\hat{\lambda}_2 = 14.01,$<br>$\hat{\nu}_2 = 2.21,$<br>$\hat{\alpha} = 0.10,$<br>$\hat{\beta} = -1.00$ | 3.24     | 9  | 0.95       | −486.80             | 985.59  | 985.59  |
| BCMPII | $\hat{\lambda}_1 = 14.52,$<br>$\hat{\nu}_1 = 2.25,$<br>$\hat{\lambda}_2 = 14.00,$<br>$\hat{\nu}_2 = 2.21,$<br>$\hat{\beta} = -1.00$                           | 3.32     | 10 | 0.97       | −486.95             | 983.89  | 983.89  |
| BGIT-I | $\hat{p}_1 = 0.25$<br>$\hat{p}_2 = 0.25$<br>$\hat{p}_3 = 0.50$                                                                                                | 20.14    | 11 | 0.04       | −502.75             | 1017.49 | 1035.64 |

Note. BGIT-I\* and BGIT-II\* denote the pgf-based estimates, while the other models are fitted by maximum-likelihood estimation. The degrees of freedom are computed after grouping the observed frequency table. Likelihood-based quantities such as log-likelihood, AIC, and BIC are not reported for the pgf-based estimators because these estimators are not obtained by maximizing a likelihood.

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**Table 12.** Parametric bootstrap standard errors and confidence intervals for the critical care and emergency room data.

| Model    | Parameter | Estimate              | Bootstrap SE          | 95% Bootstrap CI                             |
|----------|-----------|-----------------------|-----------------------|----------------------------------------------|
| BGIT-I   | $p_1$     | 0.25                  | 0.0103                | (0.2314, 0.2708)                             |
|          | $p_2$     | 0.25                  | 0.0102                | (0.2292, 0.2692)                             |
|          | $p_3$     | 0.50                  | 0.0119                | (0.4770, 0.5225)                             |
|          | $q_1$     | $6.15 \times 10^{-6}$ | $2.74 \times 10^{-6}$ | $(6.15 \times 10^{-6}, 1.48 \times 10^{-5})$ |
|          | $q_2$     | $6.15 \times 10^{-6}$ | 0.0087                | $(6.15 \times 10^{-6}, 1.42 \times 10^{-5})$ |
|          | $q_3$     | 0.99999               | 0.0087                | (0.99998, 0.99999)                           |
| BGIT-I*  | $p_1$     | 0.26                  | 0.0149                | (0.2323, 0.2839)                             |
|          | $p_2$     | 0.26                  | 0.0150                | (0.2242, 0.2813)                             |
|          | $p_3$     | 0.48                  | 0.0188                | (0.4540, 0.5295)                             |
|          | $q_1$     | $6.15 \times 10^{-6}$ | 0.0616                | $(5.19 \times 10^{-6}, 0.2039)$              |
|          | $q_2$     | $6.15 \times 10^{-6}$ | 0.0662                | $(6.15 \times 10^{-6}, 0.2541)$              |
|          | $q_3$     | 0.99999               | 0.0950                | (0.5952, 0.99999)                            |
| BGIT-II  | $p_1$     | 0.99999               | 0.0039                | (0.9835, 0.99999)                            |
|          | $p_2$     | $1.00 \times 10^{-8}$ | $2.94 \times 10^{-8}$ | $(1.00 \times 10^{-8}, 1.21 \times 10^{-7})$ |
|          | $p_3$     | $6.15 \times 10^{-6}$ | 0.0039                | $(6.15 \times 10^{-6}, 0.0165)$              |
|          | $q_1$     | 0.43                  | 0.0216                | (0.3835, 0.4694)                             |
|          | $q_2$     | 0.53                  | 0.0385                | (0.4490, 0.6013)                             |
|          | $q_3$     | 0.05                  | 0.0233                | $(3.69 \times 10^{-6}, 0.0921)$              |
|          | $r_1$     | 0.50                  | 0.0168                | (0.4693, 0.5362)                             |
|          | $r_2$     | 0.49                  | 0.0168                | (0.4626, 0.5307)                             |
|          | $r_3$     | $3.10 \times 10^{-6}$ | 0.0011                | $(3.11 \times 10^{-6}, 9.36 \times 10^{-6})$ |
| BGIT-II* | $p_1$     | 0.99999               | 0.0109                | (0.9564, 0.99999)                            |
|          | $p_2$     | $1.00 \times 10^{-8}$ | 0.0031                | $(1.00 \times 10^{-8}, 3.47 \times 10^{-7})$ |
|          | $p_3$     | $6.15 \times 10^{-6}$ | 0.0105                | $(6.15 \times 10^{-6}, 0.0391)$              |
|          | $q_1$     | 0.49                  | 0.0277                | (0.4581, 0.5683)                             |
|          | $q_2$     | 0.51                  | 0.0554                | (0.3261, 0.5419)                             |
|          | $q_3$     | $3.12 \times 10^{-6}$ | 0.0343                | $(3.09 \times 10^{-6}, 0.1245)$              |
|          | $r_1$     | 0.49                  | 0.0282                | (0.4595, 0.5685)                             |
|          | $r_2$     | 0.50                  | 0.0559                | (0.3326, 0.5405)                             |
|          | $r_3$     | $3.10 \times 10^{-6}$ | 0.0343                | $(3.09 \times 10^{-6}, 0.1134)$              |

For BGIT-II, the fitted  $p$ -parameters is close to the boundary, with  $p_1$  close to one and  $p_2$  and  $p_3$  close to zero. In contrast, the main  $q$ - and  $r$ -parameters are estimated with relatively small bootstrap standard errors under the MLE-based BGIT-II fit. The pgf-based BGIT-II\* fit shows a similar pattern, although some  $q$ - and  $r$ -parameters have larger bootstrap standard errors than those from the MLE-based BGIT-II fit.

Overall, although boundary estimates are observed in both BGIT-I and BGIT-II, the bootstrap results support the goodness-of-fit findings. BGIT-II provides a more appropriate BGIT fit for the critical care and emergency room data, with smaller bootstrap standard errors for several of the main  $q$ - and  $r$ -parameters and substantially better goodness-of-fit than BGIT-I.

## 8. Conclusions

In this paper, we extended the  $\text{GIT}_{3,1}$  distribution to the bivariate setting and proposed two flexible bivariate count distributions, namely BGIT-I and BGIT-II. The proposed models retain the simple, tractable pgf structure of the GIT family while allowing different marginal dispersion behaviors and dependence structures.

BGIT-I combines bivariate binomial and bivariate negative binomial components, allowing negative, near-zero, and positive dependence. In contrast, BGIT-II induces non-negative dependence through a common component. Thus, the two models provide complementary alternatives for bivariate count data. The positive integer parameters also provide flexibility in controlling the shape, location, and spread of the joint distribution. For limitations of bivariate distributions based on trivariate reduction, see [27,11].

Both maximum-likelihood estimation and pgf-based estimation were considered. The pgf-based estimator provides a computationally attractive alternative to MLE because the proposed BGIT models have tractable pgfs, whereas their pmfs are more complicated to evaluate. Real-data applications show that the proposed BGIT models provide competitive fits to bivariate count data with varying dispersion and dependence structures. In particular, BGIT-I performs well for the positively correlated over-dispersed data, while BGIT-II provides a good fit for the under-dispersed data with near-zero dependence. Although existing models, including the bivariate Conway–Maxwell–Poisson models in [12], may provide the best fit in some cases, the BGIT models remain useful alternatives because they offer flexible dependence structures and simple pgf forms.

Parametric bootstrap results provide uncertainty quantification for the fitted BGIT models, and the Monte Carlo simulation study further supports the finite-sample performance of the proposed estimators. Future research will aim to relax the integer constraint to simplify estimation and support further extensions. Statistical properties may be studied through Monte Carlo simulation, with the BGIT samples being easily generated by the methods examined by [28].

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