

Article

Modeling Healthcare Data with Logistic Quantile and Uniform-Based Mixture Polynomial Distributions

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Abstract

Continuous healthcare data often deviate from normality, which can substantially increase the risk of making invalid inferences, given that many inferential statistical procedures rely on normality assumption. To obviate this issue, we propose a new family of non-normal distributions based on a linear combination of the quantile functions of standard logistic and uniform (0, 1) distributions. This new family of non-normal distributions is studied within three different methods: *L*-moments, conventional moments, and percentiles. Its performance is compared among the three methods in the context of parameter estimation and data modeling. The results of Monte Carlo simulation and bootstrapping techniques indicate that the *L*-moment-based estimates of parameters of *L*-skewness and *L*-kurtosis are substantially less biased than their percentile-based estimates of left-right tail-weight ratio (a measure of skewness) and tail-weight factor (a measure of kurtosis), which in turn are superior to their moment-based counterparts of skewness and kurtosis, especially for small sample sizes and higher-order moments. On the other hand, the data modeling results indicate that the percentile-based fits of the proposed distributions provide slightly better approximations to real-world healthcare data than their *L*-moment-based counterparts, whereas both percentile- and *L*-moment-based methods are superior to their conventional moment-based counterparts.

Keywords: logistic quantile; polynomial mixture; non-normal distribution; healthcare data; Monte Carlo simulation; estimation

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1. Introduction

Real-world healthcare data (e.g., hospital charges, length of hospital stays, cholesterol, glucose, blood pressure, body mass index [BMI], waist circumference, ankle circumference, chest circumference, etc.) often exhibit non-normal characteristics (e.g., skewness, heavy tails, peaked at the center of the distribution), which may pose challenges to the classical data modeling approaches and inferential statistical procedures, such as *t*-tests, analysis of variance (ANOVA), and linear regression that rely on normality assumption of the outcome variable.

Many applied and methodological research studies demonstrate that continuous healthcare and biomedical data frequently deviate from the well-known normal distribution, suggesting the use of data transformations, non-parametric techniques, or

alternative modeling strategies. Empirical studies and reviews (e.g., [1,2]) show that clinical variables such as laboratory biomarkers, symptom scores, healthcare costs, and length of hospital stays are often skewed or heavy-tailed, rendering normality assumptions inappropriate. Systematic reviews (e.g., [3]) further suggest that non-normal distributions are typical rather than the exception across health and social science research, with right skewness and excess kurtosis being particularly prevalent. Applied methodological papers (e.g., [4,5]) highlight that reliance on mean and standard deviation for describing non-normal data can lead to misleading interpretation and inferential errors. In health economics and outcomes research (e.g., [6,7]), healthcare cost and healthcare utilization data consistently exhibit strong positive skew and non-normal variance structures, which motivates the use of generalized linear and non-parametric models.

The quest for developing new probability distributions to provide accurate modeling of non-normal data is not new. Historically, a variety of frameworks or systems have been developed as a contribution to the long-standing stream of research for modeling non-normal distributions. Some of the well-known historical contributions to modeling non-normal distributions with skewness and kurtosis include studies such as Refs. [8,9] introduced a differential equation-based framework to systematically classify, and model skewed and kurtotic distributions. Ref. [10] proposed a transformation-based distribution system that generates flexible skewed distributions through parametric transformations of normal distribution. Ref. [11] introduced the Box–Cox transformation to normalize non-normal data via power transformation. Ref. [12], later expanded by [13], proposed the *g*-and-*h* family of distributions, a quantile-based framework that allows independent control of skewness and tail weight. Ref. [14] developed a cubic polynomial transformation method to generate non-normal distributions from a standard normal variable, which was later extended by [15] to simulate correlated multivariate non-normal distributions. Ref. [16] introduced the skew-normal distribution, the first formal skew-elliptical family preserving mathematical tractability. Ref. [17] proposed the skewed *t* distribution for financial modeling, while [18] introduced skewed distributions via scale mixtures for Bayesian modeling of skewness and heavy tails. Ref. [19] introduced a family of power transformations to reduce skewness and make data closer to normal or symmetric, a feature like Box–Cox transformation but more flexible. Finally, Ref. [20] extended Fleishman’s framework with a fifth-order polynomial transformation, expanding the feasible skewness–kurtosis region.

Although each of these frameworks or distributions possesses distinct properties and can model specific form of non-normality, no single distribution is universally applicable to model all types of non-normal data. As a result, researchers continue to develop new families of distributions tailored to specific data characteristics, such as high skewness, heavy tails, and bimodality. Selected examples of recent contributions to the field of non-normal distributions include studies such as the ‘sinh-arcsinh transformation’ proposed by [21] as a four-parameter distribution that includes symmetric and skewed forms with a variety of tail weights. Ref. [22] proposed a robust mixture modeling framework based on multivariate *t* distribution with Box–Cox transformation, which can handle skewness, outliers, and data transformation. Ref. [23] proposed the Lambert $W \times F$ distribution that can help normalize heavy-tailed data.

Mixture polynomials based on quantile functions (qfs) of well-known probability distributions (e.g., normal, Cauchy) have emerged as promising alternative techniques for modeling non-normal data, particularly in applications where skewness, tail behavior, presence of outliers, and distributional heterogeneity are of concern. Ref. [24] introduced two families of mixture polynomials known as quantile mixtures of normal quantile and Cauchy quantile, respectively, studied through the methods of *L*-moments [25] and trimmed *L*-moments [26], and provided examples of their applications in modeling non-

normal continuous data, especially in the field of daily stock returns. Mixture polynomials defined in [24] as linear combinations of qfs of two distributions (e.g., normal and uniform)—referred to as quantile mixtures—offer a flexible alternative to density-based mixture models, especially when likelihood-based inference becomes intractable or unstable.

The normal-quantile and uniform (NQU) mixture polynomial (e.g., [24]) can yield a wide variety of non-normal probability distributions, which can provide flexible yet smooth approximations to non-normal continuous data. Despite their potential, these NQU mixture polynomials have their own limitations. One of the limitations of NQU mixture polynomial distributions is that they may not provide accurate fits to some non-normal data. For example, Figure 1 (panels a and c) shows two L -moment-based NQU pdfs superimposed, respectively, over the Student's $t_{(df=7)}$ distribution and Amazon (AMZN) stock daily return rate (percent change) data for 10 years between 18 August 2014 and 16 August 2024, downloaded from the website <https://www.nasdaq.com/market-activity/stocks/amzn/historical> (accessed on 17 August 2024). The AMZN stock daily return rate data were also used in [27].

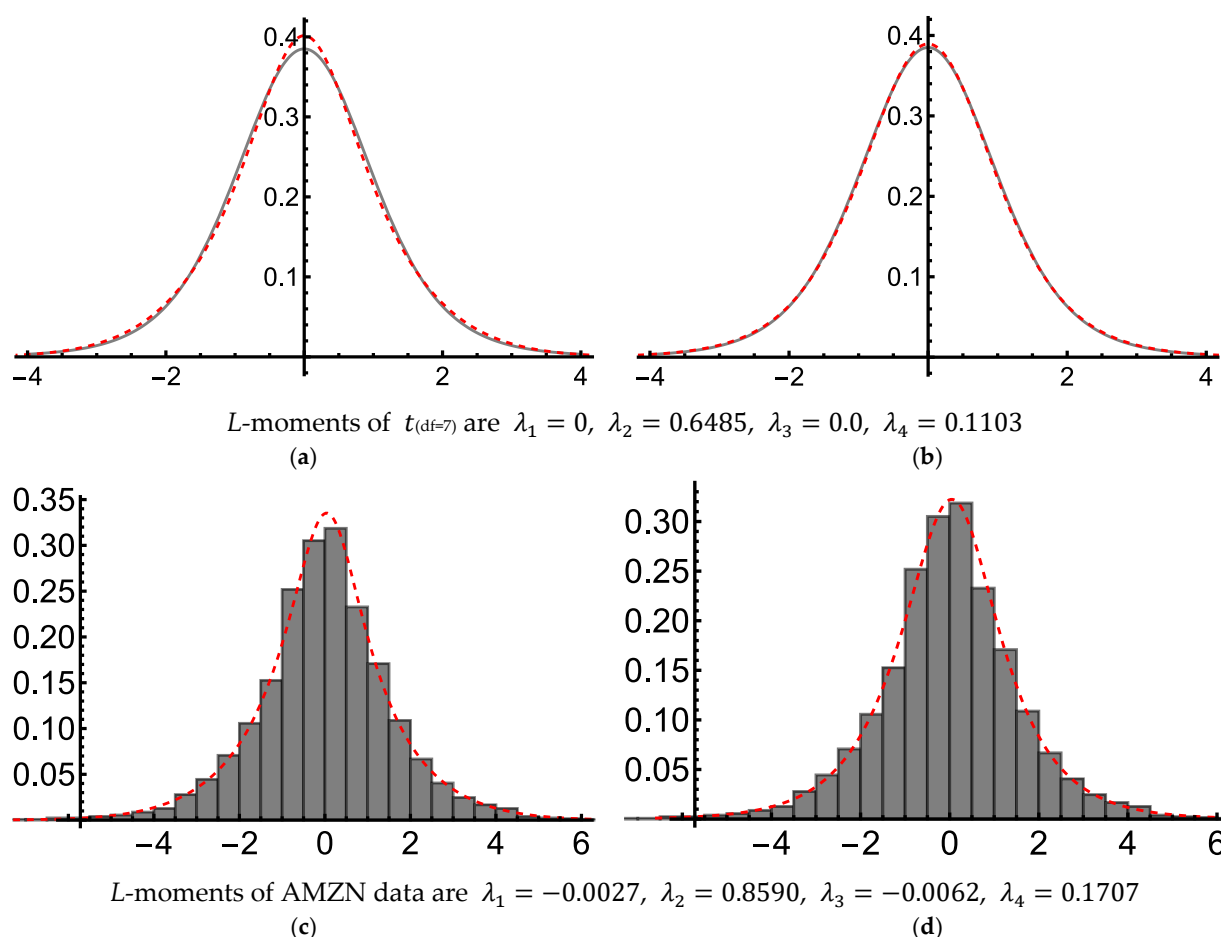


Figure 1. L -moment-based NQU (panels (a,c)) and LQU (panels (b,d)) pdfs superimposed over the Student's $t_{(df=7)}$ distribution (panels (a,b)) and AMZN stock daily return rate data (panels (c,d)). Also given are the first four L -moments ($\lambda_{1,2,3,4}$) for the $t_{(df=7)}$ distribution and AMZN stock daily return rate data, respectively.

Also provided in Figure 1 (panels b and d) are two L -moment-based pdfs of logistic-quantile and uniform (LQU) mixture polynomial distribution—the proposed family of distributions in this paper—superimposed over the two distributions, the Student's $t_{(df=7)}$ distribution and AMZN stock daily return rate data. Figure 1 shows that the LQU pdfs (panels b and d) provide a better fit for the reference distributions than the NQU pdfs

(panels a and c). Consistent with this, Table 1 shows that the percentile estimates derived from the LQU pdfs are consistently closer to the corresponding parameter values than those derived from the NQU pdfs. Similarly, the smaller Euclidean distance (d) values for the LQU fits indicate greater similarity to the reference distributions than the NQU fits.

Table 1. Percentile parameters for the $t_{(df=7)}$ distribution and AMZN stock daily return rate data in Figure 1 with model parameters and percentile estimates based on NQU and LQU approximations, respectively.

$t_{(df=7)}$		NQU		LQU	
Percentile Parameters (P)	Model Parameters	Percentile Estimates (E)	Model Parameters	Percentile Estimates (E)	
0.05	−1.8946	0.05	−1.9446	0.05	−1.9126
0.10	−1.4149	0.10	−1.4408	0.10	−1.4222
0.25	−0.7111	$\alpha_0 = 0.7524$	0.25	−0.6989	$\alpha_0 = 0.0391$
0.50	0.0000	$\alpha_1 = −1.5048$	0.50	0.0000	$\alpha_1 = −0.0782$
0.75	0.7111	$\alpha_2 = 0.0$	0.75	0.6989	$\alpha_2 = 0.0$
0.90	1.4149	$\alpha_3 = 1.5939$	0.90	1.4408	$\alpha_3 = 0.6615$
0.95	1.8956		0.95	1.9446	
$d = 0.0809$			$d = 0.0274$		
AMZN Data		NQU		LQU	
Percentile Parameters (P)	Model Parameters	Percentile Estimates (E)	Model Parameters	Percentile Estimates (E)	
0.05	−2.6082	0.05	−2.6439	0.05	−2.5944
0.10	−1.8864	0.10	−1.8994	0.10	−1.8707
0.25	−0.8552	$\alpha_0 = 1.5676$	0.25	−0.8632	$\alpha_0 = 0.4630$
0.50	0.0021	$\alpha_1 = −3.0174$	0.50	0.0127	$\alpha_1 = −0.8081$
0.75	0.9048	$\alpha_2 = −0.1848$	0.75	0.8655	$\alpha_2 = −0.1848$
0.90	1.8238	$\alpha_3 = 2.4684$	0.90	1.8656	$\alpha_3 = 1.0245$
0.95	2.5721		0.95	2.5944	
$d = 0.0735$			$d = 0.0509$		

Note: The Euclidean distance $d = \sqrt{\sum (E - P)^2}$ was used as a similarity index to measure similarity between percentile parameters (P) and corresponding estimates (E).

Another limitation of quantile mixtures is that they have not received enough attention in the fields of healthcare, biomedical sciences or public health for modeling non-normal data, particularly in conjunction with logistic-quantile framework.

In this paper, we propose a new class of probability distributions, known henceforth as logistic-quantile and uniform (LQU) mixture polynomial distribution, by combining a logistic-qf with a quadratic polynomial of uniform (0, 1)-qf to solve a problem of accurately modeling non-normal healthcare data. The proposed class of distributions incorporates flexibility of logistic distribution and simplicity of uniform distribution, yielding a coherent modeling framework capable of accommodating features such as skewness and heavy tails of non-normal data. We study the proposed distributions and compare their data modeling capabilities using three methods, namely, the method of L -moments (MoLM) [25], method of percentiles (MoP) [28], and method of moments (MoM) [29]. Also, we discuss parameter estimation for each method and illustrate their performance by modeling real-world continuous healthcare data.

The rest of the paper is organized as follows: In Section 2, we provide an overview of theoretical background by defining terminology used in the methodology. In Section 3, we present a methodology to study LQU mixture polynomial distributions through the methods of MoLM and MoP and derive systems of equations for solving model

parameters that control location, scale, and shape of the LQU distribution for each framework. In Appendix A, we detail a methodology to study LQU mixture polynomial distributions using the MoM-based framework and derive a system of equations for mean, standard deviation, skewness, and kurtosis. In Section 4, we describe the steps for implementing the proposed methodology in the context of parameter estimation and modeling healthcare data and provide results of Monte Carlo simulation and bootstrapping technique. In Section 5, we discuss the results. In Section 6, we provide concluding remarks.

2. Theoretical Background

2.1. Standard Logistic Distribution

Let Y be a random variable from a standard logistic distribution (with location and scale parameters of 0 and 1, respectively), then its probability density function (pdf) and cumulative distribution function (cdf) can be given, respectively, as [30] follows:

$$f(y) = \frac{e^{-y}}{(1+e^{-y})^2} \quad (1)$$

$$F(y) = \frac{1}{1+e^{-y}} \quad (2)$$

where $y \in (-\infty, \infty)$ and $e = 2.718$ (rounded to 3 decimal places) is Euler's number. The qf (i.e., the inverse cdf) of this standard logistic distribution is given by

$$Q_L(u) = \ln\left(\frac{u}{1-u}\right) \quad (3)$$

where $\ln(\cdot)$ denotes the natural logarithm (i.e., logarithm to the base e) and u is a realization of random variable U , also theqf, of uniform (0, 1) distribution. Although both the standard normal ($\mu = 0$, $\sigma = 1$) and standard logistic (location $\mu = 0$, scale $s = 1$) distributions are bell-shaped and symmetric about their mean ($\mu = 0$), the latter has fatter tails and shorter peak height (at $\mu = 0$) than the former (see Figure 2). To compare the differences in the tail region of the two distributions (standard normal and standard logistic), Table 2 shows their percentiles.

Table 2. Standard normal vs. standard logistic distribution.

Standard Normal ($\mu = 0$, $\sigma = 1$)			Standard Logistic (Location $\mu = 0$, Scale $s = 1$)		
Mean (μ)	SD (σ)	Peak Height	Mean (μ)	SD (σ)	Peak Height
0	1	0.3989	0	$\pi/\sqrt{3}$	0.25
Percentiles			Percentiles		
0.01		−2.3263	0.01		−4.5951
0.05		−1.6449	0.05		−2.9444
0.10		−1.2816	0.10		−2.1972
0.25		−0.6745	0.25		−1.0986
0.50		0.0000	0.50		0.0000
0.75		0.6745	0.75		1.0986
0.90		1.2816	0.90		2.1972
0.95		1.6449	0.95		2.9444
0.99		2.3263	0.99		4.5951

Note: SD = standard deviation.

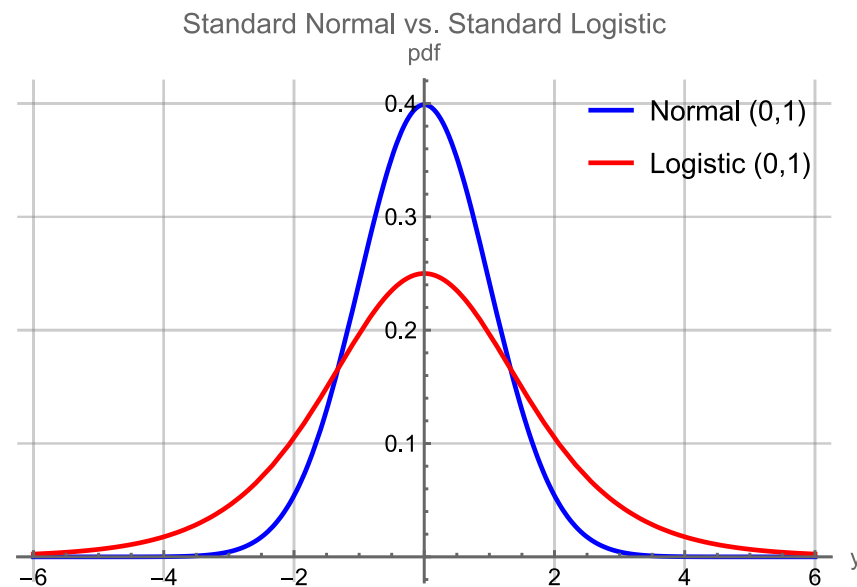


Figure 2. Standard normal distribution ($\mu = 0$, $\sigma = 1$) versus standard logistic distribution (location $\mu = 0$, scale $s = 1$).

2.2. Method of L-Moments (MoLM)

The method of L -moments (MoLM), introduced by [25], has been widely applied in modeling non-normal distributions primarily in hydrology and agriculture. Let Y be a continuous random variable with cumulative distribution function (cdf), $F(y) = u$, and qf, $Q(u)$, then its first four theoretical L -moments can be defined as [24,25] follows:

$$\lambda_1 = \int_0^1 Q(u) du \quad (4)$$

$$\lambda_2 = \int_0^1 Q(u)(2u - 1) du \quad (5)$$

$$\lambda_3 = \int_0^1 Q(u)(6u^2 - 6u + 1) du \quad (6)$$

$$\lambda_4 = \int_0^1 Q(u)(20u^3 - 30u^2 + 12u - 1) du \quad (7)$$

The values of λ_1 and λ_2 represent the location and scale parameters of the distribution. Specifically, λ_1 is the arithmetic mean, referred to as the L -mean and $\lambda_2 (\geq 0)$ denotes the L -scale, which is one-half the Gini's coefficient of mean difference (pp. 47–48, [29]). The higher order L -moments, λ_3 and λ_4 , can be converted into dimensionless L -moment ratios known as L -skewness (τ_3) and L -kurtosis (τ_4), as follows:

$$\tau_3 = \lambda_3 / \lambda_2 \quad (8)$$

$$\tau_4 = \lambda_4 / \lambda_2 \quad (9)$$

The L -moment-based methodology has been used in a variety of fields such as hydrology and meteorology (e.g., [31–38]), engineering (e.g., [39,40]), extreme value analysis (e.g., [41]), stock price modeling (e.g., [24]), and malware detection (e.g., [42]).

The L -moment-based methodology is widely recognized as a more robust and reliable alternative to conventional moment-based approach and, in many cases, maximum likelihood estimation—specifically for small sample sizes and distributions that deviate substantially from normality (e.g., [24–26,31]). Because of this advantage, several theoretical and applied studies have been conducted to compare the performance of L -moments and L -correlation with that of conventional moments and Pearson correlation in the context of various non-normal distributions. For example, Ref. [43] developed an L -moment-

based characterization of the third- and fifth-order power method distributions. Ref. [44] proposed an L -moment-based doubling method for the family of generalized lambda distribution. Ref. [45] extended this L -moment-based doubling technique for power method distributions using standard normal and logistic variables. Ref. [46] proposed an L -moment-based double power method distribution using uniform and triangular variables. Refs. [47,48] studied L -moment-based Burr (Type III and Type XII) and Dagum distributions, respectively. See [49–54] for additional L -moment-based studies.

2.3. Method of Percentiles (MoP)

Let Y be a continuous random variable with qf, $Q(u)$, as in Equation (3), then the method of percentiles (MoP)-based analogs of location, scale, skew function, and kurtosis function associated with Y are respectively defined by its median (ξ_1), inter-decile range (ξ_2), left–right tail-weight ratio (ξ_3 , a skew function), and tail-weight factor (ξ_4 , a kurtosis function), which can be written [55,56] as follows:

$$\xi_1 = Q(u)_{u=0.5} \quad (10)$$

$$\xi_2 = Q(u)_{u=0.9} - Q(u)_{u=0.1} \quad (11)$$

$$\xi_3 = (Q(u)_{u=0.5} - Q(u)_{u=0.1}) / (Q(u)_{u=0.9} - Q(u)_{u=0.5}) \quad (12)$$

$$\xi_4 = (Q(u)_{u=0.75} - Q(u)_{u=0.25}) / (Q(u)_{u=0.9} - Q(u)_{u=0.1}) \quad (13)$$

Some examples of studies that have used the percentile-based methods are the method of percentile-based generalized lambda distribution [28], the Tukey g -and- h distributions [57], the standard normal-based power method distributions [58], the Tukey distributions [59]. The MoP-based methodology has not received enough attention in the fields of healthcare, biomedical sciences, and public health.

3. Methodology

3.1. Logistic-Quantile and Uniform Mixture Polynomial

The logistic-quantile and uniform (LQU) mixture polynomial distribution in the context of this paper is defined as [24] follows:

$$Q_{LQU}(u) = \alpha_3 Q_L(u) + \{\alpha_2 u^2 + \alpha_1 u + \alpha_0\} \quad (14)$$

where $u \sim \text{Uniform}(0, 1)$ is a random variable on the interval $(0, 1)$, $Q_L(u)$ is the qf of the standard logistic distribution, as defined in Equation (3), α_0 , α_1 , α_2 , and α_3 are the real-valued model parameters that control the location, scale, and shape of the resulting LQU distribution.

3.2. The Probability Density Function (pdf) and Cumulative Distribution Function (cdf)

The pdf and cdf of the LQU mixture polynomial distribution in Equation (14) can be defined in parametric forms [24] as follows:

$$f(Q_{LQU}(u)) = 1/Q'_{LQU}(u) \quad (15)$$

$$F(Q_{LQU}(u)) = u \quad (16)$$

where $Q'_{LQU}(u)$ on the right-hand side of Equation (15) is the first derivative of the LQU mixture polynomial in Equation (14).

Proof. Let $Q_{LQU}(u)$, for all $u \in (0, 1)$, in Equation (14) be the qf of a continuous probability distribution with cdf $F(x)$ and pdf $f(x)$. Let $F(x)$ be strictly increasing and differentiable so that $Q_{LQU}(u) = F^{-1}(u)$ is differentiable.

From the inverse relationship between the cdf and the qf, it satisfies $F(Q_{LQU}(u)) = u$, which is the cdf in Equation (16).

Differentiating both sides of Equation (16) with respect to u and applying the chain rule yields the following equation:

$$F'(Q_{LQU}(u)) Q'_{LQU}(u) = 1.$$

Since $F'(x) = f(x)$, the above equation can be written as $f(Q_{LQU}(u)) Q'_{LQU}(u) = 1$, which can be simplified as

$$f(Q_{LQU}(u)) = 1/Q'_{LQU}(u),$$

which is the pdf in Equation (15). $\square$

3.3. Parameter Space for the Model Parameters of LQU Distribution

To ensure that the LQU mixture polynomial, $Q_{LQU}(u)$, in Equation (14) is a valid qf, it must be monotonically increasing. Using this property, the parameter space for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of LQU distribution was derived as $\mathcal{P} = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4: \alpha_3 \geq 0, \alpha_1 + 2\alpha_2 u + \alpha_3/(u(1-u)) \geq 0 \forall u \in (0, 1)\}$. See Appendix B for the steps used to derive this parameter space.

3.4. LQU Distribution via Method of L-Moments (MoLM)

The first four L -moments [25] of the proposed LQU mixture polynomial distribution can be derived by integrating Equations (4)–(7) after replacing $Q(u)$ with $Q_{LQU}(u)$ of Equation (14) on the right-hand sides of Equations (4)–(7). Thus, the simplified forms of the first four L -moments of the LQU mixture polynomial distribution can be written as follows:

$$\lambda_1 = \alpha_0 + (3\alpha_1 + 2\alpha_2)/6 \quad (17)$$

$$\lambda_2 = \alpha_3 + (\alpha_1 + \alpha_2)/6 \quad (18)$$

$$\lambda_3 = \alpha_2/30 \quad (19)$$

$$\lambda_4 = \alpha_3/6 \quad (20)$$

Substituting Equations (18)–(20) into Equations (8) and (9), the simplified formulae for L -skewness (τ_3) and L -kurtosis (τ_4) can be subsequently written as follows:

$$\tau_3 = \alpha_2/(5(\alpha_1 + \alpha_2 + 6\alpha_3)) \quad (21)$$

$$\tau_4 = \alpha_3/(\alpha_1 + \alpha_2 + 6\alpha_3) \quad (22)$$

The L -moment-based expressions in Equations (17)–(22) provide a clear interpretation of the model parameters. For example, the parameter α_0 controls the location, as it appears only in Equation (17), the expression for λ_1 . The parameter α_1 primarily governs the L -scale through λ_2 , in Equation (18), without directly affecting the higher-order L -moments. The parameter α_2 uniquely determines the third L -moment, λ_3 , and hence controls the skewness, while α_3 uniquely determines the fourth L -moment, λ_4 , and thus controls the tail weight (L -kurtosis). Accordingly, the proposed LQU distribution provides a nearly orthogonal parameterization, with α_0 , α_1 , α_2 , and α_3 governing the location, scale, skewness, and tail weight of the distribution, respectively.

The closed-form formulae for obtaining the values of model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of an MoLM-based LQU distribution, obtained by solving Equations (17)–(20), can be given as follows:

$$\alpha_0 = \lambda_1 - 3\lambda_2 + 5\lambda_3 + 18\lambda_4 \quad (23)$$

$$\alpha_1 = 6(\lambda_2 - 5\lambda_3 - 6\lambda_4) \quad (24)$$

$$\alpha_2 = 30\lambda_3 \quad (25)$$

$$\alpha_3 = 6\lambda_4 \quad (26)$$

where λ_1 , λ_2 , λ_3 , and λ_4 on the right-hand sides of Equations (23)–(26) can be either user-specified or computed from data samples, for example, in data modeling examples (see Figure 1 or Figure 3).

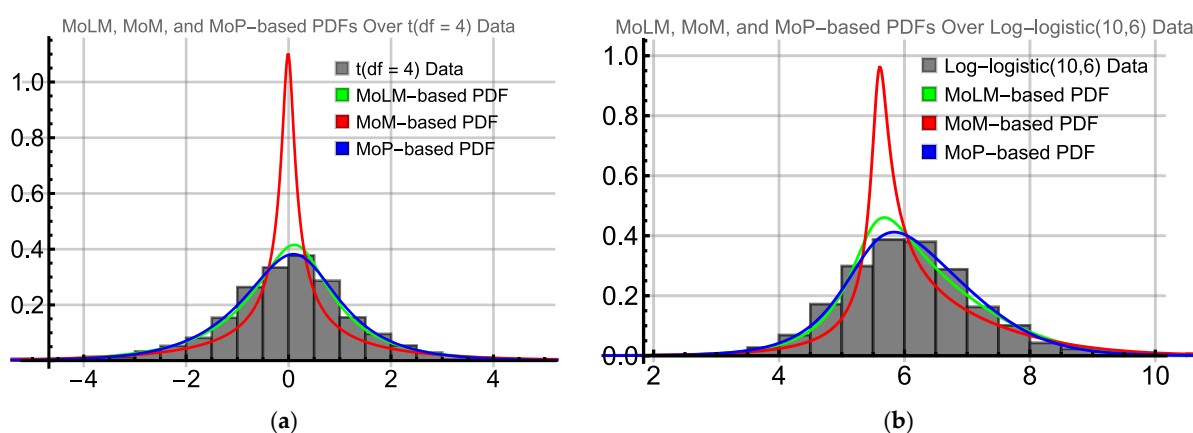


Figure 3. The pdfs of MoLM-, MoM-, and MoP-based LQU mixture polynomial distributions superimposed over the histogram of a random sample ($n = 2000$) of data taken from the (a) Student's t -distribution with 4 degrees of freedom, $t_{(df=4)}$ and (b) log-logistic distribution with shape and scale parameters of 10 and 6, $LL_{(10,6)}$, respectively.

For a standardized MoLM-based LQU distribution, the closed-form formulae for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, obtained by substituting $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$, respectively, on the right-hand sides of Equations (23) and (24), can be given as follows:

$$\alpha_0 = 5\lambda_3 + 18\lambda_4 - 3/\sqrt{\pi} \quad (27)$$

$$\alpha_1 = -6(5\lambda_3 + 6\lambda_4 - 1/\sqrt{\pi}) \quad (28)$$

$$\alpha_2 = 30\lambda_3 \quad (29)$$

$$\alpha_3 = 6\lambda_4 \quad (30)$$

Alternatively, the model parameters $\alpha_0, \alpha_1, \alpha_2$, and α_3 of an MoLM-based LQU distribution can be obtained by solving the system of Equations (17), (18), (21), and (22) after substituting user-specified values of λ_1 and λ_2 (e.g., 0 and $1/\sqrt{\pi}$ for the standardized LQU distribution) on the right-hand sides of Equations (17) and (18) and user-specified values of τ_3 and τ_4 on the right-hand sides of Equations (21) and (22), respectively. It is noted that the values of λ_1 and λ_2 for the standard logistic (SL) and standard normal (SN) distributions are SL ($\lambda_1 = 0$, $\lambda_2 = 1$) and SN ($\lambda_1 = 0$, $\lambda_2 = 1/\sqrt{\pi}$), respectively.

3.5. LQU Distribution via Method of Percentiles (MoP)

The method of percentiles (MoP)-based parameters of median (ξ_1), inter-decile range (ξ_2), left–right tail-weight ratio (ξ_3 , a skew function), and tail-weight factor (ξ_4 , a kurtosis function) associated with LQU mixture polynomial distribution of the form in Equation (14) can be obtained by simplifying Equations (10)–(13) after substituting $Q(u)$ on the right-hand sides of these equations with $Q_{LQU}(u)$ of Equation (14). Thus, the simplified forms of the MoP-based parameters of the LQU mixture polynomial distribution can be written as follows:

$$\xi_1 = \alpha_0 + (2\alpha_1 + \alpha_2)/4 \quad (31)$$

$$\xi_2 = 4 (\ln(3)\alpha_3 + (\alpha_1 + \alpha_2)/5) \quad (32)$$

$$\xi_3 = (5\alpha_1 + 3\alpha_2 + 25\ln(3)\alpha_3)/(5\alpha_1 + 7\alpha_2 + 25\ln(3)\alpha_3) \quad (33)$$

$$\xi_4 = (5(\alpha_1 + \alpha_2 + 4\ln(3)\alpha_3))/(8(\alpha_1 + \alpha_2 + 5\ln(3)\alpha_3)) \quad (34)$$

The closed-form formulae for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of an MoP-based LQU distribution, obtained by solving Equations (31)–(34), can be given as follows:

$$\alpha_0 = \xi_1 - 5\xi_2\xi_4 + 5\xi_2(11\xi_3 + 21)/(32(\xi_3 + 1)) \quad (35)$$

$$\alpha_1 = 10\xi_2\xi_4 - 5\xi_2(3\xi_3 + 13)/(8(\xi_3 + 1)) \quad (36)$$

$$\alpha_2 = 25\xi_2(1 - \xi_3)/(8(\xi_3 + 1)) \quad (37)$$

$$\alpha_3 = \xi_2(5 - 8\xi_4)/(4\ln(3)) \quad (38)$$

To demonstrate the MoLM-, MoM-, and MoP-based methodology, provided in Figure 3 (panels a and b) are the pdfs of MoLM-, MoM-, and MoP-based LQU mixture polynomial distributions superimposed over the histograms of random samples ($n = 2000$) of data taken from the (a) Student's t distribution with 4 degrees of freedom, $t_{(df=4)}$, and (b) log-logistic distribution with shape and scale parameters of 10 and 6, $LL_{(10,6)}$, respectively. Sample estimates of the first four indices for each method were obtained and used on the right-hand sides of each system of equations before solving for the values of model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. The solved values of model parameters were substituted into Equations (14) and (15) for obtaining LQU pdf for each method.

Specifically, for the MoLM-based method, the values of sample estimates $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4)$ were computed for each dataset in Figure 3 (panels a and b) using equations given in (pp. 113–114, [25]). These sample estimates were used on the right-hand sides of Equations (23)–(26) for solving for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, which were subsequently substituted into Equations (14) and (15) to obtain the qf of MoLM-based LQU distribution and its pdf.

The MoM-based methodology is presented in Appendix A. For the MoM-based method, the values of sample estimates $(\hat{\mu}, \hat{\sigma}, \hat{\gamma}_3, \hat{\gamma}_4)$ were computed for each dataset in Figure 3 (panels a and b) using Fisher's k -statistics (pp. 299–300, [29]) and Equations (A2)–(A3). These sample estimates were used on the right-hand sides of Equations (A4)–(A7) to solve for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, which were substituted into Equations (14) and (15) to obtain the qf of MoM-based LQU distribution and its pdf.

For the MoP-based methodology, the sample estimates $(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4)$ were computed for each dataset in Figure 3 (panels a and b) using methodology given in (p. 5, [55]). These sample estimates were used on the right-hand sides of Equations (35)–(38) for solving for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, which were substituted into Equations (14) and (15) to obtain the qf of MoP-based LQU distribution and its pdf.

Table 3 presents the sample estimates computed using each method, together with the corresponding solved model parameters for the LQU distributions in Figure 3.

Table 3. Sample estimates with solved values of model parameters for the MoLM-, MoM-, and MoP-based pdfs of LQU distributions in Figure 3.

Sample	Method	Sample Estimates				Model Parameters			
		Location	Scale	Skewness	Kurtosis	α_0	α_1	α_2	α_3
$t_{df=4}$	MoLM	$\hat{\lambda}_1$ −0.0082	$\hat{\lambda}_2$ 0.7631	$\hat{\lambda}_3$ −0.0180	$\hat{\lambda}_4$ 0.1796	0.8458	−1.3484	−0.5396	1.0778
	MoM	$\hat{\mu}$ −0.0082	$\hat{\sigma}$ 1.4925	$\hat{\gamma}_3$ 0.0011	$\hat{\gamma}_4$ 9.7052	2.8038	−5.6269	0.0042	1.6334
	MoP	$\hat{\xi}_1$ 0.0332	$\hat{\xi}_2$ 3.1392	$\hat{\xi}_3$ 1.0698	$\hat{\xi}_4$ 0.4728	0.3777	−0.5237	−0.3306	0.8699
$LL_{(10,6)}$	MoLM	$\hat{\lambda}_1$ 6.1255	$\hat{\lambda}_2$ 0.6228	$\hat{\lambda}_3$ 0.0559	$\hat{\lambda}_4$ 0.1100	6.5165	−1.9000	1.6771	0.6599
	MoM	$\hat{\mu}$ 6.1255	$\hat{\sigma}$ 1.1599	$\hat{\gamma}_3$ 0.9936	$\hat{\gamma}_4$ 4.2248	7.6355	−5.3004	3.4205	0.9032
	MoP	$\hat{\xi}_1$ 6.0377	$\hat{\xi}_2$ 2.6847	$\hat{\xi}_3$ 0.8247	$\hat{\xi}_4$ 0.5071	6.1432	−0.6140	0.8058	0.5760

Mathematica (Version 14.2) [60] was used for obtaining all of the figures, for solving for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, and for data modeling examples throughout this paper, whereas R (Version 4.5.2) [61] was used for Monte Carlo simulation and bootstrapping techniques.

4. Results

In Figures 4 and 5, we present four symmetric and four asymmetric MoLM-based LQU distributions, respectively, with specified values of L -mean (λ_1), L -scale (λ_2), L -skewness (τ_3), and L -kurtosis (τ_4). Presented on the right-hand sides of Tables 4 and 5 are the MoLM-based model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of each LQU distribution. These model parameters were obtained by solving Equations (17), (18), (21), and (22) after substituting the specified values of λ_1 , λ_2 , τ_3 , and τ_4 on the right-hand sides of these equations. Also presented in Tables 4 and 5 are the MoM-based parameters of mean (μ), standard deviation (σ), skewness (γ_3), and kurtosis (γ_4), and MoP-based parameters of median (ξ_1), inter-decile range (ξ_2), left–right tail-weight ratio (ξ_3), and tail-weight factor (ξ_4), which were obtained by substituting the MoLM-based model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ into Equations (A4)–(A7) and (31)–(34), respectively.

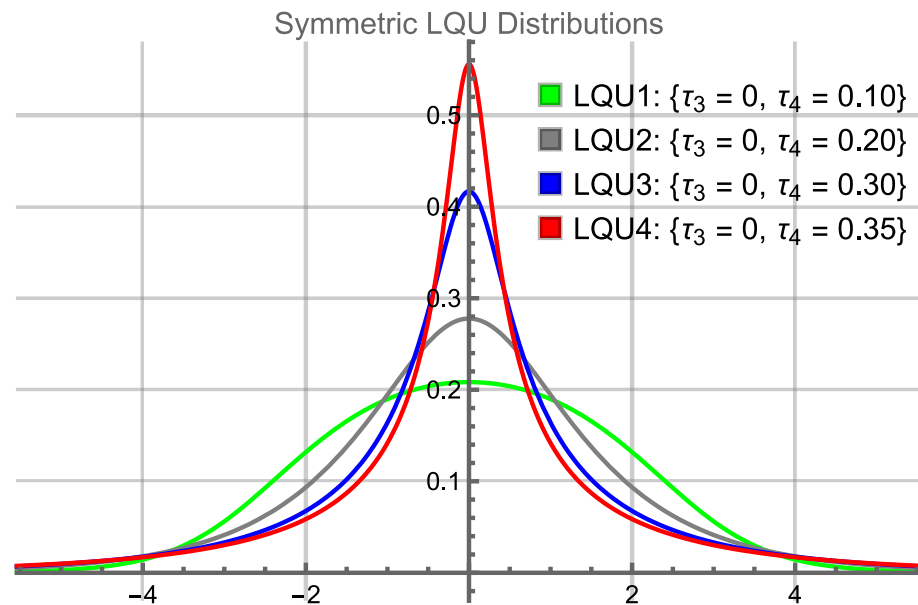


Figure 4. Four symmetric LQU distributions with specified values of L -kurtosis (τ_4). The MoLM-based model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) shown in Table 4 were obtained by substituting $\lambda_1 = 0, \lambda_2 = 1, \tau_3 = 0$ and specified values of τ_4 into Equations (17), (18), (21), and (22). Note: For a symmetric LQU distribution, $\tau_3 = 0$.

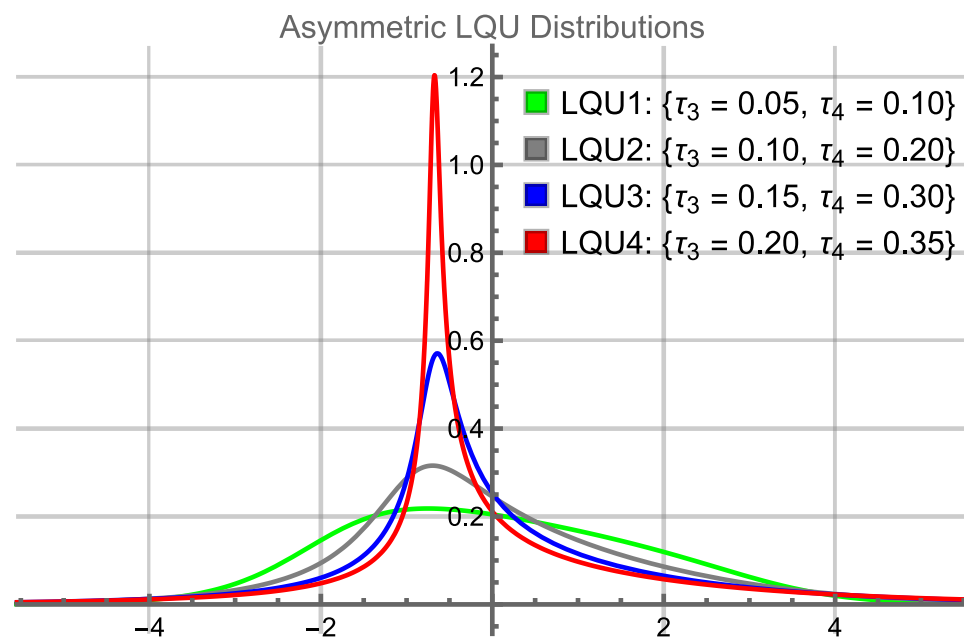


Figure 5. Four asymmetric LQU distributions with specified values of L -skewness (τ_3) and L -kurtosis (τ_4). The MoLM-based model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) shown in Table 5 were obtained by substituting $\lambda_1 = 0, \lambda_2 = 1$ and specified values of τ_3 and τ_4 into Equations (17), (18), (21), and (22).

Table 4. MoLM-, MoM-, and MoP-based parameters of location, scale, skewness, and kurtosis together with MoLM-based model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) for the four symmetric LQU distributions in Figure 4.

Dist.	Method	Parameters				MoLM-Based Model Parameters			
		Location	Scale	Skewness	Kurtosis	α_0	α_1	α_2	α_3
LQU1	MoLM	λ_1 0	λ_2 1	τ_3 0	τ_4 0.10	-1.2	2.4	0	0.6

	MoM	μ 0	σ 1.7619	γ_3 0	γ_4 −0.2126	----	----	----	----
	MoP	ξ_1 0	ξ_2 4.5567	ξ_3 1	ξ_4 0.5527	----	----	----	----
		λ_1 0	λ_2 1	τ_3 0	τ_4 0.20	0.6	−1.2	0	1.2
LQU2	MoLM	μ 0	σ 1.8486	γ_3 0	γ_4 2.1566	----	----	----	----
	MoM	ξ_1 0	ξ_2 4.3133	ξ_3 1	ξ_4 0.4722	----	----	----	----
	MoP	λ_1 0	λ_2 1	τ_3 0	τ_4 0.30	2.4	−4.8	0	1.8
LQU3	MoLM	μ 0	σ 1.9847	γ_3 0	γ_4 5.8288	----	----	----	----
	MoM	ξ_1 0	ξ_2 4.0700	ξ_3 1	ξ_4 0.3821	----	----	----	----
	MoP	λ_1 0	λ_2 1	τ_3 0	τ_4 0.35	3.3	−6.6	0	2.1
LQU4	MoLM	μ 0	σ 2.0684	γ_3 0	γ_4 7.9382	----	----	----	----
	MoM	ξ_1 0	ξ_2 3.9483	ξ_3 1	ξ_4 0.3328	----	----	----	----
	MoP	λ_1 0	λ_2 1	τ_3 0	τ_4 0.35	3.3	−6.6	0	2.1

Note: Dist. = Distribution. LQU = Logistic-quantile and Uniform. The symbol ---- means “same as above.”

Table 5. MoLM-, MoM-, and MoP-based parameters of location, scale, skewness, and kurtosis together with MoLM-based model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) for the four asymmetric LQU distributions in Figure 5.

Dist.	Method	Parameters				MoLM-Based Model Parameters			
		Location	Scale	Skewness	Kurtosis	α_0	α_1	α_2	α_3
LQU1	MoLM	λ_1 0	λ_2 1	τ_3 0.05	τ_4 0.10	−0.95	0.9	1.5	0.6
	MoM	μ 0	σ 1.7655	γ_3 0.2226	γ_4 −0.1838	----	----	----	----
	MoP	ξ_1 −0.125	ξ_2 4.5567	ξ_3 0.8094	ξ_4 0.5527	----	----	----	----
LQU2	MoLM	λ_1 0	λ_2 1	τ_3 0.10	τ_4 0.20	1.1	−4.2	3.0	1.2
	MoM	μ 0	σ 1.8621	γ_3 0.5141	γ_4 2.2401	----	----	----	----
	MoP	ξ_1 −0.25	ξ_2 4.3133	ξ_3 0.6359	ξ_4 0.4722	----	----	----	----
LQU3	MoLM	λ_1 0	λ_2 1	τ_3 0.15	τ_4 0.30	3.15	−9.3	4.5	1.8
	MoM	μ 0	σ 2.0129	γ_3 0.8108	γ_4 5.8649	----	----	----	----
	MoP	ξ_1 −0.375	ξ_2 4.0700	ξ_3 0.4773	ξ_4 0.3821	----	----	----	----
LQU4	MoLM	λ_1 0	λ_2 1	τ_3 0.20	τ_4 0.35	4.3	−12.6	6.0	2.1
	MoM	μ 0	σ 2.1162	γ_3 1.0640	γ_4 7.8575	----	----	----	----
	MoP	ξ_1 −0.5	ξ_2 3.9483	ξ_3 0.3456	ξ_4 0.3328	----	----	----	----

Note: Dist. = Distribution. LQU = Logistic-quantile and Uniform. The symbol ---- means “same as above.”

The LQU distributions in Figures 4 and 5 with their corresponding MoLM-, MoM-, and MoP-based parameters were used for Monte Carlo simulation results presented in Tables 6–11.

Table 6. Parameters, Monte Carlo estimate (MC Est) with RMSE, and bootstrap estimate (Boot Est) with SE along with 95% bootstrap confidence interval (95% CI) for the MoLM-based symmetric LQU distributions in Figure 4.

Dist.	Parameter	MC Est (RMSE)	Boot Est (SE)	95% CI	RB%
<i>n</i> = 25 (UT: 52.03 s, ST: 8.54 s, ET: 65.21 s)					
LQU1	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0007$ (0.0919)	0.0007 (0.0009)	(−0.0012, 0.0025)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1021$ (0.0693)	0.1021 (0.0007)	(0.1007, 0.1034)	2.10
LQU2	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0010$ (0.1247)	0.0009 (0.0013)	(−0.0015, 0.0034)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.1984$ (0.0847)	0.1984 (0.0008)	(0.1967, 0.2000)	----
LQU3	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0012$ (0.1702)	0.0012 (0.0017)	(−0.0021, 0.0045)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.2951$ (0.0979)	0.2950 (0.0010)	(0.2931, 0.2970)	−1.63
LQU4	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0012$ (0.1951)	0.0012 (0.0020)	(−0.0026, 0.0050)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3441$ (0.1032)	0.3441 (0.0010)	(0.3421, 0.3462)	−1.69
<i>n</i> = 100 (UT: 42.31 s, ST: 6.46 s, ET: 67.67 s)					
LQU1	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0009$ (0.0434)	0.0009 (0.0004)	(0.0000, 0.0017)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1006$ (0.0304)	0.1006 (0.0003)	(0.1000, 0.1011)	----
LQU2	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0014$ (0.0616)	0.0014 (0.0006)	(0.0002, 0.0027)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.1997$ (0.0384)	0.1997 (0.0004)	(0.1989, 0.2005)	−0.15
LQU3	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0020$ (0.0866)	0.0020 (0.0009)	(0.0003, 0.0038)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.2989$ (0.0447)	0.2989 (0.0004)	(0.2980, 0.2998)	−0.37
LQU4	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0023$ (0.1002)	0.0023 (0.0010)	(0.0003, 0.0043)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3487$ (0.0470)	0.3487 (0.0005)	(0.3477, 0.3496)	−0.37
<i>n</i> = 500 (UT: 51.19 s, ST: 7.05 s, ET: 66.24 s)					
LQU1	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0002$ (0.0191)	0.0002 (0.0002)	(−0.0002, 0.0006)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1001$ (0.0132)	0.1001 (0.0001)	(0.0998, 0.1003)	----
LQU2	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0002$ (0.0275)	0.0002 (0.0003)	(−0.0003, 0.0008)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.1999$ (0.0169)	0.1999 (0.0002)	(0.1996, 0.2002)	----
LQU3	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0002$ (0.0390)	0.0002 (0.0004)	(−0.0006, 0.0010)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.2997$ (0.0196)	0.2997 (0.0002)	(0.2994, 0.3001)	----
LQU4	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0002$ (0.0452)	0.0002 (0.0005)	(−0.0007, 0.0011)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3497$ (0.0206)	0.3497 (0.0002)	(0.3493, 0.3501)	----
<i>n</i> = 1000 (UT: 43.85 s, ST: 4.40 s, ET: 67.22 s)					
LQU1	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0002$ (0.0134)	0.0002 (0.0001)	(−0.0001, 0.0004)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1001$ (0.0093)	0.1001 (0.0001)	(0.0999, 0.1002)	----
LQU2	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0003$ (0.0194)	0.0003 (0.0002)	(−0.0001, 0.0006)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.2000$ (0.0119)	0.2000 (0.0001)	(0.1998, 0.2002)	----
LQU3	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0003$ (0.0276)	0.0003 (0.0003)	(−0.0002, 0.0009)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.3000$ (0.0138)	0.3000 (0.0001)	(0.2997, 0.3002)	----
LQU4	$\tau_3 = 0$	$\hat{\tau}_3 = 0.0004$ (0.0320)	0.0004 (0.0003)	(−0.0003, 0.0010)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3499$ (0.0145)	0.3499 (0.0001)	(0.3496, 0.3502)	----

Note: RB% = $100 \times (\text{Estimate} - \text{Parameter}) / \text{Parameter}$. The symbol ---- means “not available” when the parameter is 0 or within the 95% CI.

Table 7. Parameters, Monte Carlo estimate (MC Est) with RMSE, and bootstrap estimate (Boot Est) with SE along with 95% bootstrap confidence interval (95% CI) for the MoM-based symmetric LQU distributions in Figure 4.

Dist.	Parameter	MC Est (RMSE)	Boot Est (SE)	95% CI	RB%
<i>n</i> = 25 (UT: 41.51 s, ST: 4.61 s, ET: 63.53 s)					
LQU1	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0009$ (0.3863)	0.0009 (0.0039)	(−0.0069, 0.0083)	----
	$\gamma_4 = -0.2126$	$\hat{\gamma}_4 = -0.6013$ (0.7456)	−0.6013 (0.0063)	(−0.6137, −0.5887)	182.8
LQU2	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0004$ (0.7079)	0.0001 (0.0071)	(−0.0139, 0.0143)	----
	$\gamma_4 = 2.1566$	$\hat{\gamma}_4 = 0.5033$ (2.1777)	0.5031 (0.0142)	(0.4755, 0.5308)	−76.66
LQU3	$\gamma_3 = 0$	$\hat{\gamma}_3 = -0.0005$ (1.0666)	−0.0005 (0.0105)	(−0.0209, 0.0204)	----
	$\gamma_4 = 5.8288$	$\hat{\gamma}_4 = 1.8189$ (4.5661)	1.8192 (0.0215)	(1.7765, 1.8614)	−68.79
LQU4	$\gamma_3 = 0$	$\hat{\gamma}_3 = -0.0010$ (1.2355)	−0.0010 (0.0124)	(−0.0254, 0.0229)	----
	$\gamma_4 = 7.9382$	$\hat{\gamma}_4 = 2.4865$ (6.0007)	2.4865 (0.0249)	(2.4373, 2.5358)	−68.68
<i>n</i> = 100 (UT: 41.95 s, ST: 4.17 s, ET: 64.19 s)					
LQU1	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0040$ (0.2234)	0.0040 (0.0022)	(−0.0004, 0.0083)	----
	$\gamma_4 = -0.2126$	$\hat{\gamma}_4 = -0.3222$ (0.4644)	−0.3222 (0.0045)	(−0.3310, −0.3134)	51.55
LQU2	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0092$ (0.5276)	0.0093 (0.0053)	(−0.0010, 0.0197)	----
	$\gamma_4 = 2.1566$	$\hat{\gamma}_4 = 1.5638$ (1.7021)	1.5637 (0.0160)	(1.5324, 1.5950)	−27.49
LQU3	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0151$ (0.9163)	0.0152 (0.0091)	(−0.0028, 0.0331)	----
	$\gamma_4 = 5.8288$	$\hat{\gamma}_4 = 4.1517$ (3.5617)	4.1522 (0.0315)	(4.0904, 4.2143)	−28.77
LQU4	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0179$ (1.1089)	0.0181 (0.0111)	(−0.0035, 0.0402)	----
	$\gamma_4 = 7.9382$	$\hat{\gamma}_4 = 5.5346$ (4.5925)	5.5346 (0.0393)	(5.4586, 5.6122)	−30.28
<i>n</i> = 500 (UT: 40.27 s, ST: 3.86 s, ET: 66.51 s)					
LQU1	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0015$ (0.1057)	0.0015 (0.0011)	(−0.0006, 0.0036)	----
	$\gamma_4 = -0.2126$	$\hat{\gamma}_4 = -0.2357$ (0.2328)	−0.2356 (0.0023)	(−0.2402, −0.2311)	10.87
LQU2	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0035$ (0.2803)	0.0036 (0.0028)	(−0.0020, 0.0089)	----
	$\gamma_4 = 2.1566$	$\hat{\gamma}_4 = 2.0193$ (1.0499)	2.0193 (0.0104)	(1.9993, 2.0399)	−6.37
LQU3	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0067$ (0.5310)	0.0067 (0.0053)	(−0.0037, 0.0173)	----
	$\gamma_4 = 5.8288$	$\hat{\gamma}_4 = 5.3997$ (2.5024)	5.3996 (0.0243)	(5.3524, 5.4475)	−7.36
LQU4	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0085$ (0.6640)	0.0085 (0.0066)	(−0.0047, 0.0213)	----
	$\gamma_4 = 7.9382$	$\hat{\gamma}_4 = 7.2973$ (3.3585)	7.2969 (0.0329)	(7.2322, 7.3611)	−8.07
<i>n</i> = 1000 (UT: 44.63 s, ST: 4.22 s, ET: 69.20 s)					
LQU1	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0011$ (0.0748)	0.0011 (0.0007)	(−0.0004, 0.0025)	----
	$\gamma_4 = -0.2126$	$\hat{\gamma}_4 = -0.2219$ (0.1704)	−0.2219 (0.0017)	(−0.2252, −0.2185)	4.37
LQU2	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0029$ (0.2042)	0.0029 (0.0020)	(−0.0011, 0.0070)	----
	$\gamma_4 = 2.1566$	$\hat{\gamma}_4 = 2.1001$ (0.8120)	2.1002 (0.0082)	(2.0844, 2.1162)	−2.62
LQU3	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0057$ (0.3951)	0.0057 (0.0039)	(−0.0019, 0.0135)	----
	$\gamma_4 = 5.8288$	$\hat{\gamma}_4 = 5.6392$ (2.0211)	5.6392 (0.0203)	(5.6001, 5.6794)	−3.25
LQU4	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.0072$ (0.4983)	0.0072 (0.0049)	(−0.0025, 0.0169)	----
	$\gamma_4 = 7.9382$	$\hat{\gamma}_4 = 7.6470$ (2.7605)	7.6470 (0.0273)	(7.5944, 7.7015)	−3.67

Note: RB% = $100 \times (\text{Estimate} - \text{Parameter}) / \text{Parameter}$. The symbol ---- means “not available” when the parameter is 0 or within the 95% CI.

Table 8. Parameters, Monte Carlo estimate (MC Est) with RMSE, and bootstrap estimate (Boot Est) with SE along with 95% bootstrap confidence interval (95% CI) for the MoP-based symmetric LQU distributions in Figure 4.

Dist.	Parameter	MC Est (RMSE)	Boot Est (SE)	95% CI	RB%
<i>n</i> = 25 (UT: 42.25 s, ST: 3.60 s, ET: 63.97 s)					
LQU1	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0883$ (0.4759)	1.0883 (0.0047)	(1.0792, 1.0976)	8.83
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5509$ (0.0992)	0.5509 (0.0010)	(0.5490, 0.5529)	----
LQU2	$\xi_3 = 1$	$\hat{\xi}_3 = 1.1029$ (0.5281)	1.1030 (0.0052)	(1.0927, 1.1133)	10.29

LQU3	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4753 (0.1024)$	0.4753 (0.0010)	(0.4733, 0.4773)	0.66
	$\xi_3 = 1$	$\hat{\xi}_3 = 1.1613 (0.7066)$	1.1613 (0.0068)	(1.1479, 1.1745)	16.13
LQU4	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3961 (0.1062)$	0.3961 (0.0011)	(0.3940, 0.3981)	3.66
	$\xi_3 = 1$	$\hat{\xi}_3 = 1.2145 (0.8607)$	1.2145 (0.0083)	(1.1983, 1.2309)	21.45
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3543 (0.1081)$	0.3543 (0.0011)	(0.3523, 0.3564)	6.46
$n = 100$ (UT: 38.57 s, ST: 3.35 s, ET: 64.78 s)					
LQU1	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0176 (0.2107)$	1.0177 (0.0021)	(1.0136, 1.0217)	1.76
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5513 (0.0534)$	0.5513 (0.0005)	(0.5503, 0.5524)	−0.25
LQU2	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0216 (0.2366)$	1.0217 (0.0024)	(1.0171, 1.0263)	2.16
	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4721 (0.0554)$	0.4721 (0.0006)	(0.4710, 0.4732)	----
LQU3	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0379 (0.3134)$	1.0380 (0.0031)	(1.0319, 1.0442)	3.79
	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3850 (0.0567)$	0.3850 (0.0006)	(0.3839, 0.3861)	0.76
LQU4	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0525 (0.3716)$	1.0525 (0.0037)	(1.0453, 1.0598)	5.25
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3378 (0.0565)$	0.3378 (0.0006)	(0.3367, 0.3389)	1.50
$n = 500$ (UT: 41.06 s, ST: 2.96 s, ET: 64.78 s)					
LQU1	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0039 (0.0928)$	1.0039 (0.0009)	(1.0021, 1.0057)	0.39
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5523 (0.0242)$	0.5523 (0.0002)	(0.5518, 0.5528)	----
LQU2	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0051 (0.1035)$	1.0051 (0.0010)	(1.0031, 1.0071)	0.51
	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4720 (0.0252)$	0.4720 (0.0003)	(0.4716, 0.4725)	----
LQU3	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0088 (0.1346)$	1.0088 (0.0013)	(1.0062, 1.0115)	0.88
	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3825 (0.0256)$	0.3825 (0.0003)	(0.3820, 0.3830)	----
LQU4	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0119 (0.1567)$	1.0120 (0.0016)	(1.0088, 1.0151)	1.19
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3337 (0.0254)$	0.3337 (0.0003)	(0.3332, 0.3342)	0.27
$n = 1000$ (UT: 43.33 s, ST: 3.50 s, ET: 65.84 s)					
LQU1	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0017 (0.0650)$	1.0017 (0.0007)	(1.0004, 1.0030)	0.17
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5526 (0.0173)$	0.5526 (0.0002)	(0.5522, 0.5529)	----
LQU2	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0021 (0.0733)$	1.0021 (0.0007)	(1.0007, 1.0035)	0.21
	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4722 (0.0179)$	0.4722 (0.0002)	(0.4719, 0.4726)	----
LQU3	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0037 (0.0958)$	1.0037 (0.0010)	(1.0018, 1.0055)	0.37
	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3824 (0.0182)$	0.3824 (0.0002)	(0.3820, 0.3828)	----
LQU4	$\xi_3 = 1$	$\hat{\xi}_3 = 1.0051 (0.1115)$	1.0051 (0.0011)	(1.0029, 1.0073)	0.51
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3334 (0.0180)$	0.3334 (0.0002)	(0.3330, 0.3337)	0.18

Note: RB% = $100 \times (\text{Estimate} - \text{Parameter})/\text{Parameter}$. The symbol ---- means “not available” when the parameter is within the 95% CI.

Table 9. Parameters, Monte Carlo estimate (MC Est) with RMSE, and bootstrap estimate (Boot Est) with SE along with 95% bootstrap confidence interval (95% CI) for the MoLM-based asymmetric LQU distributions in Figure 5.

Dist.	Parameter	MC Est (RMSE)	Boot Est (SE)	95% CI	RB%
$n = 25$ (UT: 41.55 s, ST: 2.80 s, ET: 62.08 s)					
LQU1	$\tau_3 = 0.05$	$\hat{\tau}_3 = 0.0508 (0.0919)$	0.0508 (0.0009)	(0.0489, 0.0526)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1022 (0.0701)$	0.1022 (0.0007)	(0.1009, 0.1036)	2.20
LQU2	$\tau_3 = 0.10$	$\hat{\tau}_3 = 0.1005 (0.1239)$	0.1004 (0.0012)	(0.0980, 0.1029)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.1991 (0.0877)$	0.1991 (0.0009)	(0.1974, 0.2008)	----
LQU3	$\tau_3 = 0.15$	$\hat{\tau}_3 = 0.1505 (0.1680)$	0.1505 (0.0017)	(0.1472, 0.1538)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.2971 (0.1051)$	0.2971 (0.0010)	(0.2950, 0.2992)	−0.97
LQU4	$\tau_3 = 0.20$	$\hat{\tau}_3 = 0.2013 (0.1918)$	0.2012 (0.0019)	(0.1974, 0.2049)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3482 (0.1168)$	0.3482 (0.0012)	(0.3459, 0.3505)	----
$n = 100$ (UT: 53.07 s, ST: 3.71 s, ET: 61.82 s)					
LQU1	$\tau_3 = 0.05$	$\hat{\tau}_3 = 0.0509 (0.0434)$	0.0509 (0.0004)	(0.0501, 0.0518)	1.80
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1006 (0.0308)$	0.1006 (0.0003)	(0.1000, 0.1012)	----
LQU2	$\tau_3 = 0.10$	$\hat{\tau}_3 = 0.1013 (0.0611)$	0.1013 (0.0006)	(0.1002, 0.1026)	1.30

LQU3	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.1998 (0.0401)$	0.1998 (0.0004)	(0.1990, 0.2006)	----
	$\tau_3 = 0.15$	$\hat{\tau}_3 = 0.1518 (0.0853)$	0.1518 (0.0009)	(0.1501, 0.1535)	1.20
LQU4	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.2993 (0.0488)$	0.2993 (0.0005)	(0.2984, 0.3003)	----
	$\tau_3 = 0.20$	$\hat{\tau}_3 = 0.2022 (0.0978)$	0.2022 (0.0010)	(0.2003, 0.2042)	1.10
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3495 (0.0546)$	0.3495 (0.0005)	(0.3484, 0.3506)	----
$n = 500$ (UT: 43.43 s, ST: 2.67 s, ET: 64.92 s)					
LQU1	$\tau_3 = 0.05$	$\hat{\tau}_3 = 0.0502 (0.0191)$	0.0502 (0.0002)	(0.0499, 0.0506)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1001 (0.0134)$	0.1001 (0.0001)	(0.0998, 0.1003)	----
LQU2	$\tau_3 = 0.10$	$\hat{\tau}_3 = 0.1002 (0.0273)$	0.1002 (0.0003)	(0.0997, 0.1007)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.2000 (0.0176)$	0.2000 (0.0002)	(0.1996, 0.2003)	----
LQU3	$\tau_3 = 0.15$	$\hat{\tau}_3 = 0.1502 (0.0384)$	0.1502 (0.0004)	(0.1494, 0.1509)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.2999 (0.0214)$	0.2999 (0.0002)	(0.2995, 0.3004)	----
LQU4	$\tau_3 = 0.20$	$\hat{\tau}_3 = 0.2002 (0.0441)$	0.2002 (0.0004)	(0.1993, 0.2010)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3500 (0.0240)$	0.3500 (0.0002)	(0.3495, 0.3505)	----
$n = 1000$ (UT: 42.87 s, ST: 3.39 s, ET: 67.81 s)					
LQU1	$\tau_3 = 0.05$	$\hat{\tau}_3 = 0.0502 (0.0134)$	0.0502 (0.0001)	(0.0499, 0.0504)	----
	$\tau_4 = 0.10$	$\hat{\tau}_4 = 0.1001 (0.0094)$	0.1001 (0.0001)	(0.0999, 0.1002)	----
LQU2	$\tau_3 = 0.10$	$\hat{\tau}_3 = 0.1002 (0.0192)$	0.1002 (0.0002)	(0.0998, 0.1006)	----
	$\tau_4 = 0.20$	$\hat{\tau}_4 = 0.2000 (0.0124)$	0.2000 (0.0001)	(0.1998, 0.2003)	----
LQU3	$\tau_3 = 0.15$	$\hat{\tau}_3 = 0.1502 (0.0271)$	0.1502 (0.0003)	(0.1497, 0.1508)	----
	$\tau_4 = 0.30$	$\hat{\tau}_4 = 0.3000 (0.0152)$	0.3000 (0.0002)	(0.2997, 0.3003)	----
LQU4	$\tau_3 = 0.20$	$\hat{\tau}_3 = 0.2003 (0.0311)$	0.2003 (0.0003)	(0.1996, 0.2009)	----
	$\tau_4 = 0.35$	$\hat{\tau}_4 = 0.3500 (0.0170)$	0.3500 (0.0002)	(0.3497, 0.3503)	----

Note: RB% = $100 \times (\text{Estimate} - \text{Parameter}) / \text{Parameter}$. The symbol ---- means “not available” when the parameter is within the 95% CI.

Table 10. Parameters, Monte Carlo estimate (MC Est) with RMSE, and bootstrap estimate (Boot Est) with SE along with 95% bootstrap confidence interval (95% CI) for the MoM-based asymmetric LQU distributions in Figure 5.

Dist.	Parameter	MC Est (RMSE)	Boot Est (SE)	95% CI	RB%
$n = 25$ (UT: 31.83 s, ST: 3.20 s, ET: 66.56 s)					
LQU1	$\gamma_3 = 0.2226$	$\hat{\gamma}_3 = 0.1892 (0.3869)$	0.1891 (0.0039)	(0.1814, 0.1965)	−15.00
	$\gamma_4 = -0.1838$	$\hat{\gamma}_4 = -0.5778 (0.7716)$	−0.5778 (0.0066)	(−0.5907, −0.5646)	214.4
LQU2	$\gamma_3 = 0.5141$	$\hat{\gamma}_3 = 0.4002 (0.7026)$	0.4000 (0.0070)	(0.3863, 0.4139)	−22.16
	$\gamma_4 = 2.2401$	$\hat{\gamma}_4 = 0.5782 (2.2228)$	0.5780 (0.0148)	(0.5492, 0.6072)	−74.19
LQU3	$\gamma_3 = 0.8108$	$\hat{\gamma}_3 = 0.5950 (1.0483)$	0.5949 (0.0101)	(0.5753, 0.6150)	−26.62
	$\gamma_4 = 5.8649$	$\hat{\gamma}_4 = 1.9310 (4.5356)$	1.9312 (0.0223)	(1.8872, 1.9747)	−67.08
LQU4	$\gamma_3 = 1.0640$	$\hat{\gamma}_3 = 0.7686 (1.1995)$	0.7686 (0.0117)	(0.7455, 0.7912)	−27.76
	$\gamma_4 = 7.8575$	$\hat{\gamma}_4 = 2.6326 (5.8391)$	2.6327 (0.0261)	(2.5815, 2.6840)	−66.50
$n = 100$ (UT: 35.61 s, ST: 4.02 s, ET: 68.19 s)					
LQU1	$\gamma_3 = 0.2226$	$\hat{\gamma}_3 = 0.2179 (0.2225)$	0.2178 (0.0022)	(0.2135, 0.2222)	−2.11
	$\gamma_4 = -0.1838$	$\hat{\gamma}_4 = -0.2935 (0.4795)$	−0.2935 (0.0047)	(−0.3027, −0.2844)	59.68
LQU2	$\gamma_3 = 0.5141$	$\hat{\gamma}_3 = 0.4895 (0.5126)$	0.4895 (0.0051)	(0.4796, 0.4997)	−4.79
	$\gamma_4 = 2.2401$	$\hat{\gamma}_4 = 1.6529 (1.7300)$	1.6529 (0.0163)	(1.6211, 1.6850)	−26.21
LQU3	$\gamma_3 = 0.8108$	$\hat{\gamma}_3 = 0.7545 (0.8697)$	0.7545 (0.0086)	(0.7376, 0.7716)	−6.96
	$\gamma_4 = 5.8649$	$\hat{\gamma}_4 = 4.2452 (3.5433)$	4.2455 (0.0318)	(4.1833, 4.3085)	−27.62
LQU4	$\gamma_3 = 1.0640$	$\hat{\gamma}_3 = 0.9797 (1.0230)$	0.9799 (0.0103)	(0.9597, 1.0005)	−7.92
	$\gamma_4 = 7.8575$	$\hat{\gamma}_4 = 5.5998 (4.4951)$	5.5999 (0.0389)	(5.5240, 5.6766)	−28.73
$n = 500$ (UT: 34.76 s, ST: 4.09 s, ET: 70.86 s)					
LQU1	$\gamma_3 = 0.2226$	$\hat{\gamma}_3 = 0.2223 (0.1052)$	0.2223 (0.0011)	(0.2202, 0.2244)	----
	$\gamma_4 = -0.1838$	$\hat{\gamma}_4 = -0.2062 (0.2393)$	−0.2062 (0.0024)	(−0.2109, −0.2015)	12.19
LQU2	$\gamma_3 = 0.5141$	$\hat{\gamma}_3 = 0.5103 (0.2715)$	0.5103 (0.0027)	(0.5050, 0.5156)	----

LQU3	$\gamma_4 = 2.2401$	$\hat{\gamma}_4 = 2.1081 (1.0623)$	2.1082 (0.0105)	(2.0878, 2.1292)	−5.89
	$\gamma_3 = 0.8108$	$\hat{\gamma}_3 = 0.8011 (0.5016)$	0.8011 (0.0050)	(0.7913, 0.8110)	----
LQU4	$\gamma_4 = 5.8649$	$\hat{\gamma}_4 = 5.4630 (2.4827)$	5.4629 (0.0242)	(5.4161, 5.5105)	−6.85
	$\gamma_3 = 1.0640$	$\hat{\gamma}_3 = 1.0485 (0.6075)$	1.0485 (0.0061)	(1.0364, 1.0602)	−1.46
	$\gamma_4 = 7.8575$	$\hat{\gamma}_4 = 7.2774 (3.2765)$	7.2769 (0.0320)	(7.2148, 7.3402)	−7.38
$n = 1000$ (UT: 36.15 s, ST: 2.72 s, ET: 75.06 s)					
LQU1	$\gamma_3 = 0.2226$	$\hat{\gamma}_3 = 0.2227 (0.0744)$	0.2227 (0.0007)	(0.2213, 0.2242)	----
	$\gamma_4 = -0.1838$	$\hat{\gamma}_4 = -0.1928 (0.1747)$	−0.1928 (0.0017)	(−0.1962, −0.1894)	4.90
LQU2	$\gamma_3 = 0.5141$	$\hat{\gamma}_3 = 0.5131 (0.1975)$	0.5131 (0.0020)	(0.5092, 0.5170)	----
	$\gamma_4 = 2.2401$	$\hat{\gamma}_4 = 2.1857 (0.8152)$	2.1858 (0.0082)	(2.1702, 2.2019)	−2.43
LQU3	$\gamma_3 = 0.8108$	$\hat{\gamma}_3 = 0.8075 (0.3723)$	0.8076 (0.0037)	(0.8005, 0.8149)	----
	$\gamma_4 = 5.8649$	$\hat{\gamma}_4 = 5.6875 (1.9826)$	5.6875 (0.0199)	(5.6486, 5.7272)	−3.02
LQU4	$\gamma_3 = 1.0640$	$\hat{\gamma}_3 = 1.0579 (0.4544)$	1.0579 (0.0045)	(1.0490, 1.0667)	----
	$\gamma_4 = 7.8575$	$\hat{\gamma}_4 = 7.5946 (2.6528)$	7.5947 (0.0263)	(7.5435, 7.6467)	−3.35

Note: RB% = $100 \times (\text{Estimate} - \text{Parameter}) / \text{Parameter}$. The symbol ---- means “not available” when the parameter is within the 95% CI.

Table 11. Parameters, Monte Carlo estimate (MC Est) with RMSE, and bootstrap estimate (Boot Est) with SE along with 95% bootstrap confidence interval (95% CI) for the MoP-based asymmetric LQU distributions in Figure 5.

Dist.	Parameter	MC Est (RMSE)	Boot Est (SE)	95% CI	RB%
$n = 25$ (UT: 35.47 s, ST: 3.22 s, ET: 67.35 s)					
LQU1	$\xi_3 = 0.8094$	$\hat{\xi}_3 = 0.8958 (0.3975)$	0.8958 (0.0039)	(0.8883, 0.9035)	10.67
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5508 (0.1003)$	0.5508 (0.0010)	(0.5489, 0.5528)	----
LQU2	$\xi_3 = 0.6359$	$\hat{\xi}_3 = 0.7387 (0.3696)$	0.7388 (0.0036)	(0.7317, 0.7459)	16.17
	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4749 (0.1067)$	0.4749 (0.0011)	(0.4728, 0.4769)	0.57
LQU3	$\xi_3 = 0.4773$	$\hat{\xi}_3 = 0.6165 (0.4185)$	0.6165 (0.0039)	(0.6089, 0.6242)	29.16
	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3947 (0.1160)$	0.3947 (0.0012)	(0.3924, 0.3970)	3.30
LQU4	$\xi_3 = 0.3456$	$\hat{\xi}_3 = 0.5018 (0.4333)$	0.5018 (0.0040)	(0.4939, 0.5099)	45.20
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3513 (0.1260)$	0.3513 (0.0012)	(0.3489, 0.3537)	5.56
$n = 100$ (UT: 45.88 s, ST: 4.75 s, ET: 66.05 s)					
LQU1	$\xi_3 = 0.8094$	$\hat{\xi}_3 = 0.8274 (0.1730)$	0.8274 (0.0017)	(0.8241, 0.8307)	2.22
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5513 (0.0541)$	0.5513 (0.0005)	(0.5502, 0.5523)	−0.25
LQU2	$\xi_3 = 0.6359$	$\hat{\xi}_3 = 0.6588 (0.1565)$	0.6588 (0.0015)	(0.6558, 0.6618)	3.60
	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4719 (0.0583)$	0.4719 (0.0006)	(0.4708, 0.4731)	----
LQU3	$\xi_3 = 0.4773$	$\hat{\xi}_3 = 0.5095 (0.1653)$	0.5095 (0.0016)	(0.5064, 0.5128)	6.75
	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3845 (0.0639)$	0.3845 (0.0006)	(0.3833, 0.3858)	0.63
LQU4	$\xi_3 = 0.3456$	$\hat{\xi}_3 = 0.3815 (0.1572)$	0.3815 (0.0015)	(0.3785, 0.3845)	10.39
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3370 (0.0700)$	0.3370 (0.0007)	(0.3356, 0.3383)	1.26
$n = 500$ (UT: 31.13 s, ST: 2.50 s, ET: 68.50 s)					
LQU1	$\xi_3 = 0.8094$	$\hat{\xi}_3 = 0.8133 (0.0758)$	0.8133 (0.0008)	(0.8118, 0.8147)	0.48
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5523 (0.0246)$	0.5523 (0.0002)	(0.5518, 0.5527)	----
LQU2	$\xi_3 = 0.6359$	$\hat{\xi}_3 = 0.6409 (0.0672)$	0.6409 (0.0007)	(0.6396, 0.6423)	0.79
	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4719 (0.0266)$	0.4719 (0.0003)	(0.4714, 0.4725)	----
LQU3	$\xi_3 = 0.4773$	$\hat{\xi}_3 = 0.4843 (0.0684)$	0.4843 (0.0007)	(0.4830, 0.4857)	1.47
	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3823 (0.0292)$	0.3823 (0.0003)	(0.3817, 0.3829)	----
LQU4	$\xi_3 = 0.3456$	$\hat{\xi}_3 = 0.3533 (0.0631)$	0.3533 (0.0006)	(0.3521, 0.3546)	2.23
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3333 (0.0321)$	0.3333 (0.0003)	(0.3327, 0.3340)	----
$n = 1000$ (UT: 41.25 s, ST: 2.89 s, ET: 67.16 s)					
LQU1	$\xi_3 = 0.8094$	$\hat{\xi}_3 = 0.8111 (0.0530)$	0.8111 (0.0005)	(0.8101, 0.8122)	0.21
	$\xi_4 = 0.5527$	$\hat{\xi}_4 = 0.5526 (0.0175)$	0.5526 (0.0002)	(0.5522, 0.5529)	----
LQU2	$\xi_3 = 0.6359$	$\hat{\xi}_3 = 0.6381 (0.0474)$	0.6381 (0.0005)	(0.6372, 0.6390)	0.35

LQU3	$\xi_4 = 0.4722$	$\hat{\xi}_4 = 0.4722 (0.0190)$	0.4722 (0.0002)	(0.4718, 0.4726)	----
	$\xi_3 = 0.4773$	$\hat{\xi}_3 = 0.4805 (0.0484)$	0.4805 (0.0005)	(0.4795, 0.4814)	0.67
LQU4	$\xi_4 = 0.3821$	$\hat{\xi}_4 = 0.3824 (0.0208)$	0.3824 (0.0002)	(0.3819, 0.3828)	----
	$\xi_3 = 0.3456$	$\hat{\xi}_3 = 0.3492 (0.0445)$	0.3492 (0.0004)	(0.3483, 0.3501)	1.04
	$\xi_4 = 0.3328$	$\hat{\xi}_4 = 0.3333 (0.0229)$	0.3333 (0.0002)	(0.3329, 0.3338)	0.15

Note: RB% = $100 \times (\text{Estimate} - \text{Parameter})/\text{Parameter}$. The symbol ---- means “not available” when the parameter is within the 95% CI.

It can be noted from Tables 4 and 5 that for the MoLM-based LQU distributions with $\lambda_1 = 0$ and $\lambda_2 = 1$, the model parameters α_2 and α_3 can be given as $\alpha_2 = 30\tau_3$ and $\alpha_3 = 6\tau_4$, respectively, which can also be verified by substituting Equations (19) and (20) into Equations (8) and (9). Also, from Equations (17) and (18), $\alpha_2 = -3(2\alpha_0 + \alpha_1)/2$ and $\alpha_3 = 1 - (\alpha_1 + \alpha_2)/6$, which can be verified from Tables 4 and 5.

For the symmetric LQU distributions (with $\lambda_1 = 0$ and $\lambda_2 = 1$) in Figure 4, $\lambda_3 = 0$ (or $\tau_3 = 0$). As a result, $\alpha_2 = 0$ and $\alpha_3 = 6\tau_4$ from Equation (19) and substituting Equation (20) into Equation (9).

Monte Carlo simulation results, presented in Tables 6–11, were obtained with 10,000 replications using R (Version 4.5.2) [61]. The random number generator was initialized using ‘set.seed (814)’ to ensure reproducibility.

Specifically, for each LQU distribution in Figures 4 and 5, sample estimates of the higher-order parameters of skewness and kurtosis, were computed using each of the three methods (MoLM, MoM, and MoP) using four sample sizes ($n = 25, 100, 500$, and 1000). Computations of Monte Carlo estimates (MC Est) with associated root mean square error (RMSE), bootstrap estimates (Boot Est) with associated standard errors (SEs) and 95% confidence intervals (95% CIs) were performed in R (Version 4.5.2) [61] on a Windows 11 machine with Intel i7-12700H CPU (2.30 GHz, 14 cores) and 32 GB RAM. Execution time was measured using ‘proc.time()’. The user time (UT), system time (ST), and elapsed time (ET) were reported for each sample size (n) in Tables 6–11.

The values of MC Est and RMSE for each sample size (n) in Tables 6–11, were computed using the formulae: $\text{MC Est} = (\sum_{r=1}^R \hat{\theta}_r)/R$ and $\text{RMSE} = \sqrt{(\sum_{r=1}^R (\hat{\theta}_r - \theta)^2)/R}$, where R is the number of replications (i.e., 10,000), $\hat{\theta}_r$ is the value of sample statistic (or estimate) for the r^{th} replication, and θ is the corresponding parameter (e.g., skewness).

For the bootstrap resampling technique, we considered R (i.e., 10,000) values of sample statistics based on each sample size (n) as our original data. From these original data, we obtained B (i.e., 10,000) bootstrap resamples (i.e., samples with replacement). For the computations of Boot Est, its SE, and 95% CI, we used the following formulae:

$\text{Boot Est} (\hat{\theta}_{\text{Boot}}) = (\sum_{b=1}^B \hat{\theta}^{*(b)})/B$, $\text{SE} = \sqrt{(\sum_{r=1}^R (\hat{\theta}^{*(b)} - \hat{\theta}_{\text{Boot}})^2)/(B - 1)}$, and $95\% \text{ CI} = [\hat{\theta}^{*(0.025 \times B)}, \hat{\theta}^{*(0.975 \times B)}]$, where $\hat{\theta}^{*(b)}$ is the sample statistic of corresponding parameter for the b^{th} bootstrap resample, and $\hat{\theta}^{*(0.025 \times B)}$ and $\hat{\theta}^{*(0.975 \times B)}$ in the 95% CI are the 2.5th and 97.5th percentiles, respectively, of bootstrap estimates after sorting them in ascending order. We reported bootstrap estimates (Boot Est), along with their standard errors (SEs) and 95% confidence intervals (95% CIs) in Tables 6–11 to quantify estimator uncertainty and precision, thereby complementing the Monte Carlo simulation results.

4.1. Monte Carlo Simulation Results: Parameter Estimation

The results of Monte Carlo simulations, presented in Tables 6–11, in the context of parameter estimation for the logistic-quantile and uniform (LQU) mixture polynomial distributions of Figures 4 and 5 demonstrate the advantages of the MoLM-based procedure over the MoM- and MoP-based procedures, respectively. Figures 4 and 5 display pdfs of four symmetric and four asymmetric MoLM-based LQU distributions with L -mean (λ_1)

$= 0$, L -scale (λ_2) = 1, and specified values of L -skewness (τ_3) and L -kurtosis (τ_4). The MoLM-based parameters of location, scale, skewness, and kurtosis (λ_1 , λ_2 , τ_3 , and τ_4) associated with each LQU distribution in Figures 4 and 5 are displayed on the left-hand sides of Tables 4 and 5. The solved values of model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) associated with each MoLM-based LQU distribution in Figures 4 and 5 are displayed on the right-hand sides of Tables 4 and 5. These solved values of model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) for each MoLM-based LQU distribution were obtained by solving Equations (17), (18), (21), and (22) after substituting specified values of λ_1 , λ_2 , τ_3 , and τ_4 on the right-hand sides of these equations. Also presented on the left-hand sides of Tables 4 and 5 are the MoM- and MoP-based parameters of location, scale, skewness and kurtosis, which were computed by substituting solved values of ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) from the right-hand sides of Tables 4 and 5 into Equations (A4)–(A7) and Equations (31)–(34), respectively, for each LQU distribution in Figures 4 and 5.

Presented in Tables 6–11 are the values of MoLM-based parameters of L -skewness (τ_3) and L -kurtosis (τ_4), MoM-based parameters of skewness (γ_3) and kurtosis (γ_4), MoP-based parameters of left–right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4), their Monte Carlo estimates (MC Est) with root mean square errors (RMSEs), corresponding bootstrap estimates (Boot Est) with standard errors (SEs) and 95% confidence intervals (95% CIs). For obtaining these results, an R (Version 4.5.2) [61] algorithm consisting of bootstrap functions from the ‘boot’ [62] package was written to simulate 10,000 independent samples of sizes $n = 25, 100, 500$, and 1000 for computing the MoLM-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of L -skewness and L -kurtosis (τ_3 and τ_4), the MoM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skewness and kurtosis (γ_3 and γ_4), and MoP-based estimates of ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left–right tail-weight ratio and tail-weight factor (ξ_3 and ξ_4) based on the values of solved model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$) given in Tables 4 and 5 for each LQU distribution in Figures 4 and 5.

The estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of τ_3 and τ_4 were computed using the ‘samlmu()’ function of the R package ‘lmom’ [63]. The estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of γ_3 and γ_4 were computed using ‘skewness()’ and ‘kurtosis()’ functions of the R package ‘e1071’ [64]. The estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of ξ_3 and ξ_4 were computed by substituting the 10th, 25th, 50th, 75th, and 90th percentiles—obtained using ‘quantile()’ function in R (Version 4.5.2) [61]—into Equations (10)–(13) after substituting $Q(u)$ with $Q_{LQU}(u)$ of Equation (14). From each independent sample of 10,000 statistics, bootstrapped average estimates (Boot Est) with associated standard errors (SEs) and 95% confidence intervals (95% CIs) were computed for each type of estimate using 10,000 resamples via bootstrap functions of the R package ‘boot’ [62].

If a parameter was outside its associated bootstrapped 95% CI, then the percentage of relative bias (RB%) was computed for the estimate as follows:

$$RB\% = 100 \times (\text{Estimate} - \text{Parameter}) / \text{Parameter} \quad (39)$$

Inspection of Tables 6–11 indicates that each of the MoLM-based estimates is superior to its corresponding MoM- and MoP-based counterparts in terms of smaller values of RMSE, SE and smaller RB%, each indicating the relative precision of MoLM-based procedure. The superiority of MoLM-based estimation is most pronounced in the context of smaller sample sizes and higher-order moments. For example, for the fourth LQU distribution (LQU4) in Figure 5, for $n = 25$, the simulated MoLM-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of L -skewness and L -kurtosis were, on average, 100.65% and 99.49% of their respective parameters (τ_3 and τ_4). On the other hand, the simulated MoM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skewness and kurtosis were, on average, 72.24% and 33.50% of their respective parameters (γ_3 and γ_4). The simulated MoP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left–right tail-weight ratio and tail-weight factor were, on average, 145.20% and 105.56% of their respective

parameters (ξ_3 and ξ_4). Further inspection of Tables 6–11 indicates that the relative biases (RB%) of the MoLM-based estimates are substantially smaller than those associated with MoM- and MoP-based estimates.

Additionally, it can be verified from the inspection of Tables 6–11 that the bootstrapped standard error (SE) associated with each of the MoLM-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) is much smaller than the SE associated with each of the MoM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$), but similar to the MoP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$). To compare SEs, we can use relative standard error (RSE) for each type of estimate, where $RSE = 100 \times SE/Estimate$. For example, for the fourth LQU distribution (LQU4) in Figure 5, with $n = 1000$, the RSE values associated with MoLM-based estimates $\hat{\tau}_3$ and $\hat{\tau}_4$ were 0.15% and 0.06%, respectively. On the other hand, the RSE values associated with MoM-based estimates $\hat{\gamma}_3$ and $\hat{\gamma}_4$ were 0.43% and 0.35%, respectively, whereas the RSE values associated with MoP-based estimates $\hat{\xi}_3$ and $\hat{\xi}_4$ were 0.11% and 0.06%, respectively. Thus, in terms of RSE, the MoLM- and MoP-based estimates are superior to the MoM-based estimates, while both MoLM- and MoP-based estimates have similar precision.

4.2. Results from Data Modeling Examples

In Figure 6 (panels a–f), we present examples of modeling real-world healthcare data with the logistic-quantile and uniform (LQU) distributions. Specifically, in Figure 6 (panels a–d), we present histograms of chest circumference (cm), thigh circumference (cm), bodyfat percent, and abdomen circumference (cm) data superimposed, respectively, by the pdfs of MoLM-, MoM-, and MoP-based LQU distributions. These data were obtained from [65]. In Figure 6 (panels e and f), we present histograms of cholesterol and plasma fibrinogen data superimposed by the pdfs of MoLM-, MoM-, and MoP-based LQU distributions. The cholesterol data used in Figure 6e were obtained from the Heart Disease dataset from the UC Irvine Machine Learning Repository (<https://archive.ics.uci.edu/dataset/45/heart+disease>, accessed on 12 August 2025), which were collected from 303 patients at Cleveland Clinic. The plasma fibrinogen data used in Figure 6f were obtained from Mendeley Data website (<https://data.mendeley.com/datasets/r8j84ks9f9/1>, accessed on 24 February 2026) [66].

To superimpose the pdfs of MoLM-, MoM-, and MoP-based LQU distributions over each histogram in Figure 6, we first computed sample estimates of location, scale, skewness, and kurtosis for each method. Then, we substituted these sample estimates on the right-hand sides of the set of equations for each method and solved for the values of model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$), which were subsequently substituted into Equations (14) and (15) to obtain the LQU qf and pdf for each method.

For example, for the MoLM-based method, the values of sample estimates ($\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$) were computed for each dataset in Figure 6 (panels a–f) using equations given in (pp. 113–114, [25]). These values of sample estimates were used on the right-hand sides of Equations (23)–(26) for solving for the model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$), which were subsequently substituted into Equations (14) and (15) to obtain the quantile function of MoLM-based LQU distribution and its pdf.

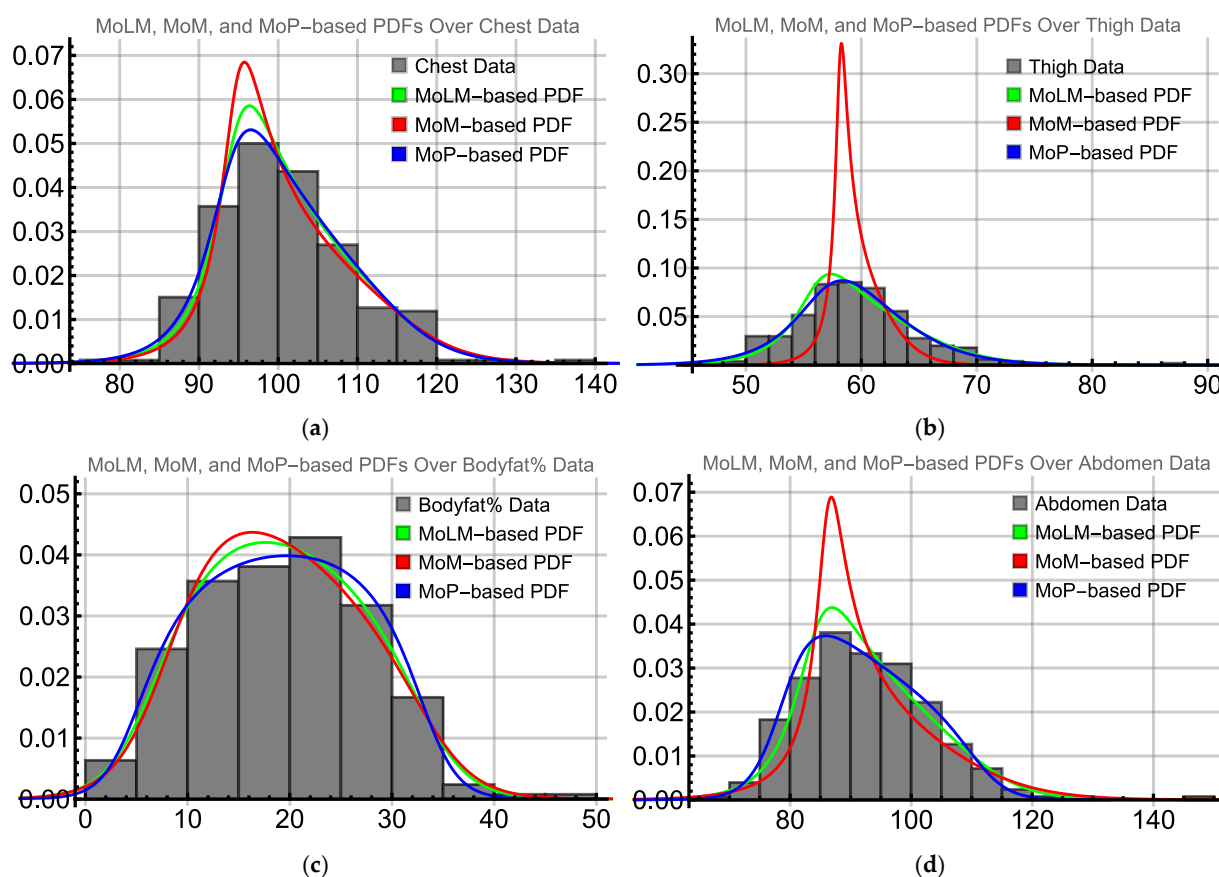
For the MoM-based method, the values of sample estimates ($\hat{\mu}, \hat{\sigma}, \hat{\gamma}_3, \hat{\gamma}_4$) were computed for each dataset in Figure 6 (panels a–f) using Fisher's k -statistics (pp. 299–300, [29]) and Equations (A2)–(A3). These sample estimates were used on the right-hand sides of Equations (A4)–(A7) to solve for the model parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$), which were substituted into Equations (14) and (15) to obtain the qf of MoM-based LQU distribution and its pdf.

For the MoP-based methodology, the sample estimates ($\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4$) were computed for each dataset in Figure 6 (panels a–f) using methodology given in (p. 5, [55]). These values of sample estimates were used on the right-hand sides of Equations (35)–(38)

for solving for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, which were subsequently substituted into Equations (14) and (15) to obtain the qf of MoP-based LQU distribution and its pdf. The sample estimates computed for each of these three methods for each dataset in Figure 6, along with the corresponding solved values of model parameters for LQU pdfs, are presented in Table 12.

Inspection of Figure 6 (panels a–f) indicates that the MoLM- and MoP-based LQU pdfs provide substantially better approximations to real-world healthcare data than that provided by MoM-based LQU pdfs. A closer look at Figure 6 indicates that MoP-based LQU pdfs provide a slightly better approximation than MoLM-based LQU pdfs. To support this claim, Table 13 provides percentile parameters obtained from each of the data variables in Figure 6 (panels a–f) and percentile estimates obtained, respectively, from the MoLM- and MoP-based LQU pdfs superimposed over each of the histograms in Figure 6. Also provided in Table 13 is the Euclidean distance (d), used as a measure of similarity (smaller means better fit), between percentile parameters and their estimates. Based on the values of d from Table 13, the MoP-based LQU pdfs provide a better fit to the chest circumference, thigh circumference, bodyfat percentage, and abdomen circumference data variables than the MoLM-based LQU pdfs, while MoLM-based LQU pdfs provide a better fit to the cholesterol and plasma fibrinogen data.

Mathematica (Version 14.2) [60] was used for computing sample estimates from each data distribution in Figure 6, solving for the model parameters for each of the methods (MoLM, MoM, and MoP), obtaining the LQU pdf for each of the three methods, and computing percentiles from data variables and LQU pdfs in Figure 6. The percentile parameters and estimates along with the values of Euclidean distance (d) for each method are presented in Table 13.



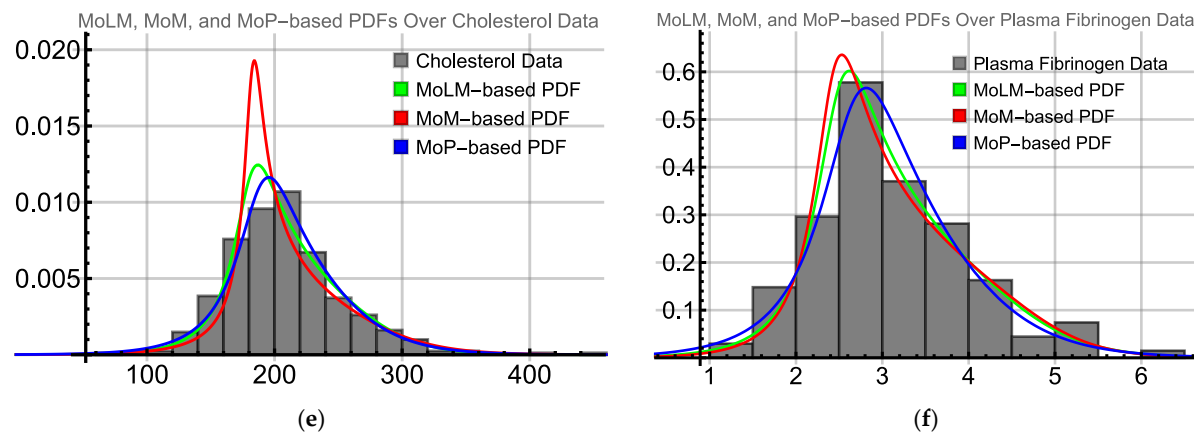


Figure 6. MoLM-, MoM-, and MoP-based pdfs of LQU distributions superimposed over the (a) Chest circumference, (b) thigh circumference, (c) bodyfat percentage, (d) abdomen circumference, (e) cholesterol, and (f) plasma fibrinogen data.

Table 12. Sample estimates with solved values of model parameters for the MoLM-, MoM-, and MoP-based pdfs of LQU distributions superimposed over histograms of data variables in Figure 6.

Data	Sample Estimates					Model Parameters			
	Method	Location	Scale	Skewness	Kurtosis	α_0	α_1	α_2	α_3
Chest	MoLM	$\hat{\lambda}_1$ 100.82	$\hat{\lambda}_2$ 4.6705	$\hat{\lambda}_3$ 0.4956	$\hat{\lambda}_4$ 0.6638	101.24	−10.743	14.868	3.9830
	MoP	$\hat{\xi}_1$ 99.650	$\hat{\xi}_2$ 21.480	$\hat{\xi}_3$ 0.6887	$\hat{\xi}_4$ 0.5249	100.07	−7.0250	12.375	3.9140
	MoM	$\hat{\mu}$ 100.82	$\hat{\sigma}$ 8.4305	$\hat{\gamma}_3$ 0.6775	$\hat{\gamma}_4$ 0.9441	102.60	−16.913	20.041	4.0966
Thigh	MoLM	$\hat{\lambda}_1$ 59.406	$\hat{\lambda}_2$ 2.8740	$\hat{\lambda}_3$ 0.2297	$\hat{\lambda}_4$ 0.4663	60.326	−6.4331	6.8907	2.7977
	MoP	$\hat{\xi}_1$ 59.000	$\hat{\xi}_2$ 13.070	$\hat{\xi}_3$ 0.8671	$\hat{\xi}_4$ 0.4935	60.152	−3.7563	2.9063	3.1289
	MoM	$\hat{\mu}$ 59.406	$\hat{\sigma}$ 5.2500	$\hat{\gamma}_3$ 0.8163	$\hat{\gamma}_4$ 2.5894	61.279	−7.6810	5.9035	1.5070
Bodyfat%	MoLM	$\hat{\lambda}_1$ 19.151	$\hat{\lambda}_2$ 4.7759	$\hat{\lambda}_3$ 0.0742	$\hat{\lambda}_4$ 0.3947	12.299	12.219	2.2257	2.3684
	MoP	$\hat{\xi}_1$ 19.200	$\hat{\xi}_2$ 22.050	$\hat{\xi}_3$ 1.0155	$\hat{\xi}_4$ 0.5839	9.8172	19.031	−0.5313	1.6498
	MoM	$\hat{\mu}$ 19.151	$\hat{\sigma}$ 8.3687	$\hat{\gamma}_3$ 0.1455	$\hat{\gamma}_4$ −0.3509	13.393	8.3491	4.7509	2.5771
Abdomen	MoLM	$\hat{\lambda}_1$ 92.556	$\hat{\lambda}_2$ 5.9433	$\hat{\lambda}_3$ 0.5712	$\hat{\lambda}_4$ 0.7867	91.743	−9.7989	17.137	4.7202
	MoP	$\hat{\xi}_1$ 90.950	$\hat{\xi}_2$ 26.480	$\hat{\xi}_3$ 0.7701	$\hat{\xi}_4$ 0.5684	84.588	7.3500	10.750	2.7307
	MoM	$\hat{\mu}$ 92.556	$\hat{\sigma}$ 10.783	$\hat{\gamma}_3$ 0.8334	$\hat{\gamma}_4$ 2.1807	100.15	−34.629	29.159	6.649
Chol	MoLM	$\hat{\lambda}_1$ 207.85	$\hat{\lambda}_2$ 24.108	$\hat{\lambda}_3$ 2.7273	$\hat{\lambda}_4$ 4.1129	223.19	−85.234	81.819	24.677
	MoP	$\hat{\xi}_1$ 204.00	$\hat{\xi}_2$ 107.70	$\hat{\xi}_3$ 0.7455	$\hat{\xi}_4$ 0.4759	229.27	−75.063	49.063	29.241
	MoM	$\hat{\mu}$ 207.85	$\hat{\sigma}$ 44.446	$\hat{\gamma}_3$ 0.9266	$\hat{\gamma}_4$ 2.5718	245.93	−164.12	131.94	28.739
Plasma fib	MoLM	$\hat{\lambda}_1$ 3.0904	$\hat{\lambda}_2$ 0.4909	$\hat{\lambda}_3$ 0.0592	$\hat{\lambda}_4$ 0.0764	3.2896	−1.5832	1.7772	0.4586
	MoP	$\hat{\xi}_1$ 2.9800	$\hat{\xi}_2$ 2.1680	$\hat{\xi}_3$ 0.7626	$\hat{\xi}_4$ 0.4797	3.4281	−1.3525	0.9125	0.5735
	MoM	$\hat{\mu}$ 3.0904	$\hat{\sigma}$ 0.8875	$\hat{\gamma}_3$ 0.6681	$\hat{\gamma}_4$ 0.7380	3.1968	−1.6276	2.1223	0.4057

Note: Chest = chest circumference, Thigh = thigh circumference, Bodyfat% = bodyfat percentage, Abdomen = abdomen circumference, Chol = cholesterol, Plasma fib = plasma fibrinogen data.

Table 13. Percentile parameters (P) obtained from data variables, percentile estimates (E) obtained, respectively, from MoLM- and MoP-based LQU pdfs superimposed over the histograms of six data variables in Figure 6 (panels a–f), and Euclidean distances (d).

Data/LQU pdf	Percentile							d
	0.05	0.10	0.25	0.50	0.75	0.90	0.95	
Chest (P)	88.80	91.10	94.20	99.60	105.30	112.30	117.00	
MoLM (E)	89.01	91.56	95.11	99.59	105.92	112.37	116.18	1.4648
MoP (E)	88.23	90.89	94.79	99.65	106.06	112.37	116.09	1.4595
Thigh (P)	51.10	53.00	56.00	59.00	62.30	66.00	68.60	
MoLM (E)	51.78	53.60	56.07	58.83	62.45	66.26	68.67	0.9753
MoP (E)	50.76	52.93	55.96	59.00	62.41	66.00	68.42	0.4082
Bodyfat% (P)	6.00	8.30	12.40	19.20	25.30	30.00	32.60	
MoLM (E)	5.94	8.34	12.89	18.97	25.32	30.30	32.89	0.6875
MoP (E)	5.91	8.09	12.73	19.20	25.60	30.14	32.28	0.6108
Abdomen (P)	76.60	79.50	84.50	90.90	99.20	105.70	111.20	
MoLM (E)	77.40	80.56	85.18	91.13	99.22	107.18	111.80	2.1977
MoP (E)	76.94	79.43	84.10	90.95	99.15	105.91	109.31	1.9753
Chol (P)	145.00	158.00	179.00	204.00	230.00	265.00	292.00	
MoLM (E)	146.47	161.26	179.88	201.03	232.40	266.97	288.72	6.5409
MoP (E)	139.54	158.00	181.44	204.00	232.69	265.70	288.34	7.5423
Plasma fib (P)	1.82	2.05	2.52	2.98	3.56	4.21	4.79	
MoLM (E)	1.86	2.14	2.50	2.94	3.61	4.31	4.74	0.1634
MoP (E)	1.67	2.04	2.52	2.98	3.56	4.21	4.66	0.1987

Note: Chest = chest circumference, Thigh = thigh circumference, Bodyfat% = bodyfat percentage, Abdomen = abdomen circumference, Chol = cholesterol, Plasma fib = plasma fibrinogen data. $d = \sqrt{\sum (E - P)^2}$.

5. Discussion

In this paper, we introduced a new family of logistic-quantile and uniform (LQU) mixture polynomial distributions as a linear combination of the quantile functions of standard logistic and uniform (0, 1) distributions. This family is studied by the method of L -moments (MoLM), method of moments (MoM), and method of percentiles (MoP). System of equations for the parameters of location, scale, skewness, and kurtosis were derived for each of these three methods (MoLM, MoM, and MoP). A methodology was provided to solve for the model parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ for each method. For each parameter estimation method, the proposed family of LQU distributions was systematically assessed to determine how accurately the model parameters could be estimated and how effectively the distributions could model healthcare data.

The MoLM-based procedure for the family of LQU distributions was superior to the MoM- and MoP-based procedures in the context of parameter estimation. The MoLM-based estimates of L -skewness and L -kurtosis can be far less biased than the MoM-based estimates of skewness and kurtosis even for small samples drawn from distributions with severe departures from normality (e.g., [25,31,43]). The simulation results in Tables 9–11 clearly indicate the superiority of the MoLM-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of L -skewness (τ_3) and L -kurtosis (τ_4) over their corresponding MoM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skewness (γ_3) and kurtosis (γ_4) and MoP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left–right tail-weight ratio (ξ_3) and tail-weight factor (ξ_4) in terms of much smaller relative biases (RB%) and smaller standard errors (SEs) in the context of asymmetric LQU distributions in Figure 5. For example, for the fourth LQU distribution (LQU4) in Figure 5, for $n = 25$, the simulated MoLM-based estimates ($\hat{\tau}_3$ and $\hat{\tau}_4$) of L -skewness and L -kurtosis were, on average, 100.65% and 99.49% of their respective parameters (τ_3 and τ_4). On the other hand,

the simulated MoM-based estimates ($\hat{\gamma}_3$ and $\hat{\gamma}_4$) of skewness and kurtosis were, on average, 72.24% and 33.50% of their respective parameters (γ_3 and γ_4). The simulated MoP-based estimates ($\hat{\xi}_3$ and $\hat{\xi}_4$) of left–right tail-weight ratio and tail-weight factor were, on average, 145.20% and 105.56% of their respective parameters (ξ_3 and ξ_4).

Another comparison of MoLM-based estimates with their MoM- and MoP-based counterparts can be made by comparing their relative standard errors (RSEs), where $RSE = 100 \times (\text{St. Error}/\text{Estimate})$. From Tables 6–11, it is evident that the MoLM-based estimates of τ_3 and τ_4 are similar in precision to the MoP-based estimates of ξ_3 and ξ_4 as their RSEs are similar, but the MoLM-based estimates are more efficient than the MoM-based estimates of γ_3 and γ_4 , as their RSEs are considerably smaller than the RSEs associated with the MoM-based estimates of γ_3 and γ_4 . For example, for the fourth LQU distribution (LQU4) in Figure 5, with $n = 1000$, the RSE values associated with MoLM-based estimates $\hat{\tau}_3$ and $\hat{\tau}_4$ were 0.15% and 0.06%, respectively. On the other hand, the RSE values associated with MoM-based estimates $\hat{\gamma}_3$ and $\hat{\gamma}_4$ were 0.43% and 0.35%, respectively, whereas the RSE values associated with MoP-based estimates $\hat{\xi}_3$ and $\hat{\xi}_4$ were 0.11% and 0.06%, respectively. Thus, in terms of RSE, the MoLM- and MoP-based estimates are superior to the MoM-based estimates, while both MoLM- and MoP-based estimates have similar precision.

In terms of modeling healthcare data, the pdfs of LQU distributions fitted using the method of percentiles (MoP) provide a slightly better approximation than the corresponding LQU distributions fitted using method of L -moments (MoLM), with each framework being superior to MoM-based approximation. Inspection of Figures 3 and 6 clearly indicates this fact. From Table 13, the percentile estimates associated with MoP-based LQU pdfs in Figure 6 (panels a–d) are much closer to their corresponding percentile parameters with smaller Euclidean distance (d) compared to the MoLM-based LQU pdfs.

6. Conclusions

In conclusion, the results indicate that, in terms of parameter estimation, the model parameters of LQU mixture polynomial distributions estimated using the method of L -moments (MoLM) demonstrated superior performance compared with those estimated using the method of moments (MoM) and method of percentiles (MoP), producing more accurate and precise parameter estimates. Conversely, with respect to data modeling, the LQU mixture polynomial distributions fitted using the method of percentiles (MoP) generally provided a closer approximation to the empirical distributions of real-world datasets than the corresponding distributions using the method of L -moments (MoLM) and the method of moments (MoM). Future work should focus on extending the LQU family to the multivariate setting, thereby enabling the generation of correlated random variables and broadening its applicability to multivariate statistical modeling.

Finally, Mathematica (Version 14.2) [60] and R (Version 4.5.2) [61] algorithms are available from the first author for implementing the MoLM-, MoM-, and MoP-based procedures.

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Conflicts of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as potential conflicts of interest.

Appendix A. Method of Moments (MoM)-Based LQU Distribution

The method of moments (MoM)-based product moments associated with the logistic-quantile and uniform (LQU) mixture polynomial distributions can be obtained by first evaluating the following integral for $r = 1, 2, 3, 4$.

$$\mu_r = \int_0^1 (Q_{LQU}(u))^r f(u) du \quad (A1)$$

where $Q_{LQU}(u)$ in Equation (A1) is the LQU mixture polynomial of the form in Equation (14), and $f(u) = 1$ is the pdf of random variable $U \sim \text{Uniform}(0, 1)$.

The first four MoM-based conventional moments ($\mu_{r=1,2,3,4}$) associated with the LQU mixture polynomial distribution can be computed by separately integrating Equation (A1) for $r = 1, 2, 3, 4$. Then, these four moments ($\mu_{r=1,2,3,4}$) can be substituted into the formulae of mean (μ) = μ_1 , variance (σ^2) = $\mu_2 - \mu_1^2$, and the following formulae of skewness (γ_3) and kurtosis (γ_4) from [29]:

$$\gamma_3 = ((\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3)/(\mu_2 - \mu_1^2)^{3/2}) \quad (A2)$$

$$\gamma_4 = ((\mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4)/(\mu_2 - \mu_1^2)^2) \quad (A3)$$

After substituting the first four moments (μ_1, μ_2, μ_3 , and μ_4)—obtained by integrating Equation (A1) for $r = 1, 2, 3, 4$ —into Equations (A2) and (A3), the formulae for the measures of skewness (γ_3) and kurtosis (γ_4), preceded by those of the mean (μ) and standard deviation (σ) for the MoM-based LQU distributions, can be given as follows:

$$\mu = \alpha_0 + (3\alpha_1 + 2\alpha_2)/6 \quad (A4)$$

$$\sigma = \sqrt{\frac{\alpha_1^2}{12} + \frac{4\alpha_2^2}{45} + \frac{\pi^2\alpha_3^2}{3} + \frac{\alpha_1\alpha_2}{6} + \alpha_1\alpha_3 + \alpha_2\alpha_3} \quad (A5)$$

$$\gamma_3 = \frac{2\sqrt{5}\alpha_2(63\alpha_1^2 + 126\alpha_1\alpha_2 + 64\alpha_2^2 + 945(\alpha_1 + \alpha_2)\alpha_3 + 3780\alpha_3^2)}{7(15\alpha_1^2 + 30\alpha_1\alpha_2 + 16\alpha_2^2 + 180(\alpha_1 + \alpha_2)\alpha_3 + 60\pi^2\alpha_3^2)^{3/2}} \quad (A6)$$

$$\gamma_4 = [-\{6(315\alpha_1^4 + 1260\alpha_1^3\alpha_2 + 1830\alpha_1^2\alpha_2^2 + 1140\alpha_1\alpha_2^3 + 256\alpha_2^4 + 1260(\alpha_1 + \alpha_2)(5\alpha_1^2 + 10\alpha_1\alpha_2 + 4\alpha_2^2)\alpha_3 + 6300(6\alpha_1^2 + 12\alpha_1\alpha_2 + 5\alpha_2^2)\alpha_3^2 - 5040\pi^4\alpha_3^4)\}/\{7(15\alpha_1^2 + 30\alpha_1\alpha_2 + 16\alpha_2^2 + 180(\alpha_1 + \alpha_2)\alpha_3 + 60\pi^2\alpha_3^2)^2\}] \quad (A7)$$

The model parameters, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, associated with the MoM-based standardized LQU mixture polynomial distribution can be determined by simultaneously solving Equations (A4)–(A7) using Mathematica (Version 14.2) (Wolfram Research Inc., 2025) function FindRoot after substituting mean (μ) = 0, standard deviation (σ) = 1, and the user-specified values of skewness (γ_3) and kurtosis (γ_4) on the right-hand-sides of Equations (A4)–(A7), respectively. The solved values of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ can then be substituted into Equations (14)–(16), respectively, to obtain the quantile function (qf), probability density function (pdf), and cumulative distribution function (cdf) of a standardized LQU mixture polynomial distribution.

Appendix B. Parameter Space for LQU Model Parameters ($\alpha_0, \alpha_1, \alpha_2, \alpha_3$)

Because the LQU mixture polynomial in Equation (14) represents a qf, it must be monotonically increasing over its domain. Consequently, its first derivative $Q'_{LQU}(u)$ must satisfy the condition $Q'_{LQU}(u) \geq 0$. This condition of nonnegativity is also necessary to ensure that the pdf defined in Equation (15) is valid. Thus,

$$Q'_{LQU}(u) = d(\alpha_3 Q_L(u) + \{\alpha_2 u^2 + \alpha_1 u + \alpha_0\})/du = (\alpha_3/(u(1-u))) + 2\alpha_2 u + \alpha_1 \geq 0. \quad (A8)$$

It is evident from Equation (A8) that the location parameter α_0 does not affect monotonicity, so it is unrestricted. Therefore, $\alpha_0 \in \mathbb{R}$. The model parameters must satisfy the following inequality:

$$(\alpha_3/(u(1-u))) + 2\alpha_2 u + \alpha_1 \geq 0 \text{ for all } u \in (0,1).$$

Since $\lim_{u \rightarrow 0} (1/(u(1-u))) = \lim_{u \rightarrow 1} (1/(u(1-u))) = \infty$, it follows that $\alpha_3 < 0$ would imply $Q'_{LQU}(u) \rightarrow -\infty$ as $u \rightarrow 0$ or $u \rightarrow 1$, contradicting monotonicity. Thus, $\alpha_3 \geq 0$ is a necessary condition.

Hence, the parameter space of the LQU mixture polynomial distribution is

$$\mathcal{P} = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4: \alpha_3 \geq 0, \alpha_1 + 2\alpha_2 u + \alpha_3/(u(1-u)) \geq 0 \text{ for all } u \in (0,1)\} \quad (A9)$$

Every 4-tuple in $\mathcal{P}$ specifies a probability distribution in terms of its qf in Equation (14). An estimation method is a mapping from the sample space to the parameter space $\mathcal{P}$ and hence to the family of distributions.

Special Cases:

1. If $\alpha_2 = 0$, then $Q'_{LQU}(u) = \alpha_1 + (\alpha_3/(u(1-u)))$. Since the minimum of $1/(u(1-u))$ is 4, the parameter space is $\{\alpha_0 \in \mathbb{R}, \alpha_2 = 0, \alpha_3 \geq 0, \alpha_1 + 4\alpha_3 \geq 0\}$.
2. If $\alpha_3 = 0$, then $Q'_{LQU}(u) = \alpha_1 + 2\alpha_2 u$. Since this is a linear function of u , the parameter space is $\{\alpha_0 \in \mathbb{R}, \alpha_1 \geq 0, \alpha_1 + 2\alpha_2 \geq 0, \alpha_3 = 0\}$.
3. Setting $\alpha_1 = 0$ yields a special case of the LQU distribution in Equation (14), whose parameter space can be obtained by substituting $\alpha_1 = 0$ into the general monotonicity condition.

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