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Modeling Various Data Structures via the New Type II Exponentiated Half Logistic-Odd Log-Logistic-G Power Series Class of Distributions

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Abstract

In this paper, we introduce the type II exponentiated half logistic-odd log-logistic-G power series class of distributions for modeling symmetric, skewed and heavy-tailed data with diverse hazard rate shapes. The proposed class of distributions is obtained by compounding the generalized family of distributions involving the type II exponentiated half logistic-G and odd log-logistic-G families with a discrete power series distribution. Various statistical properties of the proposed class of distributions, including moments, survival and hazard rate functions, order statistics, probability weighted moments, and Rényi entropy are derived. The model parameters are estimated using different estimation methods, and their performance is evaluated through Monte Carlo simulation studies. Finally, the flexibility and applicability of the proposed class of distributions are illustrated using real data sets. The results demonstrate that the proposed model provides a better fit than several existing competing models.

Keywords: type II exponentiated half logistic; odd log-logistic-G; power series distribution; heavy-tailed; estimation methods

1. Introduction

Modeling lifetime and reliability data is critical in diverse fields such as engineering, environmental science, finance, and biomedical studies. Classical lifetime distributions, including exponential, Weibull, and gamma models, have been widely used to analyzed datasets due to their mathematical tractability. However, empirical data often exhibit complex characteristics such as skewness, heavy tails, and non-monotonic hazard rates, which cannot be adequately captured by these classical models. This motivates the development of more flexible distributions that incorporate additional shape parameter(s) and transformation techniques to better reflect real-world phenomena.

One widely used approach for constructing flexible distributions is the generator-based family of distributions, which enhances flexibility in modeling skewness, tail behavior, and hazard rate function (hrf) dynamics. Two generator families that have attracted recent interest are the type II exponentiated half Logistic-G (TIEHL-G) family of distributions by [1] and odd Log-Logistic-G (OLL-G) family of distributions by [2]. These distributions control the shape of the distribution and hrf, enabling improved data fitting.



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Several extensions of the TIIHL-G family of distributions have been proposed. Examples include the type II exponentiated half-logistic exponential distribution [3], type II exponentiated half-logistic Gompertz-G family of distributions [4], type II exponentiated half-logistic-Gompertz Topp-Leone-G family of distributions [5] and generalized gamma type II exponentiated half logistic-Topp-Leone-G family of distributions [6]. Compound class of distributions, such as the type II exponentiated half logistic-odd Burr X-G power series class of distributions, have also been developed and explored [7].

The OLL-G family of distributions [2], extends any baseline distribution via the odd log-logistic transformation, adding a shape parameter to control skewness and tail weight. The OLL-G framework is particularly flexible in modeling diverse hrf shapes, including increasing, decreasing, bathtub, and unimodal patterns, and has been applied successfully in reliability, survival, hydrology, finance, and environmental studies. Related extensions include the beta odd log-logistic generalized distribution [8], odd log-logistic Weibull-G family of distributions [9], Topp-Leone-heavy-tailed-odd log-logistic-G family of distributions [10], odd log-logistic Poisson-G family of distributions [11] and the odd log-logistic generalized exponential distribution [12].

The cumulative distribution function (cdf) and probability density function (pdf) of the new type II exponentiated half logistic-odd log-logistic-G (TIIHL-OLL-G) family of distributions are given by

$$F_{TIIHL-OLL-G}(x; \beta, \alpha, \xi) = 1 - \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^\beta \quad (1)$$

and

$$f_{TIIHL-OLL-G}(x; \beta, \alpha, \xi) = 2\beta(1 - K_G(x; \alpha, \xi))^{\beta-1}(1 + K_G(x; \alpha, \xi))^{-(\beta+1)} \times \frac{\alpha g(x; \xi) [G(x; \xi)]^{\alpha-1} [\bar{G}(x; \xi)]^{\alpha-1}}{([G(x; \xi)]^\alpha + [\bar{G}(x; \xi)]^\alpha)^2}, \quad (2)$$

where $\beta, \alpha > 0$ are shape parameters, ξ is the baseline parameter vector, $\bar{G}(x; \xi) = 1 - G(x; \xi)$ denotes the survival function of the baseline distribution, and

$$K_G(x; \alpha, \xi) = \frac{[G(x; \xi)]^\alpha}{[G(x; \xi)]^\alpha + [\bar{G}(x; \xi)]^\alpha}.$$

Although these developments have enhanced flexibility, existing generator-based families still exhibit limitations in capturing complex data features. Integrating the TIIHL-G and OLL-G generator frameworks with power series compounding provides a promising approach to overcoming these limitations. Motivated by this, we propose a new class of distributions, called the type II exponentiated half logistic-odd log-logistic-G power series (TIIHL-OLL-GPS) distribution. The proposed distribution introduces additional shape parameter(s) that enhance flexibility, accommodate a wide variety of hazard rate function shapes, and generate heavy-tailed models suitable for modeling skewed data. Furthermore, it generalizes several existing distributions and yields numerous new sub-families as special cases. Owing to its increased flexibility, the proposed class is expected to provide improved goodness-of-fit compared with both classical and recently proposed competing models.

Suppose N is a discrete random variable following a power series (PS) distribution and assumed to be truncated at zero with the probability mass function (pmf) given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots, \quad (3)$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n \geq 1}$ a sequence of positive real numbers. The power series class of distributions includes Poisson, binomial, geometric and logarithmic distributions

The remainder of this paper is organized as follows. Section 2 introduces the proposed class of distributions and presents its pdf, cdf, hrf, sub-classes, asymptotes and identifiability issue. In Section 4, several structural properties are derived, including moments, moment generating function, quantile function, distribution of order statistics, probability weighted moments and entropy measures. Section 5 presents several special cases of the proposed class of distributions. Section 6 discusses parameter estimation using different estimation methods. In Section 7, a Monte Carlo simulation study is conducted to evaluate the performance of the estimators. Section 8 introduces risk measures that are useful for analyzing the potential exposure of a portfolio to losses. Section 9 illustrates the applicability of the proposed class of distributions using real data sets and compares its performance with competing models. Finally, Section 10 provides concluding remarks and possible directions for future research.

2. The New Generalized Class of Distributions

In this section, we present the new type II exponentiated half logistic-odd log-logistic-G power series (TIIHL-OLL-GPS) class of distributions and its sub-classes of generalized distributions. Let X be a random variable such that X_i , for $i = 1, 2, \dots, N$ denote the time to failure of a device due to the i^{th} defect with the assumption that the X_i 's are independent and identically distributed (iid) TIIHL-OLL-G random variables and $X_{(1)} = \min(X_1, X_2, \dots, X_N)$, with the distribution of N given by Equation (3). The TIIHL-OLL-GPS class of distributions is defined by the marginal cdf, given by

$$F(x; \beta, \alpha, \theta, \xi) = 1 - \frac{C\left(\theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^\beta\right)}{C(\theta)}, \quad (4)$$

with the corresponding pdf and hrf given as

$$\begin{aligned} f(x; \beta, \alpha, \theta, \xi) &= 2\beta\theta(1 - K_G(x; \alpha, \xi))^{\beta-1}(1 + K_G(x; \alpha, \xi))^{-(\beta+1)} \\ &\quad \times \frac{\alpha g(x; \xi) [G(x; \xi)]^{\alpha-1} [\overline{G}(x; \xi)]^{\alpha-1}}{([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha)^2} \\ &\quad \times \frac{C'\left(\theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^\beta\right)}{C(\theta)}, \end{aligned} \quad (5)$$

and

$$\begin{aligned} h(x; \beta, \alpha, \theta, \xi) &= 2\beta\theta(1 - K_G(x; \alpha, \xi))^{\beta-1}(1 + K_G(x; \alpha, \xi))^{-(\beta+1)} \\ &\quad \times \frac{\alpha g(x; \xi) [G(x; \xi)]^{\alpha-1} [\overline{G}(x; \xi)]^{\alpha-1}}{([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha)^2} \\ &\quad \times \frac{C'\left(\theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^\beta\right)}{C\left(\theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^\beta\right)}, \end{aligned} \quad (6)$$

respectively, where $\alpha, \beta, \theta > 0$, ξ is the baseline parameter vector and

$$K_G(x; \alpha, \xi) = \frac{[G(x; \xi)]^\alpha}{[G(x; \xi)]^\alpha + [\bar{G}(x; \xi)]^\alpha}.$$

The baseline hazard determines the fundamental failure behaviour, while the generator parameters modify the skewness and tail behaviour through the function $K_G(x; \alpha, \xi)$. The power-series component further adjusts the hazard by introducing an additional weighting through the ratio involving $C'(\cdot)$ and $C(\cdot)$. Consequently, the combined effect of these components allows the proposed class to exhibit increasing, decreasing, bathtub, upside-down bathtub and unimodal hazard rate shapes, depending on the baseline distribution and parameter values.

2.1. Reversed Hazard Rate Function

The reversed hazard rate function (RHRF) is an important reliability measure for lifetime data and is defined by

$$r(x) = \frac{f(x)}{F(x)}. \quad (7)$$

The reversed hazard rate function of the proposed TIIIEHL-OLL-GPS class of distributions is given by

$$\begin{aligned} r(x; \beta, \alpha, \theta, \xi) &= \frac{2\beta\theta(1 - K_G(x; \alpha, \xi))^{\beta-1}(1 + K_G(x; \alpha, \xi))^{-(\beta+1)}}{C(\theta) - C\left(\theta\left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)}\right)^\beta\right)} \\ &\quad \times \frac{\alpha g(x; \xi)[G(x; \xi)]^{\alpha-1}\bar{G}(x; \xi)^{\alpha-1}}{([G(x; \xi)]^\alpha + \bar{G}(x; \xi)^\alpha)^2} \\ &\quad \times C'\left(\theta\left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)}\right)^\beta\right). \end{aligned} \quad (8)$$

2.2. Asymptotes

In this subsection, we present the asymptotic behavior of the cdf and pdf of the TIIIEHL-OLL-GPS class of distributions.

As $x \rightarrow 0$, the baseline cdf satisfies $G(x; \xi) \rightarrow 0$, and $\bar{G}(x; \xi) \rightarrow 1$, so that

$$K_G(x; \alpha, \xi) = \frac{[G(x; \xi)]^\alpha}{[G(x; \xi)]^\alpha + [\bar{G}(x; \xi)]^\alpha} \sim [G(x; \xi)]^\alpha,$$

which gives

$$F(x; \beta, \alpha, \theta, \xi) \sim 1 - \frac{C\left(\theta\left(\frac{1 - [G(x; \xi)]^\alpha}{1 + [G(x; \xi)]^\alpha}\right)^\beta\right)}{C(\theta)},$$

$$\begin{aligned} f(x; \beta, \alpha, \theta, \xi) &\sim 2\beta\theta\alpha(1 - [G(x; \xi)]^\alpha)^{\beta-1}(1 + [G(x; \xi)]^\alpha)^{-(\beta+1)} \\ &\quad \times \frac{C'\left(\theta\left(\frac{1 - [G(x; \xi)]^\alpha}{1 + [G(x; \xi)]^\alpha}\right)^\beta\right)}{C(\theta)}. \end{aligned}$$

The asymptotes for cdf and pdf of the TIIIEHL-OLL-GPS class of distributions as $x \rightarrow \infty$, are given by

$$\begin{aligned}
F(x; \beta, \alpha, \theta, \xi) &= 1 - \frac{C\left(\theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^\beta\right)}{C(\theta)} \\
&= 1 - \frac{\sum_{n=1}^{\infty} a_n \theta^n \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^{\beta n}}{C(\theta)} \\
&\sim 1 - \frac{a_1 \theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^\beta}{C(\theta)}, \\
f(x; \beta, \alpha, \theta, \xi) &\sim \frac{\beta a_1 \theta \left(\frac{1-K_G(x; \alpha, \xi)}{1+K_G(x; \alpha, \xi)}\right)^{\beta-1}}{C(\theta)} \frac{2\alpha g(x; \xi) [G(x; \xi)]^{\alpha-1} [\overline{G}(x; \xi)]^{\alpha-1}}{\left([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha\right)^2}.
\end{aligned}$$

3. Identifiability

To establish the identifiability of the proposed TIIEHL-OLL-GPS family, we provide a formal proof based on the identifiability of the transformations involved in the construction of the model. The proof establishes identifiability through the cumulative distribution function and the injectivity of the underlying transformations.

Lemma 1. Let $T(x)$ be a function satisfying

$$0 < T(x) < 1.$$

If

$$T(x)^{\beta_1} = T(x)^{\beta_2}, \quad \forall x,$$

where $\beta_1 > 0$ and $\beta_2 > 0$, then

$$\beta_1 = \beta_2.$$

Proof. Taking logarithms gives

$$(\beta_1 - \beta_2) \log(T(x)) = 0, \quad \forall x.$$

Since $0 < T(x) < 1$, we have $\log(T(x)) \neq 0$ for every x . Hence, for any fixed x in the support,

$$(\beta_1 - \beta_2) \log(T(x)) = 0 \implies \beta_1 = \beta_2.$$

□

Define

$$K_G(x; \alpha, \xi) = \frac{G(x; \xi)^\alpha}{G(x; \xi)^\alpha + [1 - G(x; \xi)]^\alpha},$$

where $\alpha > 0$, and $0 < G(x; \xi) < 1$.

Assume that the functional mapping

$$(\alpha, \xi) \mapsto H(\cdot; \alpha, \xi)$$

is injective. This condition ensures that two different parameter vectors cannot generate the same transformed baseline function. This condition is a mild regularity requirement satisfied by identifiable baseline families whose logit-transformed representation is one-to-

one. It prevents different parameter combinations from generating the same transformed baseline function.

Then,

Lemma 2.

$$K_G(x; \alpha_1, \xi_1) = K_G(x; \alpha_2, \xi_2), \quad \forall x,$$

Implies $\alpha_1 = \alpha_2$ and $\xi_1 = \xi_2$.

Proof. Let $H(x; \alpha, \xi) = \alpha \log \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right)$. Suppose that

$$K_G(x; \alpha_1, \xi_1) = K_G(x; \alpha_2, \xi_2), \quad \forall x.$$

Let

$$G_i(x) = G(x; \xi_i), \quad i = 1, 2.$$

Then,

$$\frac{G_1(x)^{\alpha_1}}{G_1(x)^{\alpha_1} + [1 - G_1(x)]^{\alpha_1}} = \frac{G_2(x)^{\alpha_2}}{G_2(x)^{\alpha_2} + [1 - G_2(x)]^{\alpha_2}}.$$

Cross multiplication gives

$$G_1(x)^{\alpha_1} [1 - G_2(x)]^{\alpha_2} = G_2(x)^{\alpha_2} [1 - G_1(x)]^{\alpha_1}.$$

Hence,

$$\left(\frac{G_1(x)}{1 - G_1(x)} \right)^{\alpha_1} = \left(\frac{G_2(x)}{1 - G_2(x)} \right)^{\alpha_2}.$$

Taking logarithms,

$$\alpha_1 \log \left(\frac{G_1(x)}{1 - G_1(x)} \right) = \alpha_2 \log \left(\frac{G_2(x)}{1 - G_2(x)} \right).$$

Therefore,

$$H(x; \alpha_1, \xi_1) = H(x; \alpha_2, \xi_2), \quad \forall x.$$

By the injectivity assumption of $H(\cdot; \alpha, \xi)$, $\alpha_1 = \alpha_2$ and $\xi_1 = \xi_2$.

Hence, the transformation

$$K_G(x; \alpha, \xi)$$

is identifiable. $\square$

Theorem 1. Assume that:

- i The baseline family $G(x; \xi)$ is identifiable;
- ii The mapping

$$(\theta, t) \mapsto \frac{C(\theta t)}{C(\theta)}$$

is one-to-one on the parameter space $\{(\theta, t) : \theta > 0, 0 < t < 1\}$;

- iii The parameters satisfy

$$\alpha > 0, \quad \beta > 0, \quad \theta > 0;$$

- iv The transformed baseline quantity

$$H(x; \alpha, \xi) = \alpha \log \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right)$$

is identifiable, thereby excluding possible parameter redundancies arising from special sub-families or parameter combinations, then the TIIHL-OLL-GPS family of distributions is identifiable.

Proof. Let

$$\Theta_1 = (\beta_1, \alpha_1, \theta_1, \xi_1) \text{ and}$$

$$\Theta_2 = (\beta_2, \alpha_2, \theta_2, \xi_2)$$

be two parameter vectors satisfying

$$F(x; \Theta_1) = F(x; \Theta_2), \quad \forall x.$$

The cdf of the proposed family is

$$F(x; \Theta) = 1 - \frac{C \left(\theta \left[\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right]^\beta \right)}{C(\theta)}.$$

Define

$$T_i(x) = \frac{1 - K_G(x; \alpha_i, \xi_i)}{1 + K_G(x; \alpha_i, \xi_i)}, \quad i = 1, 2.$$

Equality of the two distribution functions implies

$$\frac{C(\theta_1 T_1(x)^{\beta_1})}{C(\theta_1)} = \frac{C(\theta_2 T_2(x)^{\beta_2})}{C(\theta_2)}, \quad \forall x.$$

By the injectivity of the transformation

$$(\theta, t) \mapsto \frac{C(\theta t)}{C(\theta)}$$

we obtain

$$\theta_1 = \theta_2$$

and

$$T_1(x)^{\beta_1} = T_2(x)^{\beta_2}, \quad \forall x.$$

By Lemma 1,

$$\beta_1 = \beta_2.$$

Therefore, $T_1(x) = T_2(x)$, $\forall x$, which implies

$$K_G(x; \alpha_1, \xi_1) = K_G(x; \alpha_2, \xi_2), \quad \forall x.$$

Using Lemma 2, $\alpha_1 = \alpha_2$ and $\xi_1 = \xi_2$. Therefore,

$$(\beta_1, \alpha_1, \theta_1, \xi_1) = (\beta_2, \alpha_2, \theta_2, \xi_2).$$

Hence, the TIIHL-OLL-GPS family is identifiable. $\square$

Sub-Classes and Families

Sub-classes and families of the new TIIHL-OLL-GPS class of distributions are presented in this subsection. Sub-classes of the new TIIHL-OLL-GPS class of distributions with

specified power series distributions are given in Table 1, where $\bar{F}_{TIIHL-OLL-G}(x; \beta, \alpha, \xi) = 1 - F_{TIIHL-OLL-G}(x; \beta, \alpha, \xi)$.

Table 1. Sub-classes of the TIIHL-OLL-GPS class of distributions.

Distribution	a_n	$C(\theta)$	Admissible Range	cdf
TIIHL-OLL-G Poisson	$(n!)^{-1}$	$e^\theta - 1$	$\theta > 0$	$1 - \frac{e^{\theta \bar{F}_{TIIHL-OLL-G}(x; \beta, \alpha, \xi)} - 1}{e^\theta - 1}$
TIIHL-OLL-G Geometric	1	$\frac{\theta}{1 - \theta}$	$0 < \theta < 1$	$1 - \frac{(1 - \theta) \bar{F}_{TIIHL-OLL-G}(x; \beta, \alpha, \xi)}{1 - \theta \bar{F}_{TIIHL-OLL-G}(x; \beta, \alpha, \xi)}$
TIIHL-OLL-G Logarithmic	n^{-1}	$-\log(1 - \theta)$	$0 < \theta < 1$	$1 - \frac{\log(1 - \theta \bar{F}_{TIIHL-OLL-G}(x; \beta, \alpha, \xi))}{\log(1 - \theta)}$
TIIHL-OLL-G Binomial	$\binom{m}{n}$	$(1 + \theta)^m - 1$	$\theta > 0$	$1 - \frac{(1 + \theta \bar{F}_{TIIHL-OLL-G}(x; \beta, \alpha, \xi))^m - 1}{(1 + \theta)^m - 1}$

- When $\alpha = 1$, we obtain the new type II exponentiated half logistic-G power series (TIIHL-GPS) class of distributions with the cdf

$$F(x; \beta, \theta, \xi) = 1 - \frac{C\left(\theta \left(\frac{1 - G(x; \xi)}{1 + G(x; \xi)}\right)^\beta\right)}{C(\theta)},$$

for $\beta, \theta > 0$ and parameter vector ξ .

- When $\beta = 1$, we obtain the new type II half logistic-odd log-logistic-G power series (TIIHL-OLL-GPS) class of distributions with the cdf

$$F(x; \alpha, \theta, \xi) = 1 - \frac{C(\theta \bar{F}_{TIIHL-OLL-G}(x; \alpha, \xi))}{C(\theta)},$$

for $\alpha, \theta > 0$ and parameter vector ξ , where $\bar{F}_{TIIHL-OLL-G}(x; \alpha, \xi) = \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)}\right)$.

- When $\alpha = \beta = 1$, we obtain the new type II half logistic-G power series (TIIHL-GPS) class of distributions with the cdf

$$F(x; \theta, \xi) = 1 - \frac{C\left(\theta \left(\frac{1 - G(x; \xi)}{1 + G(x; \xi)}\right)\right)}{C(\theta)},$$

for $\theta > 0$ and parameter vector ξ .

- When $\theta \rightarrow 0^+$, we obtain the new type II exponentiated half logistic-odd log-logistic-G family of distributions with the cdf

$$F(x; \beta, \alpha, \xi) = 1 - \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)}\right)^\beta,$$

for $\beta, \alpha > 0$ and parameter vector ξ .

- When $\theta \rightarrow 0^+$ and $\alpha = 1$, we obtain the type II exponentiated half logistic-G (TIIHL-G) family of distributions with the cdf

$$F(x; \beta, \xi) = 1 - \left(\frac{1 - G(x; \xi)}{1 + G(x; \xi)}\right)^\beta,$$

for $\beta > 0$ and parameter vector ξ . (See [1]).

- When $\theta \rightarrow 0^+$ and $\beta = 1$, we obtain the new type II half logistic-odd log-logistic-G (TIIHL-OLL-G) family of distributions with the cdf

$$F(x; \alpha, \xi) = 1 - \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right),$$

for $\alpha > 0$ and parameter vector ξ .

- When $\theta \rightarrow 0^+$ and $\alpha = \beta = 1$, we obtain the type II half logistic-G (TIIHL-G) family of distributions with the cdf

$$F(x; \xi) = 1 - \left(\frac{1 - G(x; \xi)}{1 + G(x; \xi)} \right),$$

for parameter vector ξ . (See [13]).

4. Some Properties

In this section, we derive several statistical properties for the TIIHL-OLL-GPS class of distributions, including density expansion, quantile function, moments, distribution of order statistics, Rényi entropy and probability weighted moments. For simplicity, we will let $F(x; \beta, \alpha, \theta, \xi) = F(x)$, and $f(x; \beta, \alpha, \theta, \xi) = f(x)$ from here onwards.

4.1. Quantile Function

The quantile function plays a key role in statistical analysis, particularly for generating random samples from a given probability distribution. For the TIIHL-OLL-GPS class of distributions, the quantile function is derived by solving the following non-linear equation:

$$1 - \frac{C\left(\theta \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)}\right)^\beta\right)}{C(\theta)} = u, \quad (9)$$

$0 \leq u \leq 1$. That is,

$$G(x; \xi) = \left(\left[\frac{1 - \frac{1-B}{1+B}}{\frac{1-B}{1+B}} \right]^{\frac{1}{\alpha}} + 1 \right)^{-1}, \quad (10)$$

where $B = \left(\left[\frac{C^{-1}((1-u)C(\theta))}{\theta} \right] \right)^{\frac{1}{\beta}}$. Therefore, the quantile function of the TIIHL-OLL-GPS class of distributions is given by

$$Q_X(u; \beta, \alpha, \theta, \xi) = G^{-1} \left[\left(\left[\frac{1 - \frac{1-B}{1+B}}{\frac{1-B}{1+B}} \right]^{\frac{1}{\alpha}} + 1 \right)^{-1} \right]. \quad (11)$$

Thus, for a specified baseline cdf $G(x; \xi)$ and function $C(\theta)$, Equation (11) can be used to generate random numbers from the TIIHL-OLL-GPS class of distributions.

4.2. Linear Representation of the Density Function

In this section, the pdf of the TIIHL-OLL-GPS class of distributions is reformulated in the structure of the exponentiated-G (Exp-G) density. By employing generalized binomial series expansion, Taylor series expansion and results on power series raised to a positive integer by [14], the pdf of the TIIHL-OLL-GPS class of distributions can be expressed as infinite linear combination of Exp-G densities. See Appendix A for details.

Considering:

$$C' \left(\theta \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^\beta \right) = \sum_{n=1}^{\infty} na_n \theta^{n-1} \left[\left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^\beta \right]^{(n-1)},$$

we can write the pdf of the TIIHL-OLL-GPS class of distributions (after simplification) as

$$f(x; \beta, \alpha, \theta, \xi) = \sum_{m=0}^{\infty} \xi_{m+1} g_{m+1}(x; \xi), \quad (12)$$

where

$$\begin{aligned} \xi_{m+1} &= \frac{2\beta\alpha\theta}{C(\theta)} \sum_{n=1}^{\infty} na_n \theta^{n-1} \sum_{i,j,k,l=0}^{\infty} \binom{n\beta-1}{i} \binom{n\beta+j}{j} \binom{-(i+j+2)}{k} \\ &\times \binom{\alpha(i+j+1+k)-1}{l} \binom{l - [\alpha(i+j+k+1)+1]}{m} \frac{(-1)^{i+j+l+m}}{(m+1)}, \end{aligned} \quad (13)$$

and $g_{m+1}(x; \xi) = (m+1)G^m(x; \xi)g(x; \xi)$ is the Exp-G pdf with power parameter $(m+1)$. Hence, the statistical properties of the TIIHL-OLL-GPS class of distributions can be obtained directly from those of the Exp-G distribution.

4.3. Infinite Series Convergence

4.3.1. First Series

Note that

$$C' \left(\theta \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^\beta \right) = \sum_{n=1}^{\infty} na_n \theta^{n-1} \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta(n-1)}.$$

Using the ratio test, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)a_{n+1}\theta^n \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta n}}{na_n \theta^{n-1} \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta(n-1)}} &= \lim_{n \rightarrow \infty} \frac{(n+1)a_{n+1}\theta \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta}}{na_n} \\ &= \theta \lim_{n \rightarrow \infty} \frac{(n+1)}{n} \frac{a_{n+1}}{a_n} \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta}. \end{aligned}$$

Since $K_G(x; \alpha, \xi)$ is a cdf, we have

$$0 < \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta} < 1.$$

Note that

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1,$$

and assuming that the power-series coefficients satisfy

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = b < \infty,$$

where $b \geq 0$, then the ratio test implies convergence provided that

$$\theta b \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta} < 1.$$

4.3.2. Second Series

$$(1 - K_G(x; \alpha, \xi))^{n\beta-1} = \sum_{i=0}^{\infty} \binom{n\beta-1}{i} (-1)^i (K_G(x; \alpha, \xi))^i.$$

Using the ratio test, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{\binom{n\beta-1}{i+1} (-1)^{i+1} (K_G(x; \alpha, \xi))^{i+1}}{\binom{n\beta-1}{i} (-1)^i (K_G(x; \alpha, \xi))^i} &= \lim_{i \rightarrow \infty} \frac{\binom{n\beta-1}{i+1} (-1) (K_G(x; \alpha, \xi))}{\binom{n\beta-1}{i}} \\ &= - \lim_{i \rightarrow \infty} \frac{\binom{n\beta-1}{i+1}}{\binom{n\beta-1}{i}} (K_G(x; \alpha, \xi)). \end{aligned}$$

Note that

$$\begin{aligned} \frac{\binom{n\beta-1}{i+1}}{\binom{n\beta-1}{i}} &= \frac{\frac{\Gamma(n\beta)}{\Gamma(i+2)\Gamma(n\beta-1-i)}}{\frac{\Gamma(n\beta)}{\Gamma(i+1)\Gamma(n\beta-i)}} \\ &= \frac{\Gamma(i+1)\Gamma(n\beta-i)}{\Gamma(i+2)\Gamma(n\beta-1-i)} \\ &= \frac{i!(n\beta-i-1)!}{(i+1)!(n\beta-2-i)!} \\ &= \frac{1}{(i+1)(n\beta-2-i)}. \end{aligned}$$

Then we have

$$\lim_{i \rightarrow \infty} \frac{\binom{n\beta-1}{i+1} (-1)^{i+1} (K_G(x; \alpha, \xi))^{i+1}}{\binom{n\beta-1}{i} (-1)^i (K_G(x; \alpha, \xi))^i} = \lim_{i \rightarrow \infty} \frac{-(K_G(x; \alpha, \xi))}{(i+1)(n\beta-2-i)} < 1.$$

Thus, the series converges for every $G(x; \alpha, \xi)$.

4.3.3. Third Series

$$(1 + K_G(x; \alpha, \xi))^{-(n\beta+1)} = \sum_{j=0}^{\infty} \binom{n\beta+j}{j} (-1)^j (K_G(x; \alpha, \xi))^j.$$

Using the ratio test, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{\binom{n\beta+j+1}{j+1} (-1)^{j+1} (K_G(x; \alpha, \xi))^{j+1}}{\binom{n\beta+j}{j} (-1)^j (K_G(x; \alpha, \xi))^j} &= \lim_{j \rightarrow \infty} \frac{\binom{n\beta+j+1}{j+1} (-1) (K_G(x; \alpha, \xi))}{\binom{n\beta+j}{j}} \\ &= - \lim_{j \rightarrow \infty} \frac{\binom{n\beta+j+1}{j+1}}{\binom{n\beta+j}{j}} (K_G(x; \alpha, \xi)). \end{aligned}$$

Note that

$$\begin{aligned} \frac{\binom{n\beta+j+1}{j+1}}{\binom{n\beta+j}{j}} &= \frac{\frac{\Gamma(n\beta+j+2)}{\Gamma(j+2)\Gamma(n\beta+1)}}{\frac{\Gamma(n\beta+j+1)}{\Gamma(j+1)\Gamma(n\beta+1)}} \\ &= \frac{\Gamma(j+1)\Gamma(n\beta+j+2)}{\Gamma(j+2)\Gamma(n\beta+j+1)} \\ &= \frac{j!(n\beta+j+1)!}{(j+1)!(n\beta+j)!} \\ &= \frac{(n\beta+j+1)}{(j+1)}. \end{aligned}$$

Then we have

$$\lim_{j \rightarrow \infty} \frac{\binom{n\beta+j+1}{j+1}(-1)^{j+1}(K_G(x; \alpha, \xi))^{j+1}}{\binom{n\beta+j}{j}(-1)^j(K_G(x; \alpha, \xi))^j} = -\lim_{j \rightarrow \infty} \frac{(n\beta+j+1)}{(j+1)}(K_G(x; \alpha, \xi)) < 1.$$

Thus, the series converges for every $G(x; \alpha, \xi)$.

4.3.4. Fourth Series

$$([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha)^{-(i+j+2)} = \sum_{k=0}^{\infty} \binom{-(i+j+2)}{k} ([G(x; \xi)]^\alpha)^k ([\overline{G}(x; \xi)]^\alpha)^{-(i+j+2)-k}.$$

Using the ratio test, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\binom{-(i+j+2)}{k+1} ([G(x; \xi)]^\alpha)^{k+1} ([\overline{G}(x; \xi)]^\alpha)^{-(i+j+2)-(k+1)}}{\binom{-(i+j+2)}{k} ([G(x; \xi)]^\alpha)^k ([\overline{G}(x; \xi)]^\alpha)^{-(i+j+2)-k}} &= \lim_{k \rightarrow \infty} \frac{\binom{-(i+j+2)}{k+1} [G(x; \xi)]^\alpha}{\binom{-(i+j+2)}{k} [\overline{G}(x; \xi)]^\alpha} \\ &= \lim_{k \rightarrow \infty} \frac{-(k+i+j+2)[G(x; \xi)]^\alpha}{(k+1)[\overline{G}(x; \xi)]^\alpha} \\ &= < 1. \end{aligned}$$

Thus, the series converges for every $G(x; \alpha, \xi)$.

4.4. Moments and Generating Function

Suppose Y_{m+1} denote the Exp-G random variable with power parameter $(m+1)$. Using Equation (12), we obtain the r^{th} moment of the TIIEHL-OLL-GPS class of distributions as follows:

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{m=0}^{\infty} \xi_{m+1} E(Y_{m+1}^r), \quad (14)$$

where $E(Y_{m+1}^r)$ is the r^{th} moment of Y_{m+1} which follows Exp-G distribution with power parameter $(m+1)$ and ξ_{m+1} is defined as Equation (13). The moment generating function is given by

$$M_X(t) = E(e^{tX}) = \sum_{m=0}^{\infty} \xi_{m+1} E(e^{tY_{m+1}}),$$

where $E(e^{tY_{m+1}})$ is the moment generating function of an Exp-G random variable with power parameter $(m+1)$ and ξ_{m+1} is given by Equation (13).

4.5. Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed TIIEHL-OLL-GPS random variables and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. Then, the pdf of the q^{th} order statistic can be written as:

$$f_{q:n}(x) = \frac{n!f(x)}{(q-1)!(n-q)!} \sum_{w=0}^{n-q} (-1)^w \binom{n-q}{w} [F(x)]^{w+q-1}.$$

Following the same steps leading to Equation (12), we can write

$$f(x)[F(x)]^{w+q-1} = \sum_{m=0}^{\infty} \Omega_{m+1} g_{m+1}(x; \xi),$$

where the expression Ω_{m+1} is provided in the Appendix A. Consequently, the pdf of the q^{th} order statistic for the TIIHL-OLL-GPS class of distributions is given by

$$f_{q:n}(x) = \frac{n!}{(q-1)!(n-q)!} \sum_{w=0}^{n-q} \sum_{m=0}^{\infty} (-1)^w \binom{n-q}{w} \Omega_{m+1} g_{m+1}(x; \xi).$$

The detailed derivation of Ω_{m+1} is given in the Appendix A.

4.6. Probability Weighted Moments

The probability weighted moments (PWMs) are useful for parameter estimation and distributional analysis. For a random variable X , the PWMs of a random variable X are defined by

$$\varepsilon_{\eta,w} = E\left(X^\eta [F(X)]^w\right) = \int_{-\infty}^{\infty} x^\eta [F(x)]^w f(x) dx.$$

Now, following the same steps leading to the distribution of order statistics, we can write

$$f(x)[F(x)]^w = \sum_{m=0}^{\infty} \varrho_{m+1} g_{m+1}(x; \xi),$$

where the coefficient ϱ_{m+1} is provided in the Appendix A. Hence, the PWMs of the TIIHL-OLL-GPS class of distributions is given by

$$\varepsilon_{\eta,w} = E\left(X^\eta [F(X)]^w\right) = \sum_{m=0}^{\infty} \varrho_{m+1} \int_{-\infty}^{\infty} x^\eta g_{m+1}(x; \xi) dx.$$

See Appendix A for additional details.

4.7. Rényi Entropy

Rényi entropy is a measure of uncertainty associated to a random variable X and is defined as

$$H_R(r) = \frac{1}{1-r} \log \left(\int_0^\infty f^r(x) dx \right),$$

for $r > 0, r \neq 1$. It is well known that Rényi entropy converges to Shannon entropy as $r \rightarrow 1$. Note that $f_{TIIHL-OLL-GPS}^r(x; \theta, \alpha, \beta, \xi) = f^r(x)$ can be written as

$$\begin{aligned} f^r(x) &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j,k,l,m=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} \\ &\times \binom{-(2r+i+j)}{k} \binom{\alpha(i+j+r+k)-r}{l} \binom{-[\alpha(r+i+j+k)+r]}{m} \\ &\times (-1)^{i+j+l+m} G^m(x; \xi) g^r(x; \xi), \end{aligned}$$

where $d_{z,r} = (zb_0)^{-1} \sum_{l=1}^z [l(r+1)-z] b_l d_{z-l,r}$ and $d_{0,r} = b_0^r$. Thus, Rényi entropy of the TIIHL-OLL-GPS family of distributions can be written as

$$\begin{aligned} I_R(r) &= \frac{1}{1-r} \log \left[\frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j,k,l,m=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} \right. \\ &\times \binom{-(2r+i+j)}{k} \binom{\alpha(i+j+r+k)-r}{l} \binom{-[\alpha(r+i+j+k)+r]}{m} \\ &\times \left. \frac{(-1)^{i+j+l+m}}{\left[1+\frac{m}{r}\right]^r} \int_0^\infty \left[1+\frac{m}{r}\right] g(x; \xi) G^{\frac{m}{r}}(x; \xi)^r dx \right] \\ &= (1-r)^{-1} \log \left(\sum_{m=0}^{\infty} C_{m+1} e^{(1-r)I_{REG}} \right), \end{aligned}$$

where

$$I_{REG} = \frac{1}{1-r} \log \left[\int_0^\infty \left(\left(1 + \frac{m}{r} \right) G^{\frac{m}{r}}(x; \xi) g(x; \xi) \right)^r dx \right]$$

is the Rényi entropy of the Exp-G distribution with power parameter $(1 + \frac{m}{r})$, and

$$\begin{aligned} C_{m+1} &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j,k,l=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} \\ &\times \binom{-(2r+i+j)}{k} \binom{\alpha(i+j+r+k)-r}{l} \binom{-[\alpha(r+i+j+k)+r]}{m} \\ &\times \frac{(-1)^{i+j+l+m}}{\left[1 + \frac{m}{r}\right]^r}. \end{aligned}$$

5. Some Special Cases

The TIIIEHL-OLL-GPS class of distributions includes numerous distributions as special cases. The following subsections discuss some special cases arising from the TIIIEHL-OLL-GPS class of distributions. We define $K_{LLoG}(x; \alpha, \lambda) = \frac{[1 - (1+x^\lambda)^{-1}]^\alpha}{[1 - (1+x^\lambda)^{-1}]^\alpha + [(1+x^\lambda)^{-1}]^\alpha}$ and

$$K_{SHL}(x; \alpha) = \frac{\left[\frac{(1-\exp(-x))}{(1+\exp(-x))} \right]^\alpha}{\left[\frac{(1-\exp(-x))}{(1+\exp(-x))} \right]^\alpha + \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))} \right]^\alpha}$$
 to simplify notations in this section.

5.1. Type II Exponentiated Half Logistic-Odd Log-Logistic-Log-Logistic Power Series (TIIIEHL-OLL-LLoG-PS) Class of Distributions

If the baseline distribution is the log-logistic distribution with cdf and pdf given by $G(x; \lambda) = 1 - (1 + x^\lambda)^{-1}$ and $g(x; \lambda) = \lambda x^{\lambda-1} (1 + x^\lambda)^{-2}$, respectively, for $\lambda, x > 0$, then the cdf and pdf of the TIIIEHL-OLL-LLoG-PS class of distributions are

$$F(x; \beta, \alpha, \theta, \lambda) = 1 - \frac{C \left(\theta \left(\frac{1 - K_{LLoG}(x; \alpha, \lambda)}{1 + K_{LLoG}(x; \alpha, \lambda)} \right)^\beta \right)}{C(\theta)},$$

and

$$\begin{aligned} f(x; \beta, \alpha, \theta, \lambda) &= 2\beta\theta(1 - K_{LLoG}(x; \alpha, \lambda))^{\beta-1} (1 + K_{LLoG}(x; \alpha, \lambda))^{-(\beta+1)} \\ &\times \frac{\alpha\lambda x^{\lambda-1} (1 + x^\lambda)^{-2} [1 - (1 + x^\lambda)^{-1}]^{\alpha-1} [(1 + x^\lambda)^{-1}]^{\alpha-1}}{\left([1 - (1 + x^\lambda)^{-1}]^\alpha + [(1 + x^\lambda)^{-1}]^\alpha \right)^2} \\ &\times \frac{C' \left(\theta \left(\frac{1 - K_{LLoG}(x; \alpha, \lambda)}{1 + K_{LLoG}(x; \alpha, \lambda)} \right)^\beta \right)}{C(\theta)}, \end{aligned}$$

respectively for $\beta, \alpha, \theta, \lambda > 0$, and $x > 0$.

5.1.1. Type II Exponentiated Half Logistic-Odd Log-Logistic-Log-Logistic Poisson (TIIIEHL-OLL-LLoGP) Distribution

By taking the Poisson distribution as the underlying power series distribution, we obtain the cdf and pdf of the TIIIEHL-OLL-LLoGP distribution as follows:

$$F(x; \beta, \alpha, \theta, \lambda) = 1 - \frac{\exp \left(\theta \left(\frac{1 - K_{LLoG}(x; \alpha, \lambda)}{1 + K_{LLoG}(x; \alpha, \lambda)} \right)^\beta \right) - 1}{\exp(\theta) - 1},$$

and

$$\begin{aligned}
 f(x; \beta, \alpha, \theta, \lambda) &= 2\beta\theta(1 - K_{LLoG}(x; \alpha, \lambda))^{\beta-1}(1 + K_{LLoG}(x; \alpha, \lambda))^{-(\beta+1)} \\
 &\times \frac{\alpha\lambda x^{\lambda-1}(1+x^\lambda)^{-2}[1 - (1+x^\lambda)^{-1}]^{\alpha-1}[(1+x^\lambda)^{-1}]^{\alpha-1}}{\left([1 - (1+x^\lambda)^{-1}]^\alpha + [(1+x^\lambda)^{-1}]^\alpha\right)^2} \\
 &\times \frac{\exp\left(\theta\left(\frac{1-K_{LLoG}(x; \alpha, \lambda)}{1+K_{LLoG}(x; \alpha, \lambda)}\right)^\beta\right)}{\exp(\theta) - 1},
 \end{aligned}$$

respectively for $\beta, \alpha, \theta, \lambda > 0$, and $x > 0$.

The pdf plots in Figure 1 can have several shapes, such as left-skewed, right-skewed, almost-symmetric and reverse J-shaped, depending on parameter values. The hrf exhibits a variety of shapes, including decreasing, increasing and upside-down bathtub shapes.

Figure 2 presents the skewness and kurtosis plots for the TIIEHL-OLL-LLoGP distribution. It is observed that, when θ and λ are held constant, skewness increases while kurtosis first decreases and then increases as β and α vary. Conversely, when β and α are fixed, both skewness and kurtosis increase with increasing values of θ and λ .

5.1.2. Type II Exponentiated Half Logistic-Odd Log-Logistic-Log-Logistic Logarithmic (TIIEHL-OLL-LLoGL) Distribution

By considering the logarithmic distribution as the underlying power series distribution, we derive the cdf and pdf of the TIIEHL-OLL-LLoGL distribution as follows:

$$F(x; \beta, \alpha, \theta, \lambda) = 1 - \frac{-\log\left(1 - \theta\left(\frac{1-K_{LLoG}(x; \alpha, \lambda)}{1+K_{LLoG}(x; \alpha, \lambda)}\right)^\beta\right)}{-\log(1 - \theta)},$$

and

$$\begin{aligned}
 f(x; \beta, \alpha, \theta, \lambda) &= 2\beta\theta(1 - K_{LLoG}(x; \alpha, \lambda))^{\beta-1}(1 + K_{LLoG}(x; \alpha, \lambda))^{-(\beta+1)} \\
 &\times \frac{\alpha\lambda x^{\lambda-1}(1+x^\lambda)^{-2}[1 - (1+x^\lambda)^{-1}]^{\alpha-1}[(1+x^\lambda)^{-1}]^{\alpha-1}}{\left([1 - (1+x^\lambda)^{-1}]^\alpha + [(1+x^\lambda)^{-1}]^\alpha\right)^2} \\
 &\times \frac{\left(1 - \theta\left(\frac{1-K_{LLoG}(x; \alpha, \lambda)}{1+K_{LLoG}(x; \alpha, \lambda)}\right)^\beta\right)^{-1}}{-\log(1 - \theta)},
 \end{aligned}$$

respectively for $\beta, \alpha, \lambda > 0$, $0 < \theta < 1$, and $x > 0$.

Figure 3 exhibits the pdf with several shapes, such as left-skewed, right-skewed, almost-symmetric, and reverse J-shaped, depending on the parameter values. The hrf exhibits a variety of shapes, including decreasing, increasing, upside-down bathtub and a bimodal hazard rate pattern with two upside-down bathtub components.

Figure 4 presents the skewness and kurtosis plots for the TIIEHL-OLL-LLoGL distribution. It is observed that, when β and θ are held constant, both skewness and kurtosis initially decrease and then increase as α and λ vary. A similar pattern is observed when λ and θ are fixed and β and α vary.

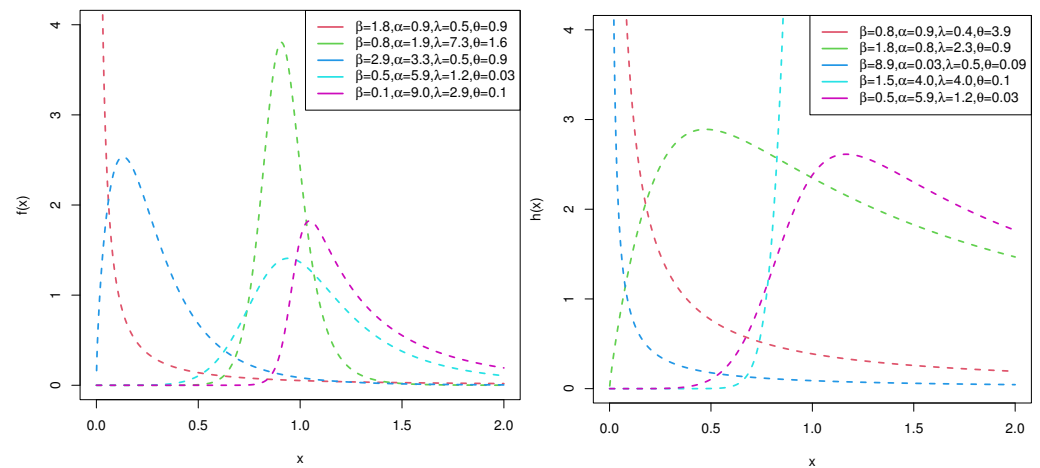


Figure 1. Plots of the Pdf and Hrf of the TIIEHL-OLL-LLoGP distribution for different combinations of the parameters β , α , and θ .

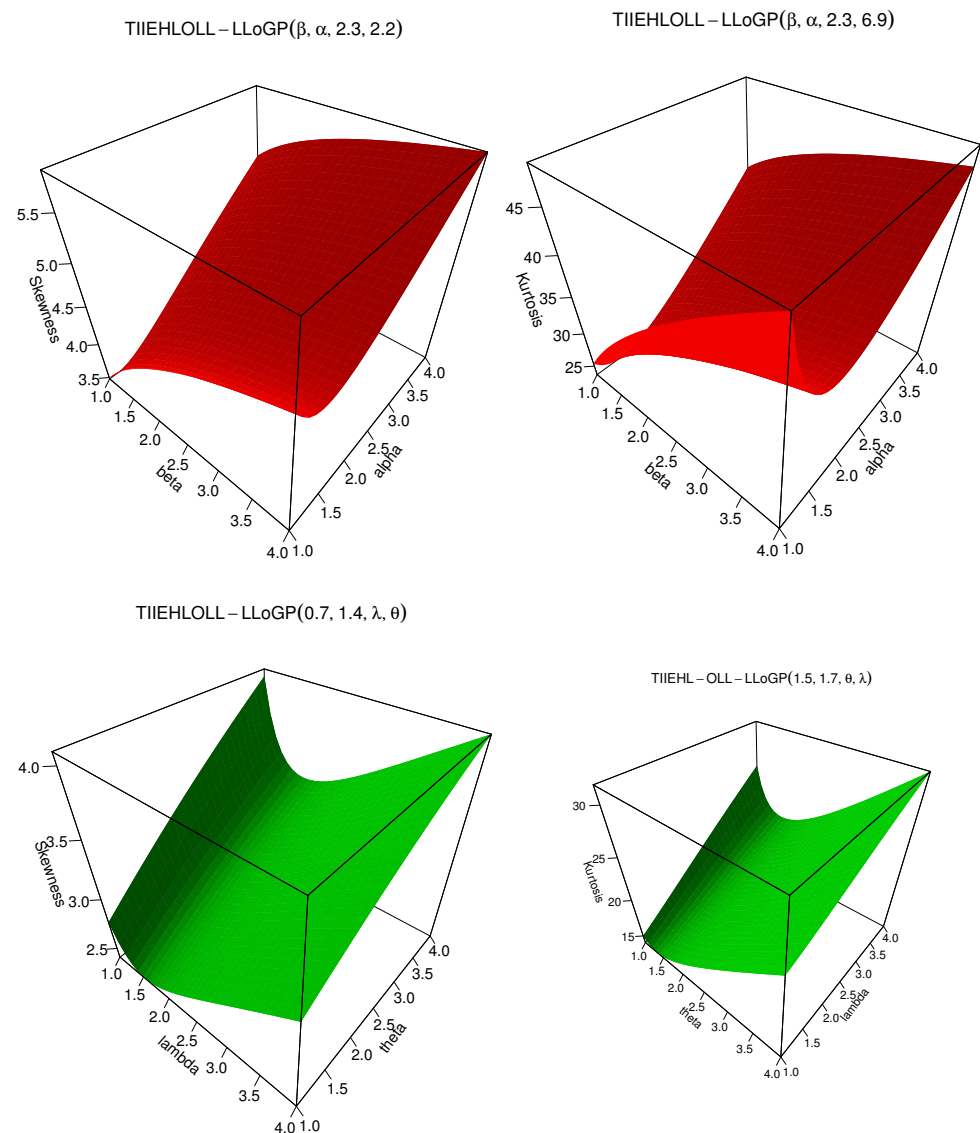


Figure 2. Skewness and Kurtosis Plots for TIIEHL-OLL-LLoGP Distribution.

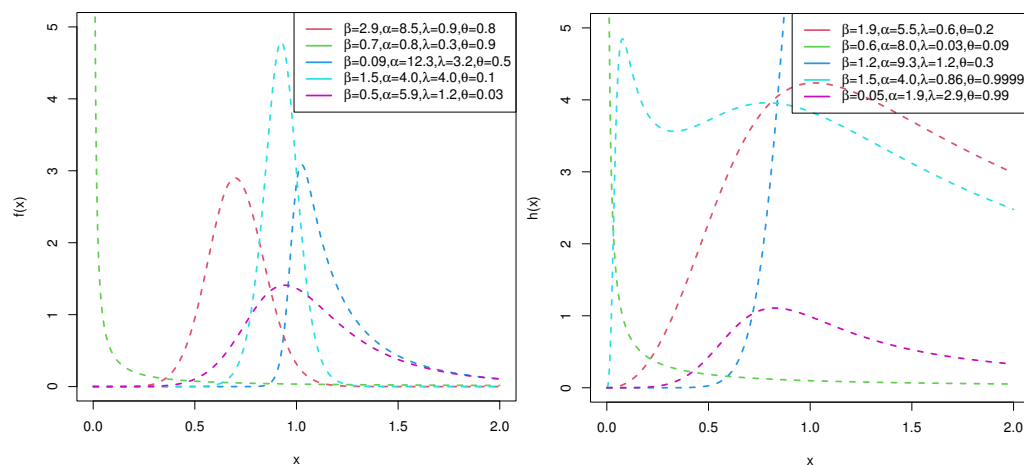


Figure 3. Plots of the Pdf and Hrf for TIIEHL-OLL-LLoGL Distribution for different combinations of the parameters β , α , λ and θ .

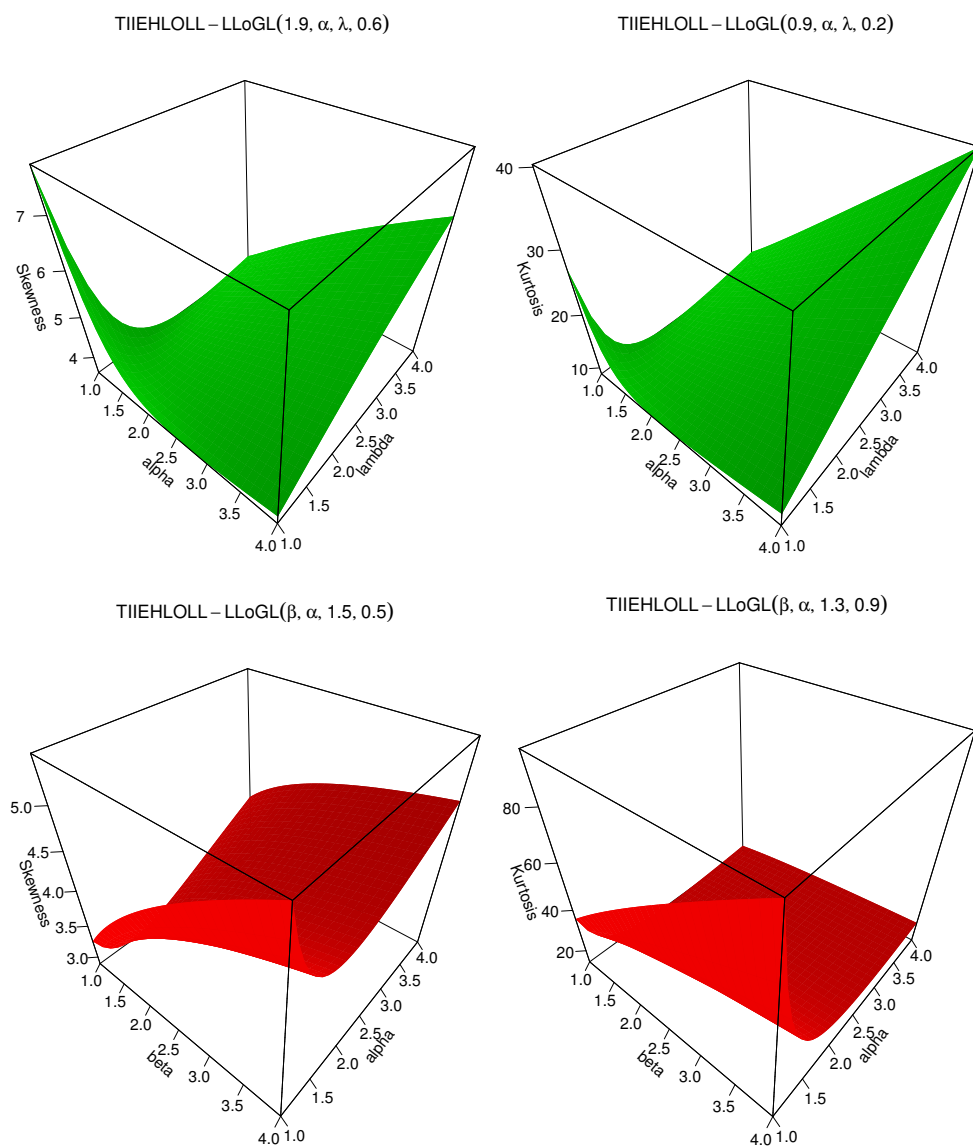


Figure 4. Skewness and Kurtosis Plots of TIIEHL-OLL-LLoGL Distribution.

5.2. Type II Exponentiated Half Logistic-Odd Log-Logistic-Standard Half Logistic Power Series (TIIHL-OLL-SHL-PS) Distribution

If the baseline distribution is the standard half logistic distribution with pdf and cdf given by $g(x) = \frac{2\exp(-x)}{(1+\exp(-x))^2}$ and $G(x) = \frac{(1-\exp(-x))}{(1+\exp(-x))}$, for $x > 0$, respectively, then the cdf and pdf of the TIIHL-OLL-SHL-PS distribution are

$$F(x; \beta, \alpha, \theta) = 1 - \frac{C\left(\theta\left(\frac{1-K_{SHL}(x;\alpha)}{1+K_{SHL}(x;\alpha)}\right)^\beta\right)}{C(\theta)}, \quad (15)$$

and

$$\begin{aligned} f(x; \beta, \alpha, \theta) &= 2\beta\theta(1-K_{SHL}(x;\alpha))^{\beta-1}(1+K_{SHL}(x;\alpha))^{-(\beta+1)} \\ &\times \frac{\alpha \frac{2\exp(-x)}{(1+\exp(-x))^2} \left[\frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^{\alpha-1} \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^{\alpha-1}}{\left(\left[\frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^\alpha + \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^\alpha\right)^2} \\ &\times \frac{C'\left(\theta\left(\frac{1-K_{SHL}(x;\alpha)}{1+K_{SHL}(x;\alpha)}\right)^\beta\right)}{C(\theta)}, \end{aligned} \quad (16)$$

for $\beta, \alpha, \theta > 0$, and $x > 0$.

5.2.1. Type II Exponentiated Half Logistic-Odd Log-Logistic-Standard Half Logistic Poisson (TIIHL-OLL-SHLP) Distribution

By assuming the Poisson distribution as the underlying power series distribution, we derive the cdf and pdf of the TIIHL-OLL-SHLP distribution as follows:

$$F(x; \beta, \alpha, \theta) = 1 - \frac{\exp\left(\theta\left(\frac{1-K_{SHL}(x;\alpha)}{1+K_{SHL}(x;\alpha)}\right)^\beta\right) - 1}{\exp(\theta) - 1}, \quad (17)$$

and

$$\begin{aligned} f(x; \beta, \alpha, \theta) &= 2\beta\theta(1-K_{SHL}(x;\alpha))^{\beta-1}(1+K_{SHL}(x;\alpha))^{-(\beta+1)} \\ &\times \frac{\alpha \frac{2\exp(-x)}{(1+\exp(-x))^2} \left[\frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^{\alpha-1} \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^{\alpha-1}}{\left(\left[\frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^\alpha + \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^\alpha\right)^2} \\ &\times \frac{\exp\left(\theta\left(\frac{1-K_{SHL}(x;\alpha)}{1+K_{SHL}(x;\alpha)}\right)^\beta\right)}{\exp(\theta) - 1}, \end{aligned} \quad (18)$$

for $\beta, \alpha, \theta > 0$, and $x > 0$.

Figure 5 shows that the pdf can take different forms depending on the parameter values, including right-skewed, almost symmetric, and reverse J-shaped distributions. The hrf exhibits a variety of shapes, including decreasing, increasing, and upside-down bathtub forms.

Figure 6 presents the skewness and kurtosis plots for the TIIHL-OLL-SHLP distribution. It is observed that, when θ is held constant, both skewness and kurtosis initially decrease and then increase as β and α are changing. An increasing pattern is observed when α is fixed, and β and θ vary.

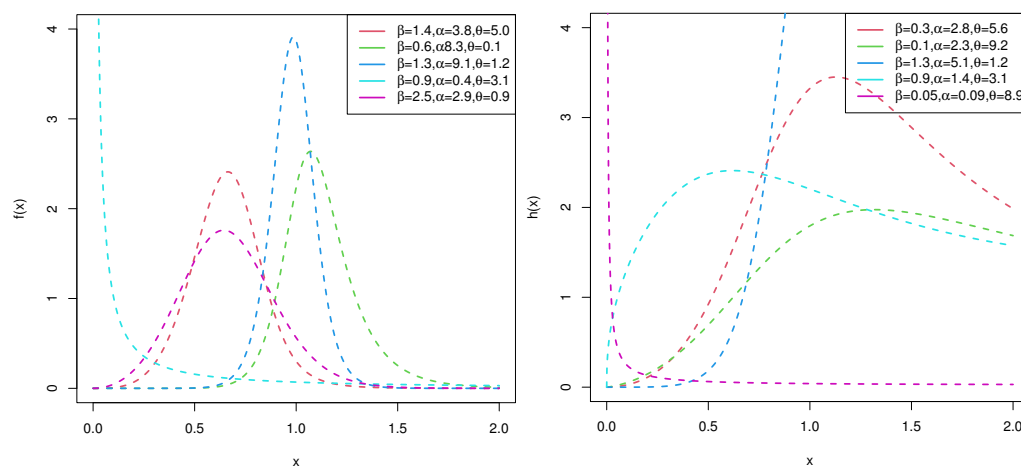


Figure 5. Plots of the Pdf and Hrf for TIIEHL-OLL-SHLP Distribution for different combinations of the parameters β , α , and θ .

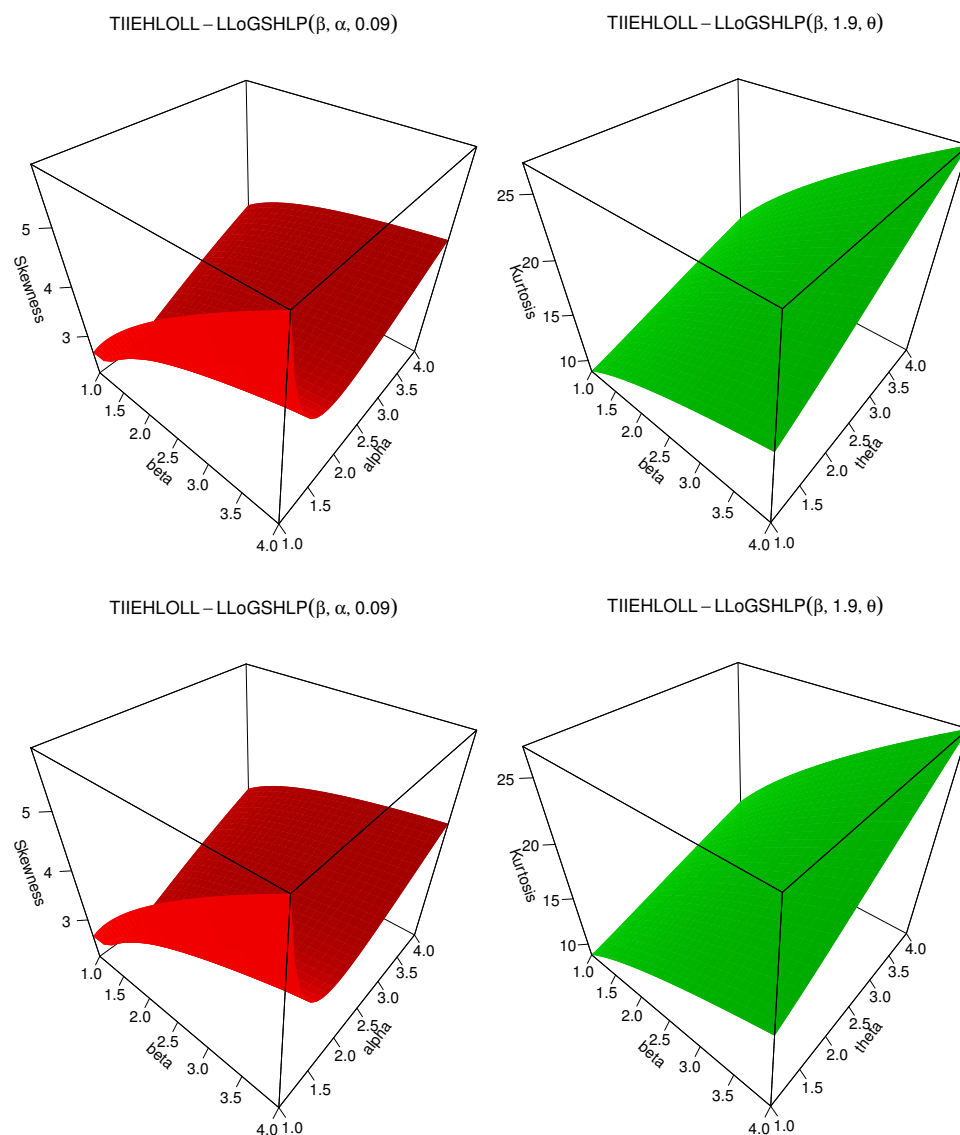


Figure 6. Skewness and Kurtosis for the TIIEHL-OLL-SHLP Distribution.

5.2.2. Type II Exponentiated Half Logistic-Odd Log-Logistic-Standard Half Logistic Logarithmic (TIIEHL-OLL-SHLL) Distribution

Assuming the logarithmic distribution as the underlying power series distribution, the cdf and pdf of the TIIEHL-OLL-SHLL distribution are derived as follows:

$$F(x; \beta, \alpha, \theta) = 1 - \frac{-\log\left(1 - \theta\left(\frac{1 - K_{SHL}(x; \alpha)}{1 + K_{SHL}(x; \alpha)}\right)^\beta\right)}{-\log(1 - \theta)}, \quad (19)$$

and

$$\begin{aligned} f(x; \beta, \alpha, \theta) &= 2\beta\theta(1 - K_{SHL}(x; \alpha))^{\beta-1}(1 + K_{SHL}(x; \alpha))^{-(\beta+1)} \\ &\times \frac{\alpha \frac{2\exp(-x)}{(1+\exp(-x))^2} \left[\frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^{\alpha-1} \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^{\alpha-1}}{\left(\left[\frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^\alpha + \left[1 - \frac{(1-\exp(-x))}{(1+\exp(-x))}\right]^\alpha\right)^2} \\ &\times \frac{\left(1 - \theta\left(\frac{1 - K_{SHL}(x; \alpha)}{1 + K_{SHL}(x; \alpha)}\right)^\beta\right)^{-1}}{-\log(1 - \theta)}, \end{aligned} \quad (20)$$

for $\beta, \alpha > 0, 0 < \theta < 1$, and $x > 0$.

Figure 7 demonstrates that the pdf can take several forms depending on the parameter values, including right-skewed, almost symmetric, and reverse J-shaped distribution. The hrf exhibits a range of behaviours, such as decreasing, increasing, upside-down bathtub followed by bathtub and upside-down bathtub shapes.

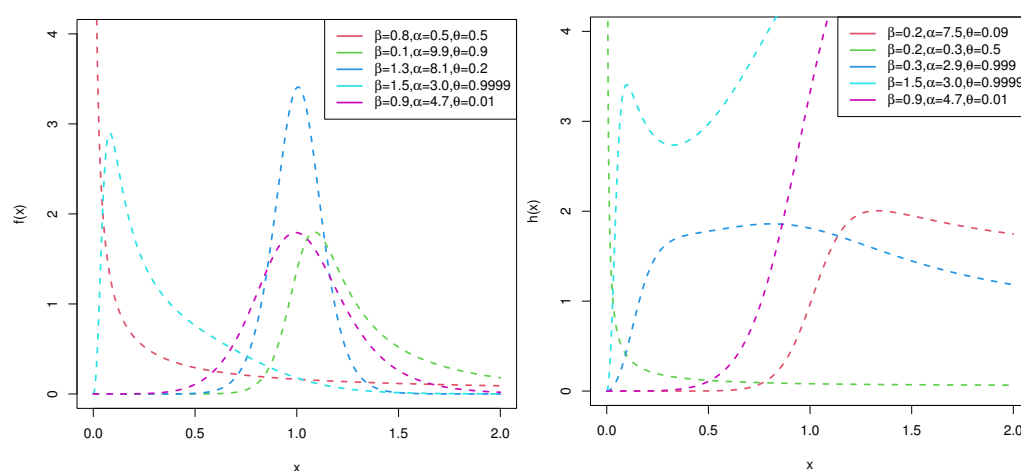


Figure 7. Plots of the Pdf and Hrf for TIIEHL-OLL-SHLL Distribution for different combinations of the parameters β , α , and θ .

Figure 8 presents the skewness and kurtosis plots for the TIIEHL-OLL-SHLL distribution. It is observed that, when θ is fixed, both skewness and kurtosis initially decrease and then increase as β and α vary.

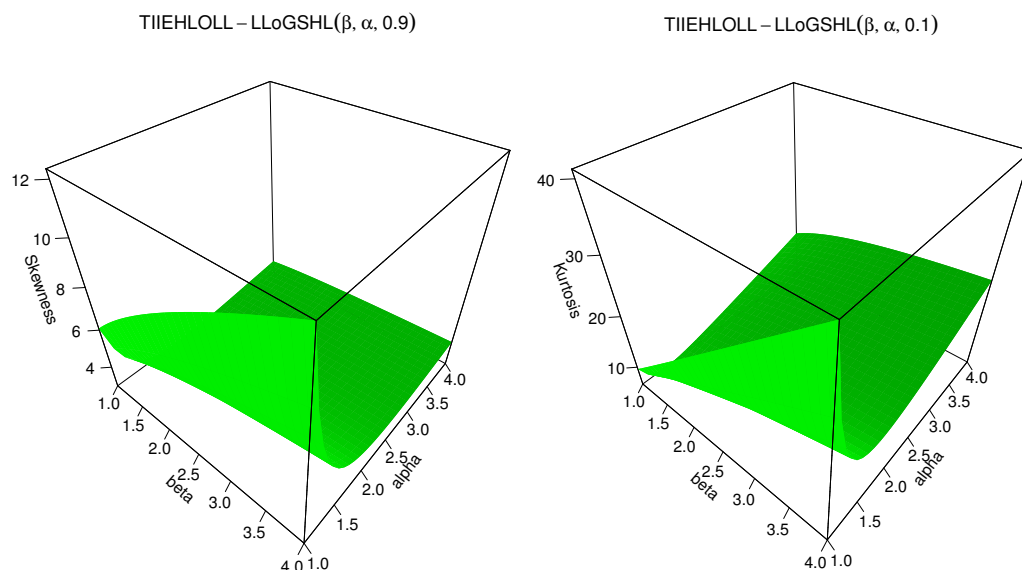


Figure 8. Skewness and Kurtosis for TIIEHL-OLL-SHLL Distribution.

6. Estimation Methods

In this section, we use different estimation methods to estimate the unknown parameters of the TIIEHL-OLL-GPS class of distributions. The estimation methods include Maximum Likelihood (ML), Ordinary Least Squares (LS), Weighted Least Squares (WLS) and Cramér-von Mises (CVM). Let $x_{(1)}, \dots, x_{(n)}$ denote the ordered sample observations drawn from the TIIEHL-OLL-GPS class of distributions. Denote $F(x_{(i)}; \beta, \alpha, \theta, \xi)$ and $S(x_{(i)}; \beta, \alpha, \theta, \xi)$ as the cdf and survival function of the i^{th} order statistic from the TIIEHL-OLL-GPS class of distributions, respectively.

6.1. Maximum Likelihood Estimation

Suppose $X \sim \text{TIIEHL} - \text{OLL} - \text{GPS}(\beta, \alpha, \theta, \xi)$. Let the parameter vector be denoted by $\Delta = (\beta, \alpha, \theta, \xi)^T$. Then, for a random sample of size n , the log-likelihood function $\ell_n = \ell_n(\Delta)$ is expressed as:

$$\begin{aligned} \ell_n(\Delta) &= (n) \ln(2\beta\alpha\theta) + (\beta - 1) \sum_{i=1}^n \ln(1 - K_G(x_i; \alpha, \xi)) \\ &\quad - (\beta + 1) \sum_{i=1}^n \ln(1 + K_G(x_i; \alpha, \xi)) + (\alpha - 1) \sum_{i=1}^n \ln[G(x_i; \xi)] \\ &\quad + \sum_{i=1}^n \ln \left(C' \left(\theta \left(\frac{1 - K_G(x_i; \alpha, \xi)}{1 + K_G(x_i; \alpha, \xi)} \right)^\beta \right) \right) - \sum_{i=1}^n \ln[C(\theta)] \\ &\quad + (\alpha - 1) \sum_{i=1}^n \ln[\overline{G}(x_i; \xi)] + \sum_{i=1}^n \ln(g(x_i; \xi)) \\ &\quad - 2 \sum_{i=1}^n \ln \left([G(x_i; \xi)]^\alpha + [\overline{G}(x_i; \xi)]^\alpha \right). \end{aligned}$$

To obtain the maximum likelihood estimates (MLEs) of the parameters of the TIIEHL-OLL-GPS class of distributions, we solve $U = \left(\frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \xi} \right)^T = \mathbf{0}$, numerically using the Newton-Raphson method. The partial derivatives of the log-likelihood function with re-

spect to each component of the parameter vector are given in the Appendix A. The observed Fisher information matrix is given by

$$J(\hat{\Delta}) = \begin{pmatrix} J_{\beta,\beta}(\hat{\Delta}) & J_{\beta,\alpha}(\hat{\Delta}) & J_{\beta,\theta}(\hat{\Delta}) & J_{\beta,\xi}(\hat{\Delta}) \\ J_{\alpha,\beta}(\hat{\Delta}) & J_{\alpha,\alpha}(\hat{\Delta}) & J_{\alpha,\theta}(\hat{\Delta}) & J_{\alpha,\xi}(\hat{\Delta}) \\ J_{\theta,\beta}(\hat{\Delta}) & J_{\theta,\alpha}(\hat{\Delta}) & J_{\theta,\theta}(\hat{\Delta}) & J_{\theta,\xi}(\hat{\Delta}) \\ J_{\xi,\beta}(\hat{\Delta}) & J_{\xi,\alpha}(\hat{\Delta}) & J_{\xi,\theta}(\hat{\Delta}) & J_{\xi,\xi}(\hat{\Delta}) \end{pmatrix}, \quad (21)$$

where $J_{i,j} = -\frac{\partial^2 \ell_n(\hat{\Delta})}{\partial \theta_i \partial \theta_j}$ for $i, j = \beta, \alpha, \theta, \xi$.

6.1.1. Standard Error Computation

The standard error of the estimate is a way to measure the accuracy of the predictions made by a model. It is also useful when calculating the confidence interval. It is given by

$$SE(\hat{\Delta}) = \sqrt{(J(\hat{\Delta}))^{-1}}, \quad (22)$$

where $(J(\hat{\Delta}))^{-1}$ is the inverse of the observed Fisher information matrix.

6.1.2. Regularity Conditions

Under the regularity conditions for maximum likelihood estimation, the MLE of Θ is consistent, asymptotically normal and asymptotically efficient. Specifically, let $\Delta = (\alpha, \beta, \theta, \xi)^T$ denote the vector of unknown parameters and let $\hat{\Delta}$ be its corresponding maximum likelihood estimator. Then, as the sample size $n \rightarrow \infty$, the MLE is consistent, asymptotically normal and asymptotically efficient. Specifically,

$$\hat{\Delta} \xrightarrow{P} \Delta, \text{ and } \sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{D} N(\mathbf{0}, J^{-1}(\Delta)),$$

where $\xrightarrow{P}$ and $\xrightarrow{D}$ denote convergence in probability and convergence in distribution, respectively, and $I(\Theta)$ is the Fisher Information Matrix.

The asymptotic theory of the maximum likelihood estimators is established under the standard regularity conditions for maximum likelihood estimation. These conditions are assumed to hold for interior parameter values, namely $\beta > 0, \alpha > 0, \theta > 0$, and ξ belonging to the interior of its admissible parameter space. These include: (i) the parameter space is an open subset of the Euclidean space containing the true parameter vector; (ii) the log-likelihood function is twice continuously differentiable with respect to the unknown parameters; (iii) the model is identifiable; (iv) differentiation and integration are interchangeable; and (v) the Fisher Information Matrix exists and is positive definite. Under these conditions, the maximum likelihood estimators are obtained numerically by solving the likelihood equations and are consistent, asymptotically normal, and asymptotically efficient. These asymptotic results do not extend to boundary parameter values (e.g., α or β approaching zero).

6.2. Ordinary Least Squares

The LS method estimates the parameters of the TIIEHL-OLL-GPS class of distributions by minimizing the objective function:

$$V(\beta, \alpha, \theta, \xi) = \sum_{i=1}^n \left[F(x_{(i)}; \beta, \alpha, \theta, \xi) - \frac{i}{n+1} \right]^2.$$

The LS estimates (LSEs) can be obtained by solving the non-linear equations:

$$\sum_{i=1}^n \left[F(x_{(i)}; \beta, \alpha, \theta, \xi) - \frac{i}{n+1} \right] \vartheta_z(x_{(i)}; \beta, \alpha, \theta, \xi) = 0, z = 1, 2, 3, 4,$$

where $\vartheta_z(x_{(i)}; \beta, \alpha, \theta, \xi)$ are defined as:

$$\vartheta_1(x_{(i)}; \beta, \alpha, \theta, \xi) = \frac{\partial F(x_{(i)}; \beta, \alpha, \theta, \xi)}{\partial \beta}, \quad (23)$$

$$\vartheta_2(x_{(i)}; \beta, \alpha, \theta, \xi) = \frac{\partial F(x_{(i)}; \beta, \alpha, \theta, \xi)}{\partial \alpha}, \quad (24)$$

$$\vartheta_3(x_{(i)}; \beta, \alpha, \theta, \xi) = \frac{\partial F(x_{(i)}; \beta, \alpha, \theta, \xi)}{\partial \theta}, \quad (25)$$

and

$$\vartheta_4(x_{(i)}; \beta, \alpha, \theta, \xi) = \frac{\partial F(x_{(i)}; \beta, \alpha, \theta, \xi)}{\partial \xi_k}. \quad (26)$$

6.3. Weighted Least Squares

The WLS method is a modified version of OLS where each term is weighted to give more importance to central observations. The WLS criterion function is:

$$W(\beta, \alpha, \theta, \xi) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}; \beta, \alpha, \theta, \xi) - \frac{i}{n+1} \right]^2.$$

The WLS estimates (WLSEs) of the parameters of the TIIHL-OLL-GPS class of distributions are obtained by solving the non-linear equations:

$$\sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}; \beta, \alpha, \theta, \xi) - \frac{i}{n+1} \right] \vartheta_z(x_{(i)}; \beta, \alpha, \theta, \xi) = 0, z = 1, 2, 3, 4,$$

where $\vartheta_z(x_{(i)}; \beta, \alpha, \theta, \xi)$ are defined in Equations (23)–(26).

6.4. Cramér-Von Mises

The CVM estimates (CVMEs) of the parameters of the TIIHL-OLL-GPS class of distributions are obtained through the minimization of the function:

$$C(\beta, \alpha, \theta, \xi) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{(i)}; \beta, \alpha, \theta, \xi) - \frac{2i-1}{2n} \right]^2,$$

with respect to β, α, θ and parameter vector ξ . The CVMEs can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^n \left[F(x_{(i)}; \beta, \alpha, \theta, \xi) - \frac{2i-1}{2n} \right] \vartheta_z(x_{(i)}; \beta, \alpha, \theta, \xi) = 0, z = 1, 2, 3, 4,$$

where $\vartheta_z(x_{(i)}; \beta, \alpha, \theta, \xi)$ are defined in Equations (23)–(26).

7. Simulation

In this section, a Monte Carlo simulation study is conducted to evaluate the performance of the ML, CVM, WLS, and LS methods for the TIIHL-OLL-GPS class of distribu-

tions. Random samples of sizes $n = 30, 60, 120, 240, 480$ and 960 are generated from the TIIHL-OLL-LLoGP distribution using two different parameter configurations presented in Tables 2 and 3. The experiment is repeated ($r = 3000$) times, and the average bias (ABIAS) and root mean square error (RMSE) of the estimators are computed to assess their performance. The results are summarized in the corresponding tables.

The ABIAS and RMSE are defined as follows:

$$\text{ABIAS}_{\hat{\Phi}}(n) = \frac{1}{r} \sum_{i=1}^r (\hat{\Phi} - \Phi), \quad \text{and} \quad \text{RMSE}_{\hat{\Phi}}(n) = \sqrt{\frac{1}{r} \sum_{i=1}^r (\hat{\Phi} - \Phi)^2}.$$

See two additional tables in the Appendix A.

The superscript shown beside an estimator in Tables 2 and 3 indicates its ranking among the competing methods. For instance, the RMSE value of 0.3334 for $\hat{\beta}$ under the ML method at $n = 30$ in Table 2 reflects the second-best performance for that case. The row labeled $\sum$ Ranks gives the cumulative ranking totals.

Table 4 summarizes both partial and overall rankings of the estimation methods for the TIIHL-OLL-LLoGP distribution. The results in Tables 2 and 3 show that both ABIAS and RMSE decrease as the sample size increases, indicating improved estimator performance. In general, larger samples yield more accurate and stable estimates across all methods. Based on the overall rankings, the ML method performs best, followed by LS and WLS, while CVM ranks last.

Figures 9 and 10 further illustrate that RMSE decreases with increasing sample size for all estimators, confirming improved estimation accuracy.

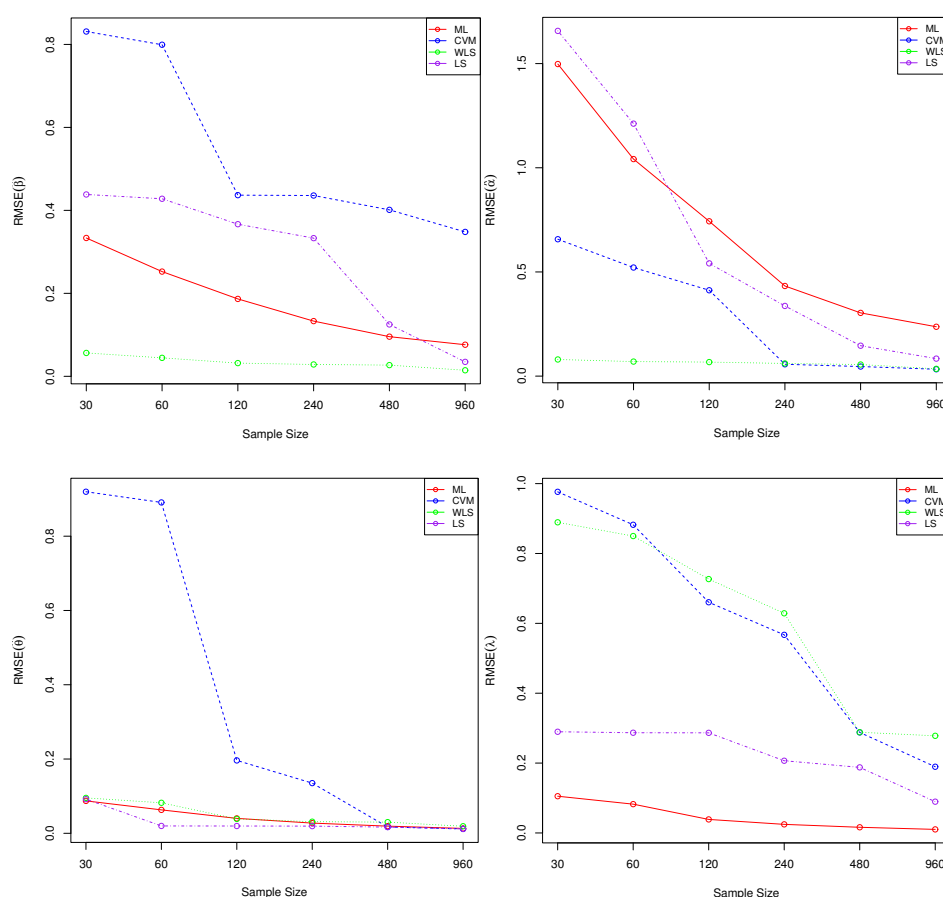
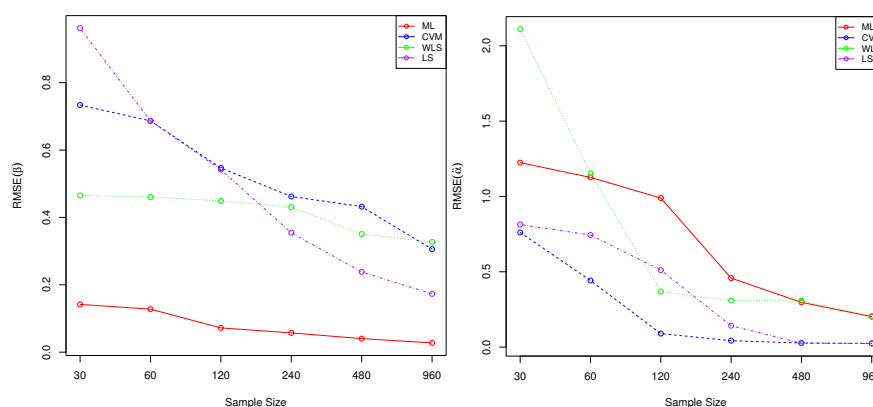


Figure 9. RMSE Plots for Table 2.

Table 2. Simulation Results for Different Estimation Methods for $\beta = 1.2, \alpha = 0.4, \theta = 0.7, \lambda = 1.2$.

<i>n</i>	Parameter	ML		CVM		WLS		LS	
		ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
30	β	−0.20761 ^{3}	0.3334 ^{2}	−0.4038 ^{4}	0.8311 ^{4}	−0.0562 ^{2}	0.0563 ^{1}	−0.0483 ^{1}	0.4383 ^{3}
	α	0.9071 ^{3}	1.4977 ^{3}	−0.4656 ^{2}	0.6569 ^{2}	−0.0788 ^{1}	0.0798 ^{1}	−0.9465 ^{4}	1.6569 ^{4}
	θ	0.0510 ^{1}	0.0872 ^{1}	−0.6919 ^{4}	0.9196 ^{4}	−0.0933 ^{3}	0.0953 ^{3}	−0.0691 ^{2}	0.0919 ^{2}
	λ	−0.0140 ^{1}	0.1051 ^{1}	−0.8765 ^{4}	0.9765 ^{4}	−0.8289 ^{3}	0.8892 ^{3}	−0.2876 ^{2}	0.2896 ^{2}
	Total Rank	15		28		17		20	
60	β	−0.1472 ^{3}	0.2524 ^{2}	−0.3799 ^{4}	0.7992 ^{4}	−0.0443 ^{2}	0.0445 ^{1}	−0.0427 ^{1}	0.4279 ^{3}
	α	0.5662 ^{4}	1.0417 ^{3}	−0.2117 ^{2}	0.5211 ^{2}	−0.0695 ^{1}	0.0697 ^{1}	−0.5211 ^{3}	1.2117 ^{4}
	θ	0.0338 ^{2}	0.0630 ^{2}	−0.5018 ^{4}	0.8909 ^{4}	−0.0392 ^{3}	0.0821 ^{3}	−0.0190 ^{1}	0.0198 ^{1}
	λ	−0.0043 ^{1}	0.0824 ^{1}	−0.6882 ^{3}	0.8822 ^{4}	−0.8049 ^{4}	0.8499 ^{3}	−0.2862 ^{2}	0.2868 ^{2}
	Total Rank	18		27		18		17	
120	β	−0.1021 ^{3}	0.1865 ^{2}	−0.3666 ^{4}	0.4366 ^{4}	−0.0301 ^{1}	0.0318 ^{1}	−0.0366 ^{2}	0.3666 ^{3}
	α	0.3752 ^{3}	0.7430 ^{4}	−0.1206 ^{2}	0.4120 ^{2}	−0.0670 ^{1}	0.06734 ^{1}	−0.4120 ^{4}	0.5412 ^{3}
	θ	0.0228 ^{3}	0.0401 ^{3}	−0.0196 ^{2}	0.1962 ^{4}	−0.0387 ^{4}	0.0387 ^{2}	−0.0162 ^{1}	0.0196 ^{1}
	λ	−0.0033 ^{1}	0.0387 ^{1}	−0.2866 ^{3}	0.6604 ^{3}	−0.6726 ^{4}	0.7267 ^{4}	−0.2804 ^{2}	0.2866 ^{2}
	Total Rank	20		24		18		18	
240	β	−0.0560 ^{3}	0.1330 ^{2}	−0.3330 ^{4}	0.4333 ^{4}	−0.0284 ^{1}	0.02854 ^{1}	−0.0333 ^{2}	0.3330 ^{3}
	α	0.2039 ^{4}	0.4325 ^{4}	−0.0533 ^{2}	0.0567 ^{1}	−0.0508 ^{1}	0.0608 ^{2}	−0.0533 ^{2}	0.3368 ^{3}
	θ	0.0130 ^{1}	0.0275 ^{2}	−0.0193 ^{3}	0.1351 ^{4}	−0.0287 ^{4}	0.0312 ^{3}	−0.0135 ^{2}	0.0193 ^{1}
	λ	−0.0025 ^{1}	0.0245 ^{1}	−0.2670 ^{3}	0.5670 ^{3}	−0.6074 ^{4}	0.6287 ^{4}	−0.1867 ^{2}	0.2067 ^{2}
	Total Rank	18		24		20		17	
480	β	−0.0334 ^{3}	0.0955 ^{2}	−0.2499 ^{4}	0.4012 ^{4}	−0.0099 ^{1}	0.0269 ^{1}	−0.0125 ^{2}	0.1250 ^{3}
	α	0.1156 ^{4}	0.3035 ^{4}	−0.0450 ^{1}	0.0459 ^{1}	−0.0465 ^{3}	0.0555 ^{2}	−0.0459 ^{2}	0.1459 ^{3}
	θ	0.0079 ^{1}	0.0192 ^{3}	−0.0167 ^{3}	0.0170 ^{2}	−0.0200 ^{4}	0.0300 ^{4}	−0.0103 ^{2}	0.0167 ^{1}
	λ	−0.0011 ^{1}	0.0162 ^{1}	−0.1773 ^{3}	0.2877 ^{3}	−0.2868 ^{4}	0.2882 ^{4}	−0.0877 ^{2}	0.1877 ^{2}
	Total Rank	19		21		23		17	
960	β	−0.0254 ^{3}	0.0761 ^{3}	−0.14807 ^{4}	0.3480 ^{4}	−0.0045 ^{1}	0.0145 ^{1}	−0.0080 ^{2}	0.03480 ^{2}
	α	0.0822 ^{4}	0.2368 ^{4}	−0.0323 ^{2}	0.0328 ^{1}	−0.0134 ^{1}	0.0349 ^{2}	−0.03283 ^{3}	0.0839 ^{3}
	θ	0.0064 ^{2}	0.0131 ^{3}	−0.0117 ^{4}	0.0117 ^{1}	−0.0102 ^{3}	0.0190 ^{4}	−0.0017 ^{1}	0.01178 ^{2}
	λ	−0.0001 ^{1}	0.01018 ^{1}	−0.0950 ^{3}	0.1895 ^{3}	−0.1780 ^{4}	0.2780 ^{4}	−0.0528 ^{2}	0.0895 ^{2}
	Total Rank	21		22		20		17	

Note: The superscript values in braces denote the ranks of the estimation methods, with rank 1 indicating the best performance (smallest value) for the corresponding criterion. The total rank is calculated by summing the ranks obtained from ABIAS and RMSE across all parameters.

**Figure 10.** Cont.

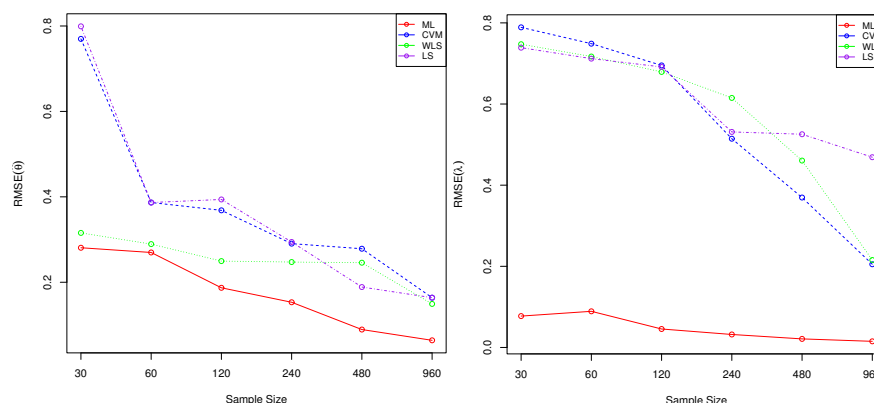


Figure 10. RMSE Plots for Table 3.

Table 3. Simulation Results for Different Estimation Methods for $\beta = 0.4, \alpha = 1.2, \theta = 3.0, \lambda = 0.7$.

<i>n</i>	Parameter	ML		CVM		WLS		LS	
		ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
30	β	−0.0272 ^{1}	0.1417 ^{1}	−0.6173 ^{3}	0.7336 ^{3}	−0.4634 ^{2}	0.4649 ^{2}	−0.9341 ^{4}	0.9617 ^{4}
	α	0.4077 ^{2}	1.2243 ^{3}	−0.3176 ^{1}	0.7608 ^{1}	−1.1100 ^{4}	2.1100 ^{4}	−0.7608 ^{3}	0.81317 ^{2}
	θ	−0.07511 ^{1}	0.2807 ^{1}	−0.4674 ^{3}	0.7698 ^{3}	−0.3155 ^{2}	0.3155 ^{2}	−0.6746 ^{4}	0.7992 ^{4}
	λ	0.0150 ^{1}	0.0773 ^{1}	−0.7388 ^{4}	0.7889 ^{4}	−0.7147 ^{3}	0.7473 ^{3}	−0.7088 ^{2}	0.7388 ^{2}
Total Rank		11		22		22		25	
60	β	−0.0151 ^{1}	0.1277 ^{1}	−0.5686 ^{3}	0.6865 ^{3}	−0.4608 ^{2}	0.4608 ^{2}	−0.6863 ^{4}	0.6865 ^{3}
	α	0.2873 ^{2}	1.1262 ^{3}	−0.0942 ^{1}	0.4420 ^{1}	−0.5355 ^{3}	1.1535 ^{4}	−0.6344 ^{4}	0.7442 ^{2}
	θ	−0.0624 ^{1}	0.2697 ^{1}	−0.3278 ^{3}	0.3866 ^{3}	−0.2891 ^{2}	0.2893 ^{2}	−0.3786 ^{4}	0.3866 ^{3}
	λ	0.0107 ^{1}	0.0890 ^{1}	−0.7118 ^{4}	0.7487 ^{4}	−0.7103 ^{3}	0.7170 ^{3}	−0.7003 ^{2}	0.7118 ^{2}
Total Rank		11		22		21		24	
120	β	0.01174 ^{1}	0.0721 ^{1}	−0.5411 ^{3}	0.5468 ^{4}	−0.4488 ^{2}	0.4489 ^{2}	−0.5411 ^{3}	0.5416 ^{3}
	α	0.0892 ^{1}	0.9894 ^{4}	0.0895 ^{2}	0.0895 ^{1}	−0.3084 ^{3}	0.3684 ^{2}	0.5089 ^{4}	0.5109 ^{3}
	θ	−0.0497 ^{1}	0.1870 ^{1}	−0.3093 ^{3}	0.3683 ^{3}	−0.2493 ^{2}	0.2495 ^{2}	−0.3693 ^{4}	0.3938 ^{4}
	λ	0.0055 ^{1}	0.0455 ^{1}	−0.6904 ^{4}	0.6951 ^{4}	−0.6217 ^{3}	0.6790 ^{2}	−0.6145 ^{2}	0.6914 ^{3}
Total Rank		11		24		18		26	
240	β	0.0113 ^{1}	0.0572 ^{1}	−0.3546 ^{2}	0.4621 ^{4}	−0.4065 ^{4}	0.4306 ^{3}	−0.3546 ^{2}	0.3546 ^{2}
	α	−0.0184 ^{1}	0.4578 ^{4}	0.0421 ^{2}	0.0426 ^{1}	−0.3070 ^{4}	0.3090 ^{3}	0.1421 ^{3}	0.1422 ^{2}
	θ	−0.03878 ^{1}	0.1531 ^{1}	−0.2204 ^{2}	0.2904 ^{3}	−0.2413 ^{3}	0.2473 ^{2}	−0.2920 ^{4}	0.2943 ^{4}
	λ	0.0022 ^{1}	0.0320 ^{1}	−0.3214 ^{2}	0.5145 ^{2}	−0.5019 ^{4}	0.6150 ^{4}	−0.3214 ^{2}	0.5312 ^{3}
Total Rank		11		18		27		22	
480	β	0.0070 ^{1}	0.0403 ^{1}	−0.3209 ^{4}	0.4320 ^{4}	−0.3085 ^{3}	0.3508 ^{3}	−0.2320 ^{2}	0.2380 ^{2}
	α	−0.0108 ^{1}	0.2972 ^{3}	0.02547 ^{2}	0.0271 ^{2}	−0.2074 ^{4}	0.3074 ^{4}	0.02547 ^{2}	0.02571 ^{1}
	θ	−0.0213 ^{1}	0.0892 ^{1}	−0.1887 ^{3}	0.2784 ^{4}	−0.2405 ^{4}	0.2457 ^{3}	−0.0887 ^{2}	0.1887 ^{2}
	λ	−0.0021 ^{1}	0.0211 ^{1}	−0.2553 ^{3}	0.3695 ^{2}	−0.2146 ^{2}	0.4606 ^{3}	−0.2553 ^{3}	0.5255 ^{4}
Total Rank		10		24		26		18	
960	β	0.0013 ^{1}	0.0275 ^{1}	−0.2305 ^{4}	0.3058 ^{3}	−0.2273 ^{3}	0.3273 ^{4}	−0.1305 ^{2}	0.1730 ^{2}
	α	0.0078 ^{1}	0.2021 ^{3}	−0.0231 ^{2}	0.0248 ^{2}	−0.1071 ^{4}	0.2071 ^{4}	−0.0231 ^{2}	0.0238 ^{1}
	θ	−0.0156 ^{1}	0.0640 ^{1}	−0.0887 ^{3}	0.1638 ^{3}	−0.1482 ^{4}	0.1492 ^{2}	−0.0638 ^{2}	0.1638 ^{3}
	λ	−0.0007 ^{1}	0.0151 ^{1}	−0.1006 ^{2}	0.2050 ^{2}	−0.1555 ^{3}	0.2155 ^{3}	−0.1900 ^{4}	0.4689 ^{4}
Total Rank		10		21		27		20	

Table 4. Partial and Overall Ranks of all Estimation Methods.

Parameters	<i>n</i>	ML	CVM	WLS	LS
$\beta = 1.2, \alpha = 0.4, \theta = 0.7, \lambda = 1.2$	30	1	4	2	3
	60	2.5	4	2.5	1
	120	3	4	1.5	1.5
	240	2	4	3	1
	480	2	3	4	1
	960	3	4	2	1
$\beta = 0.4, \alpha = 1.2, \theta = 3.0, \lambda = 0.7$	30	1	2.5	2.5	4
	60	1	3	2	4
	120	1	3	2	4
	240	1	2	4	3
	480	1	3	4	2
	960	1	3	4	2
$\beta = 1.2, \alpha = 0.4, \theta = 3.0, \lambda = 0.7$	30	1.5	4	1.5	3
	60	2	4	1	3
	120	2	4	1	3
	240	1	3	2	4
	480	1	4	2	3
	960	2	3	1	4
$\beta = 0.4, \alpha = 1.2, \theta = 0.7, \lambda = 1.2$	30	1	3.5	3.5	2
	60	1	3	4	2
	120	1	3	4	2
	240	1	3	4	2
	480	1	3	4	2
	960	1	3	4	2
Total ranks		35	80	65.5	57.5
Overall rank		1	4	3	2

8. Risk Measures and Numerical Study

This section examines several important risk measures commonly used in finance and actuarial science to assess the level of market risk associated with investment portfolios. The measures considered include Value at Risk (VaR), Tail Value at Risk (TVaR), Tail Variance (TV), and Tail Variance Premium (TVP).

8.1. Value at Risk

VaR is one of the most commonly applied measures in actuarial science and finance for quantifying exposure to market risk. It is usually evaluated at predetermined confidence levels, such as 70%, 75%, or 95%, to indicate the maximum expected loss over a given period under normal market conditions. For the TIIEHL-OLL-GPS class of distributions, the VaR is defined as follows:

$$VaR_p = G^{-1} \left[\left(\left[\frac{1-W}{W} \right]^{\frac{1}{\alpha}} + 1 \right)^{-1} \right], \quad (27)$$

where, $W = \frac{1 - \left(\left[\frac{C^{-1}((1-p)[C(\theta)])}{\theta} \right] \right)^{\frac{1}{\beta}}}{1 + \left(\left[\frac{C^{-1}((1-p)[C(\theta)])}{\theta} \right] \right)^{\frac{1}{\beta}}}$, and $p \in (0, 1)$ is a specified level of significance.

8.2. Tail Value at Risk

TVaR measures the expected loss conditional on losses exceeding the corresponding Value at Risk threshold. Unlike VaR, which only identifies the loss quantile at a specified confidence level, TVaR incorporates the severity of losses in the tail of the distribution,

thereby providing a more informative measure of extreme risk exposure. For the TIIEHL-OLL-GPS class of distributions, TVaR is defined as follows:

$$\begin{aligned} TVaR_p &= E(X | X > x_p) = \frac{1}{1-p} \int_{VaR_p}^{\infty} x f(x) dx \\ &= \frac{1}{1-p} \sum_{m=0}^{\infty} \int_{VaR_p}^{\infty} x \xi_{m+1} g_{m+1}(x; \xi) dx, \end{aligned} \quad (28)$$

where ξ_{m+1} is given in Equation (13) and $g_{m+1}(x; \xi) = (m+1)G^m(x; \xi)g(x; \xi)$ is the pdf of Exp-G distribution with the power parameter $(m+1)$.

8.3. Tail Variance

TV measures the variability of losses beyond the VaR threshold, providing insight into the dispersion of extreme losses. The TV for the TIIEHL-OLL-GPS class of distributions is defined as follows:

$$\begin{aligned} TV_p &= E(X^2 | X > x_p) - (TVaR_p)^2 \\ &= \frac{1}{1-p} \int_{VaR_p}^{\infty} x^2 f(x) dx - (TVaR_p)^2 \\ &= \frac{1}{1-p} \sum_{m=0}^{\infty} \xi_{m+1} \int_{VaR_p}^{\infty} x^2 g_{m+1}(x; \xi) dx - (TVaR_p)^2. \end{aligned} \quad (29)$$

8.4. Tail Variance Premium

The TVP is a valuable risk measure frequently employed in actuarial and insurance applications. It reflects the extra premium needed to compensate for the uncertainty associated with losses exceeding the VaR threshold. The TVP for the TIIEHL-OLL-GPS class of distributions is defined as follows:

$$TVP_p = TVaR_p + \psi(TV_p), \quad (30)$$

where $0 < \psi < 1$. The TVP of the TIIEHL-OLL-GPS class of distributions can be obtained by substituting Equations (28) and (29) into Equation (30). In the simulation study and numerical applications, the value of ψ was taken to be equal to the confidence level p used in the calculation of the risk measures.

8.5. Numerical Study for the Risk Measures

In this section, we evaluate the performance of the TIIEHL-OLL-LLoGP distribution in capturing heavy-tailed data through a numerical simulation of selected risk measures. The obtained results are further compared with those of its sub-models, as well as with equi-parameter models, namely Topp-Leone Weibull Poisson (TL-WP) distribution by [15], and the generalized Gompertz Poisson (GGP) distribution by [16].

Table 5 reports the simulated values of VaR, TVaR, TV, and TVP for the seven competing distributions. In general, larger values of these risk measures indicate heavier tail behavior. From the results presented in Table 5, it can be observed that the TIIEHL-OLL-LLoGP distribution demonstrates heavier tail characteristics compared to its sub-models, as well as the non-nested equi-parameter TLWP and GGP distributions. This suggests that the proposed model is more appropriate for modelling heavy-tailed data.

Table 5. Simulation Results of VaR, TVaR, TV and TVP.

Confidence Level		0.7	0.75	0.8	0.85	0.9	0.95
TIIHL-OLL-LLoGP ($\beta = 0.3, \alpha = 0.7, \theta = 1.9, \lambda = 3.0$)	VaR	0.9387	0.9396	0.9413	0.9441	0.9480	0.9532
	TVaR	1.2016	1.4507	1.8218	2.4371	3.6630	7.3328
	TV	1.9334	2.3277	2.9168	3.8956	5.8493	11.7036
	TVP	2.5550	3.1965	4.1552	5.7484	8.9274	18.4512
TIIHL-OLL-LLoGP ($\beta = 0.3, \alpha = 0.7, \theta = 1, \lambda = 1$)	VaR	0.6962	0.7051	0.7191	0.7377	0.7610	0.7890
	TVaR	1.1159	1.3701	1.7448	2.3608	3.5797	7.2085
	TV	1.5942	1.9371	2.4455	3.2857	4.9560	9.9463
	TVP	2.2318	2.8229	3.7012	5.1537	8.0401	16.6575
TIIHL-OLL-LLoGP ($\beta = 0.3, \alpha = 1, \theta = 1.9, \lambda = 3.0$)	VaR	0.3754	0.41034	0.44487	0.47951	0.51487	0.55149
	TVaR	0.3649	0.5243	0.7606	1.1508	1.9262	4.2416
	TV	0.2375	0.3311	0.4662	0.6848	1.1128	2.3786
	TVP	0.5312	0.7727	1.1337	1.7329	2.9278	6.5013
TIIHL-OLL-LLoGP ($\beta = 1, \alpha = 0.7, \theta = 1.9, \lambda = 3.0$)	VaR	0.4763	0.5324	0.5921	0.6559	0.7243	0.7977
	TVaR	0.9981	1.2669	1.6665	2.3282	3.6452	7.5842
	TV	1.6984	2.0906	2.6704	3.6266	5.5256	11.1992
	TVP	2.1870	2.8349	3.8029	5.4108	8.6183	18.2234
TIIHL-OLL-LLoGP ($\beta = 1, \alpha = 1, \theta = 1.9, \lambda = 3.0$)	VaR	0.4242	0.4754	0.5233	0.5698	0.6161	0.6631
	TVaR	0.5605	0.7514	0.8451	1.5462	2.5709	5.7078
	TV	0.5636	0.7672	0.9710	1.7814	2.9291	6.6408
	TVP	0.9551	1.3268	1.6219	3.0604	5.2071	12.0165
TIIHL-OLL-LLoGP ($\beta = 1, \alpha = 1, \theta = 1, \lambda = 3.0$)	VaR	0.6408	0.6809	0.7232	0.7681	0.8159	0.8673
	TVaR	0.9269	1.1667	1.5243	2.1175	3.3005	6.8430
	TV	1.6588	2.0360	2.5969	3.5258	5.3754	10.9101
	TVP	2.0881	2.6938	3.6018	5.1145	8.1384	17.2076
TLWP($\theta = 1.9, \alpha = 0.3, \beta = 0.7, \gamma = 0.2$)	VaR	0.0041	0.0093	0.0154	0.0224	0.0304	0.0395
	TVaR	0.6200	0.7494	0.9420	1.2611	1.8970	3.7999
	TV	1.2102	1.4538	1.8184	2.4253	3.6385	7.2774
	TVP	1.4672	1.8397	2.3967	3.3227	5.1716	10.7134
GGP($\theta = 1.9, \alpha = 0.7, \beta = 0.3, \gamma = 3.0$)	VaR	0.2598	0.2944	0.3342	0.3819	0.4431	0.5347
	TVaR	0.4042	0.4297	0.4587	0.4925	0.5333	0.5817
	TV	0.1798	0.2003	0.2256	0.2580	0.3018	0.3666
	TVP	0.5301	0.5800	0.6392	0.7118	0.8049	0.9299

9. Applications

The flexibility and applicability of the TIIHL-OLL-LLoGP distribution are evaluated using real data from different fields. Model performance is assessed using several goodness-of-fit measures, including the -2log-likelihood ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Corrected Akaike Information Criterion ($AICC = AIC + \frac{2p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), where n is the number of observations and p is the number of estimated parameters, Cramér-von Mises statistic (W^*), Anderson-Darling statistic (A^*), and Kolmogorov-Smirnov (K-S) statistic. A model is preferred when it yields lower values for these criteria. Although the competing models considered in this study have comparable parameter dimensions, their underlying generator structures and parameter interpretations may differ. Therefore, the comparison is based not only on the number of parameters but also on the goodness-of-fit measures and the ability of each model to adequately describe the observed data. The AIC, AICC, and BIC criteria account for the number of estimated parameters, while the likelihood-based and empirical distribution-based measures provide additional evidence of model adequacy.

The TIIHL-OLL-LLoGP distribution is compared with several distributions, including the Topp-Leone Weibull Poisson (TL-WP) distribution by [15], odd power generalized Weibull-Weibull Poisson (OPGWPP) distribution by [17], type I heavy-tailed Topp-Leone-log-logistic Poisson (TIHT-TL-LLP) distribution by [18], generalized Gompertz Poisson (GGP) distribution by [16], and odd Weibull-Topp-Leone-log-logistic Poisson (OWTLLLP)

distribution by [19]. These competing models were selected because they are flexible Poisson-based extensions that have demonstrated effectiveness in lifetime data modelling. They provide suitable benchmarks for assessing the performance of the proposed TIIEHL-OLL-LLoGP distribution while recognizing differences in their underlying construction mechanisms. The pdfs of the competing models are presented in the Appendix A.

In addition, graphical diagnostics such as fitted density plots, probability plots, the total time on test (TTT) scaled plot, and estimated hazard rate function (hrf) plots are used to further evaluate the model's adequacy.

For the probability plot, we plotted $F(x_{(j)}) = F(x_{(j)}; \hat{\beta}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares (SS):

$$SS = \sum_{j=1}^n \left[F(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

A smaller SS value indicates a better fit.

9.1. Head and Neck Cancer Data

This study assesses the survival periods (in days) of forty-four head and neck cancer patients who received combination radiation and chemotherapy. The dataset was applied by [20]. The data are:

0.1220, 0.2356, 0.2374, 0.2587, 0.3198, 0.3700, 0.4135, 0.4738, 0.5546,
0.5836, 0.6347, 0.6846, 0.7447, 0.7826, 0.8143, 0.8400, 0.9200, 0.9400,
1.1000, 1.1200, 1.1900, 1.2700, 1.3000, 1.3300, 1.4000, 1.4600, 1.5500,
1.5900, 1.7300, 1.7900, 1.9400, 1.9500, 2.0900, 2.4900, 2.8100, 3.1900,
3.3900, 4.3200, 4.6900, 5.1900, 6.3300, 7.2500, 8.1700, 17.7600.

Figure 11 show the violin plot with embedded boxplot of the of the head and neck cancer data, showing a positively skewed distribution with most observations concentrated in the central range and a few large values extending the upper tail. The identifiability of model parameters can be assessed using the log-likelihood function. Figure 12 displays the profile log-likelihood plots for the TIIEHL-OLL-LLoGP distribution, which exhibit distinct peaks. This indicates that the MLEs are unique and well-defined, thereby confirming the identifiability of the model parameters and supporting the validity of the obtained estimates.

The estimated variance-covariance matrix for TIIEHL-OLL-LLoGP model on head and neck cancer data set is given by

$$\begin{pmatrix} 3.1076 \times 10^{-7} & -0.0003 & 1.3416 \times 10^{-7} & 2.3612 \times 10^{-5} \\ -0.0003 & 0.9370 & -0.0004 & -0.1313 \\ 1.3416 \times 10^{-7} & -0.0004 & 1.7244 \times 10^{-7} & 5.7079 \times 10^{-5} \\ 2.3612 \times 10^{-5} & -0.1313 & 5.7079 \times 10^{-5} & 0.0257 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for the parameters β, α, θ and λ are given by $2.4888 \times 10^{-3} \pm 0.00109$, 2.6673 ± 1.8972 , $1.0100 \times 10^2 \pm 0.0008$ and 0.7113 ± 0.31473 , respectively. The correlation matrix for TIIEHL-OLL-LLoGP model on head and neck cancer data set is given by

$$\begin{pmatrix} 1.0000 & -0.5887 & 0.5795 & 0.2637 \\ -0.5887 & 1.0000 & -0.9997 & -0.8450 \\ 0.5795 & -0.9997 & 1.0000 & 0.8559 \\ 0.2637 & -0.8450 & 0.8559 & 1.0000 \end{pmatrix}.$$

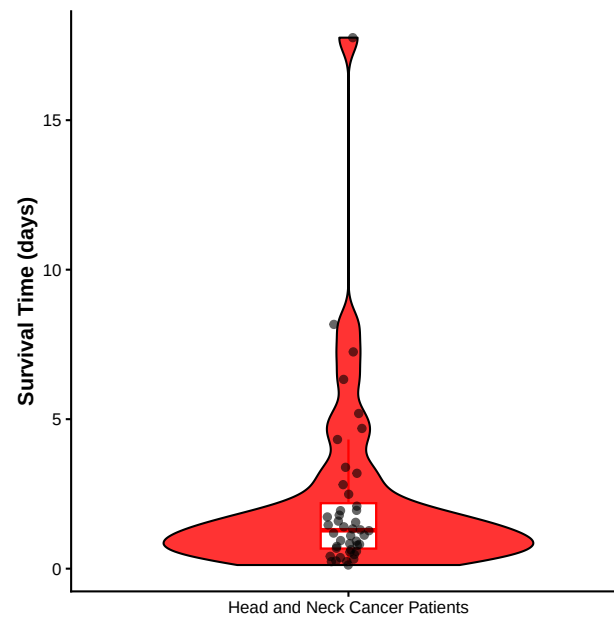


Figure 11. Violin Plot with Embedded Boxplot of the Head and Neck Cancer Data.

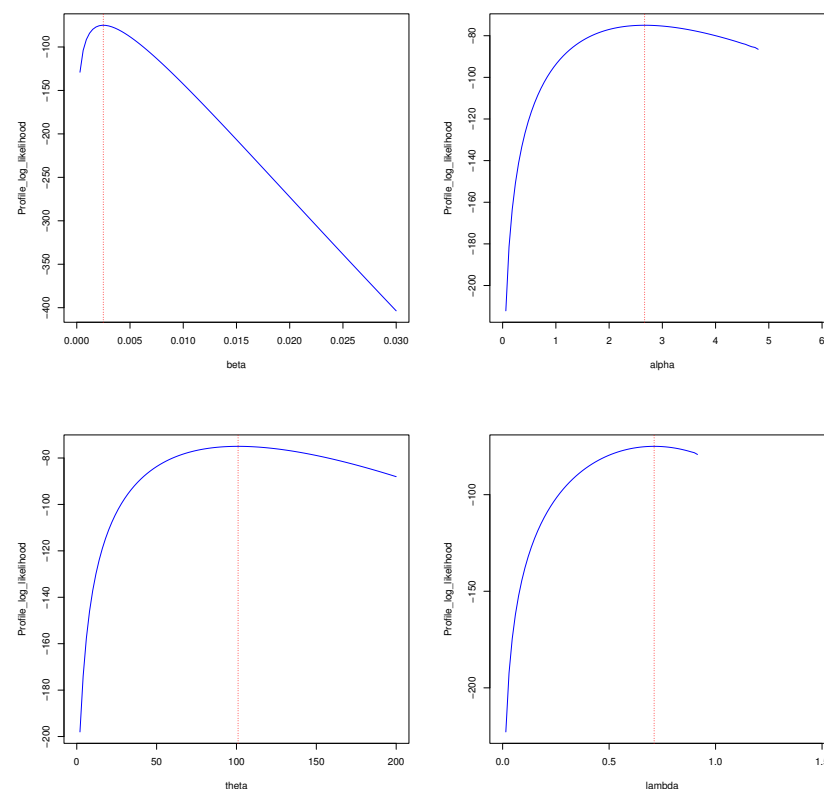


Figure 12. Profile Log-likelihood Function Plots for Parameters of TIIEHL-OLL-LLoGP Distribution on the Head and Neck Cancer Data.

Table 6 shows the maximum likelihood estimates with standard error of the estimates in parenthesis and goodness-of-fit statistics for the head and neck cancer dataset. The TIIEHL-OLL-LLoGP distribution is the best match among the competing models, as demonstrated by the smallest values of goodness-of-fit statistics, and K-S statistic. These findings show that the proposed model provides a better fit for the head and neck cancer data than the other fitted distributions.

Table 6. Parameter Estimates and Goodness-of-Fit Statistics for the Head and Neck Cancer Data.

Model	Estimates				Statistics						
	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$-2 \log(L)$	AIC	AICC	BIC	W^*	A^*	$K - S$
TIIEHL-OLL-LLoGP	2.4888×10^{-3} (5.5747×10^{-4})	2.6673 (0.9680)	1.0100×10^2 (4.1526×10^{-4})	0.7113 (0.1605)	149.9394	157.9394	158.9651	165.0762	0.0235	0.1473	0.0617
TLWP	$\hat{\theta}$ 0.3376 (0.1645)	$\hat{\alpha}$ 0.8165 (0.1817)	$\hat{\beta}$ 1.4136 (0.6548)	$\hat{\gamma}$ 6.4282×10^{-8} (0.0294)	155.9919	163.992	165.0176	171.1288	0.1094	0.6415	0.1378
OPGWWP	$\hat{\alpha}$ 25.2660 (2.1198×10^{-5})	$\hat{\beta}$ 6.1845×10^{-4} (9.4043×10^{-5})	$\hat{\lambda}$ 0.1932 (0.0206)	$\hat{\theta}$ 101.4900 (9.2852×10^{-9})	174.3072	182.3081	183.3337	189.4448	0.1362	0.8231	0.2578
TIHT-TL-LLoGP	$\hat{\theta}$ 39.0464 (0.0013)	$\hat{\alpha}$ 0.0103 (0.0066)	$\hat{\beta}$ 0.0154 (0.0272)	$\hat{\gamma}$ 2.0663 (0.9033)	155.6400	163.64	164.6656	170.7767	0.0461	0.347	0.0705
GGP	$\hat{\theta}$ 2.0894×10^{-8} (0.0185)	$\hat{\alpha}$ 0.8607 (0.1912)	$\hat{\beta}$ 0.4733 (0.1446)	$\hat{\gamma}$ 2.8066×10^{-8} (0.0408)	160.7442	168.7443	169.7699	175.881	0.1633	0.9475	0.1215
OWTLLLP	$\hat{\alpha}$ 0.1795 (0.0746)	$\hat{\lambda}$ 2.0370 (0.7371)	$\hat{\gamma}$ 6.1829 (1.3335)	$\hat{\theta}$ 4.9991×10^{-9} (0.0081)	153.9711	161.9712	162.9969	169.108	0.0570	0.3139	0.1520

Figure 13 shows that the TIIEHL-OLL-LLoGP distribution provides a better fit to the head and neck cancer data than the competing models, as it aligns closely with the histogram of the observed data and yields a lower SS value in the probability plot.

Figure 14 depicts the TTT scaled plot and the hrf plot for the head and neck cancer data. The TTT scaled plot indicates an upside-down bathtub-shaped hazard rate, and this is in agreement with the corresponding hrf plot.

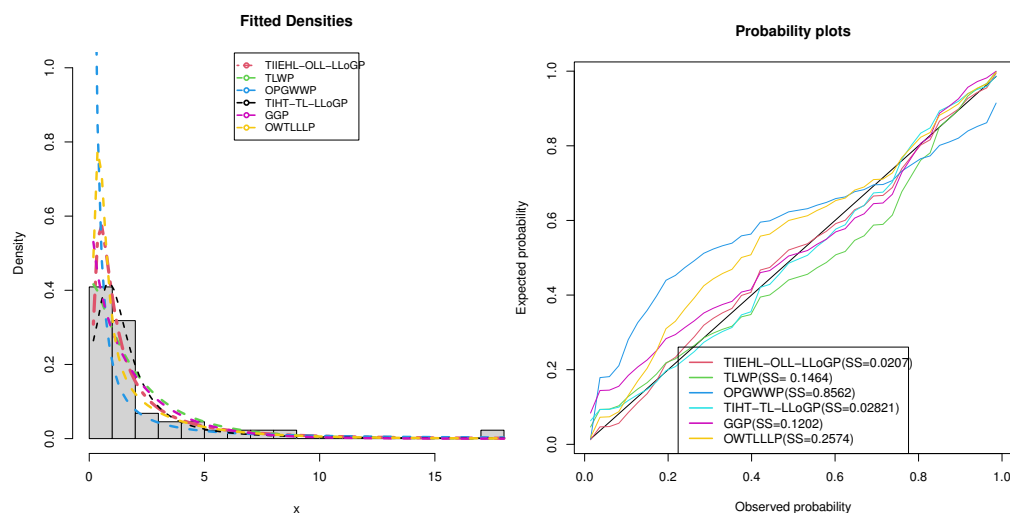


Figure 13. Histogram Superposed by Fitted Density (left) and Observed vs. Expected Probability Plots (right) for the Head and Neck Cancer Data.

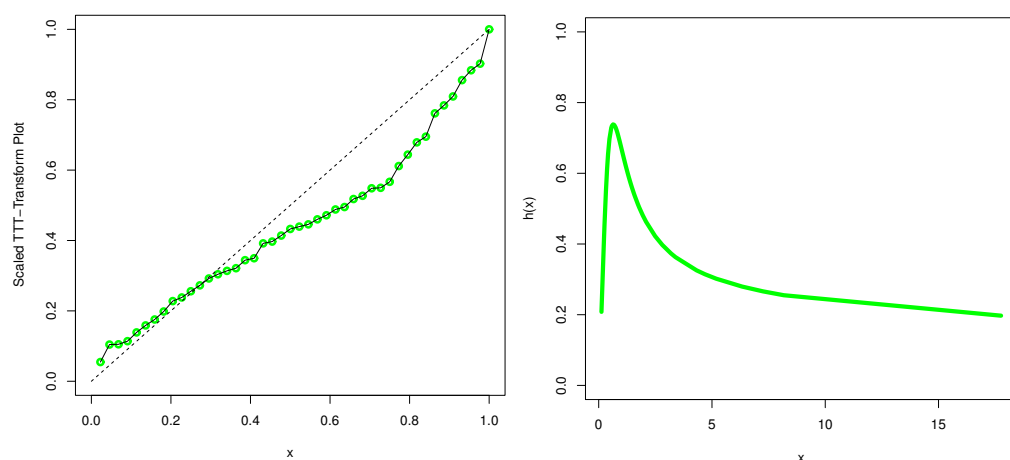


Figure 14. Scaled TTT-Transform Plots and Fitted Hrf for the TIIEHL-OLL-LLoGP Distribution for Head and Neck Cancer Data.

9.2. Airborne Communication Transceiver Data

These data correspond to maintenance on active repair times (in hours) for an airborne communication transceiver. The data was recently used by [21]. The data are:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

Figure 15 presents the violin plot with an embedded boxplot for the airborne communication transceiver data. The distribution is positively skewed, with most observations

concentrated at lower failure times and a small number of large observations producing a long right tail.

Figure 16 shows the profile log-likelihood plots for the parameters of the TIIHL-OLL-LLoGP distribution using the airborne communication transceiver data. The well-defined maxima observed in the plots suggest that the maximum likelihood estimates are unique, supporting the identifiability of the model parameters and indicating that the obtained estimates correspond to the global maxima.

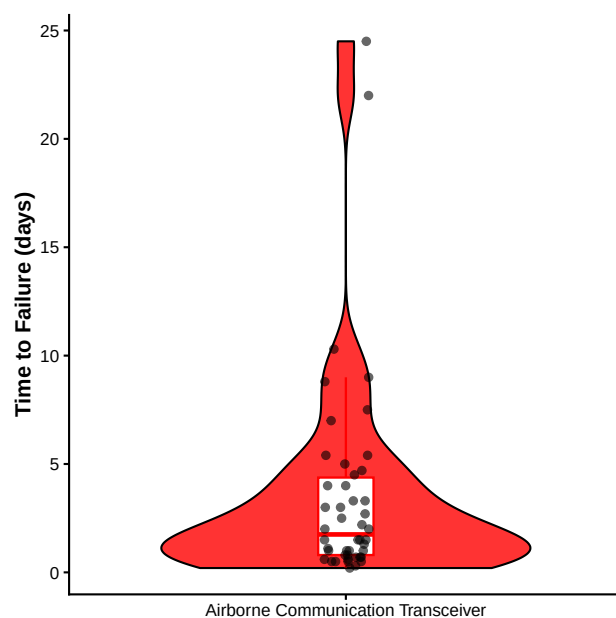


Figure 15. Violin Plot with Embedded Boxplot of the Airborne Communication Transceiver Data.

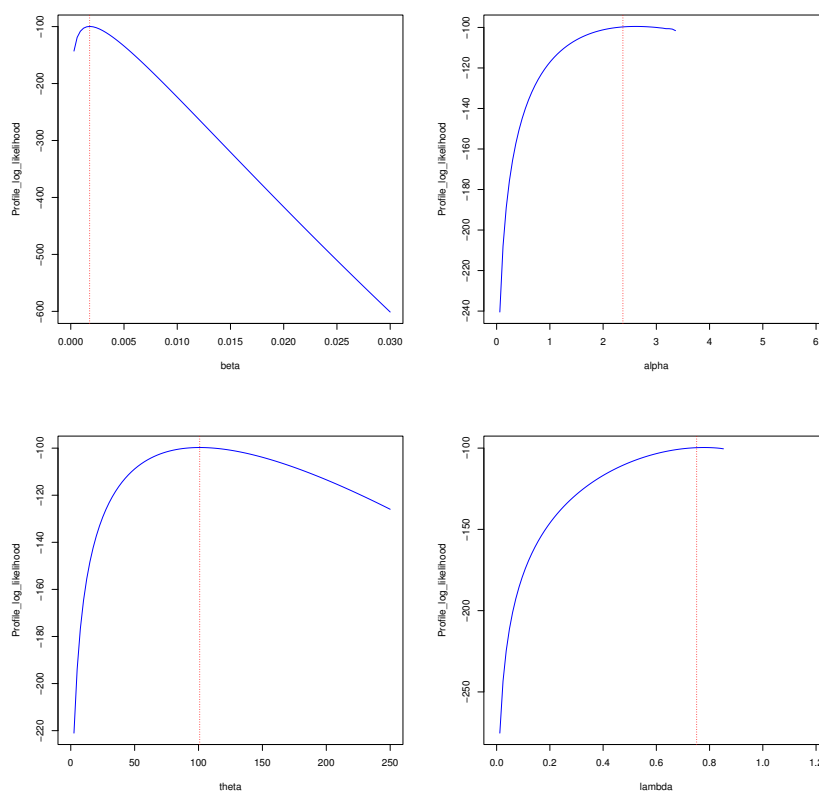


Figure 16. Profile Log-likelihood Function Plots for Parameters of TIIHL-OLL-LLoGP Distribution on the Airborne Communication Transceiver Data.

The estimated variance-covariance matrix for TIIHL-OLL-LLoGP model on airborne communication transceiver data set is given by

$$\begin{pmatrix} 1.9354 \times 10^{-7} & -1.6488 \times 10^{-4} & 2.2303 \times 10^{-8} & -6.6232 \times 10^{-6} \\ -1.6488 \times 10^{-4} & 0.4546 & -6.5199 \times 10^{-5} & -0.0439 \\ 2.2303 \times 10^{-8} & -6.5199 \times 10^{-5} & 9.3787 \times 10^{-9} & 6.7815 \times 10^{-6} \\ -6.6232 \times 10^{-6} & -0.0439 & 6.7815 \times 10^{-6} & 0.0122 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for the parameters β, α, θ and λ are given by $1.7771 \times 10^{-3} \pm 0.0008$, 2.3722 ± 1.3216 , 100.9500 ± 0.0001 and 0.7516 ± 0.2169 , respectively. The correlation matrix for TIIHL-OLL-LLoGP model on airborne communication transceiver data set is given by

$$\begin{pmatrix} 1.0000 & -0.5558 & 0.5234 & -0.13603 \\ -0.5558 & 1.0000 & -0.9984 & -0.5892 \\ 0.5234 & -0.9984 & 1.0000 & 0.6327 \\ -0.1360 & -0.5892 & 0.6327 & 1.000 \end{pmatrix}.$$

Table 7 shows the maximum likelihood estimates and goodness-of-fit statistics for the data. The TIIHL-OLL-LLoGP distribution is the best fit among competing models, as evidenced by the low values of goodness-of-fit statistics, and K-S statistics.

Figure 17 demonstrates that the TIIHL-OLL-LLoGP distribution provides a superior fit to the airborne communication transceiver data, as it closely captures the underlying data pattern and produces the smallest SS value in the probability plot.

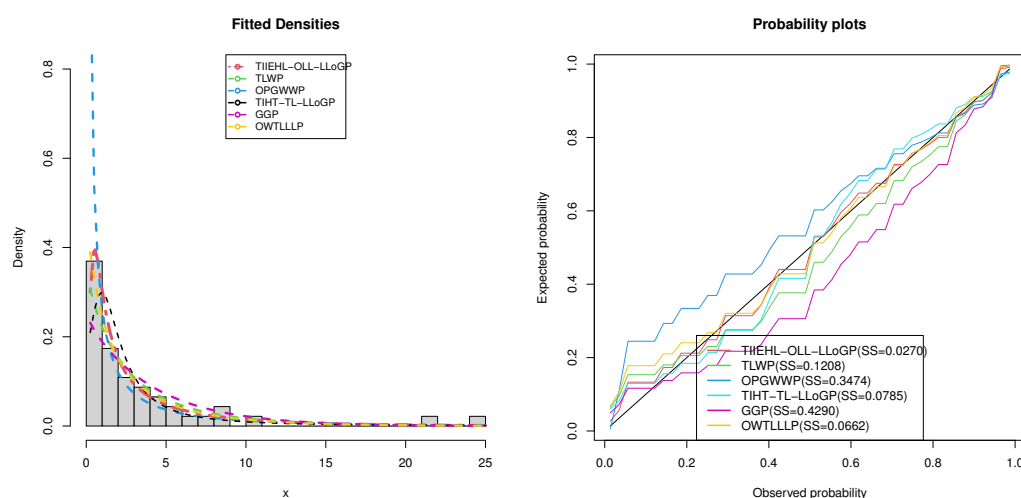


Figure 17. Histogram Superposed by Fitted Density (left) and Observed vs. Expected Probability Plots (right) for the Airborne Communication Transceiver Data.

Figure 18: TTT scaled plot and hrf plot for the data, showing an upside-down bathtub hrf.

Table 7. Parameter Estimates and Goodness-of-Fit Statistics for the Airborne Communication Transceiver Data.

Model	Estimates				Statistics						
	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K - S$
TIIEHL-OLL-LLoGP	1.7771×10^{-3} (4.3994×10^{-4})	2.3722 (0.6742)	100.9500 (9.6844×10^{-5})	0.7516 (0.1106)	199.4695	207.4696	208.4452	214.7842	0.0349	0.2621	0.0895
TLWP	$\hat{\theta}$ 0.2491 (0.1066)	$\hat{\alpha}$ 0.7535 (0.1366)	$\hat{\beta}$ 1.3756 (0.4986)	$\hat{\gamma}$ 2.4269×10^{-8} (0.0197)	206.9255	214.9256	215.9012	222.2402	0.1070	0.7391	0.1234
OPGWWP	$\hat{\alpha}$ 204.9500 (2.2728×10^{-8})	$\hat{\beta}$ 3.9994×10^{-3} (3.6704×10^{-4})	$\hat{\lambda}$ 0.2255 (4.8958×10^{-3})	$\hat{\theta}$ 2.4957×10^{-9} (6.4294×10^{-3})	205.1928	213.1929	214.1685	220.5075	0.0343	0.3380	0.2011
TIHT-TL-LLoGP	$\hat{\theta}$ 46.2022 (0.0043)	$\hat{\alpha}$ 0.0089 (0.0086)	$\hat{\beta}$ 0.0266 (0.0644)	$\hat{\epsilon}$ 1.6049 (0.9202)	207.4705	215.4705	216.4461	222.785	0.1123	0.6863	0.0962
GGP	$\hat{\theta}$ 2.4806×10^{-3} (1.9258)	$\hat{\alpha}$ 0.9803 (0.4431)	$\hat{\beta}$ 0.2369 (0.0795)	$\hat{\gamma}$ 2.9515×10^{-9} (0.0226)	210.8863	218.8865	219.8621	226.2011	0.1392	0.9672	0.1939
OWTLLP	$\hat{\alpha}$ 0.9174 (0.3386)	$\hat{\lambda}$ 0.4835 (0.1517)	$\hat{\gamma}$ 4.6548 (0.9947)	$\hat{\theta}$ 1.8379×10^{-8} (0.0210)	205.365	213.3649	214.3406	220.6795	0.0878	0.6186	0.1343

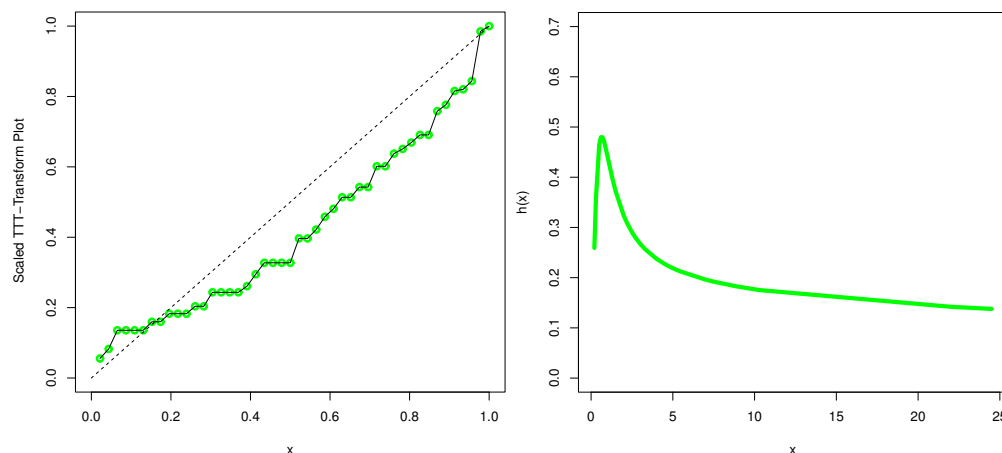


Figure 18. Scaled TTT-Transform and Fitted Hrf Plots for the TIIHL-OLL-LLoGP Distribution for Airborne Communication Transceiver Data.

9.3. Electronics Data

The data for lifetimes of 20 electronic components obtained from [22]. The data was used by [23]. The data are:

0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79, 1.80, 1.94, 2.38, 2.40, 2.87, 2.99, 3.14, 3.17, 4.72, 5.09.

Figure 19 presents the violin plot with an embedded boxplot for the electronic components failure-time data. The distribution appears to be slightly positively skewed, with a greater spread toward larger failure times and a few observations occurring at relatively high values.

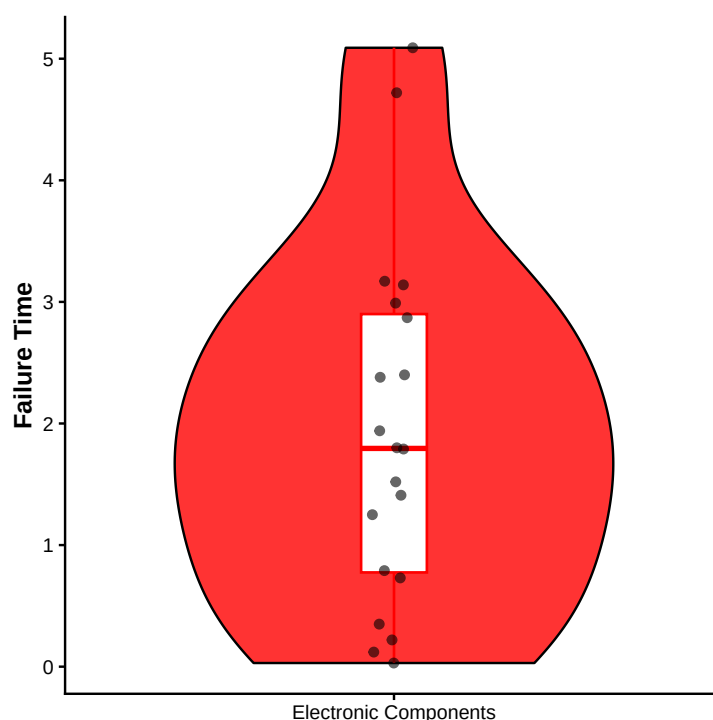


Figure 19. Violin Plot with Embedded Boxplot of the Electronics Data.

The profile log-likelihood plots for the parameters of the TIEHL-OLL-LLoGP distribution, based on the electronics data, are presented in Figure 20. The clear unimodal behavior of the curves, characterized by pronounced maxima, supports the uniqueness of the maximum likelihood estimates. These results demonstrate that the model parameters are identifiable and that the estimated values correspond to the global optimum of the likelihood surface.

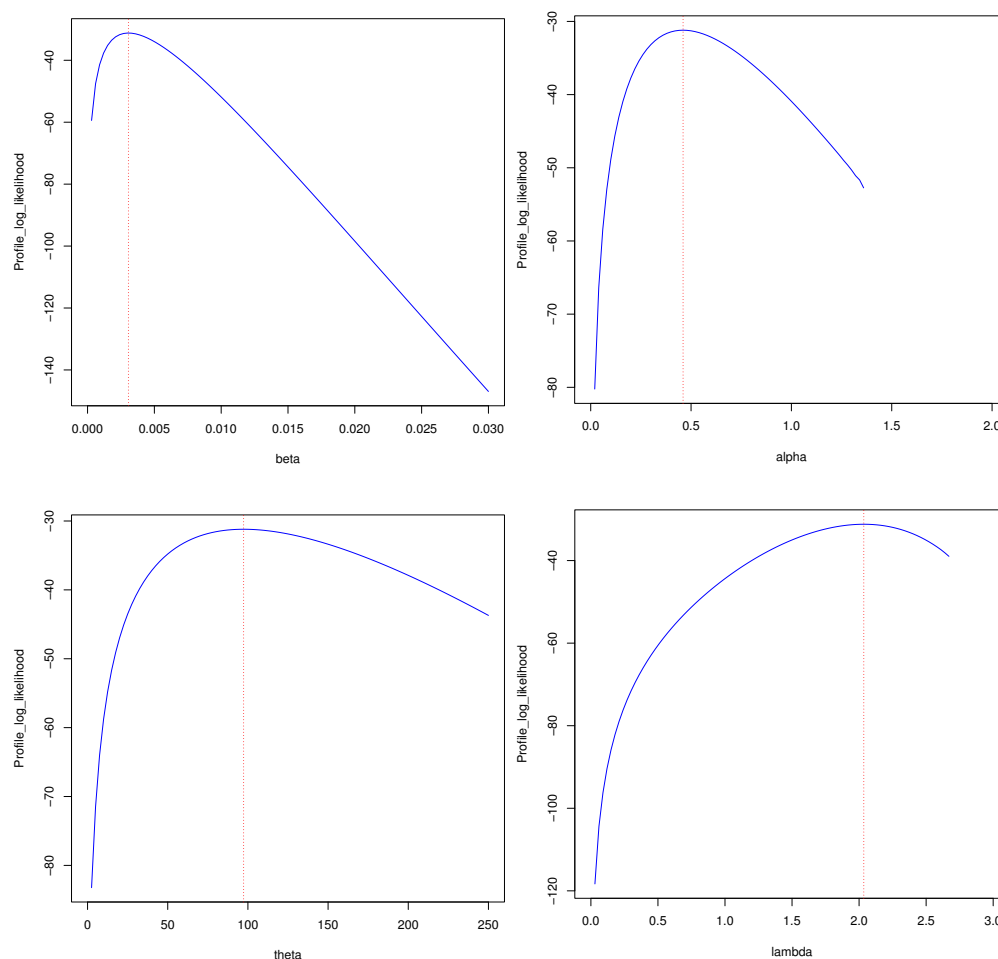


Figure 20. Profile Log-likelihood Function Plots for Parameters of TIEHL-OLL-LLoGP Distribution on the Electronics Data.

The estimated variance-covariance matrix for TIEHL-OLL-LLoGP model on electronics data set is given by

$$\begin{pmatrix} 1.0828 \times 10^{-6} & -7.3188 \times 10^{-6} & -3.1912 \times 10^{-8} & -1.9958 \times 10^{-4} \\ -7.3188 \times 10^{-6} & 0.0358 & -1.0919 \times 10^{-5} & -0.0596 \\ -3.1912 \times 10^{-8} & -1.0919 \times 10^{-5} & 5.2615 \times 10^{-9} & 3.0162 \times 10^{-5} \\ -1.9958 \times 10^{-4} & -0.0596 & 3.0162 \times 10^{-5} & 0.1735 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for the parameters β, α, θ and λ are given by $3.0586 \times 10^{-3} \pm 0.0020, 0.4602 \pm 0.3708, 97.3280 \pm 0.0001$ and 2.0352 ± 0.8165 , respectively. The correlation matrix for TIEHL-OLL-LLoGP model on electronics data set is given by

$$\begin{pmatrix} 1.0000 & -0.0371 & -0.4227 & -0.4603 \\ -0.0371 & 1.0000 & -0.7955 & -0.7564 \\ -0.4227 & -0.7955 & 1.0000 & 0.9980 \\ -0.4603 & -0.7564 & 0.9980 & 1.0000 \end{pmatrix}.$$

The maximum likelihood estimates and corresponding goodness-of-fit statistics are presented in Table 8. Based on the model comparison criteria, the TIIEHL-OLL-LLoGP distribution outperforms the competing models, yielding the lowest values for the goodness-of-fit measures. This evidence indicates that the TIIEHL-OLL-LLoGP distribution offers the best representation of the electronics data.

Figure 21 demonstrates that the TIIEHL-OLL-LLoGP distribution provides a superior fit to the electronics data, as it closely captures the underlying data pattern and produces the smallest SS value in the probability plot.

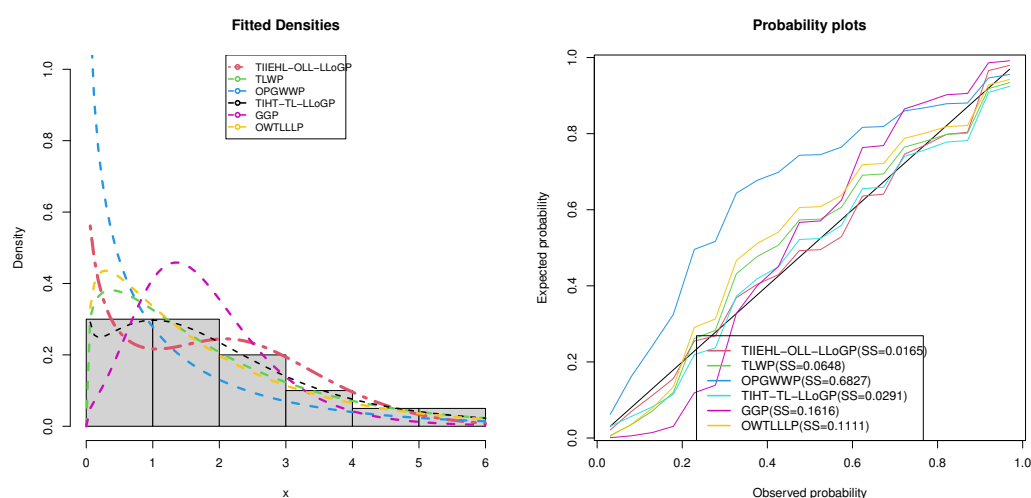


Figure 21. Histogram Superposed by Fitted Density (left) and Observed vs. Expected Probability Plots (right) for the Electronics Data.

Figure 22 displays the scaled TTT plot together with the estimated hrf for the electronics data. Both plots indicate a bathtub-shaped hazard rate, where the failure rate decreases in the early stage and subsequently increases in the later stage of the lifetime.

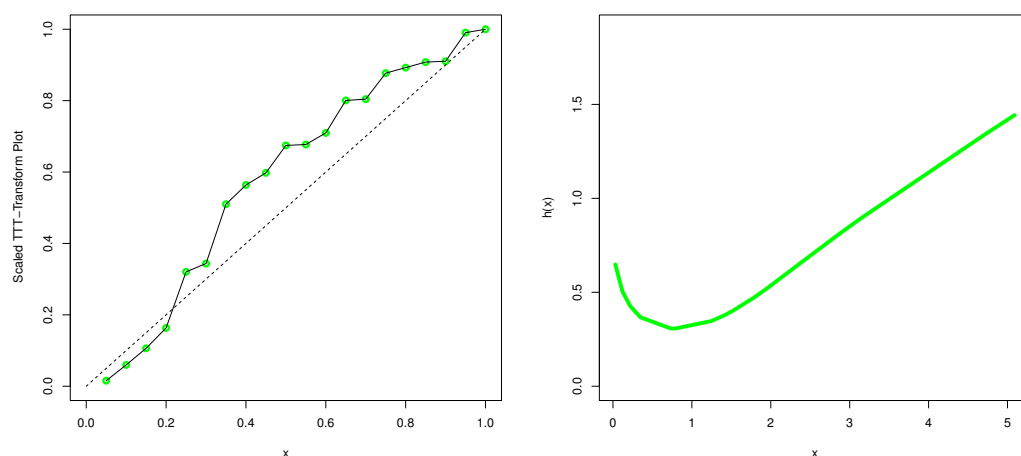


Figure 22. Scaled TTT-Transform and Fitted Hrf Plots for the TIIEHL-OLL-LLoGP Distribution for Electronics Data.

Table 8. Parameter Estimates and Goodness-of-Fit Statistics for the Electronics Data.

Model	Estimates				Statistics						
	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$-2\log(L)$	AIC	AICC	BIC	W^*	A^*	$K - S$
TIIEHL-OLL-LLoGP	3.0586×10^{-3} (1.0410×10^{-3})	0.4602 (0.1892)	97.3280 (7.2604×10^{-5})	2.0352 (0.4166)	62.39617	70.3961	73.0628	74.3791	0.0229	0.1730	0.0964
TLWP	$\hat{\theta}$ 0.2672 (0.6704)	$\hat{\alpha}$ 1.0343 (1.3388)	$\hat{\beta}$ 1.1775 (2.4422)	$\hat{\gamma}$ 1.9286×10^{-8} (0.0293)	66.1620	74.1619	76.8286	78.1448	0.0829	0.4846	0.1317
OPGWWP	$\hat{\alpha}$ 1.8273 (1.3642)	$\hat{\beta}$ 0.4795 (0.2539)	$\hat{\lambda}$ 0.3510 (0.1470)	$\hat{\theta}$ 1.2453×10^{-8} (0.0182)	72.09834	80.0983	82.7649	84.0812	0.0899	0.5226	0.3436
TIHT-TL-LLoGP	$\hat{\theta}$ 483.7000 (1.1257×10^{-4})	$\hat{\alpha}$ 0.0035 (0.0037)	$\hat{b}$ 0.0528 (0.1343)	$\hat{c}$ 1.1323 (0.2358)	66.0247	74.0249	76.6916	78.0078	0.0446	0.2919	0.1183
GGP	$\hat{\theta}$ 4.0462 (3.2342)	$\hat{\alpha}$ 1.3652 (1.6829)	$\hat{\beta}$ 1.2732 (0.3429)	$\hat{\gamma}$ 6.0639×10^{-9} (0.0172)	73.8360	81.8370	84.5036	85.8199	0.0477	0.3002	0.1693
OWTLLLP	$\hat{\alpha}$ 1.4787 (0.9497)	$\hat{\lambda}$ 0.4195 (0.2250)	$\hat{\gamma}$ 3.3481 (0.6142)	$\hat{\theta}$ 6.3535×10^{-8} (0.0432)	66.6955	74.6956	77.3623	78.6785	0.0937	0.5436	0.1663

10. Concluding Remarks

In this study, the TIIEHL-OLL-GPS class of distributions was obtained by combining the TIIEHL-G and OLL-G families with a discrete power series distribution. This new distribution can adequately model skewed and heavy-tailed data, as well as data with various hazard rate function shapes. The study develops essential statistical features, such as moments, survival function, order statistics, probability weighted moments and Rényi entropy and uses a variety of methods to estimate model parameters. Monte Carlo simulations were used to examine the estimator's performance. The model was evaluated using key risk measures, including VaR, TVaR, TV, and TVP, which demonstrated its effectiveness in capturing extreme risk behaviour. When fitted to real datasets, the proposed model outperformed several competing distributions, indicating its suitability for complex lifetime and reliability data modeling. The TIIEHL-OLL-GPS distribution is therefore promising for applications in reliability analysis and survival studies. Future work may focus on regression-based extensions and Bayesian estimation approaches.

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Data Availability Statement: The datasets used in this study were obtained from publicly available sources. The first dataset was obtained from [20], the second dataset was obtained from [21], and the third dataset was obtained from [22]. These datasets are publicly accessible through their respective sources and can be used for research purposes. The authors did not generate new datasets during the current study.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Appendix A.1. Linear Representation of the Density Function

Let $K_G(x; \alpha, \xi) = \frac{[G(x; \xi)]^\alpha}{[G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha}$, then

$$\begin{aligned}
 f(x; \beta, \alpha, \theta, \xi) &= \frac{2\beta\theta}{C(\theta)} \sum_{n=1}^{\infty} na_n \theta^{n-1} (1 - K_G(x; \alpha, \xi))^{n\beta-1} \\
 &\times (1 + K_G(x; \alpha, \xi))^{-(n\beta+1)} \\
 &\times \frac{\alpha g(x; \xi) [G(x; \xi)]^{\alpha-1} [\overline{G}(x; \xi)]^{\alpha-1}}{\left([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha\right)^2} \\
 &= \frac{2\beta\theta}{C(\theta)} \sum_{n=1}^{\infty} na_n \theta^{n-1} \sum_{i,j=0}^{\infty} \binom{n\beta-1}{i} \binom{n\beta+j}{j} (-1)^{i+j} \\
 &\times \frac{\alpha g(x; \xi) [G(x; \xi)]^{\alpha(i+j+1)-1} [\overline{G}(x; \xi)]^{\alpha-1}}{\left([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha\right)^{i+j+2}} \\
 &= \frac{2\beta\alpha\theta}{C(\theta)} \sum_{n=1}^{\infty} na_n \theta^{n-1} \sum_{i,j,k=0}^{\infty} \binom{n\beta-1}{i} \binom{n\beta+j}{j} \binom{-(i+j+2)}{k} (-1)^{i+j} \\
 &\times g(x; \xi) [G(x; \xi)]^{\alpha(i+j+1+k)-1} [\overline{G}(x; \xi)]^{-[\alpha(i+j+k+1)+1]} \\
 &= \frac{2\beta\alpha\theta}{C(\theta)} \sum_{n=1}^{\infty} na_n \theta^{n-1} \sum_{i,j,k,l=0}^{\infty} \binom{n\beta-1}{i} \binom{n\beta+j}{j} \binom{-(i+j+2)}{k} \\
 &\times (-1)^{i+j+l} \binom{\alpha(i+j+1+k)-1}{l} g(x; \xi) [\overline{G}(x; \xi)]^{l-[\alpha(i+j+k+1)+1]} \\
 &= \frac{2\beta\alpha\theta}{C(\theta)} \sum_{n=1}^{\infty} na_n \theta^{n-1} \sum_{i,j,k,l,m=0}^{\infty} \binom{n\beta-1}{i} \binom{n\beta+j}{j} \binom{-(i+j+2)}{k} \\
 &\times \binom{\alpha(i+j+1+k)-1}{l} \binom{l-[\alpha(i+j+k+1)+1]}{m} (-1)^{i+j+l+m} \\
 &\times \frac{(m+1)}{(m+1)} g(x; \xi) G^m(x; \xi) \\
 &= \sum_{m=0}^{\infty} \xi_{m+1} g_{m+1}(x; \xi).
 \end{aligned}$$

Appendix A.2. Probability Weighted Moments

$$\begin{aligned}
 \varrho_{m+1} &= \frac{2\beta\alpha}{[C(\theta)]^{u+1}} \sum_{n=1}^{\infty} \sum_{u,z,i,j,k,l=0}^{\infty} d_{z,u} na_n \theta^{z+n} \binom{w}{u} \binom{\beta(u+n)-1}{i} \\
 &\times \binom{\beta(u+n)+j}{j} \binom{-(i+j+2)}{k} \binom{\alpha(i+j+1+k)-1}{l} \\
 &\times \binom{l-[\alpha(i+j+k+1)+1]}{m} \frac{(-1)^{u+i+j+l+m}}{(m+1)}. \tag{A1}
 \end{aligned}$$

Appendix A.3. Distribution of Order Statistics

We can write

$$\begin{aligned}
 f(x)[F(x)]^{w+q-1} &= 2\beta\theta(1-K_G(x;\alpha,\xi))^{\beta-1} \\
 &\times (1+K_G(x;\alpha,\xi))^{-(\beta+1)} \\
 &\times \frac{\alpha g(x;\xi)[G(x;\xi)]^{\alpha-1}[\overline{G}(x;\xi)]^{\alpha-1}}{([G(x;\xi)]^\alpha + [\overline{G}(x;\xi)]^\alpha)^2} \\
 &\times \frac{C'\left(\theta\left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^\beta\right)}{C(\theta)} \\
 &\times \left[1 - \frac{C\left(\theta\left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^\beta\right)}{C(\theta)}\right]^{w+q-1}.
 \end{aligned}$$

Using the generalized binomial series expansion, we have

$$\begin{aligned}
 \left[1 - \frac{C\left(\theta\left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^\beta\right)}{C(\theta)}\right]^{w+q-1} &= \sum_{u=0}^{\infty} (-1)^u \binom{w+q-1}{u} \\
 &\times \left[\frac{C\left(\theta\left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^\beta\right)}{C(\theta)}\right]^u.
 \end{aligned}$$

Note that

$$\left[\frac{C\left(\theta\left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^\beta\right)}{C(\theta)}\right]^u = \sum_{z=0}^{\infty} \frac{d_{z,u} \theta^z \left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^{\beta z}}{C^u(\theta)},$$

where $d_{z,u} = (zb_0)^{-1} \sum_{r=1}^z [u(r+1) - z] b_r d_{z-r,u}$ and $d_{0,u} = b_0^u$ and

$$\begin{aligned}
 \Omega_{m+1} &= \frac{2\beta\alpha}{[C(\theta)]^{u+1}} \sum_{n=1}^{\infty} \sum_{u,z,i,j,k,l=0}^{\infty} d_{z,u} n a_n \theta^{z+n} \binom{w+q-1}{u} \binom{\beta(u+n)-1}{i} \\
 &\times \binom{\beta(u+n)+j}{j} \binom{-(i+j+2)}{k} \binom{\alpha(i+j+1+k)-1}{l} \\
 &\times \binom{l - [\alpha(i+j+k+1)+1]}{m} \frac{(-1)^{u+i+j+l+m}}{(m+1)}.
 \end{aligned} \tag{A2}$$

Appendix A.4. Rényi Entropy

Note that $f_{TIIHL-OLL-GPS}^r(x; \theta, \alpha, \beta, \xi) = f^r(x)$ can be written as

$$\begin{aligned}
 f^r(x) &= (2\beta\alpha\theta)^r (1-K_G(x;\alpha,\xi))^{r\beta-r} \\
 &\times (1+K_G(x;\alpha,\xi))^{-r(\beta+1)} \\
 &\times \frac{g^r(x;\xi)[G(x;\xi)]^{r\alpha-r}[\overline{G}(x;\xi)]^{r\alpha-r}}{([G(x;\xi)]^\alpha + [\overline{G}(x;\xi)]^\alpha)^{2r}} \\
 &\times \frac{\left[C'\left(\theta\left(\frac{1-K_G(x;\alpha,\xi)}{1+K_G(x;\alpha,\xi)}\right)^\beta\right)\right]^r}{C^r(\theta)}.
 \end{aligned}$$

Applying the power series raised to a positive integer as given by [14], results in

$$\left[C' \left(\theta \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^\beta \right) \right]^r = \sum_{z=0}^{\infty} d_{z,r} \theta^z \left(\frac{1 - K_G(x; \alpha, \xi)}{1 + K_G(x; \alpha, \xi)} \right)^{\beta z},$$

where $d_{z,r} = (zb_0)^{-1} \sum_{l=1}^z [l(r+1) - z] b_l d_{z-l,r}$ and $d_{0,r} = b_0^r$. We can write

$$\begin{aligned} f^r(x) &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z (1 - K_G(x; \alpha, \xi))^{\beta(r+z)-r} \\ &\times (1 + K_G(x; \alpha, \xi))^{-(\beta(r+z)+r)} \\ &\times \frac{g^r(x; \xi) [G(x; \xi)]^{r\alpha-r} [\overline{G}(x; \xi)]^{r\alpha-r}}{([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha)^{2r}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f^r(x; \beta, \alpha, \xi) &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} (-1)^{i+j} \\ &\times \frac{g^r(x; \xi) [G(x; \xi)]^{\alpha(i+j+r)-r} [\overline{G}(x; \xi)]^{r(\alpha-1)}}{([G(x; \xi)]^\alpha + [\overline{G}(x; \xi)]^\alpha)^{2r+i+j}} \\ &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j,k=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} \\ &\times (-1)^{i+j} \binom{-(2r+i+j)}{k} g^r(x; \xi) [G(x; \xi)]^{\alpha(i+j+r+k)-r} \\ &\times [\overline{G}(x; \xi)]^{-[\alpha(r+i+j+k)+r]} \\ &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j,k,l=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} \\ &\times (-1)^{i+j+l} \binom{-(2r+i+j)}{k} \binom{\alpha(i+j+r+k)-r}{l} g^r(x; \xi) \\ &\times [\overline{G}(x; \xi)]^{-[\alpha(r+i+j+k)+r]} \\ &= \frac{(2\beta\alpha\theta)^r}{C^r(\theta)} \sum_{z=0}^{\infty} d_{z,r} \theta^z \sum_{i,j,k,l,m=0}^{\infty} \binom{\beta(r+z)-r}{i} \binom{\beta(r+z)+r+j-1}{j} \\ &\times \binom{-(2r+i+j)}{k} \binom{\alpha(i+j+r+k)-r}{l} \binom{-[\alpha(r+i+j+k)+r]}{m} \\ &\times (-1)^{i+j+l+m} G^m(x; \xi) g^r(x; \xi). \end{aligned}$$

Appendix B. Competing Models

The competing models are listed below and they are:

The generalized Gompertz Poisson (GGP) distribution with the pdf

$$f(x; \theta, \alpha, \beta, \gamma) = \theta \alpha \beta \exp(\gamma x - \theta) (1-x) x^{\alpha-1} \exp(\theta x^\alpha),$$

for $\theta, \alpha, \beta, \gamma > 0, x > 0$,

The Topp-Leone Weibull Poisson (TLWP) distribution with the pdf

$$f(x; \theta, \alpha, \beta, \gamma) = \frac{2\gamma\theta\alpha\beta x^{\beta-1} e^{-2\alpha x^\beta} (1 - e^{-2\alpha x^\beta})^{\gamma-1} e^{\theta(1 - (1 - e^{-2\alpha x^\beta})^\gamma)}}{e^\theta - 1}$$

for $\theta, \alpha, \beta, \gamma > 0, x > 0$,

Odd power generalized Weibull-Weibull Poisson (OPGWWP) distribution with the pdf

$$\begin{aligned} f(x; \alpha, \beta, \lambda, \theta) &= \theta\alpha\beta \left[1 + \left(\frac{(1 - \exp(-x^\lambda))}{\exp(-x^\lambda)} \right)^\alpha \right]^{\beta-1} \left(\frac{(1 - \exp(-x^\lambda))}{\exp(-x^\lambda)} \right)^{\alpha-1} \\ &\times \exp \left(1 - \left[1 + \left(\frac{(1 - \exp(-x^\lambda))}{\exp(-x^\lambda)} \right)^\alpha \right]^\beta \right) \frac{\lambda x^{\lambda-1} \exp(-x^\lambda)}{(\exp(-x^\lambda))^2} \\ &\times \frac{\exp \left(\theta \left(\exp \left(1 - \left[1 + \left(\frac{(1 - \exp(-x^\lambda))}{\exp(-x^\lambda)} \right)^\alpha \right]^\beta \right) \right) \right)}{(\exp(\theta) - 1)}, \end{aligned}$$

for $\alpha, \beta, \lambda, \theta > 0, x > 0$,

Odd Weibull-Topp-Leone-log-logistic Poisson (OWTLLLP) distribution with the pdf

$$\begin{aligned} f(x; \alpha, \lambda, \gamma, \theta) &= \frac{2\theta\gamma\alpha\lambda x^{\lambda-1} (1 + x^\lambda)^{-3} [1 - (1 + x^\lambda)^{-2}]^{\gamma\alpha-1}}{[1 - (1 - (1 + x^\lambda)^{-2})^\gamma]^{\alpha+1}} \\ &\times \exp \left\{ - \left[\frac{[1 - (1 + x^\lambda)^{-2}]^\gamma}{[1 - (1 - (1 + x^\lambda)^{-2})^\gamma]} \right]^\alpha \right\} \\ &\times \frac{\exp \left(\theta \left(\exp \left\{ - \left[\frac{[1 - (1 + x^\lambda)^{-2}]^\gamma}{[1 - (1 - (1 + x^\lambda)^{-2})^\gamma]} \right]^\alpha \right\} \right) \right)}{\exp(\theta) - 1}, \end{aligned}$$

for $\alpha, \lambda, \gamma, \theta > 0, x > 0$, and

Type I heavy-tailed Topp-Leone-log-logistic Poisson (TIHT-TL-LLP) distribution with the pdf

$$\begin{aligned} f(x; \theta, a, b, c) &= 2a^2\theta b c x^{c-1} (1 + x^c)^{-3} (1 - [1 + x^c]^{-2})^{b-1} \left[1 - (1 - [1 + x^c]^{-2})^b \right]^{a-1} \\ &\times \left[1 - (1 - a)(1 - [1 + x^c]^{-2})^b \right]^{-(a+1)} \frac{\exp \left(\theta \left[\frac{1 - (1 - [1 + x^c]^{-2})^b}{1 - (1 - a)(1 - [1 + x^c]^{-2})^b} \right]^a \right)}{e^\theta - 1}, \end{aligned}$$

for $\theta, a, b, c > 0, x > 0$.

Appendix B.1. Simulation Results

Table A1. Simulation Results for Different Estimation Methods for $\beta = 1.2, \alpha = 0.4, \theta = 3.0, \lambda = 0.7$.

<i>n</i>	Parameter	ML		CVM		WLS		LS	
		ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
30	β	−0.20761 ^{3}	0.3334 ^{2}	−0.4038 ^{4}	0.8311 ^{4}	−0.0562 ^{2}	0.0563 ^{1}	−0.0483 ^{1}	0.4383 ^{3}
	α	0.9071 ^{3}	1.4977 ^{3}	−0.4656 ^{2}	0.6569 ^{2}	−0.0788 ^{1}	0.0798 ^{1}	−0.9465 ^{4}	1.6569 ^{4}
	θ	−0.07511 ^{1}	0.2807 ^{1}	−0.4674 ^{3}	0.7698 ^{3}	−0.3155 ^{2}	0.3155 ^{2}	−0.6746 ^{4}	0.7992 ^{4}
	λ	0.0150 ^{1}	0.0773 ^{1}	−0.7388 ^{4}	0.7889 ^{4}	−0.7147 ^{3}	0.7473 ^{3}	−0.7088 ^{2}	0.7388 ^{2}
Total Rank		15		26		15		24	
60	β	−0.1472 ^{3}	0.2524 ^{2}	−0.3799 ^{4}	0.7992 ^{4}	−0.0443 ^{2}	0.0445 ^{1}	−0.0427 ^{1}	0.4279 ^{3}
	α	0.5662 ^{4}	1.0417 ^{3}	−0.2117 ^{2}	0.5211 ^{2}	−0.0695 ^{1}	0.0697 ^{1}	−0.5211 ^{3}	1.2117 ^{4}
	θ	−0.0624 ^{1}	0.2697 ^{1}	−0.3278 ^{3}	0.3866 ^{3}	−0.2891 ^{2}	0.2893 ^{2}	−0.3786 ^{4}	0.3866 ^{3}
	λ	0.0107 ^{1}	0.0890 ^{1}	−0.7118 ^{4}	0.7487 ^{4}	−0.7103 ^{3}	0.7170 ^{3}	−0.7003 ^{2}	0.7118 ^{2}
Total Rank		16		26		15		22	
120	β	−0.1021 ^{3}	0.1865 ^{2}	−0.3666 ^{4}	0.4366 ^{4}	−0.0301 ^{1}	0.0318 ^{1}	−0.0366 ^{2}	0.3666 ^{3}
	α	0.3752 ^{3}	0.7430 ^{4}	−0.1206 ^{2}	0.4120 ^{2}	−0.0670 ^{1}	0.06734 ^{1}	−0.4120 ^{4}	0.5412 ^{3}
	θ	−0.0497 ^{1}	0.1870 ^{1}	−0.3093 ^{3}	0.3683 ^{3}	−0.2493 ^{2}	0.2495 ^{2}	−0.3693 ^{4}	0.3938 ^{4}
	λ	0.0055 ^{1}	0.0455 ^{1}	−0.6904 ^{4}	0.6951 ^{4}	−0.6217 ^{3}	0.6790 ^{2}	−0.6145 ^{2}	0.6914 ^{3}
Total Rank		16		26		13		25	
240	β	−0.0560 ^{3}	0.1330 ^{2}	−0.3330 ^{4}	0.4333 ^{4}	−0.0284 ^{1}	0.02854 ^{1}	−0.0333 ^{2}	0.3330 ^{3}
	α	0.2039 ^{4}	0.4325 ^{4}	−0.0533 ^{2}	0.0567 ^{1}	−0.0508 ^{1}	0.0608 ^{2}	−0.0533 ^{2}	0.3368 ^{3}
	θ	−0.03878 ^{1}	0.1531 ^{1}	−0.2204 ^{2}	0.2904 ^{3}	−0.2413 ^{3}	0.2473 ^{2}	−0.2920 ^{4}	0.2943 ^{4}
	λ	0.0022 ^{1}	0.0320 ^{1}	−0.3214 ^{2}	0.5145 ^{2}	−0.5019 ^{4}	0.6150 ^{4}	−0.3214 ^{2}	0.5312 ^{3}
Total Rank		17		20		18		23	
480	β	−0.0334 ^{3}	0.0955 ^{2}	−0.2499 ^{4}	0.4012 ^{4}	−0.0099 ^{1}	0.0269 ^{1}	−0.0125 ^{2}	0.1250 ^{3}
	α	0.1156 ^{4}	0.3035 ^{4}	−0.0450 ^{1}	0.0459 ^{1}	−0.0465 ^{3}	0.0555 ^{2}	−0.0459 ^{2}	0.1459 ^{3}
	θ	−0.0213 ^{1}	0.0892 ^{1}	−0.1887 ^{3}	0.2784 ^{4}	−0.2405 ^{4}	0.2457 ^{3}	−0.0887 ^{2}	0.1887 ^{2}
	λ	−0.0021 ^{1}	0.0211 ^{1}	−0.2553 ^{3}	0.3695 ^{2}	−0.2146 ^{2}	0.4606 ^{3}	−0.2553 ^{3}	0.5255 ^{4}
Total Rank		17		22		19		21	
960	β	−0.0254 ^{3}	0.0761 ^{3}	−0.14807 ^{4}	0.3480 ^{4}	−0.0045 ^{1}	0.0145 ^{1}	−0.0080 ^{2}	0.03480 ^{2}
	α	0.0822 ^{4}	0.2368 ^{4}	−0.0323 ^{2}	0.0328 ^{1}	−0.0134 ^{1}	0.0349 ^{2}	−0.03283 ^{3}	0.0839 ^{3}
	θ	−0.0156 ^{1}	0.0640 ^{1}	−0.0887 ^{3}	0.1638 ^{3}	−0.1482 ^{4}	0.1492 ^{2}	−0.0638 ^{2}	0.1638 ^{3}
	λ	−0.0007 ^{1}	0.0151 ^{1}	−0.1006 ^{2}	0.2050 ^{2}	−0.1555 ^{3}	0.2155 ^{3}	−0.1900 ^{4}	0.4689 ^{4}
Total Rank		18		21		17		23	

Table A2. Simulation Results for Different Estimation Methods for $\beta = 0.4, \alpha = 1.2, \theta = 0.7, \lambda = 1.2$.

<i>n</i>	Parameter	ML		CVM		WLS		LS	
		ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
30	β	−0.0272 ^{1}	0.1417 ^{1}	−0.6173 ^{3}	0.7336 ^{3}	−0.4634 ^{2}	0.4649 ^{2}	−0.9341 ^{4}	0.9617 ^{4}
	α	0.4077 ^{2}	1.2243 ^{3}	−0.3176 ^{1}	0.7608 ^{1}	−1.1100 ^{4}	2.1100 ^{4}	−0.7608 ^{3}	0.81317 ^{2}
	θ	0.0510 ^{1}	0.0872 ^{1}	−0.6919 ^{4}	0.9196 ^{4}	−0.0933 ^{3}	0.0953 ^{3}	−0.0691 ^{2}	0.0919 ^{2}
	λ	−0.0140 ^{1}	0.1051 ^{1}	−0.8765 ^{4}	0.9765 ^{4}	−0.8289 ^{3}	0.8892 ^{3}	−0.2876 ^{2}	0.2896 ^{2}
Total Rank		11		24		24		21	
60	β	−0.0151 ^{1}	0.1277 ^{1}	−0.5686 ^{3}	0.6865 ^{3}	−0.4608 ^{2}	0.4608 ^{2}	−0.6863 ^{4}	0.6865 ^{3}
	α	0.2873 ^{2}	1.1262 ^{3}	−0.0942 ^{1}	0.4420 ^{1}	−0.5355 ^{3}	1.1535 ^{4}	−0.6344 ^{4}	0.7442 ^{2}
	θ	0.0338 ^{2}	0.0630 ^{2}	−0.5018 ^{4}	0.8909 ^{4}	−0.0392 ^{3}	0.0821 ^{3}	−0.0190 ^{1}	0.0198 ^{1}
	λ	−0.0043 ^{1}	0.0824 ^{1}	−0.6882 ^{3}	0.8822 ^{4}	−0.8049 ^{4}	0.8499 ^{3}	−0.2862 ^{2}	0.2868 ^{2}
Total Rank		13		23		24		19	

Table A2. Cont.

<i>n</i>	Parameter	ML		CVM		WLS		LS	
		ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
120	β	0.01174 ^{1}	0.0721 ^{1}	−0.5411 ^{3}	0.5468 ^{4}	−0.4488 ^{2}	0.4489 ^{2}	−0.5411 ^{3}	0.5416 ^{3}
	α	0.0892 ^{1}	0.9894 ^{4}	0.0895 ^{2}	0.0895 ^{1}	−0.3084 ^{3}	0.3684 ^{2}	0.5089 ^{4}	0.5109 ^{3}
	θ	0.0228 ^{3}	0.0401 ^{3}	−0.0196 ^{2}	0.1962 ^{4}	−0.0387 ^{4}	0.0387 ^{2}	−0.0162 ^{1}	0.0196 ^{1}
	λ	−0.0033 ^{1}	0.0387 ^{1}	−0.2866 ^{3}	0.6604 ^{3}	−0.6726 ^{4}	0.7267 ^{4}	−0.2804 ^{2}	0.2866 ^{2}
	Total Rank	15		22		23		19	
240	β	0.0113 ^{1}	0.0572 ^{1}	−0.3546 ^{2}	0.4621 ^{4}	−0.4065 ^{4}	0.4306 ^{3}	−0.3546 ^{2}	0.3546 ^{2}
	α	−0.0184 ^{1}	0.4578 ^{4}	0.0421 ^{2}	0.0426 ^{1}	−0.3070 ^{4}	0.3090 ^{3}	0.1421 ^{3}	0.1422 ^{2}
	θ	0.0130 ^{1}	0.0275 ^{2}	−0.0193 ^{3}	0.1351 ^{4}	−0.0287 ^{4}	0.0312 ^{3}	−0.0135 ^{2}	0.0193 ^{1}
	λ	−0.0025 ^{1}	0.0245 ^{1}	−0.2670 ^{3}	0.5670 ^{3}	−0.6074 ^{4}	0.6287 ^{4}	−0.1867 ^{2}	0.2067 ^{2}
	Total Rank	12		22		29		16	
480	β	0.0070 ^{1}	0.0403 ^{1}	−0.3209 ^{4}	0.4320 ^{4}	−0.3085 ^{3}	0.3508 ^{3}	−0.2320 ^{2}	0.2380 ^{2}
	α	−0.0108 ^{1}	0.2972 ^{3}	0.02547 ^{2}	0.0271 ^{2}	−0.2074 ^{4}	0.3074 ^{4}	0.02547 ^{2}	0.02571 ^{1}
	θ	0.0079 ^{1}	0.0192 ^{3}	−0.0167 ^{3}	0.0170 ^{2}	−0.0200 ^{4}	0.0300 ^{4}	−0.0103 ^{2}	0.0167 ^{1}
	λ	−0.0011 ^{1}	0.0162 ^{1}	−0.1773 ^{3}	0.2877 ^{3}	−0.2868 ^{4}	0.2882 ^{4}	−0.0877 ^{2}	0.1877 ^{2}
	Total Rank	12		23		30		14	
960	β	0.0013 ^{1}	0.0275 ^{1}	−0.2305 ^{4}	0.3058 ^{3}	−0.2273 ^{3}	0.3273 ^{4}	−0.1305 ^{2}	0.1730 ^{2}
	α	0.0078 ^{1}	0.2021 ^{3}	−0.0231 ^{2}	0.0248 ^{2}	−0.1071 ^{4}	0.2071 ^{4}	−0.0231 ^{2}	0.0238 ^{1}
	θ	0.0064 ^{2}	0.0131 ^{3}	−0.0117 ^{4}	0.0117 ^{1}	−0.0102 ^{3}	0.0190 ^{4}	−0.0017 ^{1}	0.01178 ^{2}
	λ	−0.0001 ^{1}	0.01018 ^{1}	−0.0950 ^{3}	0.1895 ^{3}	−0.1780 ^{4}	0.2780 ^{4}	−0.0528 ^{2}	0.0895 ^{2}
	Total Rank	13		22		30		14	

Appendix B.2. Application Code

```

rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
x<-c(0.1220,0.2356, 0.2374, 0.2587, 0.3198, 0.3700, 0.4135,
      0.4738,
      0.5546,0.5836, 0.6347, 0.6846, 0.7447, 0.7826, 0.8143, 0.8400,
      0.9200,0.9400, 1.1000, 1.1200, 1.1900, 1.2700, 1.3000, 1.3300,
      1.4000,1.4600, 1.5500, 1.5900, 1.7300, 1.7900, 1.9400, 1.9500,
      2.0900,2.4900, 2.8100, 3.1900, 3.3900, 4.3200, 4.6900, 5.1900,
      6.3300,7.2500, 8.1700, 17.7600)

```

```

TIIHLOLLSHLPS_LL<- function(a,alpha,theta,lambda) {

  -sum(log(((exp(theta*(((1-((1-exp(-x^lambda))^alpha)
/((1-exp(-x^lambda))^alpha+
(exp(-x^lambda))^alpha))/(1+((1-exp(-x^lambda))^alpha)/
((1-exp(-x^lambda))^alpha+(exp(-x^lambda))^alpha))
)^a)))/(exp(theta)-1))*(2*theta*a*alpha*(lambda*(x^(lambda-1))
*exp(-x^lambda))*((1-exp(-x^lambda))^(alpha-1))
*((exp(-x^lambda))^(alpha-1))*((1-((1-exp(-x^lambda))^alpha)/

```

```

((1 - exp(-x^lambda))^alpha + (exp(-x^lambda))^alpha)^(a-1))
*((1 + ((1 - exp(-x^lambda))^alpha) / ((1 - exp(-x^lambda))^alpha
+ (exp(-x^lambda))^alpha))^(-(a+1))) * (((1 - exp(-x^lambda))^alpha +
(exp(-x^lambda))^alpha)^(-2))))}
TIEHLOLLSHLPS.result <- mle2(TIEHLOLLSHLPS_LL, hessian =
NULL, start = list(a = .1090892, alpha = 1.9881, theta = 101.009, lambda
= 0.61),
optimizer = "nlnmb", lower = 0)
summary(TIEHLOLLSHLPS.result)

vcov(TIEHLOLLSHLPS.result)
outputTIEHLOLLSHLPS.result <- vcov(TIEHLOLLSHLPS.result)
1.96 * sqrt(diag(outputTIEHLOLLSHLPS.result))

cov2cor(vcov(TIEHLOLLSHLPS.result))

```

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