Article

Ultimate Limit State Reliability-Based Optimization of MSE Wall Considering External Stability

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Abstract: We present reliability-based optimization (RBO) of the Mechanically Stability Earth (MSE) walls, using constrained optimization, considering the external stability, under ultimate limit state conditions of sliding, eccentricity, and bearing capacity. The design is optimized for a target reliability index of 3 that corresponds to an approximate failure probability of 1 in 1000. Reliability index is calculated by the first-order reliability method (FORM). The MSE wall, founded on cohesionless soil, with horizontal backfill and uniform live traffic surcharge, is studied. The RBO results are reported for the height of MSE wall ranging from 1.5 m to 20 m. For target reliability index of 3, the optimized length to height ratio, \( L_{opt} / H \), of the MSE wall is greater than 0.7 (the minimum length to height ratio requirement of AASHTO) for \( H \leq 4.5 \) m, and then it decreases below the minimum required value of 0.7 for \( H > 4.5 \) m. The RBO approach presented in this study will help practitioners to achieve cost-effectiveness in design.

Keywords: MSE wall; external stability; constrained optimization; reliability-based optimization; first-order reliability method

1. Introduction

The geotechnical engineering systems inherit risks and uncertainties and in order to apply sustainable design approach it is essential to quantify these uncertainties to rationalize the practice [1–7]. Traditional deterministic methods in geotechnical engineering are considered insufficient due to uncertainties inherently associated with geotechnical materials [8]. Empirical safety factors used in traditional deterministic design cannot incorporate effect of the uncertainties of design variables on the overall performance.

The uncertainties in the design of geotechnical systems come from applied loading, geotechnical properties of soil, and the models used in calculations [8,9]. Reliability-based design (RBD) of geotechnical systems is an alternative method to the practice of allowable stress design (ASD). Reliability-based optimization (RBO) is a great technique for optimizing geotechnical-related design problems satisfying to a predefined criteria (such as economy in construction) while explicitly satisfying the design requirements and accommodating the unavoidable uncertainties [10].

In recent past, several studies report the reliability-based design (RBD) as well as reliability-based optimization (RBO) of retaining structures. Sayed et al. [11] performed a parametric sensitivity analysis of reinforced soil wall to investigate the effect of material uncertainties under static and dynamic loading. Basha and Babu [12] used inverse FORM to study reliability-based design optimization of anchored sheet pile wall. Babu and Basha [13] used inverse reliability approach for design optimization of cantilever sheet pile wall. Basha and Babu [14] used reliability based approach for design optimization...

Mechanically stabilized earth (MSE) wall, also called reinforced earth wall, was proposed by Henry Vidal in the early 1960s [22]. In United Stated, more than fifty percent of the retaining structures used in transportation infrastructure consist of MSE walls [23]. The earthen materials are reinforced to support their weight and other loads. These walls consist of facing, reinforcement, reinforced soil and retained soil. Welded wire mats, metal bars, geosynthetics, or other anchorage systems are used as reinforcement to improve the mechanical properties of the soil mass. The MSE walls must satisfy both external and internal stability requirements. The external stability includes sliding, eccentricity, bearing capacity and overall stability checks. The internal stability includes pullout and structural failure of the reinforcement. Each stability criteria represents a separate limit state. The MSE walls used in transportation applications are designed according to the load and resistance factor (LRFD) design as per AASHTO specifications [24].

The aim of present study is to find an optimum design of the MSE wall using constrained optimization considering the external stability for a target reliability (or target failure probability). The failure of probability for earth retaining structures ranges from 0.1 to 0.0001. [25,26]. The target reliability of 3 corresponding to failure probability of 0.0013 (or an approximate probability of a failure of 1 in 1000) is used. The limit states of sliding, eccentricity, and bearing capacity failure are checked. LRFD design procedure of AASTHO [24] is used. The first-order reliability method (FORM) is used to determine reliability index. Constrained optimization with linear approximation (COBYLA) and reliability index calculation is implemented in Python language and Open TURNS [27,28], open-source software for probabilistic modeling and uncertainty management. RBO results of MSE wall for heights ranging from $H = 1.5$ to $20$ m are presented.

2. Methodology

In this section, the reliability index, constrained optimization algorithm, the algorithm for constrained optimization, the uncertainty of variables, and limit state equations are discussed.

2.1. Reliability Index

Due to the inherent uncertainty of input variables related to material properties and loads, the stability of an engineering system can be expressed by the reliability index. According to Hasofer and Lind [29], “the reliability index $\beta$ is the shortest distance from the mean value point of the random variables to the limit state surface” which can be expressed as follow:

$$\beta = \min_{x \in F} \sqrt{(x - \mu)^T C^{-1}(x - \mu)}$$  \hspace{1cm} (1)

where $x = $ vector of random variables, $\mu = $ vector of mean values, $F = $ failure domain, and $C = $ covariance matrix.
In many engineering problems, random variables are not normally distributed, which makes reliability index calculations not so easy. Therefore, using different techniques [30–34], the non-normal random variables, in the physical space, can be transformed into normal independent variables, in the standard space.

To calculate $\beta$ in the standard normal space $U$, a most probable design point $u^*$ should be found by minimizing $(u^T u = \sum_{i=1}^{n} u_i^2)$, under the constraint $G(u) = 0$, where $G(u)$ is the limit state function in standard normal space. Therefore, as shown in Figure 1 [35], $\beta$ is the minimum distance from the origin of the standardized normal space of random variable to the limit state surface at the failure point $u^*$, which can be expressed as follows:

$$\beta = \min \sqrt{u^T u} = \min \sqrt{\sum_{i=1}^{n} u_i^2} = \min \sqrt{(u^*)^T u^*}$$

where $u_i$ = transformed random variable in the standard normal space $U$. According to Low & Tang [34], random variable $x_i$ can be obtained from transformed random variable $u_i$ by Equations (3) and (4) for normal and lognormal distribution respectively.

$$x_i = \mu_i + u_i \sigma_i$$

$$x_i = \exp(\lambda + \zeta u_i), \quad \zeta = \sqrt{\log_e \left[ \frac{1 + (\sigma_i / \mu_i)^2}{2} \right]}, \quad \lambda = \log_e \mu_i - 0.5 \zeta^2$$

where $\mu_i = \text{mean of random variable } x_i$, $\sigma_i = \text{standard deviation of random variable } x_i$.

![Figure 1. First- and second-order reliability methods.](image)

2.2. Constrained Optimization with Linear Approximation (COBYLA)

Powell [36, 37] developed a gradient-free, constrained optimization algorithm, which is capable of handling inequality and equality constraints. Constrained optimization is method of finding a vector $x$ that is local minimum to a scalar function $f(x)$ subject to constraints on the allowable $x$:

$$\min_{x} f(x) \text{ such that } \begin{cases} g(x) = 0 \\ h(x) \geq 0 \\ lb \leq x \leq ub \end{cases}$$

where $g(x) = \text{equality constraint}$, $h(x) = \text{inequality constraint}$, $lb = \text{lower bound}$, $ub = \text{upper and lower bounds}$, and $f(x) = \text{objective function that returns a scalar}$. COBYLA algorithm in this study is implemented with Open TURNS [27, 28], which is open-source software for probabilistic modeling and uncertainty management.
2.3. Process of Reliability-Based Optimization (RBO)

Minimizing the cost of a geotechnical structure, satisfying the requirements of the target reliability $\beta_T$ and design equations is essentially a constrained optimization problem. RBO becomes a multi-objective (double loop) optimization problem because the determination of reliability index itself is an optimization problem [20]. Constrained optimization is used in two loops: searching the design variables while minimizing the cost of the geotechnical problem with target reliability as a constraint in the outer loop, and computing $\beta$ with design requirements as constraints in the inner loop (Figure 2).

![Flowchart for reliability-based optimization](image)

The cost function of geotechnical problem is minimized in the outer loop. The calculation sequence is as follows:

1. Initialize dimensions of the geotechnical problem (i.e., design variables).
2. Run COBYLA to minimize the cost function (i.e., MSE wall area). Target reliability index $\beta_T$ is the constraint. $\beta_T$ should be less than or equal to the calculated value of $\beta$ (determined in the inner loop).
3. If the cost function is minimum and $\beta - \beta_T \geq 0$, all requirements are satisfied, stop the execution. If cost function is not minimum, change design variables, and repeat calculations from Step 2.

Reliability index for a limit state is calculated in the inner loop by searching transformed random variable $u_i$. The calculation sequence is as follows:

1. Initialize vector $u$.
2. Run COBYLA algorithm. The objective is minimization of reliability index. The constraint is geotechnical design equation (or limit state).
Step 3: Calculate $\beta$. If $\beta$ is minimum and constraint is satisfied, stop execution, and pass control to the outer loop. If $\beta$ is not minimum, change vector $u$, and repeat the process from step 2.

2.4. Reliability-Based Optimization of MSE Wall Considering External Stability

Limit state design (LSD) method deals with uncertainty of loads and material properties by incorporating load and resistance factors. The load and resistance factored design (LRFD) compares factored resistance to the sum of factored loads:

$$(RF)R_n \geq (LF_i)Q_{n,i}$$

where $RF = \text{resistance factor } (\leq 1)$; $R_n = \text{nominal resistance}$; and $LF_i = \text{load factors } (\geq 1)$ for particular type of design loads $Q_{n,i}$ (e.g., dead load and live load). Tables 1 and 2 show the load and resistance factors, respectively, for MSE walls as specified by AASHTO [24].

Table 1. Summary of load factors.

<table>
<thead>
<tr>
<th>Load Factor</th>
<th>Description</th>
<th>Value and Its Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LF$_{EV}$</td>
<td>Vertical pressure from dead load of earth fill</td>
<td>1.00 For sliding &amp; eccentricity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.35 For bearing capacity</td>
</tr>
<tr>
<td>LF$_{EH}$</td>
<td>Horizontal earth load</td>
<td>1.50 For sliding &amp; eccentricity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.50 For bearing capacity</td>
</tr>
<tr>
<td>LF$_{LS,V}$</td>
<td>Vertical pressure from live surcharge load</td>
<td>1.75 For bearing capacity</td>
</tr>
<tr>
<td>LF$_{LS,H}$</td>
<td>Horizontal pressure from live surcharge load</td>
<td>1.75 For sliding, eccentricity &amp; bearing capacity</td>
</tr>
</tbody>
</table>

Table 2. Summary of resistance factors.

<table>
<thead>
<tr>
<th>Resistance Factor</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>RF$_{SL}$</td>
<td>Sliding resistance</td>
<td>1.0</td>
</tr>
<tr>
<td>RF$_{BC}$</td>
<td>Bearing resistance</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Reliability-based optimization (RBO) is implemented on the external stability of the MSE wall. Load and resistance factor design (LRFD) of MSE walls is adopted from AASHTO [24].

The reinforced soil mass of the MSE wall acts as a rigid block when checked for external stability [38]. The design constraints for external stability of MSE wall are sliding, eccentricity, and bearing capacity which are the limit states (or failure modes). The objective is to find a solution with minimum wall area while fulfilling the reliability and design requirements.

Figure 3 shows a schematic diagram of MSE wall with horizontal backfill and uniform live traffic surcharge. For sliding stability and eccentricity, continuous traffic surcharge loads are considered acting beyond the end of a reinforced zone. In contrast, the traffic surcharge loads are considered acting over reinforced and retained backfill zones for bearing capacity, as shown in Figure 3 [24].

Lateral forces acting on the MSE wall can be expressed as:

$$F_1 = 0.5\gamma_f H^2 K_a$$  \hspace{1cm} (7)

$$F_2 = qHK_a$$  \hspace{1cm} (8)

where $F_1 = \text{lateral force due to self-weight of the retained soil}$; $F_2 = \text{lateral force due to uniform live surcharge load, } q, \text{ on the top of the retained soil}$; $K_a = \tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \text{active}$
earth pressure coefficient; \( \gamma_F = \) unit weight of retained fill; \( \phi_F = \) friction angle of retained fill; and \( H = \) MSE wall height.

![Diagram](image)

**Figure 3.** External stability for MSE wall with horizontal backfill and traffic surcharge.

### 2.4.1. Limit State Equation for Sliding Failure

Factored sliding force, \( R_{UL} \), and factored resistance, \( R_R \), can be expressed as:

\[
R_{UL} = 1.5F_1 + 1.75F_2
\]

\[
R_R = 1.0W_R \tan \delta
\]

where \( W_R = \gamma_R LH \) = self-weight of reinforced soil, \( \delta = \min(\phi_R, \phi_{FN}) \) = interface friction angle at the base of the MSE wall, \( \phi_R = \) reinforced soil friction angle, \( \phi_{FN} = \) foundation soil friction angle, and \( L = \) reinforcement length. Interface friction angle \( \delta \) is taken as the minimum of reinforced soil friction angle \( \phi_R \) and foundation soil friction angle \( \phi_{FN} \) (Article 11.10.5.3 of AASHTO [24]). Load and resistance factors given in Tables 1 and 2 are used in Equations (9) and (10).

The limit state equation for sliding failure is

\[
G_1(x) = R_R - R_{UL} = 1.0W_R \tan \delta - (1.5F_1 + 1.75F_2) - \gamma_R LH \tan \delta - (0.75\gamma_F H^2 + 1.75qH)K_a
\]

The factor of safety at ultimate limit state, also termed as capacity/demand ratio (CDR), for sliding failure, is

\[
CDR_{SL} = \frac{R_R}{R_{UL}} = \frac{\gamma_R LH \tan \delta}{(0.75\gamma_F H^2 + 1.75qH)K_a}
\]

### 2.4.2. Limit State Equation for Eccentricity Failure

The location of the base reaction, \( R \), is limited to the middle two-thirds of the base width for a soil foundation (Article 11.6.3.3 of AASHTO [24]). Therefore, the maximum eccentricity is

\[
e_{\text{max}} = \frac{L}{3}
\]

Considering the moment equilibrium at point \( A \) (Figure 3), the base reaction eccentricity is given by

\[
e = \frac{(\sum M_D)_A - (\sum M_R)_A}{W_R}
\]
where \((\sum M_O)_A\) and \((\sum M_R)_A\) are the summation of overturning and resisting moments, respectively, about point \(A\); \(W_R\) = self-weight of reinforced soil. Factored overturning and resisting moments are given by

\[
(\sum M_O)_A = 1.5(F_1 H/3) + 1.75(F_2 H/2), \quad (\sum M_R)_A = 0
\]  

(15)

Substituting factored overturning and resisting moments in Equation (14) modifies to the following form:

\[
e = \frac{(0.25\gamma_F H^2 + 0.875qH)K_a}{\gamma_R L}
\]  

(16)

The limit state equation for eccentricity is given by

\[
G_2(x) = e_{\text{max}} - e = \frac{L}{3} \frac{(0.25\gamma_F H^2 + 0.875qH)K_a}{\gamma_R L}
\]  

(17)

The capacity/demand ratio (CDR) for eccentricity failure is

\[
\text{CDR}_e = \frac{e_{\text{max}}}{e} = \frac{\gamma_R}{(0.75\gamma_F H^2 + 2.625qH)K_a}
\]  

(18)

2.4.3. Limit State Equation for Bearing Capacity Failure

The traffic surcharge loads are considered acting over reinforced as well as retained backfill, therefore, the eccentricity, \(e_B\), of the base reaction is modified to following form.

\[
e_B = \frac{1.5(F_1 H/2) + 1.75(F_2 H/2)}{1.35\gamma_R LH + 1.75qL} = \frac{(0.25\gamma_F H^2 + 0.875qH^2)K_a}{1.35\gamma_R LH + 1.75qL}
\]  

(19)

The factored ultimate bearing stress, \(\sigma_V\), at the base of the MSE wall is given by

\[
\sigma_V = \frac{1.35\gamma_R LH + 1.75qL}{L - 2e_B}
\]  

(20)

A uniform stress distribution can be assumed over a reduced area at the wall base instead of nonuniform distribution due to eccentric loading. According to Article 11.10.5.4 of AASHTO [24], this area is defined by a length equal to MSE wall length minus twice the eccentricity as shown in Figure 3.

The nominal bearing resistance for granular foundation soil with no cohesion is given by

\[
q_n = 0.5(L - 2e_B)\gamma_F N_\gamma
\]  

where \(\gamma_F = \) unit weight of foundation soil, \(N_\gamma = \) dimensionless bearing capacity factor. The dimensionless bearing capacity factors, based on Vesic [39], are given by

\[
N_\gamma = 2 \left[ e^{x \tan (\phi_FN)} \frac{\pi}{4} \left( x \frac{\pi}{2} + \frac{\phi_FN}{2} \right) + 1 \right] \tan (\phi_FN)
\]  

(22)

The factored bearing resistance for granular foundation soil is expressed as

\[
q_R = 0.65q_n = 0.325(L - 2e_B)\gamma_F N_\gamma
\]  

(23)

The limit state equation for bearing resistance is given by

\[
G_3(x) = q_R - \sigma_V = 0.325(L - 2e_B)\gamma_F N_\gamma - \frac{1.35\gamma_R LH + 1.75qL}{L - 2e_B}
\]  

(24)
The capacity/demand ratio for bearing resistance is expressed as

\[
CDR_{BC} = \frac{qR}{\sigma_V}
\]  

(25)

2.5. Assessment of Uncertainties

Uncertain variables in sliding, overturning, and bearing capacity limit states are (1) unit weight, (2) friction angle, and (3) uniform live surcharge load.

Uncertainty of unit weight: Past studies [3,40–44] have reported the coefficient of variation (COV) of unit weight of soil, \( \gamma_r \), in the range of 0.03 and 0.2. In this study, COV is taken as 0.05 for both compacted and uncompacted soils. A normal distribution of unit weight is assumed.

Uncertainty of friction angle: Active earth pressure coefficients and bearing capacity factors are calculated from critical-state friction angle, \( \phi_c \). Previous studies [45–47] have reported the COV of friction angle of soil, \( \phi_c \), in the range of 0.008 and 0.1. In this study, COV is taken as 0.025 following a lognormal distribution.

Uncertainty of uniform live surcharge \( q \): To include vehicular load, an equivalent height of soil, \( h_{eq} \), is assumed on the top of MSE wall [24], an equivalent height of soil, \( h_{eq} \), is considered on the top of the MSE wall to include vehicular load. Table 3 shows the equivalent soil height, \( h_{eq} \), for vehicular load on an abutment perpendicular to traffic, while Table 4 shows the same for vehicular load on MSE wall running parallel to traffic. Table 5 shows the live traffic surcharge values for different wall heights used in this study. In this study, COV is taken as 0.2 following a lognormal distribution.

<table>
<thead>
<tr>
<th>MSE Wall Height ( H ) (m)</th>
<th>( h_{eq} ) (m)</th>
<th>( q_0^a ) (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.2</td>
<td>24</td>
</tr>
<tr>
<td>3.0</td>
<td>0.9</td>
<td>18</td>
</tr>
<tr>
<td>( \geq 6.0 )</td>
<td>0.6</td>
<td>12</td>
</tr>
</tbody>
</table>

\( a \) unit weight of soil is taken as 20 kN/m\(^3\) for vehicular loading.

<table>
<thead>
<tr>
<th>MSE Wall Height ( H ) (m)</th>
<th>( h_{eq} ) (m)</th>
<th>Distance from Wall Backface to Edge of Traffic = 0 m</th>
<th>Distance from Wall Backface to Edge of Traffic ( \geq 0.3 ) m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>3.0</td>
<td>1.05</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>( \geq 6.0 )</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MSE Wall Height ( H ) (m)</th>
<th>( h_{eq} ) (m)</th>
<th>( q^a ) (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.20</td>
<td>24</td>
</tr>
<tr>
<td>2.0</td>
<td>1.09</td>
<td>21.8</td>
</tr>
<tr>
<td>2.5</td>
<td>0.99</td>
<td>19.8</td>
</tr>
<tr>
<td>3.0</td>
<td>0.90</td>
<td>18</td>
</tr>
<tr>
<td>3.5</td>
<td>0.82</td>
<td>16.4</td>
</tr>
<tr>
<td>4.0</td>
<td>0.76</td>
<td>15.2</td>
</tr>
<tr>
<td>4.5</td>
<td>0.70</td>
<td>14</td>
</tr>
<tr>
<td>5.0</td>
<td>0.66</td>
<td>13.2</td>
</tr>
<tr>
<td>5.5</td>
<td>0.62</td>
<td>12.4</td>
</tr>
<tr>
<td>( \geq 6.0 )</td>
<td>0.60</td>
<td>12</td>
</tr>
</tbody>
</table>

\( a \) unit weight of soil is taken as 20 kN/m\(^3\) for vehicular loading.
Seven uncertain variables considered in the study are friction angle of reinforced fill, $\phi_R$; friction angle of retained fill, $\phi_F$; friction angle of the foundation soil, $\phi_{FN}$; unit weight of reinforced fill, $\gamma_R$; unit weight of retained fill, $\gamma_F$, unit weight of foundation soil, $\gamma_{FN}$; and uniform live surcharge load, $q$. Table 6 summarizes the distribution and statistics of these variables.

**Table 6. Distribution and statistics of uncertain variables.**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Distribution</th>
<th>Mean</th>
<th>COV</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_R$ ($^\circ$)</td>
<td>Lognormal</td>
<td>36</td>
<td>0.025</td>
<td>0.9</td>
</tr>
<tr>
<td>$\gamma_R$ (kN/m$^3$)</td>
<td>Normal</td>
<td>20</td>
<td>0.050</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_F$ ($^\circ$)</td>
<td>Lognormal</td>
<td>30</td>
<td>0.025</td>
<td>0.75</td>
</tr>
<tr>
<td>$\gamma_F$ (kN/m$^3$)</td>
<td>Normal</td>
<td>18</td>
<td>0.050</td>
<td>0.9</td>
</tr>
<tr>
<td>$\phi_{FN}$ ($^\circ$)</td>
<td>Lognormal</td>
<td>33</td>
<td>0.025</td>
<td>0.825</td>
</tr>
<tr>
<td>$\gamma_{FN}$ (kN/m$^3$)</td>
<td>Normal</td>
<td>18</td>
<td>0.050</td>
<td>0.9</td>
</tr>
<tr>
<td>$q$ (kN/m2)</td>
<td>Lognormal</td>
<td>varies</td>
<td>0.2</td>
<td>varies</td>
</tr>
</tbody>
</table>

2.6. Process for Reliability-Based Optimization of MSE Wall

Considering a MSE wall of height, $H$, its length, $L$, is the design variable which needs to be optimized to obtain the minimum area. Past studies [25,26] suggest failure probability in the range of 0.1 to 0.0001 for retaining structures. The target reliability of 3, corresponding to failure probability of 0.0013 (or an approximate probability of a failure of 1 in 1000) is used in this study. The design space is considered as $0.4H < L < 2H$. The problem can be expressed as

$$
\min C(d) = LH \\
\text{subject to } \beta_{SL}, \beta_e, \beta_{BC} \geq 3
$$

(26)

where $C(d)$ = area of the MSE wall and $\beta_{SL}, \beta_e, \beta_{BC}$ are the reliability constraints for limit states of sliding, eccentricity, and bearing capacity, respectively. Constrained optimization by linear approximation (COBYLA) is used in multi-objective constrained optimization problem. The area of the MSE wall is minimized in the outer loop with the constraint as $\beta_i - \beta_T \geq 0$. The reliability index $\beta_i$ is minimized in the inner loop with the constraint as limit state corresponding to the geotechnical design requirement.

3. Results and Discussion

Reliability-based optimization (RBO) of the MSE wall for different heights ranging from $H = 1.5$ to 20 m is performed. The case 6-m high MSE wall is discussed to illustrate the process of RBO. The optimization process presented in Methodology section is implemented in Python 3.8 and open TURNS 1.16 [27,28]. The optimization results for the case of $H = 6$ m are shown in Table 7 and Figure 4, respectively. As seen in Table 7, three optimal scenarios are considered for limit states of sliding, eccentricity, and bearing capacity failures, respectively. These optimal scenarios are discussed as follows.

- **Optimal 1** is the optimized solution for limit state of sliding (i.e., $\beta_{SL} - \beta_T \geq 0$). Figure 4a shows the convergence of cost function. From the optimal 1 solution, reliability indices $\beta_{SL}, \beta_{BC}$ for other limit states (i.e., overturning and bearing capacity failures, respectively) are calculated. It is noted that both $\beta_e$ and $\beta_{BC}$ are less than $\beta_T$, therefore, this scenario does not satisfy all reliability constraints (i.e., all $\beta_i \geq \beta_T$) and thus design requirements.

- **Optimal 2** is the optimized solution for limit state of eccentricity (i.e., $\beta_e - \beta_T \geq 0$). Figure 4b shows the convergence of cost function. From optimal 2 solution, reliability indices $\beta_{SL}, \beta_{BC}$ for other limit states are calculated. In this case, all reliability constraints (i.e., all $\beta_i \geq \beta_T$) and design requirements are satisfied.

- **Optimal 3** is the optimized solution for limit state of bearing capacity (i.e., $\beta_{BC} - \beta_T \geq 0$). Figure 4c shows the convergence of the cost function. From this optimal solution, reliability indices $\beta_{SL}, \beta_e$ for other limit states are calculated. It is noted that $\beta_e$ is less
than \( \beta_T \), therefore, this solution also does not satisfy all reliability constraints (i.e., all \( \beta_i \neq \beta_T \)) and thus design requirements.

Table 7. Optimization results of MSE wall for \( H = 6 \) m height.

<table>
<thead>
<tr>
<th>Optimal Scenario</th>
<th>Design Constraint</th>
<th>( d )</th>
<th>( L/H )</th>
<th>Reliability Index</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal 1</td>
<td>Sliding</td>
<td>3.412</td>
<td>20.470</td>
<td>0.569</td>
<td>3.000</td>
</tr>
<tr>
<td>Optimal 2</td>
<td>Eccentricity</td>
<td>3.845</td>
<td>23.069</td>
<td>0.641</td>
<td>4.271</td>
</tr>
<tr>
<td>Optimal 3</td>
<td>Bearing Capacity</td>
<td>3.636</td>
<td>21.816</td>
<td>0.606</td>
<td>3.684</td>
</tr>
</tbody>
</table>

\( \beta_{SL} \) is reliability index for limit state of sliding, \( \beta_e \) is reliability index for limit state of overturning, and \( \beta_{BC} \) is reliability index for limit state of bearing capacity.

Figure 4. Convergence of objective or cost (i.e., area) of the MSE wall of height \( H = 6 \) m for design constraints of (a) sliding, (b) overturning, and (c) bearing capacity failures, respectively.

Only optimal 2 is the solution where all reliability constraint requirements and thus geotechnical design requirements are satisfied. Thus, the optimized solution obtained with the design constraint of eccentricity failure satisfies all reliability constraints requirements. It can be concluded that design constraint of eccentricity is critical for MSE wall with the height of 6 m because \( \beta_e = \beta_T \) whereas other \( \beta \) values are greater than \( \beta_T \). Table 8 shows a comparison between the initial and optimized design for this case.

Table 8. Summary of optimization results of MSE wall for \( H = 6 \) m height.

<table>
<thead>
<tr>
<th>State</th>
<th>( d )</th>
<th>( L/H )</th>
<th>Reliability Index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( L ) (m)</td>
<td>( C(d) ) (( m^3 ))</td>
<td>( \beta_{SL} )</td>
</tr>
<tr>
<td>Initial</td>
<td>6</td>
<td>36</td>
<td>1</td>
</tr>
<tr>
<td>Optimized</td>
<td>3.845</td>
<td>23.069</td>
<td>0.641</td>
</tr>
</tbody>
</table>

The evolution of absolute and relative error during the convergence of cost function as well as constraint error, \( (\beta_i - \beta_T) \), during RBO of the MSE wall of height \( H = 6 \) m is shown in Figure 5. The absolute and relative error can be defined, respectively, as \( |X_{n+1} - X_n| \) and \( |X_{n+1} - X_n| / |X_{n+1}| \) where \( X_{n+1} \) and \( X_n \) are two consecutive approximations of the optimum.
Figure 5. Evolution of absolute, relative error of cost function and constraint error during RBO of the MSE wall of height \( H = 6 \) m for design constraints of (a) sliding, (b) overturning, and (c) bearing capacity failures, respectively.

Results of reliability-based Optimization (RBO) of the MSE wall for different heights ranging from \( H = 1.5 \) to \( 20 \) m are shown in Figure 6 and Table 9, respectively. For the MSE wall of height \( H \leq 3 \) m, sliding is governing limit state, whereas for \( H > 3 \) m, eccentricity is governing limit state (Figure 7). Since live traffic surcharge values (Table 5) are high for this height range (i.e., \( H \leq 3 \) m) of the MSE wall, the sliding is governing limit state. Figure 8 shows that the capacity/demand ratio of the optimized design for each limit state is greater than the critical value of 1.

The importance factors (expressed as a percentage) of uncertain variables of the optimized design for different heights of the MSE wall are shown in Figure 9. The importance factors \( \alpha_i^2 \) of uncertain variables are defined as \( \alpha_i^2 = \left(\frac{\gamma_i}{\beta}\right)^2 \times 100\% \) [27] where \( \gamma_i \) = design point in standard normal space and \( \beta = \) reliability index. It implies from the definition that \( \sum \alpha_i^2 = 100\% \). The conclusions drawn from this figure are: (a) at low heights of the MSE wall, the influence of live traffic surcharge, \( q \), is dominant, which makes sliding as governing limit state, (b) the influence of live traffic surcharge, \( q \), decreases with the increase in the MSE wall height, (c) influence of reinforced-fill unit weight, \( \gamma_R \), increases with the increase in the MSE wall height and is prominent when design is based on limit state of eccentricity, (d) internal friction angle of reinforced-fill, \( \phi_R \), does not influence external stability of the MSE wall at all, and (e) internal friction angle, \( \phi_{FN} \), and unit weight, \( \gamma_{FN} \), of
foundation soil has no influence as limit state of bearing capacity does not dictate design for any case of the MSE wall from \( H = 1.5 \) to 20 m.

Table 9. Optimization results of the MSE wall for \( \beta_T = 3 \) for heights ranging from \( H = 1.5 \) m to 20 m.

<table>
<thead>
<tr>
<th>( H ) (m)</th>
<th>Reliability Index</th>
<th>Critical Limit State</th>
<th>( L_{opt}/H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>3</td>
<td>6.468</td>
<td>8.98</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5.228</td>
<td>8.246</td>
</tr>
<tr>
<td>2.5</td>
<td>3</td>
<td>4.265</td>
<td>7.312</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3.461</td>
<td>6.268</td>
</tr>
<tr>
<td>3.5</td>
<td>3.134</td>
<td>3</td>
<td>5.472</td>
</tr>
<tr>
<td>4</td>
<td>3.452</td>
<td>3</td>
<td>5.188</td>
</tr>
<tr>
<td>4.5</td>
<td>3.737</td>
<td>3</td>
<td>4.905</td>
</tr>
<tr>
<td>5</td>
<td>3.953</td>
<td>3</td>
<td>4.669</td>
</tr>
<tr>
<td>5.5</td>
<td>4.142</td>
<td>3</td>
<td>4.442</td>
</tr>
<tr>
<td>6</td>
<td>4.271</td>
<td>3</td>
<td>4.272</td>
</tr>
<tr>
<td>7</td>
<td>4.417</td>
<td>3</td>
<td>4.061</td>
</tr>
<tr>
<td>7.5</td>
<td>4.475</td>
<td>3</td>
<td>3.971</td>
</tr>
<tr>
<td>10</td>
<td>4.669</td>
<td>3</td>
<td>3.631</td>
</tr>
<tr>
<td>12.5</td>
<td>4.776</td>
<td>3</td>
<td>3.416</td>
</tr>
<tr>
<td>15</td>
<td>4.842</td>
<td>3</td>
<td>3.271</td>
</tr>
<tr>
<td>17.5</td>
<td>4.887</td>
<td>3</td>
<td>3.168</td>
</tr>
<tr>
<td>20</td>
<td>4.919</td>
<td>3</td>
<td>3.091</td>
</tr>
</tbody>
</table>

Figure 7. Reliability indices of the optimized design.

Figure 8. Capacity/demand ratio of the optimized design.
According to article 11.10.2 of AASHTO [24], the $L/H$ ratio should not be less than 0.7. If the $L/H$ ratio is set equal to 0.7 for $H > 4.5$ m, as per AASHTO [24] requirements, then actual reliability index of the design is well above the reliability index value of 3.0.

Table 10 and Figure 10 show the optimized results modified for AASHTO’s minimum $L/H$ ratio requirement. The relation between optimal length, $L_{opt}$, and height, $H$, of the MSE wall can be expressed by Equation (27) which is valid for external stability of MSE wall with horizontal backfill and live traffic surcharge founded on the cohesionless soils.

$$L_{opt} = 0.0954H^2 - 0.369H + 2.894 \text{ for } H \leq 4.5 \text{ m } (\beta = 3)$$
$$= 0.7H \text{ for } H > 4.5 \text{ m } (\beta > 3)$$

Table 10. Optimization results for $\beta_T = 3$ modified for AASHTO’s minimum $L/H$ ratio requirements.

<table>
<thead>
<tr>
<th>$H$ (m)</th>
<th>$L_{opt}/H$</th>
<th>Reliability Index, $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.695</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1.277</td>
<td>3</td>
</tr>
<tr>
<td>2.5</td>
<td>1.033</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0.878</td>
<td>3</td>
</tr>
<tr>
<td>3.5</td>
<td>0.786</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>0.740</td>
<td>3</td>
</tr>
<tr>
<td>4.5</td>
<td>0.703</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0.7</td>
<td>3.595</td>
</tr>
<tr>
<td>5.5</td>
<td>0.7</td>
<td>4.230</td>
</tr>
<tr>
<td>6</td>
<td>0.7</td>
<td>4.701</td>
</tr>
<tr>
<td>7</td>
<td>0.7</td>
<td>5.292</td>
</tr>
<tr>
<td>7.5</td>
<td>0.7</td>
<td>5.549</td>
</tr>
<tr>
<td>10</td>
<td>0.7</td>
<td>6.470</td>
</tr>
<tr>
<td>12.5</td>
<td>0.7</td>
<td>6.858</td>
</tr>
<tr>
<td>15</td>
<td>0.7</td>
<td>7.105</td>
</tr>
<tr>
<td>17.5</td>
<td>0.7</td>
<td>7.274</td>
</tr>
<tr>
<td>20</td>
<td>0.7</td>
<td>7.397</td>
</tr>
</tbody>
</table>
Figure 10. Reliability-based optimized results modified for AASHTO’s minimum length requirement (a) relationship between optimal length, $L_{opt}$, and height, $H$ (b) relationship between $L_{opt}/H$ and height, $H$.

4. Conclusions

Reliability-based optimization (RBO) of the MSE wall, for target reliability index of $\beta_T = 3$, under ultimate limit state, considering external stability is presented. Constrained optimization with linear approximation (COBYLA) is used for minimization of the cost function (i.e., area) of the MSE wall. The limit states corresponding to external stability, i.e., sliding, eccentricity, and bearing capacity, respectively, are used. RBO is carried out as a two-loop constrained optimization problem with the outer loop to minimize area of MSE wall with constraint as target reliability index, and the inner loop to minimize the reliability index with constraints as geotechnical design requirements (i.e., limit states of sliding, eccentricity, and bearing capacity, respectively). RBO of the MSE wall, founded on cohesions soil, for different heights ranging from $H = 1.5$ to 20 m with horizontal back fill and live traffic surcharge are studied. The findings of this study summarized as follows:

1. At the low height of the MSE wall (i.e., $H \leq 3$ m), the influence of live traffic surcharge, $q$, is dominant (due to high $q$ values recommended by AASHTO), making sliding as the governing limit state. The influence of traffic surcharge decreases with the increase in the MSE wall height. For $H > 3$ m, eccentricity is the governing limit state.
2. The influence of reinforced-fill unit weight, $\gamma_R$, increases with the increase in the MSE wall height and is prominent when the limit state of eccentricity governs the design.
3. For $H \leq 4.5$ m, length to height ratio of the optimized solution, $L_{opt}/H$, is greater than 0.7 (minimum requirement of AASHTO), and then it decreases below the minimum required value of 0.7 for $H > 4.5$ m.
4. If $L/H$ is kept as 0.7, as per AASHTO’s requirement, for $H > 4.5$ m, the actual reliability index of the design will be well above the target reliability index of 3.

Present study focuses on the external stability only. However, inclusion of limit states for internal stability (pullout and reinforcement failure) along with external stability is future interest of the authors.

Author Contributions: Conceptualization, Z.M.; methodology, Z.M. and M.U.Q.; software, Z.M.; validation, M.U.Q. and Z.A.M.; formal analysis, Z.M.; investigation, Z.M.; resources, Z.A.M.; data curation, M.U.Q. and Q.B.a.I.L.; writing—original draft preparation, Z.M.; writing—review and editing, M.U.Q. and Q.B.a.I.L.; visualization, Z.M.; supervision, Z.M.; project administration, M.U.Q.; funding acquisition, Z.A.M. All authors have read and agreed to the published version of the manuscript.

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