Modeling to Factor Productivity of the United Kingdom Food Chain: Using a New Lifetime-Generated Family of Distributions

Salem A. Alyami 1,*, Ibrahim Elbatal 1,†, Naif Alotaibi 1,‡, Ehab M. Almetwally 2,3,†© and Mohammed Elgarhy 4,†©

1 Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia; ielbatal@imamu.edu.sa (I.E.); nmaalotaibi@imamu.edu.sa (N.A.)
2 Faculty of Business Administration, Delta University of Science and Technology, Gamasa 11152, Egypt; elah.metwaly@deltacollege.edu.eg
3 Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt
4 The Higher Institute of Commercial Sciences, Al Mahalla Al Kubra 31951, Egypt; m_elgarhy85@sva.edu.eg
* Correspondence: saalyami@imamu.edu.sa
† These authors contributed equally to this work.

Abstract: This article proposes a new lifetime-generated family of distributions called the sine-exponentiated Weibull-H (SEW-H) family, which is derived from two well-established families of distributions of entirely different nature: the sine-G (S-G) and the exponentiated Weibull-H (EW-H) families. Three new special models of this family include the sine-exponentiated Weibull exponential (SEWE_x), the sine-exponentiated Weibull Rayleigh (SEWR) and sine-exponentiated Weibull Burr X (SEWBX) distributions. The useful expansions of the probability density function (pdf) and cumulative distribution function (cdf) are derived. Statistical properties are obtained, including quantiles (Q_x), moments (M_x), incomplete M_x (IM_x), and order statistics (O_x) are computed. Six numerous methods of estimation are produced to estimate the parameters: maximum likelihood (M_L), least-square (L_S), maximum product of spacing (MPS_P), weighted L_S (WL_S), Cramér–von Mises (C_VM), and Anderson–Darling (A_D). The performance of the estimation approaches is investigated using Monte Carlo simulations. The total factor productivity (TFP) of the United Kingdom food chain is an indication of the efficiency and competitiveness of the food sector in the United Kingdom.

TFP growth suggests that the industry is becoming more efficient. If TFP of the food chain in the United Kingdom grows more rapidly than in other nations, it suggests that the sector is becoming more competitive. TFP, also known as multi-factor productivity in economic theory, estimates the fraction of output that cannot be explained by traditionally measured inputs of labor and capital employed in production. In this paper, we use five real datasets to show the relevance and flexibility of the suggested family. The first dataset represents the United Kingdom food chain from 2000 to 2019, whereas the second dataset represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP; the third dataset contains the tensile strength of single carbon fibers (in GPa); the fourth dataset is often called the breaking stress of carbon fiber dataset; the fifth dataset represents the TFP growth of agricultural production for thirty-seven African countries from 2001–2010. The new suggested distribution is very flexible and it outperforms many known distributions.

Keywords: total factor productivity; food chain; time series; forecast; maximum likelihood estimation; maximum product spacing; sine class of distributions; exponentiated Weibull class of distributions

1. Introduction

In the last few years, various techniques of adding a parameter to distributions have been proposed and discussed. These extended distributions give the flexibility in particular applications, such as economics, engineering, biomedical, biological studies, engineering,
physics, food, environmental sciences, COVID-19, and many more. Several famous families are the Marshall–Olkin-G given in [1], odd Fréchet-G [2], beta-G [3], logarithmic-X family of distributions [4], the extended cosine-G [5], the arcsine-exponential-X family [6], truncated Cauchy power Weibull-G [7], the odd-exponentiated half logistic-G [8], generalized transmuted-G [9], the generalized odd log-logistic-G [10], transmuted odd Fréchet-G [11], the logistic-X [12], beyond the Sin-G family [13], Cos-G class of distributions [14], odd Perks-G [15], U family of distributions [16], the extended odd Fréchet-G [17], exponentiated M-G [18], transmuted geometric-G [19], half logistic Burr X-G [20], a new sine-G in [21], exponentiated-truncated inverse Weibull-G [22], Burr X-G [23], sec-G [24], odd Nadarajah-Haghighi-G [25], Topp-Leone-G [26], sine Topp-Leone-G family of distributions by [27], a new power Topp-Leone-G by [28], truncated inverted Kumaraswamy generated-G by [29], among others.

The authors of [30] proposed the EW-H family; this class extended the Weibull-H family of distribution introduced by [31]. The cdf of the EW-H family is provided via

$$G_{EW-H}(x; \lambda, \theta, \beta, \delta) = \left[1 - e^{-\lambda \left(\frac{H(x; \delta)}{\mu(x; \delta)}\right)^\theta}\right]^\beta, \lambda, \theta, \beta > 0, x \in R, \delta \in R,$$

and the pdf reduces to

$$g_{EW-H}(x; \lambda, \theta, \beta, \delta) = \lambda \theta \beta h(x; \delta) \frac{H(x; \delta)^{\theta-1}}{\mu(x; \delta)^{\theta+1}} e^{-\lambda \left(\frac{H(x; \delta)}{\mu(x; \delta)}\right)^\theta} \left[1 - e^{-\lambda \left(\frac{H(x; \delta)}{\mu(x; \delta)}\right)^\theta}\right]^\beta-1.$$

where $H(x; \delta)$, $h(x; \delta)$, and $\mu(x; \delta)$ signify the cdf, pdf, and survival function (sf) of a baseline model considering a vector of parameters $\delta$.

The creation of trigonometric classes of distributions has recently garnered considerable attention. These families have the benefit of maintaining a balance between their definitions’ relative simplicity, which makes it possible to fully understand their mathematical features, and their broad application for modeling numerous sorts of real-world datasets. These two conclusions result from the proper use of adaptable trigonometric functions. The authors of [32] presented another idea of generating a new life distributions by modification of trigonometric functions to give new statistical distributions. They transformed the sine function into a new statistical distribution called the S-G family where the cdf and pdf are provided via

$$F(x) = \sin\left(\frac{\pi}{2} G(x)\right), \quad x \in R,$$

and

$$f(x) = \frac{\pi}{2} g(x) \cos\left(\frac{\pi}{2} G(x)\right), \quad x \in R.$$

The associated hazard rate function (hrf) is provided via

$$\xi(x) = \frac{\pi}{2} g(x) \tan\left(\frac{\pi}{4}(1 + G(x))\right).$$

There are several further trigonometric families of distributions. For illustration, consider the beta trigonometric distribution by [33], sine square distribution by [34], a cosine approximation to the normal distribution by [35], odd hyperbolic cosine exponential-distribution by [36], new trigonometric classes of probabilistic distributions by [37], odd hyperbolic cosine family of lifetime distributions by [38], transmuted arcsine distribution by [39], among others.

In the article under consideration, our primary focus lies in introducing a new family of sine-generated distributions by considering the exponentiated Weibull-H family as the baseline distribution in the sine family. This new family is referred to as the SEW-H family of distributions. The following arguments give enough motivation to study the proposed model. We specify it as follows: (i) the new suggested family of distributions is very flexible and contains many generated family of distributions (see Table 1); (ii) the
shapes of the probability density function (pdf) for the new models can be decreasing, right skewness, left skewness, unimodal, and heavy-tailed; (iii) the new suggested model has a closed form for quantile function and this makes the calculation of some properties such as skewness and kurtosis very easy; also to generate random numbers from the new suggested family becomes easy; (iv) some statistical and mathematical properties of the new suggested family such as $Q_U$, $M_0$, $IM_0$ and $O_S$ are explored; (v) six different methods of estimation, including $ML$, $LS$, $MPRS$, $WL$, $CRM$, and $AD$, are produced to estimate the parameters. We hope that the proposed model can be implemented to fit data in diverse scientific entities. This ability of the model is explored using five real life datasets proving the practical utility of the model being featured:

The first dataset: It represents the food chain in the United Kingdom from 2000 to 2019. The food sector plays a significant part in our economy, accounting for about 9 per cent of the Gross Value Added of the UK non-financial business economy. Four sectors make up the food chain: manufacture, wholesale, retail and non-residential catering. Both alcoholic and non-alcoholic drinks are included in food. Total factor productivity is a measure of the efficiency with which inputs are converted into outputs. For example, TFP increases if the volume of outputs increases while the volume of inputs stays the same. Similarly, TFP increases if the volume of inputs decreases while the volume of outputs stays the same. Although there is a practical limit on how much food people want to buy, the volume of output can increase due to increases in quality of products and by increases in exports.

The second dataset: It represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP.

The third dataset: It is called the Single carbon fiber data and it contains the tensile strength of single carbon fibers (in GPa).

The fourth dataset: It is often called the breaking stress of carbon fibers dataset.

The fifth dataset: It represents the TFP growth agricultural production for thirty-seven African countries from 2001–2010 as reported in Figure 1. Increasing the efficiency of agricultural production—getting more output from the same amount of resources—is critical for improving food security. To measure the efficiency of agricultural systems, we use TFP. TFP is an indicator of how efficiently agricultural land, labor, capital, and materials (agricultural inputs) are used to produce a country’s crops and livestock (agricultural output)—it is calculated as the ratio of total agricultural output to total production inputs. When more output is produced from a constant amount of resources, meaning that resources are being used more efficiently, TFP increases. Measures of land and labor productivity—partial factor productivity (PFP) measures—are calculated as the ratio of total output to total agricultural area (land productivity) and to the number of economically active persons in agriculture (labor productivity). Because PFP measures are easy to estimate, they are often used to measure agricultural production performance. These measures normally show higher rates of growth than TFP because growth in land and labor productivity can result not only from increases in TFP but also from a more intensive use of other inputs (such as fertilizer or machinery). Indicators of both TFP and PFP contribute to the understanding of agricultural systems needed for policy and investment decisions by enabling comparisons across time and across countries and regions. These TFP and PFP estimates were generated using the most recent data from Economic Research Service of the United States Department of Agriculture (ERS-USDA), the FAOSTAT database of the Food and Agriculture Organization of the United Nations (FAO), and national statistical sources.
This paper is organized as follows. In Section 2, we present a new extended generator of the exponentiated Weibull-H family and its submodels. In Section 3, we demonstrate that the SEW-H density is given by a linear combination of exponentiated-H (exp-H) densities. Three new special models of this family include SEWEx, the SEWR, and SEWBX distributions. They are introduced in Section 4. Some statistical features of the SEW-H family including the $Q_U$ function, $M_{OS}$, $M_{OS}$, and $O_S$ are provided in Section 5. Six numerous methods of estimation, including $M_L$, $L_S$, $MP_{SP}$, $WL_S$, $CR_{VM}$, and $AD$, are produced to estimate the parameters in Section 6. In Section 7, simulation results to assess the performance of the different estimate procedures are discussed. In Section 8, we provide application to five real datasets to illustrate the importance and flexibility of the new family. Finally, some concluding remarks are presented in Section 9.

2. The Sine-Exponentiated Weibull-H Family

Here, in this section, we construct a new flexible family of distributions called the SEW-H family of distributions. By inserting Equation (1) into Equation (3), we obtain the cdf as follows

$$F_{SEW-H}(x; \lambda, \theta, \beta, \delta) = \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{M_{OS}} \right)^\theta} \right]^{\beta} \right\}, x \in \mathbb{R},$$

as well as the associated pdf is provided via

$$f_{SEW-H}(x; \lambda, \theta, \beta, \delta) = \frac{\pi}{2} \lambda \beta \theta h(x; \delta) \frac{H(x; \delta)^{\theta-1}}{H(x; \delta)^{\theta+1}} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{M_{OS}} \right)^\theta} \right]^{\beta-1} \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{M_{OS}} \right)^\theta} \right]^{\beta} \right\}.$$ 

For a random variable $(R_V)$ $X$ that has the pdf given in Equation (7) is indicated with $X \sim SEW (\lambda, \theta, \beta, \delta)$. 

Figure 1. TFP growth for African countries.
The reliability functions, the sf, hrf, and reversed with for the SEW-H family are respectively provided via

\[ T_{SEW-H}(x; \lambda, \theta, \beta, \delta) = 1 - \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta} \right\}, \]

\[ \xi_{SEW-H}(x) = \frac{\pi}{2} \theta \beta h(x; \delta) \frac{H(x; \delta)}{H(x; \delta)}^{\beta-1} e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta-1} \]

\[ \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta} \right\} \times \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta-1} \]

and

\[ \tau_{SEW-H}(x) = \frac{\pi}{2} \theta \beta h(x; \delta) \frac{H(x; \delta)}{H(x; \delta)}^{\beta-1} e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta-1} \]

\[ \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta} \right\} \times \cot \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta} \right\}. \]

Several functions could be used in a variety of mathematical techniques within the family. Table 1 lists certain special models of the SEW-H family.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \lambda )</th>
<th>( \theta )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SW-H family</td>
<td>-</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>SBX-H family</td>
<td>1</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>SEE-H family</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>SE-H family</td>
<td>-</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

3. Linear Representations

The accompanying result demonstrates the growth of the SEW-H family’s pdf and cdf via parent function modifications. We presume that integration and differentiation term by term under the infinite sum are technically conceivable. By using the Taylor expansion of the cosine function,

\[ \cos \left[ \frac{\pi}{2} G(x) \right] = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \left( \frac{\pi}{2} G(x) \right)^{2i}, \]

we have

\[ \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{\beta} \right\} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \left( \frac{\pi}{2} \right)^{2i} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\theta}} \right]^{2i}. \]
Inserting Equation (8) in Equation (7), the SEW-H density function reduces to
\[
f_{\text{SEW-H}}(x; \lambda, \theta, \beta, \delta) = \sum_{i=0}^{\infty} \left( \frac{(-1)^i}{i!} \frac{(\pi \lambda \beta h(x; \delta))^{2i+1}}{2} \right) e^{-\lambda H(x; \delta)}
\]
and we can write
\[
\frac{H(x; \delta) \beta^{-1} - \lambda H(x; \delta)}{H(x; \delta)^{\beta+1}} e^{-\lambda H(x; \delta)} \left[ 1 - e^{-\lambda H(x; \delta)} \right]^{\beta(2i+1)-1}.
\]

Let \(|z| < 1\) and \(a > 0\) is a real non-integer, the generalized binomial series expansion holds
\[
(1 - z)^a = \sum_{j=0}^{\infty} (-1)^j \binom{a}{j} z^j,
\]
and we can write
\[
\left[ 1 - e^{-\lambda \frac{H(x; \delta)}{H(x; \delta)^{\delta+1}}} \right]^{\beta(2i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\beta(2i+1)-1}{j} e^{-\lambda \frac{H(x; \delta)}{H(x; \delta)^{\delta+1}}}.
\]

Inserting the above expression in Equation (11), the SEW-H density reduces to
\[
f_{\text{SEW-H}}(x; \lambda, \theta, \beta, \delta) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)! \beta(2i+1)-1} \frac{(\pi \lambda \beta h(x; \delta))^{2i+1}}{2} e^{-\lambda H(x; \delta)}
\]
\[
\frac{H(x; \delta) \beta^{-1} - \lambda H(x; \delta)}{H(x; \delta)^{\beta+1}} e^{-\lambda H(x; \delta)} \left[ 1 - e^{-\lambda H(x; \delta)} \right]^{\beta(2i+1)-1}.
\]

By expanding \(e^{-\lambda (j+1) \frac{H(x; \delta)}{H(x; \delta)^{\delta+1}}}\) in a power series, we get
\[
e^{-\lambda (j+1) \frac{H(x; \delta)}{H(x; \delta)^{\delta+1}}} = \sum_{m=0}^{\infty} \frac{(-1)^m \lambda (j+1)^m}{m!} \frac{H(x; \delta)^{\delta m}}{H(x; \delta)^{\delta m}}.
\]

By applying Equation (13) to the last term in Equation (12) gives
\[
f_{\text{SEW-H}}(x; \lambda, \theta, \beta, \delta) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)! \beta(2i+1)-1} \frac{(\pi \lambda \beta h(x; \delta))^{2i+1}}{2} e^{-\lambda H(x; \delta)}
\]
\[
\times \sum_{m=0}^{\infty} \frac{(-1)^m \lambda (j+1)^m}{m!} \frac{H(x; \delta)^{\delta(m+1)-1}}{H(x; \delta)^{\delta m + (m+1) + 1}}.
\]
The generalized binomial expansion is used to obtain \((1 - H(x; \delta))^{-[\theta(m+1)+1]}\). We are able to write
\[
(1 - H(x; \delta))^{-[\theta(m+1)+1]} = \sum_{k=0}^{\infty} \frac{\Gamma(\theta(m+1)+k+1)}{k! \Gamma(\theta(m+1)+1)} H(x; \delta)^k.
\]

Inserting Equation (15) in Equation (14), the SEW-H pdf may be written as an infinite linear combination of exponentiated-H pdfs
\[
f_{\text{SEW-H}}(x) = \sum_{m,k=0}^{\infty} \varphi_{m,k} \Omega_{\theta(m+1)+k}(x).
\]
where

\[
\varphi_{m,k} = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+m} \lambda \beta |\lambda(j+1)|^m \Gamma(\theta(m+1)+k+1)}{m!k!(2i)!\Gamma(\theta(m+1)+1)|\theta(m+1)+k|} \\
\left(\lambda \beta (j+1)\right)^{2i+1}.\]

\[
\Omega_{\varphi}(x) = \rho h(x; \delta) H^{\rho-1}(x; \delta) \]

is the exponentiated-G pdf with the power parameter \(\rho\). As a result, the SEW-H pdf may be communicated as a finite combination of exponentiated-H pdfs with parameter \((\theta(m+1)+k)\). Similarly, the cdf of the SEW-H family may also be communicated as a mixture of exponentiated-H cdfs with

\[
F_{SEW-H}(x) = \sum_{m,k=0}^{\infty} \varphi_{m,k} \Omega_{\theta(m+1)+k}(x).
\]

4. Some Special Models of the SEW-H Family

Naturally, the features of any special distribution of the SEW-H family depend on those of the parent distribution. In this spirit, we focus our attention on the three new SEW-H family special distributions represented by the accompanying pliant mother distributions: the exponential, Rayleigh, and Burr X distributions.

The first special distribution: Sine-exponentiated Weibull exponential (SEWE\(_X\)) distribution with cdf and pdf as well as

\[
F_{SEWE_X}(x; \lambda, \theta, \beta, \rho) = \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda(e^{\beta x} - 1)^{\theta}} \right]^{\beta} \right\},
\]

and

\[
f_{SEWE_X}(x; \lambda, \theta, \beta, \rho) = \frac{\pi \lambda \beta \rho}{2} \frac{\left(1 - e^{-\lambda e^{\beta x}}\right)^{\theta-1}}{e^{-\rho \theta x}} e^{-\lambda(e^{\beta x} - 1)^{\theta}} \left[1 - e^{-\lambda(e^{\beta x} - 1)^{\theta}}\right]^{\beta-1} \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda(e^{\beta x} - 1)^{\theta}} \right]^{\beta} \right\}.
\]

Different pdf forms of the SEWE\(_X\) distribution are shown in Figure 2.

![Figure 2. Density for the SEWE\(_X\) distribution.](image)
The second special distribution: Sine-exponentiated Weibull Rayleigh (SEWR) distribution with cdf and pdf as follows

\[ F_{\text{SEWR}}(x; \lambda, \theta, \beta, \rho) = \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( e^{\rho x^2} - 1 \right)^{\theta}} \right]^{\beta} \right\}, \]

and

\[ f_{\text{SEWR}}(x; \lambda, \theta, \beta, \rho) = \frac{\pi \lambda \theta \beta \rho x}{\left( e^{\rho x^2} \right)^{\theta + 1}} e^{-\lambda \left( e^{\rho x^2} - 1 \right)^{\theta}} \left[ 1 - e^{-\lambda \left( e^{\rho x^2} - 1 \right)^{\theta}} \right]^{\beta - 1} \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( e^{\rho x^2} - 1 \right)^{\theta}} \right]^{\beta} \right\}. \]

Different pdf forms of the SEW-H Rayleigh distribution are shown in Figure 3.

![Figure 3. Density function for the SEWR distribution.](image)

The third special distribution: Sine-exponentiated Weibull Burr X (SEWBX) distribution with cdf and pdf provided via

\[ F_{\text{SEWBX}}(x; \lambda, \theta, \beta, \eta) = \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \left( 1 - e^{-x^2} \right)^{\eta} - 1 \right)^{\theta}} \right]^{\beta} \right\}, \]

and

\[ f_{\text{SEWBX}}(x; \lambda, \theta, \beta, \eta) = \pi \lambda \beta \eta x e^{-x^2} \left( \frac{1 - e^{-x^2} \eta^{\theta - 1}}{1 - \left( 1 - e^{-x^2} \eta \right)^{\theta + 1}} \right)^{\theta - 1} e^{-\lambda \left( \left( 1 - e^{-x^2} \right)^{-\eta} - 1 \right)^{\theta}} \left[ 1 - e^{-\lambda \left( \left( 1 - e^{-x^2} \right)^{-\eta} - 1 \right)^{\theta}} \right]^{\beta - 1} \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \left( 1 - e^{-x^2} \right)^{-\eta} - 1 \right)^{\theta}} \right]^{\beta} \right\}. \]
Different pdf forms of the SEW-H Burr distribution are shown in Figure 4.

![Figure 4. Density function for the SEWBX distribution.](image)

5. Statistical Properties

We looked at the statistical features of the SEW-H family of distributions in this part, specifically the $Q_U$ function, $M_O$s, $I_M$Os, and $O_S$.

5.1. Quantile Function

Theoretical considerations, statistical applications, and Monte Carlo techniques all make use of $Q_U$ functions. $Q_U$ functions are used in Monte Carlo simulations to generate simulated RVs for classical and novel continuous distributions. By inverting Equation (6), we may derive the SEW-H $Q_U$ function, $x = Q(u)$.

$$F^{-1}(u) = Q_H(u) = H^{-1}\left[\frac{\left\{-\frac{1}{\lambda} \log\left[1 - \frac{1}{2} \arcsin\left(u - \frac{1}{2}\right)\right]\right\}^{\frac{1}{\beta}}}{1 + \left\{-\frac{1}{\lambda} \log\left[1 - \frac{1}{2} \arcsin\left(u - \frac{1}{2}\right)\right]\right\}^{\frac{1}{\theta}}}\right].$$

(17)

Here, $Q_H(u)$ signifies the $Q_U$ function corresponding to the baseline distribution. Let us consider the nonlinear equation $F(Q(u)) = Q(F(u)) = u$, $u \in (0, 1)$ distinguishes $Q(u)$. The median is computed by putting $u = 0.5$,

$$Median = H^{-1}\left[\frac{\left\{-\frac{1}{\lambda} \log\left[1 - \frac{1}{2} \arcsin(0.5)\right]\right\}^{\frac{1}{\beta}}}{1 + \left\{-\frac{1}{\lambda} \log\left[1 - \frac{1}{2} \arcsin(0.5)\right]\right\}^{\frac{1}{\theta}}}\right].$$

5.2. Various Types of Moments

In this part, we obtain the expressions for the ordinary and moment generating functions of the SEW-H family of distributions. The $M_O$s of different orders will aid in
calculating the predicted lifetime of a device, as well as the dispersion, skewness, and kurtosis in a given collection of observations occurring in dependability applications.

Let \( W_{\theta(m+1)+k} \) be a \( R_V \) having the exponentiated-H pdf \( \Omega_{\theta(m+1)+k} \) with power parameter \( \theta(m+1)+k \). The \( r_{th} M_O \) of the SEW-H family of distributions can be computed from Equation (16)

\[
\mu' = E(X^r) = \sum_{m,k=0}^{\infty} \varphi_{mk} E(W_{\theta(m+1)+k}^r),
\]

(18)

where \( W_{\theta(m+1)+k} \) signifies the exponentiated-H distribution with power parameter \( \theta(m+1)+k \).

Another formula for the \( r_{th} M_O \) follows from Equation (16) as

\[
\mu' = E(X^r) = \sum_{m,k=0}^{\infty} \varphi_{mk} E(W_{\theta(m+1)+k}^r),
\]

where

\[
E(W_{\theta}^r) = \kappa \int_{-\infty}^{\infty} x^r h(x) H(x)^{\kappa-1}, \nu > 0
\]

could be computed mathematically in relation to the baseline \( Q_U \) function, i.e., \( Q_H(u) = H^{-1}(u) \) as

\[
E(W_{\theta}^r) = \kappa \int_0^1 u^{\kappa-1} Q_H(u) \nu du.
\]

Some numerical values of the first four moments \( \mu_1', \mu_2', \mu_3', \mu_4' \), variance (V), skewness (CS), kurtosis (CK), and coefficient of variation (CV) for the SEWE\( _{\chi} \) and SEWR models are mentioned in Tables 2 and 3.

In this step, we present two formulas for the \( M_O \) generating function \( (M_O G_F) \). The first formula may be determined using Equation (16), as shown below

\[
M_X(t) = E(e^{tX}) = \sum_{m,k=0}^{\infty} \varphi_{mk} M_{k+1}(t),
\]

(19)

where \( M_{\theta(m+1)+k}(t) \) is the \( M_O G_F \) of \( W_{im(m+1)+k} \). Consequently, we can easily compute \( M_X(t) \) from the exp-G generating function. The second formula for the \( M_X(t) \) follows from Equation (16) as

\[
M_X(t) = E(e^{tX}) = \sum_{m,k=0}^{\infty} \varphi_{mk} M_{\theta(m+1)+k}(t),
\]

where \( M_v(t) \) is the \( M_O G_F \) of \( R_V W_v \) provided via

\[
M_v(t) = \int_{-\infty}^{\infty} e^{tX} h(x) h(x)^{\nu-1}, \nu > 0
\]

\[
= \nu \int_0^1 u^{\nu-1} e^{t\nu} Q_H(u) du,
\]

(20)

which could be investigated numerically from the baseline \( R_V \) function, i.e., \( Q_H(u) = G^{-1}(u) \).

The \( s_{th} I M_O \) of \( X \) defined by \( \eta_s(t) \) for any real \( s > 0 \) can be computed from Equation (16) as

\[
\eta_s(t) = \int_{-\infty}^{t} x^s f(x) dx = \sum_{m,k=0}^{\infty} \varphi_{mk} \int_{-\infty}^{t} X^s \eta_{s,\theta(m+1)+k}(t) dx,
\]

(21)

where

\[
\eta_{s,\omega}(t) = \int_0^{H(t)} u^{\omega-1} Q_H(u) du,
\]

and \( \eta_{s,\omega}(t) \) can be investigated numerically.
Table 2. Results of $\mu_1^\prime$, $\mu_2^\prime$, $\mu_3^\prime$, $\mu_4^\prime$, $V$, $CS$, $CK$, and $CV$ for the SEWX model at $\lambda = 1.5$ and $\rho = 0.5$.

<table>
<thead>
<tr>
<th>$\beta$ ↓</th>
<th>$\theta$ ↓</th>
<th>$\mu_1^\prime$ ↓</th>
<th>$\mu_2^\prime$ ↓</th>
<th>$\mu_3^\prime$ ↓</th>
<th>$\mu_4^\prime$ ↓</th>
<th>$V$ ↓</th>
<th>$CS$ ↑</th>
<th>$CK$ ↑</th>
<th>$CV$ ↑</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.2</td>
<td>0.287</td>
<td>0.234</td>
<td>0.273</td>
<td>0.402</td>
<td>0.151</td>
<td>2.02</td>
<td>8.056</td>
<td>1.352</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.333</td>
<td>0.299</td>
<td>0.370</td>
<td>0.567</td>
<td>0.189</td>
<td>1.773</td>
<td>6.64</td>
<td>1.305</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.368</td>
<td>0.360</td>
<td>0.468</td>
<td>0.742</td>
<td>0.225</td>
<td>1.607</td>
<td>5.75</td>
<td>1.289</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.386</td>
<td>0.397</td>
<td>0.533</td>
<td>0.862</td>
<td>0.248</td>
<td>1.527</td>
<td>5.327</td>
<td>1.289</td>
</tr>
<tr>
<td></td>
<td>2.3</td>
<td>0.409</td>
<td>0.448</td>
<td>0.627</td>
<td>1.044</td>
<td>0.281</td>
<td>1.437</td>
<td>4.853</td>
<td>1.297</td>
</tr>
<tr>
<td></td>
<td>2.6</td>
<td>0.426</td>
<td>0.495</td>
<td>0.718</td>
<td>1.227</td>
<td>0.313</td>
<td>1.375</td>
<td>4.513</td>
<td>1.312</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.444</td>
<td>0.550</td>
<td>0.834</td>
<td>1.469</td>
<td>0.353</td>
<td>1.322</td>
<td>4.194</td>
<td>1.339</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>1.27</td>
<td>0.223</td>
<td>0.233</td>
<td>0.296</td>
<td>0.142</td>
<td>1.656</td>
<td>5.9</td>
<td>1.322</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>0.315</td>
<td>0.273</td>
<td>0.305</td>
<td>0.404</td>
<td>0.174</td>
<td>1.496</td>
<td>5.051</td>
<td>1.328</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.335</td>
<td>0.316</td>
<td>0.372</td>
<td>0.513</td>
<td>0.204</td>
<td>1.405</td>
<td>4.541</td>
<td>1.348</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.355</td>
<td>0.345</td>
<td>0.415</td>
<td>0.585</td>
<td>0.223</td>
<td>1.368</td>
<td>4.311</td>
<td>1.369</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.362</td>
<td>0.386</td>
<td>0.432</td>
<td>0.617</td>
<td>0.249</td>
<td>1.336</td>
<td>4.072</td>
<td>1.403</td>
</tr>
<tr>
<td></td>
<td>2.3</td>
<td>0.369</td>
<td>0.436</td>
<td>0.481</td>
<td>0.708</td>
<td>0.272</td>
<td>1.324</td>
<td>3.918</td>
<td>1.44</td>
</tr>
<tr>
<td></td>
<td>2.6</td>
<td>0.386</td>
<td>0.500</td>
<td>0.560</td>
<td>0.829</td>
<td>0.301</td>
<td>1.326</td>
<td>3.798</td>
<td>1.49</td>
</tr>
<tr>
<td>1.9</td>
<td>1.2</td>
<td>0.255</td>
<td>0.200</td>
<td>0.196</td>
<td>0.222</td>
<td>0.135</td>
<td>1.512</td>
<td>4.795</td>
<td>1.443</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.266</td>
<td>0.232</td>
<td>0.242</td>
<td>0.288</td>
<td>0.161</td>
<td>1.467</td>
<td>4.414</td>
<td>1.505</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.271</td>
<td>0.256</td>
<td>0.283</td>
<td>0.351</td>
<td>0.182</td>
<td>1.468</td>
<td>4.245</td>
<td>1.573</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.273</td>
<td>0.273</td>
<td>0.307</td>
<td>0.391</td>
<td>0.195</td>
<td>1.482</td>
<td>4.2</td>
<td>1.619</td>
</tr>
<tr>
<td></td>
<td>2.3</td>
<td>0.273</td>
<td>0.286</td>
<td>0.340</td>
<td>0.447</td>
<td>0.211</td>
<td>1.514</td>
<td>4.194</td>
<td>1.685</td>
</tr>
<tr>
<td></td>
<td>2.6</td>
<td>0.271</td>
<td>0.299</td>
<td>0.369</td>
<td>0.500</td>
<td>0.225</td>
<td>1.554</td>
<td>4.237</td>
<td>1.749</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.268</td>
<td>0.312</td>
<td>0.403</td>
<td>0.564</td>
<td>0.240</td>
<td>1.612</td>
<td>4.34</td>
<td>1.828</td>
</tr>
<tr>
<td>1.2</td>
<td>1.2</td>
<td>0.211</td>
<td>0.170</td>
<td>0.163</td>
<td>0.176</td>
<td>0.126</td>
<td>1.654</td>
<td>4.929</td>
<td>1.677</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.211</td>
<td>0.188</td>
<td>0.192</td>
<td>0.218</td>
<td>0.143</td>
<td>1.701</td>
<td>4.901</td>
<td>1.792</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.208</td>
<td>0.200</td>
<td>0.216</td>
<td>0.257</td>
<td>0.157</td>
<td>1.77</td>
<td>5.017</td>
<td>1.902</td>
</tr>
</tbody>
</table>
Table 3. Results of $\mu'_1, \mu'_2, \mu'_3, \mu'_4, V, CS, CK,$ and $CV$ for the SEWR model at $\lambda = 2.2$ and $\rho = 0.1$.

<table>
<thead>
<tr>
<th>$\beta$ ↓</th>
<th>$\theta$ ↓</th>
<th>$\mu'_1$ ↓</th>
<th>$\mu'_2$ ↓</th>
<th>$\mu'_3$ ↓</th>
<th>$\mu'_4$ ↓</th>
<th>$V$ ↓</th>
<th>$CS$ ↓</th>
<th>$CK$ ↓</th>
<th>$CV$ ↓</th>
</tr>
</thead>
</table>
| 5.3. Order Statistics

Order statistics ($O_S$) is a very important statistical dimension that deals with the order in data. It is defined as follows. If $X_1, X_2, \ldots, X_n$ are the independent RVs following a SEW-H family of distributions of size $n$ and if we arrange these variables in ascending order as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, then the variables $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are $O_S$s of RVs. $O_S$ have many applications in survival, reliability, failure analysis, and it is a natural way to perform a reliability analysis of a system. The cdf of $i$th $O_S$ can provided via...
\[ F_{i,n}(x) = \frac{1}{B(i, n - i + 1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} i^{j+i}(x) \]

\[ = \frac{1}{B(i, n - i + 1)(i + j)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[ \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\beta}} \right] \right\} \right]^{i+j}. \quad (22) \]

The corresponding pdf is provided via

\[ f_{i,n}(x) = \frac{f(x)}{B(i, n - i + 1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} i^{j+i-1}(x) \]

\[ = \sum_{j=0}^{n-i} \frac{\pi \lambda}{2} (-1)^j \binom{n-i}{j} \pi \lambda \beta h(x; \delta) \frac{H(x; \delta)^{\beta-1} - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\beta}}}{1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\beta}}} \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\beta}} \right] \right\} \left[ \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{H(x; \delta)}{H(x; \delta)} \right)^{\beta}} \right] \right\} \right]^{i+j}. \quad (23) \]

The \( r \)th \( M_{ij} \) of the \( i \)th \( v \) is provided via

\[ \mu_r = E(X_{ij}^r) = \int_{-\infty}^{\infty} x^r f_{i,n}(x) \, dx = \]

\[ = \frac{1}{B(i, n - i + 1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \int_{-\infty}^{\infty} x^r f(x) F^{j+i-1}(x) \, dx \]

\[ = \frac{1}{B(i, n - i + 1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \mu_r^{j+i-1} \]

where the integral can be investigated numerically.

6. Estimation Methods

This section employs six estimation techniques to assess the estimation problem of the SEW-H family parameters: \( M_L \), \( L_S \), \( MP_{RP} \), \( WL_S \), \( C_{RV} \), \( VM \), and \( A_D \). For more details, see \([40–43]\).

6.1. Maximum Likelihood Estimation

In this section, we examine the estimation of the SEW-H family’s \( \lambda, \theta, \beta, \) and \( \delta \) parameters using the \( M_L \) method while ensuring the \( M_L \) estimates \( (M_L \) Es) have nice convergence features. \( M_L \) Es have useful qualities, and they can be applied to test statistics as well as the construction of confidence intervals and regions. The following is a presentation of the method’s key components as they relate to the SEW-H family: assume \( x_1, \ldots, x_n \) be a random sample of size \( n \) from the SEW-H family given by \( (7) \). Then, the total log-likelihood function for the vector \( \Omega = (\lambda, \theta, \beta, \delta) \) is provided via

\[ L_n(\Omega) = n \log\left( \frac{\pi}{2} \right) + n \log(\lambda) + n \log(\theta) + n \log(\beta) + \sum_{i=1}^{n} \log h(x_i; \delta) + (\theta - 1) \sum_{i=1}^{n} \log H(x_i; \delta) \]

\[ - (\theta + 1) \sum_{i=1}^{n} \log(H(x_i; \delta)) - \lambda \sum_{i=1}^{n} d_i^\delta + (\beta - 1) \sum_{i=1}^{n} \log \left[ 1 - e^{-\lambda d_i^\delta} \right] \]

\[ + \sum_{i=1}^{n} \log \left\{ \cos \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\delta} \right] \right) \right\}. \quad (24) \]
where \( d_i = \frac{H(x_i; \delta)}{H(x; \delta)} \). The corresponding score vector components, say \( U_n(\Omega) = \left( \frac{\partial L_n(\Omega)}{\partial \lambda}, \frac{\partial L_n(\Omega)}{\partial \theta}, \frac{\partial L_n(\Omega)}{\partial \beta}, \frac{\partial L_n(\Omega)}{\partial \delta} \right)^T \) are given by

\[
\frac{\partial L_n(\Omega)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} d_i^\beta + (\beta - 1) \sum_{i=1}^{n} \frac{d_i^\beta e^{-\lambda d_i^\beta}}{1 - e^{-\lambda d_i^\beta}} - \frac{\pi}{2} \sum_{i=1}^{n} \frac{\beta d_i^\beta e^{-\lambda d_i^\beta} \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\} \left[ 1 - e^{-\lambda d_i^\beta} \right]^{\beta - 1}}{\cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\}},
\]

\[
\frac{\partial L_n(\Omega)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log H(x_i; \delta) - \sum_{i=1}^{n} \log (\overline{H}(x_i; \delta)) - \lambda \sum_{i=1}^{n} d_i^\beta \log d_i + (\beta - 1) \sum_{i=1}^{n} \frac{d_i^\beta e^{-\lambda d_i^\beta} \log d_i}{1 - e^{-\lambda d_i^\beta}} - \frac{\pi}{2} \sum_{i=1}^{n} \frac{\beta d_i^\beta e^{-\lambda d_i^\beta} \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\} \left[ 1 - e^{-\lambda d_i^\beta} \right]^{\beta - 1} \log d_i}{\cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\}},
\]

\[
\frac{\partial L_n(\Omega)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log \left[ 1 - e^{-\lambda d_i^\beta} \right] - \frac{\pi}{2} \sum_{i=1}^{n} \left[ 1 - e^{-\lambda d_i^\beta} \right]^{\beta - 1} \log \left[ 1 - e^{-\lambda d_i^\beta} \right] \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\} \cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\},
\]

and

\[
\frac{\partial L_n(\Omega)}{\partial \delta} = \sum_{i=1}^{n} \frac{h'(x_i; \delta)}{h(x_i; \delta)} + (\theta - 1) \sum_{i=1}^{n} \frac{H'(x_i; \delta)}{H(x_i; \delta)} - (\theta + 1) \sum_{i=1}^{n} \overline{H}'(x_i; \delta) - \lambda \sum_{i=1}^{n} d_i^{\beta - 1} \left( \frac{\partial \lambda}{\partial \delta} \right) + (\beta - 1) \sum_{i=1}^{n} \frac{\lambda d_i^{\beta - 1} e^{-\lambda d_i^\beta} (\frac{\partial \lambda}{\partial \delta})}{1 - e^{-\lambda d_i^\beta}} - \frac{\pi}{2} \sum_{i=1}^{n} \frac{\beta \lambda d_i^{\beta - 1} e^{-\lambda d_i^\beta} \sin \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\} \left[ 1 - e^{-\lambda d_i^\beta} \right]^{\beta - 1} \left( \frac{\partial \lambda}{\partial \delta} \right)}{\cos \left\{ \frac{\pi}{2} \left[ 1 - e^{-\lambda d_i^\beta} \right] \right\}}.
\]

where \( h'(x_i; \delta) = \frac{\partial h(x_i; \delta)}{\partial \delta}, h'(x_i; \delta) = \frac{\partial H(x_i; \delta)}{\partial \delta}, \overline{H}'(x_i; \delta) = \frac{\partial \overline{H}(x_i; \delta)}{\partial \delta} \). Setting the nonlinear system of equations \( \frac{\partial L_n(\Omega)}{\partial \lambda} = \frac{\partial L_n(\Omega)}{\partial \theta} = \frac{\partial L_n(\Omega)}{\partial \beta} = \frac{\partial L_n(\Omega)}{\partial \delta} = 0 \) and solving these equations simultaneously, we can obtain the \( M L E (\hat{\Theta}) \). These equations can be numerically solved using iterative techniques using statistical software since analytical solutions are not possible.

### 6.2. Weighted and Ordinary Least Square

To determine the parameters of different distributions by \( L_S \), the \( WL_S \), and ordinary \( L_S (OL_S) \) approaches are utilized. If \( \Theta = (\lambda, \beta, \theta)^T \) parameters from the SEW-H family class have parameters, then let \( x_1, x_2, \ldots, x_n \) be a random sample. By minimizing the following estimators of \( OL_S (OL_S E) \) and \( WL_S (WL_S E) \) of the \( \Omega = (\Theta, \delta)^T \), distribution parameters of SEW-H family could be derived.
\[ V(\Omega) = \sum_{i=1}^{n} H_i \left[ \sin \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{\nu(x_i)}{\mu(x_i)} \right)^a} \right] \right) - \frac{j}{n+1} \right]^2, \] \quad (29)

\( H_i \) is equal to one for OL\(_3\)E and \( H_i \) is \( \frac{(n+1)^2(n+2)}{[n(n-1)]} \) with respect to \( \Omega \) for WL\(_3\)E. Furthermore, the OL\(_3\)E and WL\(_3\)E with regard to \( \Omega \) are obtained by solving the nonlinear equations.

### 6.3. Product Spacing’s Method

In the case of a random sample of size \( n, x_{1:n} < \cdots < x_{n:n} \), the uniform spacing of the SEW-H family can be described as:

\[ PS_i(\Omega) = F(x_{i:n}, \Omega) - F(x_{i-1:n}, \Omega); \quad i = 1, \ldots, n + 1. \] \quad (30)

In this case, \( PS_i(\Omega) \) stands for the uniform spacings, \( F(x_{i:n}, \Omega) = 0, F(x_{n+1:n}, \Omega) = 1 \), and \( \sum_{i=1}^{n+1} PS_i(\Omega) = 1 \). By \( MP_R S_P \) of the SEW-H family parameters,

\[ G(\Omega) = \frac{1}{n+1} \sum_{i=1}^{n+1} \left\{ \sin \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{\nu(x_i)}{\mu(x_i)} \right)^a} \right] \right) - \sin \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{\nu(x_{i-1})}{\mu(x_{i-1})} \right)^a} \right] \right) \right\}, \] \quad (31)

with regard to \( \Omega \). Further, the \( MP_R S_P \) estimates \( (MP_R S_P E) \) of the SEW-H family can also be computed by solving the nonlinear Equation (31) of derivatives of \( G(\Omega) \) with respect to \( \Omega \).

### 6.4. Cramér-von-Mises

By minimizing the following function with respect to \( \Omega \), the \( C_R VM \) estimators \( (C_R VM E) \) of the SEW-H family with vector parameters \( \Omega \) are derived.

\[ C(\Omega) = \frac{1}{12} + \sum_{i=1}^{n} \left[ \sin \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{\nu(x_i)}{\mu(x_i)} \right)^a} \right] \right) - \frac{2i-1}{2n} \right]^2. \] \quad (32)

In addition, one can solve the nonlinear equations of derivatives of \( C(\Omega) \) with respect to \( \Omega \).

### 6.5. Anderson–Darling Method

Different kinds of minimal distance estimators in \( A_D \) are the \( A_D \) estimators \( (A_D E) \). By minimizing, the \( A_D E \) of the SEW-H family’s parameters is obtained.

\[ AD(\Omega) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left( \ln \left\{ \sin \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{\nu(x_i)}{\mu(x_i)} \right)^a} \right] \right) \right\} - \ln \left\{ 1 - \sin \left( \frac{\pi}{2} \left[ 1 - e^{-\lambda \left( \frac{\nu(x_{i-1})}{\mu(x_{i-1})} \right)^a} \right] \right) \right\} \right)^2, \] \quad (33)

The nonlinear equations of the derivatives of \( A_D(\Omega) \) with respect to each parameter of the vector \( \Omega \) may also be solved to yield the \( A_D E \).

### 7. Simulation

To assess the consistency and accuracy of the six estimating techniques used by the new class, Monte Carlo simulations are performed in this section. For illustration purposes, the simulations are carried out using estimators for the parameters of the SEW-H Exp distribution. Using the inverse transformation, samples of sizes \( n = 40, n = 80 \), and \( n = 160 \) are produced for the simulated replication with 1000 iterations.

\[ x_i = \frac{1}{\lambda} \left[ - \log \left( 1 - \frac{1}{1 + \left( -\frac{1}{\beta \log(2+\gamma)} \right)^{1/\beta}} \right) \right]^{1/\beta}, i = 1, 2, \ldots, n, \] \quad (34)
where a uniform distribution on \((0, 1)\) is represented by \(U\). The mean square error values (MSEV) and estimated relative bias values (RBV) are used to analyze the numerical results. The estimated RBV and the MSEV for the parameter estimators are shown in Tables A1–A3. We establish four arbitrary true values for \((\beta, \theta, \lambda, \rho)\), such as:

In Table A1, set I: \((3, 0.75, 0.75, 0.5)\) and set II: \((3, 0.75, 0.75, 3)\);
In Table A2, set III: \((3, 0.75, 3, 0.5)\), and set IV: \((3, 0.75, 3, 3)\);
In Table A3, set III: \((3, 3, 3, 0.5)\), and set IV: \((3, 3, 3, 3)\).

Numerous calculations were made using the R statistical programming language, with the 'stats' package, which used the Conjugate-gradient maximization algorithm being the most helpful statistical package. We can draw the following conclusions from Tables A1–A3. The results showed that, as the sample size increases, RBV and MSEV decrease, which is consistent with expectations. The proposed estimates of \(\hat{\beta}, \hat{\theta}, \hat{\lambda}\), and \(\hat{\rho}\) perform better in terms of their RBV and MSEV as \(n\) increases. These results unequivocally show the reliability and consistency of the estimating techniques. In order to estimate the parameters of the SEW-H Exp distribution, the six estimation approaches perform effectively.

8. Applications

8.1. Food Chain Data

The first dataset represents the food chain in the United Kingdom from 2000 to 2019, see https://www.gov.uk/government/statistics/food-chain-productivity, accessed on 30 June 2022. The data are as follows: 100, 99.9, 98.5, 100.1, 101.9, 101.4, 103.1, 103.2, 104.2, 102.9, 104.1, 104.8, 104.7, 105.8, 103.4, 104.1, 105.5, 107.2, 108.6, 109. For the first dataset, the numerical values of \(\hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\mu}, \) and \(\hat{\rho}\) are provided in Table 4. From the numerical comparison of the competing distributions in Table 4, we observe that the proposed SEWE\(_x\) model is the best choice to implement for fitting of the food chain data. For the SEWE\(_x\) distribution, the values of the analytical measures are AIC = 105.5160, BIC = 109.4989, CVMV = 0.0316, ADV = 0.2317, and KSD = 0.0973, with PVKS = 0.9915.

To support the best fitting power of the SEWE\(_x\) model, a visual illustration is provided in Figure 5. From the visual illustration in Figure 5, we can see that the SEWE\(_x\) distribution follows the fitted pdf, cdf, PP and QQ plot very closely. To support the results of Table 4, a visual illustration is provided in Figures 5, 6 and 7.

| Table 4. \(M_1\) with SEs and different measures for food chain data. |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|
| SEWE\(_x\)      | \(\hat{\beta}\) | \(\hat{\theta}\) | \(\hat{\lambda}\) | \(\hat{\rho}\) | \(\hat{\mu}\) | AIC      | BIC      | CVMV    | ADV    | KSD    | PVKS    |
| 25.4578         | 5.8544 | 0.0969 | 0.0100 |       |       | 105.5160 | 109.4989 | 0.0316  | 0.2317 | 0.0973 | 0.9915  |
| 2.5656          | 0.6520 | 0.0416 | 0.0002 |       |       | 119.7390 | 124.7177 | 0.0325  | 0.2321 | 0.1969 | 0.4202  |
| EGWGP           | 12.9987 | 0.0028 | 0.2820 | 0.1226 | 0.9072 | 137.3374 | 142.3161 | 0.0329  | 0.2400 | 0.3543 | 0.0132  |
| 6.9675          | 0.0003 | 1.1229 | 0.0319 | 0.1267 |       | 108.0184 | 112.0013 | 0.0679  | 0.4812 | 0.1416 | 0.8177  |
| KEBXII          | 200.4707 | 104.7107 | 1151.0364 | 44.4507 | 0.0386 | 255.0566 | 259.0395 | 0.1066  | 0.6499 | 0.5813 | 0.0000  |
| 91.5733         | 47.5651 | 823.2539 | 7.5603 |       | 0.0005 | 108.9627 | 112.9457 | 0.0486  | 0.3695 | 0.1314 | 0.8800  |
| WL              | 39.6383 | 94.6265 | 0.2092 | 4.3605 |       | 12.1874  | 3.3482   | 3.8432  |       |       |       |
| 500.3255        | 5.4938 | 0.1859 | 1.3285 |       |       | 9.0714   | 0.0250   | 9.9170  | 0.5157 |       |       |
| MOAPW           | 8.6847 | 13.4815 | 14.5558 | 94.1638 |       | 108.9627 | 112.9457 | 0.0486  | 0.3695 | 0.1314 | 0.8800  |
| 12.1874         | 3.3482 | 14.2031 | 3.8432 |       |       | 108.9627 | 112.9457 | 0.0486  | 0.3695 | 0.1314 | 0.8800  |
| KW              | 1.0714 | 0.0250 | 9.9170 | 0.5157 |       | 255.0566 | 259.0395 | 0.1066  | 0.6499 | 0.5813 | 0.0000  |
| 0.0069          | 0.0056 | 0.0022 | 0.0014 |       |       | 19.3720  | 1.2484   | 0.0054  |       |       |       |
| ESW             | 7.4081 | 0.0472 | 0.0011 |       |       | 174.1453 | 177.1325 | 0.0331  | 0.2310 | 0.4250 | 0.0015  |
Figure 5. Plots of empirical cdf with fitted cdf, histogram with fitted pdf, PP and Q-Q plot of the SEWE_X model for the food chain data.

Figure 6. Plots of empirical cdf with fitted cdf of the models for the food chain data.

Figure 7. Plots of histogram with fitted pdf of the models for the food chain data.
8.2. Wholesale Data

The second dataset represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP, see https://www.gov.uk/government/statistics/food-chain-productivity, accessed on 30 June 2022. The data are as follows: 100, 101.7, 99.6, 101, 102.7, 101.1, 104.2, 104.6, 106.3, 104.8, 105.6, 107.1, 107.5, 108.6, 107.5, 106.6, 109.1, 112, 114.4, 112.5.

For second dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 5.

From the numerical comparison of the competing distributions in Table 5, we observe that the proposed SEWE$_X$ model is the best choice to implement for fitting the Wholesale data. For the SEWE$_X$ distribution, the values of the analytical measures are AIC = 121.2337, BIC = 125.2169, CVMV = 0.0292, ADV = 0.2512, and KSD = 0.0937, with PVKS = 0.9947.

To support the best fitting power of the SEWE$_X$ model, a visual illustration is provided in Figure 8. From the visual illustration in Figure 8, we can see that the SEWE$_X$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely. To support the results of Table 4, a visual illustration is provided in Figures 9 and 10.

Table 5. $M_1$E with SEs and different measures for Wholesale data.

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>$\mu$</th>
<th>AIC</th>
<th>BIC</th>
<th>CVMV</th>
<th>ADV</th>
<th>KSD</th>
<th>PVKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEWE$_X$</td>
<td>27.5666</td>
<td>2.6193</td>
<td>0.0172</td>
<td>0.0201</td>
<td></td>
<td>121.2337</td>
<td>125.2167</td>
<td>0.0292</td>
<td>0.2512</td>
<td>0.0937</td>
<td>0.9947</td>
</tr>
<tr>
<td></td>
<td>2.5646</td>
<td>1.5169</td>
<td>0.0073</td>
<td>0.0166</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EGWGP</td>
<td>39.1871</td>
<td>0.0038</td>
<td>8.5649</td>
<td>0.6128</td>
<td>0.9686</td>
<td>162.5654</td>
<td>167.5440</td>
<td>0.1286</td>
<td>0.7957</td>
<td>0.4030</td>
<td>0.0030</td>
</tr>
<tr>
<td></td>
<td>16.1161</td>
<td>0.0004</td>
<td>0.0057</td>
<td>0.0018</td>
<td>0.0012</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KEBXII</td>
<td>537.5850</td>
<td>649.1720</td>
<td>375.1731</td>
<td>4.2158</td>
<td>0.5565</td>
<td>135.8559</td>
<td>140.8345</td>
<td>0.0318</td>
<td>0.2694</td>
<td>0.2525</td>
<td>0.1561</td>
</tr>
<tr>
<td></td>
<td>5.6995</td>
<td>21.6522</td>
<td>194.6549</td>
<td>0.3778</td>
<td>0.0499</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WL</td>
<td>0.0025</td>
<td>45.0467</td>
<td>0.3496</td>
<td>13.7505</td>
<td></td>
<td>124.2758</td>
<td>128.2587</td>
<td>0.0720</td>
<td>0.5231</td>
<td>0.1491</td>
<td>0.7653</td>
</tr>
<tr>
<td></td>
<td>0.0007</td>
<td>3.7874</td>
<td>0.1985</td>
<td>1.3843</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MOAPW</td>
<td>378.1695</td>
<td>5.1839</td>
<td>449.6787</td>
<td>71.0195</td>
<td></td>
<td>123.1665</td>
<td>127.1494</td>
<td>0.0369</td>
<td>0.3183</td>
<td>0.1064</td>
<td>0.9774</td>
</tr>
<tr>
<td></td>
<td>1288.6846</td>
<td>1.5231</td>
<td>948.4074</td>
<td>11.2737</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KW</td>
<td>1.0993</td>
<td>0.0279</td>
<td>10.0478</td>
<td>0.5114</td>
<td></td>
<td>256.5495</td>
<td>260.5324</td>
<td>0.0632</td>
<td>0.4927</td>
<td>0.6143</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.0032</td>
<td>0.0062</td>
<td>0.0034</td>
<td>0.0013</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ESW</td>
<td>20.8703</td>
<td>1.3592</td>
<td>0.0032</td>
<td></td>
<td></td>
<td>171.1671</td>
<td>174.1543</td>
<td>0.0268</td>
<td>0.2254</td>
<td>0.4003</td>
<td>0.0033</td>
</tr>
<tr>
<td></td>
<td>8.1773</td>
<td>0.0295</td>
<td>0.0003</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 8. Plots of empirical cdf with fitted cdf, histogram with fitted pdf, PP and Q-Q plot of the SEWE$_X$ model for the Wholesale data.
8.3. Single Carbon Fiber Data

The source of this dataset is given in [44]. It contains the tensile strength of single carbon fibers (in GPa). This information is provided by 0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.997, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585. For the third dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 6. From the numerical comparison of the competing distributions in Table 6, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting the food chain data. For the SEWE$_x$ distribution, the values of the analytical measures are AIC = 105.2991, BIC = 114.2356, CVMV = 0.0172, ADV = 0.1557, and KSD = 0.0403, with PVKS = 0.9999. To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figures 11–13. From the visual illustration in Figures 11–13, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely.
Table 6. $M_1E$ with SEs and different measures for single carbon fiber data.

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimates</th>
<th>$\beta$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>AIC</th>
<th>BIC</th>
<th>KSD</th>
<th>PVKS</th>
<th>CVMV</th>
<th>ADV</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEWEX</td>
<td>estimates</td>
<td>14.2450</td>
<td>0.2028</td>
<td>1.1062</td>
<td>2.9008</td>
<td>105.2991</td>
<td>114.2356</td>
<td>0.0403</td>
<td>0.9999</td>
<td>0.0172</td>
<td>0.1557</td>
</tr>
<tr>
<td>SE</td>
<td>7.1414</td>
<td>0.1436</td>
<td>1.0753</td>
<td>1.4375</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OLLMW</td>
<td>estimates</td>
<td>28.7708</td>
<td>0.0560</td>
<td>0.6093</td>
<td>0.0125</td>
<td>106.6756</td>
<td>115.6121</td>
<td>0.0468</td>
<td>0.9982</td>
<td>0.0222</td>
<td>0.1625</td>
</tr>
<tr>
<td>SE</td>
<td>39.8918</td>
<td>0.0814</td>
<td>0.1193</td>
<td>0.0310</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KW</td>
<td>estimates</td>
<td>0.7560</td>
<td>0.1473</td>
<td>1.0575</td>
<td>3.4344</td>
<td>105.5197</td>
<td>114.4561</td>
<td>0.0487</td>
<td>0.9967</td>
<td>0.0220</td>
<td>0.1947</td>
</tr>
<tr>
<td>SE</td>
<td>0.0286</td>
<td>0.2002</td>
<td>0.2090</td>
<td>0.0151</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GMW</td>
<td>estimates</td>
<td>0.4879</td>
<td>4.4450</td>
<td>0.8570</td>
<td>0.3466</td>
<td>105.4390</td>
<td>114.3755</td>
<td>0.0422</td>
<td>0.9997</td>
<td>0.0185</td>
<td>0.1654</td>
</tr>
<tr>
<td>SE</td>
<td>0.7596</td>
<td>8.9394</td>
<td>0.2829</td>
<td>1.0786</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EOWL</td>
<td>estimates</td>
<td>2.6361</td>
<td>0.1152</td>
<td>8.7047</td>
<td>19.2595</td>
<td>105.5875</td>
<td>114.5240</td>
<td>0.0435</td>
<td>0.9995</td>
<td>0.0211</td>
<td>0.1878</td>
</tr>
<tr>
<td>SE</td>
<td>0.4997</td>
<td>0.2885</td>
<td>43.9097</td>
<td>100.6442</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EOWINH</td>
<td>estimates</td>
<td>3.2607</td>
<td>0.0821</td>
<td>0.5014</td>
<td>2.9358</td>
<td>109.0365</td>
<td>117.9729</td>
<td>0.0424</td>
<td>0.9997</td>
<td>0.0405</td>
<td>0.3270</td>
</tr>
<tr>
<td>SE</td>
<td>1.2895</td>
<td>0.3173</td>
<td>0.2179</td>
<td>1.9671</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 11. Plots of empirical cdf with fitted cdf of the models for the third dataset.

Figure 12. Plots of histogram with fitted pdf of the models for the third dataset.
8.4. Breaking Stress Dataset

The fourth dataset, often called breaking stress of carbon fiber dataset, was used by [45]. This dataset is given by: “3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 2.87, 2.55, 3.31, 3.1, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.99, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53” . For the fourth dataset, the numerical values of \( \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\mu}, \text{and} \hat{\rho} \) are provided in Table 7. From the numerical comparison of the competing distributions in Table 7, we observe that the proposed SEWE\(_x\) model is the best choice to implement for fitting the food chain data. For the SEWE\(_x\) distribution, the values of the analytical measures are AIC = 178.4290, BIC = 187.1876, CVMV = 0.0631, ADV = v, and KSD = 0.0707, with PVKS = 0.8967.

To support the best fitting power of the SEWE\(_x\) model, a visual illustration is provided in Figures 14–16. From the visual illustration in Figures 14–16, we can see that the SEWE\(_x\) distribution follows the fitted pdf, cdf, PP and QQ plot very closely.

Table 7. \( M_j \) E with SEs and different measures for carbon fiber dataset.

<table>
<thead>
<tr>
<th></th>
<th>( \hat{\beta} )</th>
<th>( \hat{\theta} )</th>
<th>( \hat{\lambda} )</th>
<th>( \hat{\rho} )</th>
<th>AIC</th>
<th>BIC</th>
<th>KSD</th>
<th>PVKS</th>
<th>CVMV</th>
<th>ADV</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEWE(_x) estimates</td>
<td>24.8869</td>
<td>0.0945</td>
<td>1.4235</td>
<td>3.0217</td>
<td>178.4290</td>
<td>187.1876</td>
<td>0.0707</td>
<td>0.8967</td>
<td>0.0631</td>
<td>0.3704</td>
</tr>
<tr>
<td>SE</td>
<td>75.6618</td>
<td>0.1957</td>
<td>2.3635</td>
<td>4.4106</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OLLMW estimates</td>
<td>2.761189</td>
<td>0.140337</td>
<td>0.054904</td>
<td>1.73303</td>
<td>182.6582</td>
<td>191.4168</td>
<td>0.0835</td>
<td>0.7467</td>
<td>0.1416</td>
<td>0.7459</td>
</tr>
<tr>
<td>SE</td>
<td>25.9085</td>
<td>0.0357</td>
<td>0.0977</td>
<td>0.0260</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KW estimates</td>
<td>0.7246</td>
<td>0.1677</td>
<td>0.5057</td>
<td>3.8408</td>
<td>179.2803</td>
<td>188.0390</td>
<td>0.0841</td>
<td>0.7392</td>
<td>0.0730</td>
<td>0.4549</td>
</tr>
<tr>
<td>SE</td>
<td>0.0144</td>
<td>0.0244</td>
<td>0.0102</td>
<td>0.0170</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GMW estimates</td>
<td>0.4364</td>
<td>5.4989</td>
<td>0.5161</td>
<td>0.1485</td>
<td>178.7462</td>
<td>187.5048</td>
<td>0.0761</td>
<td>0.8398</td>
<td>0.0654</td>
<td>0.3940</td>
</tr>
<tr>
<td>SE</td>
<td>0.6527</td>
<td>8.0561</td>
<td>0.1731</td>
<td>0.5409</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EOWINH estimates</td>
<td>4.5935</td>
<td>0.0028</td>
<td>0.2527</td>
<td>21.5314</td>
<td>180.3045</td>
<td>189.0631</td>
<td>0.0832</td>
<td>0.7515</td>
<td>0.0946</td>
<td>0.5382</td>
</tr>
<tr>
<td>SE</td>
<td>1.9504</td>
<td>0.2126</td>
<td>0.1221</td>
<td>24.5623</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 14. Plots of histogram with fitted pdf of the models for the carbon fiber dataset.

Figure 15. Plots of empirical cdf with fitted cdf of the models for the carbon fiber dataset.

Figure 16. Plots of PP and Q-Q plot of the SEWE<sub>x</sub> model for the carbon fiber dataset.
8.5. TFP Growth Dataset

The fifth dataset represents the TFP growth agricultural production for thirty-seven African countries from 2001–2010, see https://dataverse.harvard.edu/dataset.xhtml?persistentId=doi:10.7910/DVN/9IOAKR, accessed on 30 June 2022. The dataset is given as 4.6, 0.9, 1.8, 1.4, 0.2, 3.9, 1.8, 0.8, 2.0, 0.8, 1.6, 0.8, 2.0, 1.6, 0.5, 0.1, 2.5, 2.4, 0.6, 1.1, 0.7, 1.7, 1.0, 1.7, 2.5, 3.5, 0.3, 0.9, 2.3, 0.5, 1.5, 5.1, 0.2, 1.5, 0.8, 2.0, 1.6, 0.5, 0.1, 2.5, 2.4.

For the fifth dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 8.

From the numerical comparison of the competing distributions in Table 8, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting the TFP growth data. For the SEWE$_x$ distribution, the values of the analytical measures are $\text{AIC} = 114.7737$, $\text{BIC} = 116.0237$, $\text{CVMV} = 0.0329$, $\text{ADV} = 0.1988$, and $\text{KSD} = 0.0826$, with $\text{PVKS} = 0.9622$.

To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figure 14. From the visual illustration in Figures 14–16, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely. To support results of Table 8, a visual illustration is provided in Figures 14 and 15.

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>$\mu$</th>
<th>$\text{AIC}$</th>
<th>$\text{BIC}$</th>
<th>$\text{CVMV}$</th>
<th>$\text{ADV}$</th>
<th>$\text{KSD}$</th>
<th>$\text{PVKS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEWE$_x$ estimates</td>
<td>18.9511</td>
<td>0.2121</td>
<td>2.6809</td>
<td>0.6078</td>
<td></td>
<td>114.7737</td>
<td>116.0237</td>
<td>0.0329</td>
<td>0.1988</td>
<td>0.0826</td>
<td>0.9622</td>
</tr>
<tr>
<td>SE</td>
<td>686.6481</td>
<td>3.1465</td>
<td>27.8125</td>
<td>5.5706</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EGWGP estimates</td>
<td>0.9855</td>
<td>0.6312</td>
<td>1.3171</td>
<td>0.4959</td>
<td>0.0860</td>
<td>116.8016</td>
<td>118.7371</td>
<td>0.0332</td>
<td>0.1994</td>
<td>0.0830</td>
<td>0.9618</td>
</tr>
<tr>
<td>SE</td>
<td>4.0934</td>
<td>4.3180</td>
<td>0.4487</td>
<td>0.2928</td>
<td>0.9245</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KEBXII estimates</td>
<td>39.6448</td>
<td>1.1351</td>
<td>569.6257</td>
<td>0.1379</td>
<td>2.7644</td>
<td>117.6760</td>
<td>119.6115</td>
<td>0.0385</td>
<td>0.2376</td>
<td>0.1061</td>
<td>0.7991</td>
</tr>
<tr>
<td>SE</td>
<td>81.7057</td>
<td>2.3102</td>
<td>1917.6428</td>
<td>0.1862</td>
<td>4.2189</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WL estimates</td>
<td>2.6661</td>
<td>1.3872</td>
<td>1.1264</td>
<td>4.3821</td>
<td></td>
<td>114.9927</td>
<td>116.2427</td>
<td>0.0330</td>
<td>0.1989</td>
<td>0.0828</td>
<td>0.9622</td>
</tr>
<tr>
<td>SE</td>
<td>73.0566</td>
<td>0.5367</td>
<td>4.0920</td>
<td>102.3363</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MOAPW estimates</td>
<td>0.9331</td>
<td>1.4697</td>
<td>0.8656</td>
<td>2.0064</td>
<td></td>
<td>114.9835</td>
<td>116.2335</td>
<td>0.0340</td>
<td>0.1989</td>
<td>0.0843</td>
<td>0.9550</td>
</tr>
<tr>
<td>SE</td>
<td>9.1323</td>
<td>0.4757</td>
<td>4.3685</td>
<td>1.1887</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KW estimates</td>
<td>3.1522</td>
<td>0.1048</td>
<td>5.1531</td>
<td>1.0782</td>
<td></td>
<td>116.6211</td>
<td>117.8711</td>
<td>0.0566</td>
<td>0.3540</td>
<td>0.1279</td>
<td>0.5805</td>
</tr>
<tr>
<td>SE</td>
<td>0.0914</td>
<td>0.0174</td>
<td>0.0097</td>
<td>0.0095</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0655</td>
<td>1.2939</td>
<td>0.2672</td>
<td></td>
<td></td>
<td>113.0408</td>
<td>113.7681</td>
<td>0.0289</td>
<td>0.1795</td>
<td>0.0793</td>
<td>0.9741</td>
</tr>
</tbody>
</table>

Table 8. $M_1E$ with SEs and different measures for TFP growth data.

Figure 17. Plots of empirical cdf with fitted cdf of the models for the TFP growth dataset.
Figure 18. Plots of histogram with fitted pdf of the models for the TFP growth dataset.

Figure 19. Plots of PP and Q-Q plot of the SEWE$_X$ model for the TFP growth dataset.

9. Conclusions and Summary

In this article, a new lifetime-generated family of distributions called the sine-exponentiated Weibull-H family is proposed; this family is obtained from two well-established families of distributions of completely different nature: the sine-G and the exponentiated Weibull-H families. Three new sub-models were proposed and discussed, including the sine-exponentiated Weibull Rayleigh (SEWR), sine-exponentiated Weibull Burr X (SEWBX), and Sine-exponentiated Weibull exponential (SEWE$_x$) distributions. Some important statistical features of the new family of distributions are investigated, such as quantiles, moments, incomplete moments, and order statistics. Six methods of estimation, namely $ML\, L_S\, MP\, RS\, WL\, C\, VM$, and $AD$, are produced to estimate the parameters. The performance of the estimation approaches is investigated using Monte Carlo simulation. In this article, we use five real datasets to show the relevance and flexibility of the suggested family. The first dataset represents the United Kingdom food chain from 2000 to 2019, whereas the second dataset represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP; the third dataset contains the tensile strength of single carbon fibers (in GPa); the fourth dataset is often called the breaking stress of carbon fiber dataset; the fifth dataset represent the TFP growth agricultural production for thirty-seven African countries from 2001–2010. The SEWE$_x$ model as example of the suggested family...
gives the best fit for all datasets against all competitive models. In the future, we hope to introduce many new statistical models from the suggested family of distributions and study their statistical properties. We also hope that these models have many applications in different fields, including agricultural sciences, environmental sciences, biomedical sciences, engineering sciences, economics, and lifetime data.

**Author Contributions**: Data curation, S.A.A.; Formal analysis, E.M.A.; Investigation, N.A.; Methodology, I.E.; Resources, N.A.; Software, E.M.A.; Supervision, I.E.; Writing—original draft, M.E.; Writing—review & editing, M.E. All authors have read and agreed to the published version of the manuscript.

**Funding**: The authors extend their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University for funding this work through Research Group no. RG-21-09-15.

**Data Availability Statement**: Datasets are available in the application section.

**Acknowledgments**: The authors extend their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University for funding this work through Research Group no. RG-21-09-15.

**Conflicts of Interest**: The authors declare no conflicts of interest.

**Appendix A**

### Table A1. RBv and MSEV for different estimation techniques of the parameters of SEWEX distribution: set-I and set-II.

<table>
<thead>
<tr>
<th>ρ</th>
<th>n</th>
<th>M_{1}E RBV</th>
<th>M_{1}E MSEV</th>
<th>L_{3}E RBV</th>
<th>L_{3}E MSEV</th>
<th>W_{L_{4}}E RBV</th>
<th>W_{L_{4}}E MSEV</th>
<th>MP_{457P_{4}}E RBV</th>
<th>MP_{457P_{4}}E MSEV</th>
<th>C_{4}VM E RBV</th>
<th>C_{4}VM E MSEV</th>
<th>A_{3}E RBV</th>
<th>A_{3}E MSEV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>40</td>
<td>−0.0051</td>
<td>0.0065</td>
<td>0.0005</td>
<td>0.0008</td>
<td>−0.0014</td>
<td>0.0013</td>
<td>−0.0051</td>
<td>0.0048</td>
<td>0.0023</td>
<td>0.0008</td>
<td>0.0005</td>
<td>0.0009</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>−0.0032</td>
<td>0.0038</td>
<td>−0.0005</td>
<td>0.0003</td>
<td>0.0009</td>
<td>0.0005</td>
<td>−0.0032</td>
<td>0.0027</td>
<td>0.0003</td>
<td>0.0004</td>
<td>−0.0003</td>
<td>0.0005</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>0.1263</td>
<td>0.0441</td>
<td>0.0135</td>
<td>0.0024</td>
<td>0.0141</td>
<td>0.0055</td>
<td>0.1259</td>
<td>0.0496</td>
<td>0.0108</td>
<td>0.0023</td>
<td>0.2533</td>
<td>0.0068</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>−0.0008</td>
<td>0.0016</td>
<td>−0.0001</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0003</td>
<td>−0.0008</td>
<td>0.0016</td>
<td>0.0002</td>
<td>0.0003</td>
<td>−0.0001</td>
<td>0.0005</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>−0.0298</td>
<td>0.0087</td>
<td>−0.0021</td>
<td>0.0048</td>
<td>−0.0001</td>
<td>0.0003</td>
<td>−0.0300</td>
<td>0.0091</td>
<td>0.0048</td>
<td>0.0050</td>
<td>−0.0025</td>
<td>0.0054</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>0.0567</td>
<td>0.0494</td>
<td>0.0027</td>
<td>0.0049</td>
<td>0.0108</td>
<td>0.0073</td>
<td>0.0406</td>
<td>0.0490</td>
<td>0.0006</td>
<td>0.0069</td>
<td>0.0035</td>
<td>0.0127</td>
</tr>
<tr>
<td>ρ</td>
<td></td>
<td>0.0483</td>
<td>0.0223</td>
<td>0.0129</td>
<td>0.0024</td>
<td>0.0046</td>
<td>0.0025</td>
<td>0.0486</td>
<td>0.0230</td>
<td>0.0131</td>
<td>0.0022</td>
<td>0.0203</td>
<td>0.0086</td>
</tr>
</tbody>
</table>

Sustainability 2022, 14, 8942
Table A2. RBv and MSEV for different estimation techniques of the parameters of SEWE$_X$ distribution:
set-III and set-IV.

<table>
<thead>
<tr>
<th></th>
<th>$M_{L1}E$</th>
<th>$L_{S1}E$</th>
<th>$W_{L1}E$</th>
<th>$M_{P1,0.1}E$</th>
<th>$C_{x}$ VME</th>
<th>$A_{x}$ VME</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>0.0118</td>
<td>0.0440</td>
<td>0.0741</td>
<td>0.00003</td>
<td>0.00013</td>
<td>0.00022</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.0145</td>
<td>0.0898</td>
<td>0.0013</td>
<td>0.00002</td>
<td>0.00418</td>
<td>0.00019</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0105</td>
<td>0.0025</td>
<td>0.0001</td>
<td>0.00001</td>
<td>0.00002</td>
<td>0.00002</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.0133</td>
<td>0.0024</td>
<td>0.0001</td>
<td>0.00001</td>
<td>0.00002</td>
<td>0.00002</td>
</tr>
</tbody>
</table>

Table A3. RBv and MSEV for different estimation techniques of the parameters of SEWE$_X$ distribution:
set-V and set-VI.

<table>
<thead>
<tr>
<th></th>
<th>$M_{L1}E$</th>
<th>$L_{S1}E$</th>
<th>$W_{L1}E$</th>
<th>$M_{P1,0.1}E$</th>
<th>$C_{x}$ VME</th>
<th>$A_{x}$ VME</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>0.0112</td>
<td>0.0921</td>
<td>0.0057</td>
<td>0.00089</td>
<td>0.03810</td>
<td>0.01597</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.02074</td>
<td>0.0942</td>
<td>0.0010</td>
<td>0.00024</td>
<td>0.00143</td>
<td>0.00143</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.01018</td>
<td>0.00050</td>
<td>0.0027</td>
<td>0.00012</td>
<td>0.00016</td>
<td>0.00016</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.01047</td>
<td>0.0018</td>
<td>0.0023</td>
<td>0.00016</td>
<td>0.00014</td>
<td>0.00014</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.00600</td>
<td>0.00168</td>
<td>0.0037</td>
<td>0.00039</td>
<td>0.00005</td>
<td>0.00005</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.0237</td>
<td>0.0015</td>
<td>0.0002</td>
<td>0.00012</td>
<td>0.00012</td>
<td>0.00012</td>
</tr>
</tbody>
</table>

Sustainability 2022, 14, 8942  
26 of 28
### Table A3. Cont.

<table>
<thead>
<tr>
<th>$M_r E$</th>
<th>$L_1 E$</th>
<th>$ WL_1 E$</th>
<th>$MP_q S_r E$</th>
<th>$C_q VME$</th>
<th>$A_p E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>$\beta$</td>
<td>$-0.00514$</td>
<td>$0.61948$</td>
<td>$-0.00496$</td>
<td>$0.02314$</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>$0.00867$</td>
<td>$0.31488$</td>
<td>$-0.00473$</td>
<td>$0.05100$</td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>$-0.00316$</td>
<td>$0.01230$</td>
<td>$-0.00070$</td>
<td>$0.00042$</td>
</tr>
<tr>
<td>160</td>
<td>$\rho$</td>
<td>$-0.00795$</td>
<td>$0.00375$</td>
<td>$-0.00301$</td>
<td>$0.00445$</td>
</tr>
</tbody>
</table>

References

6. He, W.; Ahmad, Z.; Afify, A.Z.; Goual, H. The arcsine exponentiated-X family: Validation and insurance application. *Complexity* 2020, 2020, 8394815. [CrossRef]


