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Optimal Extended Warranty Strategy: Provided by Manufacturer or E-Commerce Platform?

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Abstract: To meet the different needs of consumers in an E-commerce market environment, more and more enterprises have launched extended warranty services, which can reduce consumer risk and stimulate market demand in E-commerce supply chains. However, knowing who will launch the extended warranty services has become particularly important. To propose an optimal strategy of the extended warranty service provider and seller, three game models are constructed in an E-commerce supply chain consisting of a manufacturer and an E-commerce platform. In addition, this study investigated the effect of extended warranty strategy on consumers. The results show that when the extended warranty coverage and product failure probability are small, the extended warranty provided by the E-commerce platform can simulate the demand of products and extended warranty. The extended warranty provided by the manufacturer is the best choice for the manufacturer. On the other hand, the extended warranty provided by the E-commerce platform is always beneficial for the E-commerce platform and E-commerce supply chain system. The E-commerce platform should take some measures and control some parameters to make up for the losses of the manufacturer, so that the strategy of the extended warranty provided and sold by the E-commerce platform becomes the consistent choice of E-commerce supply-chain members, system, and consumers.

Keywords: extended warranty strategy; E-commerce supply chain; E-commerce platform

1. Introduction

With the rapid development of information technology, the Internet has penetrated into each aspect of social life, and directly affected the business activities of enterprises and promoted the development of their business modes. More and more offline physical manufacturers sell products through E-commerce platforms, such as Amazon, eBay, and JD [1]. However, increasingly fierce competition among companies has led to a shrinking profit margin for traditional product sales in E-commerce platform markets. Extended warranty services can effectively widen the profit margin for companies. For example, retailers’ margins are usually between 60% and 70% from selling extended warranties [2]. The global extended warranty market for consumer electronics amounted to USD 29.4 billion in 2017 and is estimated to grow to USD 50.2 billion by 2026 [3]. The extended warranty can also enhance consumers’ shopping experience. For instance, during the “Double Eleven” shopping carnival in 2021, several brands such as JD jointly customized extended warranty strategies for 3C and digital products to enhance consumers’ shopping experience and reduce the cost of repairing or replacing risks for electronic products and home appliances (https://www.sohu.com/a/499222207_161105, accessed on 4 December 2021). TaoBao launched an extended warranty strategy for home appliances, digital products, and medical equipment that cannot guarantee the repair problems after the repair period...
In the E-commerce platform market environment, whether the extended warranty services are provided by manufacturers or E-commerce platforms has attracted widespread attention [4,5]. Many domestic extended warranty services are provided by foreign extended warranty service providers [6], such as Gome, Suning, and JD (https://baike.baidu.com/item/%E5%BB%B6%E4%BF%9D%E6%9C%8D%E5%8A%A1/2162876?fr=aladdin, accessed on 8 September 2022). Under this extended warranty service mode, the providers of extended warranty and the providers of products are independent of each other; they are unable to provide uniform and comprehensive service to consumers [7]. More and more manufacturers provide the extended warranty services. For example, Apple Inc. has launched an extended warranty service “Apple Care+”, which provides an opportunity for consumers to replace old iPhones with new products (https://www.apple.com/legal/sales-support/iphoneupgrade_us, accessed on 8 September 2022). Similarly, as a manufacturer, EVGA.com has provided consumers a 3-year extended warranty (https://eu.evga.com/support/stepup, accessed on 8 September 2022). Moreover, E-commerce platforms may also directly provide extended warranty services. For example, JD took the lead in launching a new project of extended warranty services (https://tech.sina.com.cn/i/2009-03-25/09082940217.shtml, accessed on 25 March 2009).

In the operations of E-commerce supply chains, manufacturers wholesale products to E-commerce platforms, and then E-commerce platforms sell these products to consumers through an E-commerce platform channel. Under this selling mode, the extended warranty services can be provided and sold by manufacturers or E-commerce platforms [5,8]. For example, some domestic manufacturers provide extended warranty services for their products, including Haier, Hisense, Lenovo, Dell, etc. (https://baike.baidu.com/item/%E5%BB%B6%E4%BF%9D%E6%9C%8D%E5%8A%A1/2162876?fr=aladdin, accessed on 8 September 2022). In some cases, the extended warranty services may be provided by manufacturers while being sold by E-commerce platforms, and E-commerce platforms share a certain proportion revenue of selling extended warranty services with manufacturers [9]. For instance, the AppleCare+ extended warranty service is provided by Apple and sold by the E-commerce platform JD (https://item.jd.com/10024589712063.html, accessed on 8 September 2022).

Motivated by the existence of different extended warranty providers and sellers in E-commerce supply chains, it is worth investigating who will provide the extended warranty service. Therefore, by considering different extended warranty providers, we construct three models of an E-commerce supply chain consisting of a manufacturer and an E-commerce platform. In this study, we investigated the following issues: (1) What are the effects of the factors such as product failure probability and extended warranty coverage on the demands of products and extended warranty, and the decisions of supply-chain members? (2) How do the different extended warranty strategies affect the profits of supply-chain members? (3) How do you propose an extended warranty strategy by controlling some parameters to benefit the E-commerce supply-chain members, supply chain system, and consumers?

The main contributions of this paper are as follows. Firstly, we theoretically investigated the extended warranty provider strategy in E-commerce supply chains. Secondly, we considered the situation where providers and sellers are different entities, which is rarely studied in existing research. Finally, we proposed a method of selecting an extended warranty strategy that is beneficial for E-commerce supply-chain members, system, and consumers.

The remainder of this paper is organized as follows: Section 2 reviews the related literature. In Section 3, we describe the problem and model assumptions. In Section 4, the Stackelberg equilibrium results of the three models are given. Section 5 analyzes and compares the equilibrium results of the three models. Section 6 illustrates the numerical analysis. The conclusions are summarized in Section 7.
2. Literature Review

There are two streams of literature related to this paper: E-commerce supply-chain operations and extended warranty service strategies.

2.1. E-Commerce Supply-Chain Operations

With the rapid development of E-commerce platforms, many scholars have paid attention to research E-commerce supply-chain operations [1,10,11]. Some researchers investigated the issue of sales mode selection in E-commerce supply chains. For examples, Qin et al. [12] studied the issue of sales mode and logistics service strategy of an E-commerce platform and investigated the impact of the interaction term between sales mode and service strategy. Chang et al. [13] considered two channel modes for E-commerce platforms: the resale mode or the marketplace mode, and then investigated the interaction between the financing strategies of online retailers and the channel sales of E-commerce platforms. He et al. [14] investigated the impacts of a manufacturer’s channel encroachment and an E-commerce platform’s logistics integration in an E-commerce platform service supply chain. The issue of financing in E-commerce supply chains and E-commerce supply-chain decision-making under financial constraints was also addressed. Qin et al. [15] studied the strategies of financing and carbon emission reduction for manufacturers under the consideration of capital constraint in E-commerce supply chains. In addition, the pricing decisions of the E-commerce supply chain were also investigated. Xie et al. [16] constructed a platform supply chain consisting of a third-party seller–E-commerce platform–strategic consumer and analyzed the pricing decisions and studied those who benefit from the E-consumer credit. A number of scholars studied the different operations of E-commerce supply chains. Fent and Jin [17] studied firms’ innovation strategies under different selling structures and found that manufacturer innovation performs better than platform innovation. Prajapati et al. [18] integrated the blockchain, Internet of things, sustainability, and circular economy and developed a framework for normal and virtual closed-loop supply chains. Zha et al. [19] investigated the platform’s information-sharing strategy and the overseas supplier’s logistics mode choice and found that information sharing can influence the overseas supplier’s logistics mode choice under certain conditions. Most of the existing literature has investigated the pricing and other decisions of E-commerce supply chains. Few studies have considered the extended warranty services and their providers.

2.2. Extended Warranty Service Strategies

Many scholars researched the extended warranty service strategies from different aspects [5,8,20]. For instance, Liu et al. [5] investigated the emerging trend of the complimentary extended warranty in industries, and it turned out that the free pre-registration warranty mode is better than the traditional warranty mode. Huang et al. [21] proposed extended warranty programs of lump sum or monthly payment to consumers with different attitudes toward risk with consideration of multiple dependent competing failure processes. Mitra [22] investigated the issue of extended warranty for consumers that includes consumer preference after the initial coverage period has ended. Dan et al. [23] considered a dual-channel supply chain where both the manufacturer and the retailer can sell homogeneous products bundled with extended warranty that is provided by the manufacturer. Ma et al. [24] examined how manufacturers can control the retail channel to gain more profit from the supply-chain system in the context of supply-chain competition and retailers offering extended warranties. Zhang et al. [25] considered the outsource warranty strategies in a dual-channel supply chain in which either the retailer or a third party may provide an extended warranty. Zhang et al. [26] investigated the optimal pricing decisions of manufacturers for products and extended warranty and the transferability of extended warranty in the presence of P2P markets. Huang et al. [27] found that when a service provider’s cost advantage is high or the competition is more intense, the manufacturers will be more willing to choose the strategy of outsourcing the warranty. Some scholars have studied the pricing of extended warranty services [9]. For example, Wang et al. [2]
examined the design and pricing of an extended warranty strategy that offers different kinds of coverage and prices. Liu et al. [28] investigated the optimal pricing strategy of extended warranty service in a closed-loop supply chain and showed that the extended warranty strategy of retailers does not always decrease the manufacturer’s profitability. The benefits of extended warranty services to enterprises have also been paid attention to [5]. Chen et al. [29] constructed two extended warranty channels to compare whether the manufacturer or retailer offers the warranty service. Zhang et al. [30] found that the total profits of the supply chain increase if more customers purchase extended warranty from the retailers when both the retailer and manufacturer provide the extended warranty. Liu et al. [31] considered an extended producer responsibility and investigated the relationship between fairness concerns and the pricing-based responsibility transmission. Zhang et al. [32] investigated whether the E-commerce platform provides extended warranty service and found that the new and remanufactured products with extended warranty service strategy are the best choice. The existing literature on extended warranty service strategies has mostly investigated the impacts of extended warranty services provided by a specific provider on supply-chain decisions. Few literatures have paid attention to the issue of providers of extended warranty services in E-commerce supply chains.

### 3. Problem Description and Model Assumptions

We consider an E-commerce supply chain consisting of a manufacturer (M) and an E-commerce platform (P). In this E-commerce supply chain, M wholesales products to P, and then P sells products to consumers through the E-commerce platform channel. The extended warranty is an optional additional service, which can be voluntarily purchased by consumers when they purchase the products. By considering different providers and sellers of the extended warranty, three models are constructed: extended warranty provided and sold by M (Model MD), extended warranty provided by M and sold by P (Model MA), and extended warranty provided and sold by P (Model PD), which are shown in Figure 1.

![Figure 1. Three models in an E-commerce supply chain.](image)

In the model MD, the extended warranty is provided and sold by M, so the extended warranty cost is undertaken by M, and the revenue of extended warranty belongs to M. In the model MA, the extended warranty is provided by M and sold by P, so the extended warranty cost is undertaken by M, while P shares a certain proportion revenue of extended warranty with M. In the model PD, the extended warranty is provided and sold by P, so the extended warranty cost is undertaken by P, and the revenue of extended warranty belongs to P.
Based on the above problem description, we employ the symbols and notation given in Table 1 throughout this paper.

Table 1. Parameters and decision variables.

<table>
<thead>
<tr>
<th>Notations</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$v$</td>
<td>The value for the usage during the repair period</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Product failure probability</td>
</tr>
<tr>
<td>$T$</td>
<td>Extended warranty coverage</td>
</tr>
<tr>
<td>$r$</td>
<td>The sharing ratio of extended warranty service revenue</td>
</tr>
<tr>
<td>$c_e$</td>
<td>Unit cost of extended warranty</td>
</tr>
<tr>
<td>$w_n$</td>
<td>Unit wholesale price of product</td>
</tr>
<tr>
<td>$p_n^i/p_e^i$</td>
<td>Unit retail price of product/extended warranty in model $i$, $i \in {MD, MA, PD}$</td>
</tr>
<tr>
<td>$u_n^i/u_e^i$</td>
<td>Utility to purchase products/both products and extended warranty in model $i$, $i \in {MD, MA, PD}$</td>
</tr>
<tr>
<td>$D_n^i/D_e^i$</td>
<td>Demand of products/extended warranty in model $i$, $i \in {MD, MA, PD}$</td>
</tr>
<tr>
<td>$\pi_j^i$</td>
<td>Profit of supply chain member $j$ in model $i$, $i \in {MD, MA, PD}$, $j \in {M, P}$, and $T$ refers to the E-commerce supply-chain system</td>
</tr>
<tr>
<td>$CS^i$</td>
<td>Customer surplus in model $i$, $i \in {MD, MA, PD}$</td>
</tr>
</tbody>
</table>

To make the analysis tractable, we introduce the following assumptions.

**Consumer preference.** Following Bian et al. [8], Mai et al. [33], and Jiang and Zhang [34], we assume that when consumers do not buy the extended warranty, the consumers’ utility function is $u_n^i = v(1 - \alpha) - p_n^i$, $0 < \alpha < 1$. When consumers buy an extended warranty with coverage $T$, $0 < T < 1$, the consumers’ utility function is $u_e^i = v(1 - \alpha) + v\alpha T - p_n^i - p_e^i$ [8]. Consumers decide whether to buy the extended warranty service according to the utility [35]. When $u_e^i > u_n^i$, consumers will buy the extended warranty; otherwise, they will not buy the extended warranty. Thus, the demand of the product is $D_n^i = \frac{1}{2} - \frac{p_n^i}{1 - \alpha} + \frac{p_e^i}{2\alpha T}$, and the demand of the extended warranty is $D_e^i = \frac{1}{2} - \frac{p_e^i}{2\alpha T}$.

**Cost structure.** The cost of the extended warranty is dependent on the product failure probability and extended warranty coverage [8,32]. The cost of providing extended warranty is a convex function of the extended warranty coverage, and it is formulated as $c_e = aT^2$ [36,37]. In addition, we assume that the production cost or operation cost of the E-commerce platform is zero [38,39].

**Game specification.** In the E-commerce supply chain, a Stackelberg game is modeled between M and P, where M is the leader and P is the follower [32,40]. Firstly, M determines the wholesale price of products. When the extended warranty is provided by M, M determines the retail price of the extended warranty. Then, P sets the retail price of products and determines the retail price of the extended warranty when the extended warranty is provided by P.

If the extended warranty coverage is very small, i.e., $T \leq \max\{\frac{1 - \alpha}{16\alpha(1 - r)}, \frac{1 - \alpha}{8\alpha}\}$, then the profits of M and P can reach any positive value, thereby generating a trivial and interesting case [32,41]. Thus, to obtain the E-commerce supply-chain members’ optimal decisions and generate interesting managerial insights, this study focused only on the case of $\max\{\frac{1 - \alpha}{16\alpha(1 - r)}, \frac{1 - \alpha}{8\alpha}\} < T < 1$ (which is elaborated and proven in the proof of Theorems 1–3).
4. Model Formulations

4.1. Extended Warranty Provided and Sold by M (Model MD)

In the model MD, the extended warranty is provided and sold by M. M wholesales products to P, and then P sells these products to consumers. The optimization models of profit maximization for M and P are formulated as

$$\max_{w_n^{MD}, p_e^{MD}} \pi_M^{MD} = w_n^{MD} D_n^{MD} + (p_e^{MD} - c_r) D_e^{MD}$$

subject to

$$\max_{p_n^{MD}} \pi_P^{MD} = (p_n^{MD} - w_n^{MD}) D_n^{MD}$$

By using the reverse induction, we obtain the optimal results for the model MD, which are described in the following Theorem 1.

**Theorem 1.** In the model MD, the optimal prices are given as

$$w_n^{MD} = \frac{2aT(3+T)(1-a)}{a+16aT-1}$$

and

$$p_e^{MD} = \frac{3aT(3+T)(1-a)}{a+16aT-1},$$

Proof. See Appendix A.

Thus, the optimal demands of products and extended warranty are given as

$$D_n^{MD*} = \frac{aT(1-a^n + 8aT + 8aT^2)}{a+16aT-1}$$

and

$$D_e^{MD*} = \frac{4aT(1-a^n)}{a+16aT-1},$$

respectively.

The optimal profits of M, P, and supply-chain system are given as

$$\pi_M^{MD*} = \frac{aT^2(1-a)(T+3)^2}{(a+16aT-1)^2},$$

and

$$\pi_P^{MD*} = \pi_M^{MD*} + \pi_P^{MD*},$$

respectively. □

4.2. Extended Warranty Provided by M and Sold by P (Model MA)

In the model MA, the extended warranty is provided by M and sold by P, and P shares a certain proportion revenue of extended warranty with M. The optimization models of profit maximization for M and P are formulated as

$$\max_{w_n^{MA}, p_e^{MA}} \pi_M^{MA} = w_n^{MA} D_n^{MA} + (p_e^{MA} - c_r) D_e^{MA} - c_r D_e^{MA}$$

subject to

$$\max_{p_n^{MA}} \pi_P^{MA} = (p_n^{MA} - w_n^{MA}) D_n^{MA} + r p_e^{MA} D_e^{MA}$$

By using the reverse induction, we obtain the optimal results for the model MA, which are described in the following Theorem 2.

**Theorem 2.** In the model MA, the optimal prices are given as

$$w_n^{MA*} = \frac{2aT(1-a)(T+3-3r)}{a+16aT-16arT-1},$$

and

$$p_n^{MA*} = \frac{3aT(1-a)(T+3-3r)}{a+16aT-16arT-1},$$

Proof. See Appendix B.

Thus, the optimal demands of products and extended warranty are given as

$$D_n^{MA*} = \frac{aT(1-a^n + 8aT + 8aT^2)}{a+16aT-16arT-1}$$

and

$$D_e^{MA*} = \frac{4aT(1-a^n)}{a+16aT-16arT-1},$$

respectively.

The optimal profits of M, P, and supply-chain system are given as

$$\pi_M^{MA*} = \frac{aT^2(1-a)(T+3-3r)^2}{(a+16aT-16arT-1)^2},$$

$$\pi_P^{MA*} = \pi_M^{MA*} + \pi_P^{MA*},$$

respectively. □
4.3. Extended Warranty Provided and Sold by P (Model PD)

In the model PD, the extended warranty is provided and sold by P. M wholesales products to P, and P sells these products to consumers. The optimization models of profit maximization for M and P are formulated as

\[ \begin{align*}
\text{max}_{\pi_M^{PD}} &= w_{n}^{PD} D_n^{PD} \\
\text{s.t. max}_{\pi_p^{PD}} &= (p_{n}^{PD} - w_{n}^{PD}) D_n^{PD} + (p_{e}^{PD} - c_e) D_e^{PD}
\end{align*} \]  

By using the reverse induction, we obtain the optimal results for the model PD, which are described in the following Theorem 3.

**Theorem 3.** In the model PD, the optimal prices are given as $w_n^{PD^*} = \frac{(1-a)(T+3)}{8(a+8aT-1)}$, $p_n^{PD^*} = \frac{(1-a)(T+3)(a+12aT-1)}{8(a+8aT-1)}$, and $p_e^{PD^*} = \frac{aT[16aT(1+T)+(1-a)(1-T)]}{4(a+8aT-1)}$.

**Proof.** See Appendix C.

Thus, the optimal demands of products and extended warranty are given as $D_n^{PD^*} = \frac{aT(T+3)}{2(a+8aT-1)}$ and $D_e^{PD^*} = \frac{15aT(1-T)+T(1-a)T-5(1-a)}{8aT(a+8aT-1)}$, respectively.

The optimal profits of M, P, and supply-chain system are given as $\pi_M^{PD^*} = \frac{aT(1-a)(T+3)^2}{16(a+8aT-1)}$, $\pi_P^{PD^*} = \frac{aT(1-a)(T+3)^2}{16(a+8aT-1)}$, and $\pi_T^{PD^*} = \pi_M^{PD^*} + \pi_P^{PD^*}$, respectively.

5. Sensitive and Comparative Analysis of Equilibrium Results

By analyzing and comparing the equilibrium results in Theorems 1–3, the following conclusions can be drawn.

By analyzing the effects of parameters $a$ and $T$ on the optimal wholesale prices of products in the models $MD$, $MA$, and $PD$, and comparing them in the three models, we have the following Proposition 1.

**Proposition 1.** The optimal wholesale prices in the three models satisfy

1. $\frac{\partial w_{n}^{PD^*}}{\partial a} < 0$, $i \in \{MD, MA, PD\}$;
2. $\frac{\partial w_{n}^{PD^*}}{\partial T} > 0$; if $a > \frac{2T+3}{16T^2+2T+3}$, then $\frac{\partial w_{n}^{MA^*}}{\partial T} > 0$; if $a > \frac{2T+3(1-r)}{2T+(16T^2+3)(1-r)}$, then $\frac{\partial w_{n}^{PD^*}}{\partial T} > 0$;
3. $w_{n}^{MA^*} > w_{n}^{MD^*} > w_{n}^{PD^*}$.

**Proof.** See Appendix D.

Proposition 1(1) shows that with the increase in the product failure probability $a$, the optimal wholesale prices of the products in the three models all decrease. That is to say, regardless of whether the extended warranty is provided by M or P, with the increase in the product failure probability, consumers will obtain less utility from the product, and M should decrease the wholesale price of the product to encourage P to wholesale more products and sell to consumers. On the other hand, the lower wholesale price of the products will encourage consumers to purchase extended warranty after they have purchased the products.

Proposition 1(2) indicates that with the increase in the extended warranty coverage, M will increase the wholesale price of the products in the model PD. On the other hand, when the product failure probability is more than a certain value, with the increase in the extended warranty coverage, the optimal wholesale prices of the products all increase in the models $MD$ and $MA$. As a result, manufacturers should set appropriate wholesale prices of the products according to the extended warranty coverage.
Proposition 1(3) shows that the wholesale price of the products in the model MA is the highest, while that in the model PD is the lowest. When extended warranty is provided by M and sold by P, M undertakes the extended warranty cost, while P only shares a certain proportion revenue of extended warranty with M. M will increase the wholesale price of the products to obtain more profit. □

By analyzing the effects of parameters α and T on the optimal retail prices of the products and extended warranty in the models MD, MA, and PD, and comparing them in the three models, we have the following Proposition 2.

Proposition 2. The optimal retail prices in the three models satisfy

1. \( \frac{\partial p^{MA^*}}{\partial \alpha} < 0, \quad \frac{\partial p^{MA^*}}{\partial T} < 0; \) if \( \alpha + \frac{4T}{T + 20} \), then \( \frac{\partial \alpha}{\partial p} < 0; \)
2. \( \frac{\partial p^{MD^*}}{\partial \alpha} < 0, \quad \frac{\partial p^{MD^*}}{\partial T} > 0; \) if \( \alpha < \frac{2T + 3}{16T^2 + 2T + 3} \), then \( \frac{\partial \alpha}{\partial p} > 0; \)
3. \( p^{MA^*}_n > p^{PD^*}_n, \quad p^{MA^*}_n > p^{MD^*}_n; \) if \( \alpha < \frac{1}{1 + 4\alpha} \), then \( p^{PD^*}_n < p^{MD^*}_n; \)
4. \( p^{MA^*}_e > p^{PD^*}_e > p^{MD^*}_e. \)

Proof. See Appendix E.

Proposition 2(1) indicates that with the increase in the product failure probability α, the optimal retail prices of the products in the models MD and MA all decrease. When the product failure probability is small, with the increase in the product failure probability α, the optimal retail prices of the products in the model PD increases. That is to say, the product failure probability has a negative effect on the retail prices of the products. When the product failure probability increases, the consumer’s utility decreases, so P decreases the retail prices of the products to encourage consumers to buy the products. Therefore, E-commerce platforms should control the product failure probability to increase the retail prices of the products.

Proposition 2(2) shows that, when the product failure probability satisfies some conditions, with the increase in the extended warranty coverage T, the optimal retail prices of the products in the models MD and MA all increase. In order to increase the retail prices of the products, E-commerce platforms should extend the warranty coverage.

Proposition 2(3) shows that the retail price of the products in the model MA is higher than that in the model MD or PD. In other words, comparing the strategy of the extended warranty provided and sold by M or P, under the strategy of the extended warranty provided by M and sold by P, P can set the higher retail price of the products. When the product failure probability is low, P will set a higher retail price of the products in the model PD than that in the model MD.

From Proposition 2(4), we can find that, similar to the relationship of the retail prices of the products, the retail price of extended warranty in the model MA is the highest, while that in the model MD is the lowest. When the extended warranty is provided by M and sold by P, P can set a high retail price of the extended warranty. On the other hand, when the extended warranty is provided and sold by M, M sets a low retail price of the extended warranty. Therefore, E-commerce supply-chain members set the retail prices of the extended warranty and products according to the extended warranty provider strategy. □

By analyzing the effects of parameters α and T on the optimal demands of the products and extended warranty in the models MD, MA, and PD, and comparing them in the three models, we have the following Proposition 3.

Proposition 3. The optimal demands in the three models satisfy

1. \( \frac{\partial D^*_i}{\partial \alpha} < 0, \quad \frac{\partial D^*_i}{\partial T} > 0, \) \( i \in \{MD, MA, PD\}; \)
(2) if $a > \frac{2T+3}{6T+2T+3}$, then $\frac{\partial D^{MD^*}}{\partial a} > 0$, if $a > \frac{2T+3(1-\epsilon)}{2T+(16T^2+3)(1-\epsilon)}$, then $\frac{\partial D^{MA^*}}{\partial a} > 0$,

if $a > \frac{2T+3}{6T+2T+3}$, the $\frac{\partial D^{PD^*}}{\partial a} > 0$;

(3) if $0 < a < \frac{2T+3}{16T+2T+3}$, then $\frac{\partial D^{MD^*}}{\partial a} > 0$, if $0 < a < \frac{2T+3(1-\epsilon)}{2T+(16T^2+3)(1-\epsilon)}$, then $\frac{\partial D^{MA^*}}{\partial a} > 0$,

if $\frac{1}{25} < a < \frac{1}{4T+1}$, then $\frac{\partial D^{PD^*}}{\partial a} > 0$;

(4) $D_N^{PD^*} > D_N^{MD^*}$, $D_M^{MA^*} > D_N^{MA^*}$, if $a > \frac{3+T}{3+T+6T+16T^2}$, then $D_N^{PD^*} > D_N^{MA^*}$;

(5) $D_e^{MD^*} > D_e^{PD^*} > D_e^{MA^*}$.

**Proof.** See Appendix F.

Proposition 3(1) shows that with the increase in the product failure probability $a$, the optimal demands of the products in the three models all decrease, while the optimal demands of the extended warranty in the three models all increase. That is to say, the product failure probability has a negative effect on the demands of the products, while it has a positive effect on the demands of the extended warranty. The higher the product failure probability is, the more willing the consumers are to buy extended warranty services.

Proposition 3(2) and 3(3) indicate that, when the product failure probability satisfies some conditions, with the increase in the extended warranty coverage, the optimal demands of the products and extended warranty in the three models all increase. Due to the fact that, with the increase in the product failure probability and extended warranty coverage, the wholesale and retail prices of the products increase, from the perspective of consumers, with the increase in the prices of the products, the utility of purchasing the products decreases in the three models, while the demands of the extended warranty increase in the three models. Consumers will purchase more extended warranty with the increase in the product failure probability and extended warranty coverage in order to obtain more out of the products.

Proposition 3(4) indicates that the demand of the products in the model $MD$ is the lowest. Moreover, when the product failure probability is high, the demand of the products in the model $MA$ is lower than that in the model $PD$. From the perspective of expanding demands of the products, the strategy of extended warranty provided and sold by $P$ or the extended warranty provided by $M$ and sold by $P$ is more effective.

From Proposition 3(5), we can find that, different from the relationship of demands of the products, the demand of the extended warranty in the model $MD$ is the highest, while that in the model $MA$ is the lowest. That is to say, comparing the strategy of extended warranty sold by $P$, the strategy of extended warranty provided and sold by $M$ is more effective on expanding the demand of the extended warranty. $\square$

By analyzing the effects of parameters $a$ and $T$ on the optimal profits of $M$ and $P$ in the models $MD$, $MA$, and $PD$, we have the following Proposition 4.

**Proposition 4.** The optimal profits in the three models satisfy

(1) $\frac{\partial n_P^{PD^*}}{\partial a} < 0$, $\frac{\partial n_P^{MD^*}}{\partial a} < 0$;

(2) if $a > \max\left\{\frac{2T-1}{2T-1+T', \frac{1}{4T-4T'}}\right\}$, then $\frac{\partial n_P^{MA^*}}{\partial a} < 0$; if $\frac{1}{3+T+16T^2} < a < 1$, then $\frac{\partial n_P^{PD^*}}{\partial T} > 0$; if $a > \frac{2T+3}{16T^2+27+3}$, then $\frac{\partial n_P^{MD^*}}{\partial T} > 0$.

**Proof.** See Appendix G.

Proposition 4(1) shows that, with the increase in the product failure probability, $M$’s profit in the model $PD$ and $P$’s profit in the model $MD$ all decrease. In other words, the product failure probability will damage the profits of $M$ and $P$. That is because when the extended warranty is provided and sold by $M$, $P$ can only obtain the profit by selling the product, whereas, when the retail price of the product decreases with the increase in the product failure probability, $P$’s profit will decrease. In addition, when the extended warranty is provided and sold by $P$, the decrease in the retail prices of the product will
Therefore, E-commerce supply-chain members can extend the extended warranty coverage to boost more profits. Proposition 4(2) indicates that, when the product failure probability $\alpha$ is high, with the increase in the extended warranty coverage, the profit of $M$ in the model $MD$ decreases. In addition, when the product failure probability satisfies some conditions, with the increase in the extended warranty coverage, the profit of $M$ in the model $PD$ and the profit of $P$ in the model $MD$ will increase. That is to say, under some conditions, the increase in the extended warranty coverage can bring more profits for E-commerce supply-chain members. Therefore, E-commerce supply-chain members can extend the extended warranty coverage to boost more profits. $\square$

6. Numerical Analysis

In order to illustrate the impact of the relevant parameters in the supply chain on the decisions of the E-commerce supply-chain members, a numerical experiment is used in this section. Following Bian et al. [8], Zhang and Gao [26], and Dong et al. [41], the parameters are set as $r = 0.2$. Substituting $r = 0.2$ into the demands of the extended warranty in the three models, in order to ensure that all the optimal results are positive and satisfy the assumption $\max\{\frac{1-\alpha}{16(1-T)^2}, \frac{1-\alpha}{8}\alpha\} < T < 1$, $\alpha$ varied within the range of $[0.6,0.9]$ and $T$ varied within the range of $[0.5,1]$. The impacts of the product failure probability $\alpha$ and extended warranty coverage $T$ on the demands of products are described in Figure 2.

![Figure 2. Impact of $\alpha$ and $T$ on the demand of products.](image)

Figure 2 shows that, with the increase in the extended warranty coverage and the decrease in the product failure probability, the demands of products in three models all increase. In reality, when the product failure probability is increasing, consumers will choose to give up purchasing the product, which will lead to a decrease in the demand of the products. Moreover, when the extended warranty coverage increases, consumers will feel that they can have a longer product life, which will increase the demand of the products. In addition, the demand of the products under the strategy of extended warranty provided and sold by $M$ is the lowest. When the extended warranty coverage and product failure probability are both small, the demand of the products under the strategy of extended warranty provided and sold by $P$ is the highest. When the extended warranty coverage and product failure probability are both large, the strategy of extended warranty provided by $M$ and sold by $P$ is the most effective on expanding the demand of the products.

The impacts of the product failure probability $\alpha$ and extended warranty coverage $T$ on the demands of the extended warranty are described in Figure 3.
The product failure probability has a negative effect on the profits of M. Therefore, manufacturers should control the product failure probability to obtain more profits. Moreover, the decrease in the product failure probability, the profits of M in the three models all increase. That is to say, when the product failure probability is high, consumers prefer to buy extended warranty services. On the other hand, when the extended warranty coverage is low, the demand of the extended warranty is high. For consumers, the cost of replacing a product is much higher than the cost of purchasing extended warranty, so the consumers will purchase more extended warranty when the extended warranty coverage increases. In addition, the demand of the extended warranty in the model MA is the lowest, while the demand of the extended warranty in the model MD is the highest. However, the demand of the extended warranty in the model PD is only a little smaller than that in the model MD. As a result, in order to expand the demand of the extended warranty, the strategy of extended warranty provided and sold by M or P is the best choice for E-commerce supply-chain members.

The impacts of the product failure probability $\alpha$ and extended warranty coverage $T$ on the profits of M are described in Figure 4.

**Figure 3.** Impact of $\alpha$ and $T$ on the demand of extended warranty.

From Figure 3, we can find that, with the increase in the product failure probability and the decrease in the extended warranty coverage, the demands of the extended warranty in the three models all increase. That is to say, when the product failure probability is high, consumers prefer to buy extended warranty services. On the other hand, when the extended warranty coverage is low, the demand of the extended warranty is high. For consumers, the cost of replacing a product is much higher than the cost of purchasing extended warranty, so the consumers will purchase more extended warranty when the extended warranty coverage increases. In addition, the demand of the extended warranty in the model MA is the lowest, while the demand of the extended warranty in the model MD is the highest. However, the demand of the extended warranty in the model PD is only a little smaller than that in the model MD. As a result, in order to expand the demand of the extended warranty, the strategy of extended warranty provided and sold by M or P is the best choice for E-commerce supply-chain members.

**Figure 4.** Impact of $\alpha$ and $T$ on the profit of M.

Figure 4 indicates that, with the increase in the extended warranty coverage and the decrease in the product failure probability, the profits of M in the three models all increase. The product failure probability has a negative effect on the profits of M. Therefore, manufacturers should control the product failure probability to obtain more profits. Moreover,
when the extended warranty coverage is short, the profit of M in the model MD is the highest. On the other hand, when the extended warranty coverage is long, the profit of M in the model MA is the highest. In other words, the extended warranty provided by M is beneficial for M. When the extended warranty coverage is low, manufacturers prefer the extended warranty provided and sold by themselves. When the extended warranty coverage is high, the strategy of the extended warranty provided by themselves and sold by E-commerce platforms is the best choice for manufacturers.

The impacts of the product failure probability $\alpha$ and extended warranty coverage $T$ on the profits of P are described in Figure 5.

![Figure 5. Impact of $\alpha$ and $T$ on the profit of P.](image)

Figure 5 indicates that, with the increase in the extended warranty coverage and the decrease in the product failure probability, the profits of P in the three models all increase. That is because the increase in the demands of the products due to the increase in the extended warranty coverage will increase P’s profits. The high product failure probability will also damage the profits of P. In addition, the profit of P in the model PD is the highest. Different from the manufacturers’ choice, the strategy of extended warranty provided and sold by P is the best choice for E-commerce platforms.

The impacts of the product failure probability $\alpha$ and extended warranty coverage $T$ on the profits of E-commerce supply-chain system are described in Figure 6.

![Figure 6. Impact of $\alpha$ and $T$ on the profit of supply-chain system.](image)
Figure 6 shows that, with the increase in the extended warranty coverage and the decrease in the product failure probability, the profits of the E-commerce supply-chain system in the three models all increase. The product failure probability also damages the profits of the E-commerce supply-chain system. As a result, E-commerce supply-chain members should control the product failure probability and expand the extended warranty coverage. In addition, similar to the profits of P, the profit of the E-commerce supply-chain system in the model PD is always the highest. The strategy of extended warranty provided and sold by P is the most beneficial for the E-commerce supply-chain system.

Following Ding et al. [36], Yenipazarli [42], Yazdekhasti et al. [43], and Liu et al. [44], consumer surplus of products and extended warranty in an E-commerce supply chain can be calculated as $CS^i = \int \frac{\mu^2}{2\sigma} u(p) dv + \int \frac{\mu^2}{2\sigma} u(e) dv = \frac{\mu^2}{2\sigma} + \frac{\mu^2}{2\sigma} + \frac{1}{2\sigma^2} - p_e - p_i$. In addition, the impacts of the product failure probability $\alpha$ and extended warranty coverage $T$ on the consumer surplus are described in Figure 7.

Figure 7. Impact of $\alpha$ and $T$ on consumer surplus.

Figure 7 shows that, with the increase in the extended warranty coverage, the consumer surplus in the model MA firstly decreases and then increases. When the product failure probability is small, with the increase in the extended warranty coverage, the consumer surplus in the models PD and MD firstly decreases and then increases. On the other hand, when the product failure probability is large, with the increase in the extended warranty coverage, the consumer surplus in the models PD and MD always decreases. That is to say, the high extended warranty coverage is not always beneficial for consumers. Moreover, when the extended warranty coverage is short, the consumer surplus in the model PD is the highest. On the other hand, when the extended warranty coverage is long, the consumer surplus in the model MA is the highest. In other words, when the extended warranty coverage is short, the strategy of extended warranty provided and sold by P is beneficial for consumers. On the contrary, the strategy of extended warranty provided by M and sold by P is the best choice for consumers.

From Figures 4–7, we can find that manufacturers choose the extended warranty strategy between the strategy of extended warranty provided and sold by M and the strategy of extended warranty provided by M and sold by P. Manufacturers will not select the strategy of extended warranty provided and sold by P. However, from the perspective of E-commerce platforms and the E-commerce supply-chain system, the strategy of extended warranty provided and sold by P is the best choice. Moreover, when the extended warranty coverage is short, the strategy of extended warranty provided and sold by P is beneficial for consumers. As a result, when the extended warranty provider sets appropriate extended warranty coverage, the strategy of extended warranty provided and sold by P should
be the consistent choice of E-commerce supply-chain members, which is also beneficial for E-commerce supply-chain system and consumers. Therefore, E-commerce platforms should take some measures to motivate manufacturers to better implement the strategy of extended warranty provided and sold by P.

7. Conclusions

This paper explored the provider and seller of extended warranty services and put forward an extended warranty strategy in an E-commerce supply chain. We constructed three models to analyze the decisions of E-commerce supply-chain members and investigated the impact of extended warranty services on the benefits of E-commerce supply-chain members. The results show the following: (1) The product failure probability and extended warranty coverage affect the decisions of E-commerce supply-chain members. In addition, under most conditions, the extended warranty provided and sold by P can effectively expand the demands of the products and extended warranty. (2) E-commerce supply-chain members prefer different extended warranty strategies under different conditions. When the extended warranty coverage is short, the manufacturer prefers the strategy of extended warranty provided and sold by M. On the contrary, the strategy of extended warranty provided by M and sold by P is beneficial for the manufacturer. The E-commerce platform always prefers the strategy of extended warranty provided and sold by P. (3) From the perspective of E-commerce supply-chain-system operation, the strategy of extended warranty provided and sold by P is the best choice, which is also beneficial for consumers when the extended warranty is appropriately set.

Based on these findings, some management insights can be drawn. Firstly, extended warranty services affect the pricing decisions and demands of products and extended warranty. An appropriate extended warranty strategy can promote the sales of products and extended warranty, which can solve the product failure problem and extend the service life of products. This is successfully practiced by the automobile industry (https://news.yiche.com/hao/wenzhang/57547604/, accessed on 19 December 2021). Secondly, manufacturers hope that the extended warranty will be provided by themselves. On the other hand, the strategy of extended warranty provided and sold by E-commerce platforms is beneficial for E-commerce platforms. E-commerce supply-chain members should cooperate to choose an appropriate extended warranty strategy. Thirdly, to attract more potential consumers, E-commerce platforms directly provide and sell the extended warranty services. For instance, JD and Suning have launched their own extended warranty services (https://tech.sina.com.cn/i/2013-09-24/15508765090.shtml?utm_source=tuicool, accessed on 24 September 2013). Therefore, E-commerce supply-chain members should set an appropriate extended warranty coverage to implement the strategy of extended warranty provided and sold by E-commerce platforms, which is the best choice for E-commerce supply-chain system and consumers.

There are still some limitations of this paper. Firstly, we only considered a manufacturer and an E-commerce platform in an E-commerce supply chain. In reality, there are two or more manufacturers or E-commerce platforms, and there are competitions among supply-chain members. In further research, the competition between manufacturers and E-commerce platforms can be considered. Secondly, we only consider the E-commerce platform selling a single product and one channel structure. It is worth to investigate the extended warranty strategy by considering different products and channel structures in future research.

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Appendix A

Proof of Theorem 1. By substituting demands $D_n^{MD}$ and $D_p^{MD}$ into the profit functions $π_n^{MD}$ and $π_p^{MD}$, since $0 < α < 1$ and $\frac{∂π_n^{MD}}{∂(p_n^{MD})} = \frac{2}{α} < 0$, $π_n^{MD}$ is concave on $p_n^{MD}$. It can be obtained from $\frac{∂π_n^{MD}}{∂p_n} = 0$ that $p_n^{MD} = 2\frac{aT_n^{MD} + (1-α)p_n^{MD} + αT(1-α)}{4αT}$. By substituting $p_n^{MD}$ into $π_n^{MD}$, the Hessian matrix of $π_n^{MD}$ in terms of $w_n^{MD}$ and $p_e^{MD}$ is $H^{MD} = \left[ \begin{array}{c} \frac{1}{α} & \frac{1}{4α} \\ \frac{1}{4α} & \frac{1}{α} \end{array} \right]$. From the assumption $0 < α < 1$, we have $|H^{MD}| = \frac{1}{α} - \frac{1}{16} < 0$ and $|H_1^{MD}| = \frac{α - 16α^2}{16α^2 T^2 (1-α)}$. If $|H_1^{MD}| > 0$, then we have $\frac{1}{α} - \frac{1}{16α^2 T^2 (1-α)} < 1$, then M’s profit $π_n^{MD}$ is strictly concave with respect to $w_n^{MD}$ and $p_e^{MD}$. If $|H_1^{MD}| \leq 0$, M can make any positive profit. Therefore, this study only considered the case when $\frac{1}{α} - \frac{1}{16α^2 T^2 (1-α)} < 1$ to obtain optimal results. It can be obtained from $\frac{∂π_n^{MD}}{∂w_n} = \frac{∂π_n^{MD}}{∂p_e} = 0$ that $w_n^{MD^*} = \frac{2αT(3+T)(1-α)}{α + 16α^2 T (1-α)}$ and $p_e^{MD^*} = \frac{α(1+8αT + 8αT^2)}{α + 16α^2 T (1-α)}$. □

Appendix B

Proof of Theorem 2. By substituting demands $D_n^{MA}$ and $D_p^{MA}$ into the profit functions $π_n^{MA}$ and $π_p^{MA}$, since $0 < α < 1$ and $\frac{∂π_n^{MA}}{∂(p_n^{MA})} = \frac{2}{α} < 0$, $π_n^{MA}$ is concave on $p_n^{MA}$. It can be obtained from $\frac{∂π_n^{MA}}{∂p_n} = 0$ that $p_n^{MA} = 2\frac{aT_n^{MA} + (1-α)p_n^{MA} + αT(1-α)}{4αT}$. By substituting $p_n^{MA}$ into $π_n^{MA}$, the Hessian matrix of $π_n^{MA}$ in terms of $w_n^{MA}$ and $p_e^{MA}$ is $H^{MA} = \left[ \begin{array}{c} \frac{1}{α} & \frac{1}{4α} \\ \frac{1}{4α} & \frac{1}{α} \end{array} \right]$. From the assumption $0 < α < 1$, we have $|H^{MA}| = \frac{1}{α} - \frac{1}{16} < 0$ and $|H_1^{MA}| = \frac{16α^2 T(1-α)^2 - (1-α)}{16α^2 T^2 (1-α)}$. If $|H_1^{MA}| > 0$, then we have $\frac{1}{α} - \frac{1}{16α^2 T^2 (1-α)} < 1$, then M’s profit $π_n^{MA}$ is strictly concave with respect to $w_n^{MA}$ and $p_e^{MA}$. If $|H_1^{MA}| \leq 0$, M can make any positive profit. Therefore, this study only considered the case when $\frac{1}{α} - \frac{1}{16α^2 T^2 (1-α)} < 1$ to obtain optimal results. Under this assumption, $π_n^{MA}$ is strictly concave with respect to $w_n^{MA}$ and $p_e^{MA}$. It can be obtained from $\frac{∂π_n^{MA}}{∂w_n} = \frac{∂π_n^{MA}}{∂p_e} = 0$ that $w_n^{MA^*} = \frac{2αT(3+T)(1-α)}{α + 16α^2 T (1-α)}$ and $p_e^{MA^*} = \frac{α(1+8αT + 8αT^2)}{α + 16α^2 T (1-α)}$. □

Appendix C

Proof of Theorem 3. By substituting demands $D_n^{PD}$ and $D_p^{PD}$ into the profit functions $π_n^{PD}$ and $π_p^{PD}$, the Hessian matrix of $π_n^{PD}$ in terms of $p_n^{PD}$ and $p_e^{PD}$ is $H^{PD} = \left[ \begin{array}{c} \frac{2}{α} & \frac{2}{α} \\ \frac{2}{α} & \frac{2}{α} \end{array} \right]$. From the assumption $0 < α < 1$, we have $|H_1^{PD}| = \frac{1}{α} - \frac{1}{16} < 0$ and $|H_2^{PD}| = \frac{α - 16α^2 T^2 (1-α)}{16α^2 T (1-α)}$. If $|H_2^{PD}| > 0$, then we have $\frac{1}{α} - \frac{1}{16α^2 T^2 (1-α)} < 1$, then P’s profit $π_p^{PD}$ is strictly concave with respect to $p_n^{PD}$ and $p_e^{PD}$. If $|H_2^{PD}| \leq 0$, P can make any positive profit. Therefore, this study only
considered the case when $\frac{1-a}{8a} < T < 1$ to obtain optimal results. Combined with the assumption $\frac{1-a}{16a(1-r)} < T < 1$ in the proof of Theorem 1 and the assumption $\frac{1-a}{16a(1-r)} < T < 1$ in the proof of Theorem 2, max $\{\frac{1-a}{16a(1-r)} \frac{1-a}{8a}\} < T < 1$ is assumed in this study to obtain optimal results. Under this assumption, $\pi^*_P$ is strictly concave with respect to $p^*_n$ and $p^*_c$. It can be obtained from $\frac{\partial \pi^*_P}{\partial p^*_n} = \frac{\partial \pi^*_P}{\partial p^*_c} = 0$ that $p^*_n = \frac{(a+4aT-1)w^*_n + aT(1-a)T+3}{a+8aT-1}$ and $p^*_c = aT/(4aT+4aT-a+2w^*_n)$.

By substituting $p^*_n$ and $p^*_c$ into $\pi^*_P$, since $0 < \alpha < 1$ and max $\{\frac{1-a}{16a(1-r)} \frac{1-a}{8a}\} < T < 1$, we have $\frac{\partial^2 \pi^*_P}{\partial w^*_n^2} = \frac{8aT(a-1)}{a+8aT-1} < 0$, and thus, $\pi^*_P$ is concave on $w^*_n$. It can be obtained from $\frac{\partial \pi^*_P}{\partial w^*_n} = 0$ that $w^*_n = (1-a)(T+3)$. Substituting $w^*_n$ into $p^*_n$ and $p^*_c$, we have $p^*_n = \frac{(1-a)(T+3)(a+12aT-1)}{8(a+8aT-1)}$ and $p^*_c = \frac{aT(16aT(1+T)+1-a)(1-T)}{4(a+8aT-1)}$. □

Appendix D

Proof of Proposition 1. From the assumptions $0 < \alpha < 1$, max $\{\frac{1-a}{16a(1-r)} \frac{1-a}{8a}\} < T < 1$, and $0 < r < 1$, by examining the optimal wholesale prices of products in the models MD, MA, and PD, it can be easily verified that

(1) $\frac{\partial \pi^*_M}{\partial a} = -2T(T+3)[16aT^2+(1-a)^2]/(a+16aT-1)^2 < 0$, $\frac{\partial \pi^*_M}{\partial a} = -2T(T+3-3r)/(1-a)^2+16a^2T(1-r)/\alpha(a+16aT-1)^2 < 0$, $\frac{\partial \pi^*_M}{\partial a} = T+3 < 0$;

(2) $\frac{\partial \pi^*_M}{\partial T} = \frac{8aT}{(a+8aT-1)^2} > 0$; $\frac{\partial \pi^*_M}{\partial T} = 2a(1-a)(3a-2T+3aT+16aT-3)/(a+16aT-1)^2$, if $\alpha > \frac{T+3}{16T+2T+3}$, then $\frac{\partial \pi^*_M}{\partial T} > 0$;

(3) $w^*_n - w^*_M = 16aT^2[1-a]+(1-a)^2 < 0$, then $w^*_n < w^*_M$; $w^*_n - w^*_M = -(1-a)^2(T+3)/[a(a+16aT-1)] < 0$, then $w^*_n < w^*_M$; $w^*_n - w^*_M = 2a(1-a)(16aT^2+16aT-1)> 0$, then $w^*_n > w^*_M$; therefore, $w^*_n > w^*_M > w^*_P$. □

Appendix E

Proof of Proposition 2. From the assumptions $0 < \alpha < 1$, max $\{\frac{1-a}{16a(1-r)} \frac{1-a}{8a}\} < T < 1$, and $0 < r < 1$, by examining the optimal retail prices of products and extended warranty in the three models, it can be easily verified that

(1) $\frac{\partial \pi^*_P}{\partial a} = -3T(T+3)[16a^2T+(1-a)^2]/(a+16aT-1)^2 < 0$, $\frac{\partial \pi^*_P}{\partial a} = -3T(T+3r)[16a^2T(1-r)+(1-a)^2]/(a+16aT-16arT-1)^2 < 0$, $\frac{\partial \pi^*_P}{\partial a} = T+3 < 0$;

(2) $\frac{\partial \pi^*_P}{\partial T} = 3a(1-a)(3a-2T+3aT+16aT-3)/(a+16aT-1)^2$, if $\alpha > \frac{T+3}{16T+2T+3}$, then $\frac{\partial \pi^*_P}{\partial T} > 0$; $\frac{\partial \pi^*_P}{\partial T} = 3a(1-a)(3a-2T+3aT+16aT-3)/(a+16aT-16arT-1)^2$, if $\alpha > \frac{T+3}{16T+2T+3}$, then $\frac{\partial \pi^*_P}{\partial T} > 0$; $\frac{\partial \pi^*_P}{\partial T} = 3a(1-a)(3a-2T+3aT+16aT-3)/(a+16aT-16arT-1)^2$, if $\alpha < \frac{T+3}{16T+2T+3}$, then $\frac{\partial \pi^*_P}{\partial T} > 0$. □
Appendix F

Proof of Proposition 3. From the assumptions $0 < \alpha < 1$, max$\left\{ \frac{1-a}{16a^2(1-\gamma)}, \frac{1-a}{8a^2} \right\} < T < 1$, and $0 < r < 1$, by examining the optimal demands of products and extended warranty in the three models, it can be easily verified that

\[ p_{PD}^* - p_{MA}^* = \left(1-a\right)\left(\frac{aT(\alpha T-2a+1+8a^2T^2)}{a+8aT-1}\right) \leq 0, \text{ then } p_{PD}^* > p_{MA}^*; \]
\[ p_{PD}^* - p_{MA}^* = \left(1-a\right)\left(\frac{aT(\alpha T-2a+1+8a^2T^2)}{a+8aT-1}\right) > 0, \text{ then } p_{PD}^* < p_{MA}^*; \]
\[ p_{PD}^* - p_{MA}^* = \left(1-a\right)\left(\frac{aT(\alpha T-2a+1+8a^2T^2)}{a+8aT-1}\right) = 0, \text{ then } p_{PD}^* = p_{MA}^*. \]

Appendix G

Proof of Proposition 4. From the assumptions $0 < \alpha < 1$, max$\left\{ \frac{1-a}{16a^2(1-\gamma)}, \frac{1-a}{8a^2} \right\} < T < 1$, and $0 < r < 1$, by examining the optimal profits in the models $MD$, $MA$, and $PD$, it can be easily verified that

\[ \left(\frac{\partial p_{PD}^*}{\partial \alpha}\right) = -\frac{T(\alpha T-2a+1+8a^2T^2)}{2(a+8aT-1)} \leq 0, \text{ then } \left(\frac{\partial p_{PD}^*}{\partial \alpha}\right) > 0; \]
\[ \left(\frac{\partial p_{PD}^*}{\partial \alpha}\right) = -\frac{T(\alpha T-2a+1+8a^2T^2)}{2(a+8aT-1)} > 0, \text{ then } \left(\frac{\partial p_{PD}^*}{\partial \alpha}\right) < 0; \]
\[ \left(\frac{\partial p_{PD}^*}{\partial \alpha}\right) = -\frac{T(\alpha T-2a+1+8a^2T^2)}{2(a+8aT-1)} = 0, \text{ then } \left(\frac{\partial p_{PD}^*}{\partial \alpha}\right) = 0. \]
\[ \frac{\partial n_{MD}^{∗∗}}{\partial \alpha} = \frac{a(-2\alpha - 6\alpha + 24\alpha T^2 + 4\alpha T - 4\alpha - 1)}{(\alpha + 16\alpha T - 1)^3}, \text{if } \alpha > \max\left\{ \frac{2T - 1}{24T^2 - 6T + 1}, \frac{1}{1 + 4T - 4T^2} \right\}, \text{then} \]

\[ \frac{\partial n_{MD}^{∗∗}}{\partial \alpha} < 0; \quad \frac{\partial n_{MD}^{∗∗}}{\partial \alpha} = \frac{T + 3}{1 - \alpha} (3a + 3\alpha T + 16\alpha T^2 - 3 - 3T), \text{if } \frac{3 + 3\alpha}{3 + 3\alpha + 16\alpha T^2} < \alpha < 1, \text{then} \]

\[ \frac{\partial n_{MD}^{∗∗}}{\partial \alpha} > 0; \quad \frac{\partial n_{MD}^{∗∗}}{\partial \alpha} = 2a^2T(1 - \alpha)(T + 3)(3a - 2T + 2aT^2 + 16\alpha T^2 - 3), \text{if } \alpha > \frac{2T + 3}{16\alpha T^2 + 2T + 3}, \text{then} \]

\[ \frac{\partial n_{MD}^{∗∗}}{\partial \alpha} > 0. \]
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