

Article

Optimal Maintenance Policy for Equipment Submitted to Multi-Period Leasing as a Circular Business Model

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Abstract: The leasing of various types of equipment plays a significant role in reducing resource consumption, reducing the need for frequent replacements, and lessening the environmental impact of equipment manufacturing and disposal. This paper examines a maintenance policy for equipment that is leased multiple times throughout its lifespan. If the equipment fails to perform as expected within the basic and extended warranty durations, the lessor makes minimal repairs at its own expense. Once the warranty period has elapsed, the lessor is still responsible for carrying out any necessary repairs, but the lessee is required to pay for them. The warranty periods are not uniform. To reduce the frequency of breakdowns, the lessor carries out preventive maintenance (PM) between successive lease periods, with the aim of reducing the age of the equipment to some extent. The costs associated with PM depend on the set of actions to be performed and their associated efficiency in terms of age reduction. A mathematical model is proposed to simultaneously find the optimal efficiency levels of PM to be carried out between successive lease periods and the optimal extended warranty periods to be offered to lessees in order to maximize the lessor's expected total profit throughout the equipment's lifecycle. To demonstrate the use of the developed model, a numerical example and a sensitivity study are discussed. Our model demonstrates its ability to provide valuable insights and facilitate decision-making in the establishment of leasing contracts.

Keywords: closed-loop supply chains; leasing; circular economy; sustainability; imperfect preventive maintenance; warranty



Citation: Ben Mabrouk, A.; Chelbi, A.; Aguir, M.S.; Dellagi, S. Optimal Maintenance Policy for Equipment Submitted to Multi-Period Leasing as a Circular Business Model. *Sustainability* **2024**, *16*, 5238. <https://doi.org/10.3390/su16125238>

Academic Editor: Davide Settembre-Blundo

Received: 21 April 2024

Revised: 9 June 2024

Accepted: 12 June 2024

Published: 20 June 2024



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1. Introduction

While the rapid growth of technology has undoubtedly brought many benefits, it has also contributed to the shortening of the lifespan of products, especially electronic products [1]. This has led to the generation of a large amount of electrical and electronic waste, also recognized as e-waste [2]. Therefore, it is crucial to address and reduce the negative impact of the short life cycle of products through sustainable manufacturing, proper recycling, and responsible consumption practices. Adopting the principles of reduce, reuse, and recycle has a significant impact on creating a more sustainable and environmentally friendly future [3]. It is important for consumers to consider these factors when making purchasing decisions, and to be mindful of the environmental impact of constantly replacing products.

Due to environmental awareness, numerous studies have focused on the issue of recovery in a closed-loop supply chain (CLSC) system including both forward and reverse flows. In line with Guide and Wassenhove [4], CLSC aims primarily to take back products from customers and recover their value by reusing the entire product or some of its components. This approach not only reduces waste but also mitigates environmental harm,

aligning closely with the sustainability objectives of our study. A recent study presented by Zheng and Xu [5] highlighted the critical role of efficient CLSC management in sustainable waste power battery recycling. They offered insights for government policy aimed at promoting formal recycling of retired batteries and enhancing sustainability in the new energy vehicle sector.

When it comes to expensive equipment, leasing has proven to be a successful and efficient strategy that helps to improve the management of the return process [6]. As noted by Fleischmann et al. [7] in the case of IBM's closed-loop supply chain, equipment leasing can also serve as a catalyst for green initiatives. The equipment may be leased for a specified period of time, after which it becomes the property of the lessee. Alternatively, the lessee may return the equipment to the lessor after the lease period has elapsed, after which the equipment may be leased to other lessees for various periods over its lifetime. Consequently, leasing is in accordance with the tenets of the circular business model, as it facilitates the efficient use of resources by allowing multiple lessees to share the same equipment throughout its lifecycle. This reduces the environmental impact. A case study on the lease and refurbishment of baby strollers is presented by Sumter et al. [8]. The authors emphasized the shift from selling products to offering them following circular business models, such as the one considered in this work, which is also known as multi-period leasing.

Leases typically include a warranty. If the equipment does not perform as expected, the lessor must repair or replace it during the warranty period. By transferring the risk of equipment failure to the lessor, lessees can focus on their core business activities and reduce the financial burden of maintaining and replacing equipment. The warranty can be divided into a basic warranty (BW) and an extended warranty (EW). The latter is a service agreement that allows the lessee to extend coverage after the BW expires. Extended warranties are typically purchased separately and may cover repairs or replacement of the product at an additional cost. An overview on warranties is provided in [9].

Maintenance strategies are often included in lease contracts to mitigate the damage caused by equipment failure to manufacturers. By implementing effective maintenance strategies, the lessor may ensure that the leased equipment remains operational for longer periods of time. This contributes to the circular economy by prolonging the service life of equipment through maintenance, reducing the frequency of replacements, and promoting resource conservation by optimizing the utility of existing equipment. Several papers have been published in the context of leasing contracts that include maintenance and monitoring [10–13].

Maintenance is a critical component of the warranty. Much attention has been paid in the literature to the optimization of warranty and maintenance strategies [14–18]. However, studies that integrate warranty contracts and maintenance strategies for leased equipment are less common [19–21]. Most of them focus on choosing the best PM policy within a given leasing period and a predetermined warranty, which is also one of the objectives of our study. Moreover, our research delves into the idea of leasing over multiple periods.

The concept of multi-period leasing has a significant impact on the profitability of companies which produce or acquire a particular piece of equipment and intend to lease it to different users (lessees) for different durations during its life cycle. When equipment is leased for multiple periods, it becomes classified as used (second-hand) equipment at the end of each lease term. Generally, it is rented to the next lessee for a certain period with a warranty after it has been upgraded by the lessor to a specified level. Such an upgrade is performed preventively to help identify and address potential problems before they become major issues. This reduces the risk of equipment failure and costly repairs during the next lease term. Husniah et al. [22] developed a lease contract model for remanufactured products. They emphasized the roles of the lessor and lessee in deciding maintenance, pricing, and usage rates. Tlili et al. [23] developed an imperfect preventive maintenance model for equipment leased under a free leasing arrangement, where the lessor provides the equipment to the lessee at no cost for a specified duration, with the lessee agreeing to procure the equipment's consumables entirely from the lessor. Their aim was to find the

optimal timing and effectiveness of PM actions to adopt, enabling the rental company to maximize its profit by taking into account the equipment's usage rate, which corresponds to the consumption rate of its consumables. However, their profit model overlooked the warranty. Ben Mabrouk et al. [24] conducted a study on service companies in the oil and gas industry. These companies specialize in multi-period leasing of equipment such as separators, gas scrubbers, pumps and generators to oil companies engaged in time-limited production activities, including well testing, short production tests and gas recovery. The authors examined the profitability of multi-period leasing. For that, they assumed the same warranty period across all leasing periods. However, they did not consider the expected penalty cost, or the expected income earned during the post-warranty period when calculating the expected total profit of the lessor.

To our knowledge, no research has combined leasing with a variable warranty period throughout the lease terms over the equipment's lifecycle. This paper proposes such a combination and considers the possibility of offering an extended warranty to the lessee. In order to maximize the total expected profit over the equipment's lifecycle, the lessor must determine the optimal combination of maintenance levels to use between lease periods and the length of the extended warranty to provide.

In this study, the main research question was as follows: what is the financial impact of the new combination of preventive maintenance with variable warranty periods and extended warranty? More specifically, how would the profit of the lessor vary if the latter were to offer the possibility to perform more or less efficient preventive maintenance between lease periods, along with the possibility for the lessees to purchase an extended warranty. This research question corresponds to a question raised by a service company in the oil and gas industry in Tunisia. As an important part of its services, this company leases specific equipment (such as three-phase and two-phase separators, gas scrubbers, oil tanks, pumps, and generators) to oil companies for production activities in Tunisia and North and West Africa. Lease terms generally range from 1 to 4 years.

The remaining sections of the paper are organized as follows. Section 2 presents the mathematical model. In Section 3, a numerical example and a sensitivity analysis are presented. Finally, the conclusions and research perspectives are drawn in the last section.

The following notation will be adopted throughout the paper:

$r(t)$	rate of occurrence of failures with no PM;
n	number of lease periods;
T_i	duration of lease period i ($i = 1, \dots, n$);
L	length of the equipment's life cycle; $L = \sum_{i=1}^n T_i$;
w_i	warranty period of lease period i ($i = 1, \dots, n$);
w_{bi}	base warranty period of lease period i ($i = 1, \dots, n$);
w_{ei}	extended warranty period of lease period i ($i = 1, \dots, n$);
x_b	factor of proportionality for the base warranty period as a fraction of T_i ($0 < x_b < 1$);
x_e	factor of proportionality for the extended warranty period as a fraction of $T_i - w_{bi}$ ($0 < x_e < 1$);
τ_j	instant of the j th PM action ($j = 1, 2, \dots, n - 1$);
K	possible PM packages;
M	maximum level of maintenance effort;
m_k	PM level (integer decision variable) [$0 \leq m_k$ ($k = 1, 2, \dots, K$) $\leq M$];
A_j	equipment's virtual age after PM executed at instant τ_j ($j = 1, n - 1$);
N_i	expected number of minimal repairs during the lease period i ($i = 1, \dots, n$);
N_w	expected number of minimal repairs throughout the warranty durations for the n lease periods;
Nr_{pw}	expected number of minimal repairs to be performed over the post-warranty periods;
N_{Tmax}	maximum number of allowed failures indicated in the lease contract;
c_r	minimal repair cost;
c_p	penalty cost per unit;
c_{m_k}	cost of a PM action with level m_k ;
C_a	equipment acquisition cost;

c_{we}	extended warranty's price per time unit;
C_{rep}	expected total cost of minimal repairs during the warranty periods;
C_{PM}	expected total cost of PM actions during the lifecycle L ;
C_p	expected total penalty cost;
Re_L	expected total leasing revenue during the lifecycle L ;
Re_{pw}	expected total revenue generated during the post-warranty periods;
Re_{we}	expected revenue from the extended warranty;
PR	expected total profit during the lifecycle L ;
d_L	depreciation rate;
β	annual interest rate;
α_r	profit margin of each repair action carried out during the post-warranty period;
k_0	immediate decrease in re-lease value after the rent of a new equipment.

2. The Mathematical Model

We assume that a new piece of repairable equipment is leased for n periods of time during its lifecycle L . These leasing periods, T_i ($i = 1, 2, \dots, n$), include a base warranty period w_{bi} ($w_{bi} = x_b T_i$ where $0 < x_b < 1$). The option to purchase an extended warranty w_{ei} ($i = 1, 2, \dots, n$) is presented to the lessee for each lease period ($w_{ei} = x_e (T_i - w_{bi})$ where $0 < x_e < 1$).

For the proposed strategy (Figure 1), we assume that minimal repairs (with a negligible duration) are performed following failures that may occur throughout the warranty period [$w_i = w_{bi} + w_{ei}$ ($i = 1, 2, \dots, n$)]. These repairs will be carried out free of charge for the lessee, meaning that the lessor will be responsible for covering the cost of any necessary repair during the warranty period. This will relieve the lessee from the financial burden associated with unexpected maintenance expenses. During the post-warranty period, the lessor is in charge for all repairs, but the lessee will cover the costs plus an additional profit margin for the lessor. If the expected number of breakdowns during the lease period exceeds a certain threshold, the lessor incurs a penalty cost.

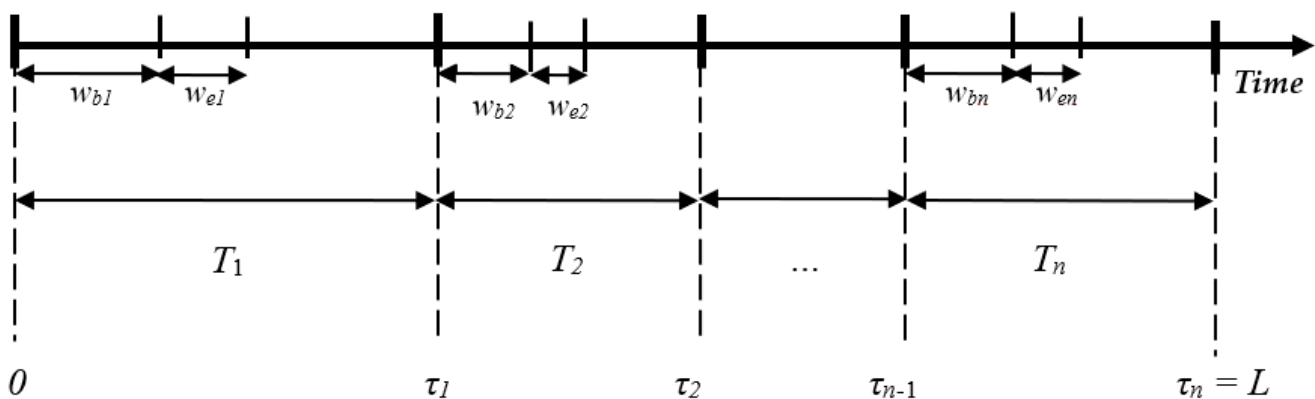


Figure 1. The proposed model over the equipment's life cycle.

The lessor performs preventive maintenance actions (with a negligible duration) at discrete time instants τ_j ($j = 1, 2, \dots, n - 1$) with $\tau_0 = 0$ at the end of each lease period (excluding the last one). Once the equipment has reached the end of its useful life, it can be disassembled in order to recover the expensive components, which can then be used again to maintain other similar equipment.

PM actions induce age reduction based on the corresponding upgrade level m_k . The degree to which a PM action revitalizes the equipment serves as an indicator for its effectiveness. The level m_k that corresponds to each PM can be any of the K potential maintenance packages ($k = 1, 2, \dots, K$), with a cost of c_{m_k} . A package is associated with a collection of maintenance tasks (e.g., component overhauls, cleaning, lubrication, and/or replacement of various sets of components).

The PM level depends on the applied maintenance package. This provides more flexibility to the decision makers, as they are not limited to applying the same maintenance level in all PMs. Hence, it is necessary to consider all possible combinations of PM levels m_k ($k = 1, 2, \dots, K$) for each of the $(n - 1)$ PM actions (K^{n-1} possible combinations).

Only one PM level is chosen for any particular PM action:

$$\sum_{k=1}^{k=K} X_{k,j} = 1 \text{ for } (j=1, \dots, n-1) \quad (1)$$

where $X_{k,j}$ is a binary variable:

$$X_{k,j} = \begin{cases} 1 : \text{PM level } m_k \text{ is applied in the } j\text{th PM} \\ 0 : \text{otherwise} \end{cases}$$

We implement the same modeling approach as in [24–26] to model the equipment's virtual age. The virtual age A_j after the j th PM can be obtained as follows:

$$A_j = A_{j-1} + \sum_{k=1}^{k=K} \delta(m_k) X_{k,j} (\tau_j - \tau_{j-1}) \text{ with } A_0 = 0 \text{ (new equipment) } (j = 1, \dots, n-1) \quad (2)$$

where $\delta(m_k)$ is a decreasing function of m_k provided by:

$$\delta(m_k) = (1 + m_k) e^{-m_k} \quad \text{with} \quad 0 \leq \delta(m_k) \leq 1 \quad (3)$$

In order to optimize the total expected profit of the lessor during the equipment's lifecycle, we aim to simultaneously find the optimal levels m_k ($k = 1, \dots, K$) of PM to be made on the rented equipment at the end of each of the $(n - 1)$ lease periods, and the extended warranty periods to be provided for each lease period.

That is, our model aims to allow the lessor, for any given configuration of costs and revenues and equipment reliability, to trade off, on one hand, the cost of repairs during the warranty periods, the cost of PM actions and the penalty cost with, on the other hand, the revenue derived from extended warranty sales, leasing fees, and maintenance services provided after the warranty has expired.

To do this, the following input parameters are considered: the equipment acquisition cost C_a , the average total cost C_{rep} of a minimal repair, the average total cost of PM actions C_{PM} which depends on the maintenance level to be adopted, the expected total penalty cost C_p , the expected total leasing revenue Re_L , the expected total revenue Re_{pw} generated during the post-warranty periods related to repairs performed and billed, and the average revenue Re_{we} from the extended warranty offered to the lessee.

2.1. The Expected Total Cost C_{rep} of Minimal Repairs during the Warranty Periods

During each lease period, if the equipment malfunctions during the warranty period, the lessor incurs an average cost denoted c_r .

C_{rep} is expressed as follows:

$$C_{rep} = c_r N_w \quad (4)$$

where N_w is the expected number of minimal repairs during the warranty periods. It can be obtained by:

$$N_w = \int_0^{w_1} r(t) dt + \sum_{j=1}^{n-1} \int_{\tau_j}^{\tau_j + w_{j+1}} r(A_j + t - \tau_j) dt \quad (5)$$

2.2. The Expected PM Total Cost C_{PM}

The following is the expression of the expected total cost of PM actions executed across the equipment life cycle:

$$C_{PM} = \sum_{j=1}^{n-1} X_{k,j} c_{m_k} \quad (6)$$

2.3. The Expected Total Penalty Cost C_P

The lessor pays a penalty cost C_{Pi} in case the expected number of failures (minimal repairs) N_i during the lease period i ($i = 1, \dots, n$) exceeds a prespecified threshold N_{Tmax} that both parties agreed upon when establishing the contract. This penalty cost is given by:

$$C_{Pi} = \varphi c_p (N_i - N_{Tmax}) \quad (7)$$

where φ is the following binary variable:

$$\varphi = \begin{cases} 1 & \text{if } N_i - N_{Tmax} > 0 \\ 0 & \text{otherwise} \end{cases}$$

The expected number of minimal repairs N_1 throughout the first lease period is provided by:

$$N_1 = \int_0^{T_1} r(t) dt \quad (8)$$

The expected number of minimal repairs N_{i+1} ($i = 1, \dots, n - 1$) during the lease period $i + 1$ is given by:

$$N_{i+1} = \int_{\tau_i}^{\tau_i + T_{i+1}} r(A_i + t - \tau_i) dt \quad (9)$$

The expressed equation for the expected total penalty cost C_P is as follows:

$$C_P = \sum_{i=1}^n C_{Pi} \quad (10)$$

2.4. The Expected Total Leasing Revenue Re_L

Re_L over the equipment lifecycle is given by the following expression [24]:

$$Re_L = 12 \left(P(A_0)T_1 + P(A_1)T_2 + \sum_{i=2}^{n-1} P(A_i)T_{i+1} \right) \quad (11)$$

$P(A_0)$ represents the monthly lease income over the first lease period. It is calculated as follows:

$$P(A_0) = f_L C_a = \left(\frac{d_L}{12L} + \left(1 - \frac{d_L}{2}\right) \frac{\beta}{12} \right) C_a \quad (12)$$

where f_L is a factor that is often used in the leasing business (see [6]).

The first term of Equation (12) is the number of monthly payments related to depreciation, and the second term is due to financing.

$P(A_1)$ of Equation (11) is the monthly lease revenue during the second lease period based on the virtual age A_1 [27]. It is given by:

$$P(A_1) = k_0 P(A_0) \left(1 - \frac{A_1}{L}\right) \quad \text{where } 0 < k_0 < 1 \quad (13)$$

$P(A_i)$ in Equation (11), given by the following equation, is the monthly lease revenue during periods $i + 1$ ($i = 2, n - 1$):

$$P(A_i) = P(A_0) \left(1 - \frac{A_i}{L}\right) \quad (14)$$

This equation translates the fact that the lower the virtual age at the start of the next period (more reliable equipment), the greater the lease revenue in that period.

2.5. The Expected Revenue Re_{pw} Generated during the Post-Warranty Periods

Repairs are always performed by the lessor throughout the post-warranty periods, but the lessee is responsible for paying for them. The lessor will apply a profit margin, α_r .

The expected number of minimal repairs Nr_{pw} to be performed during the post-warranty periods can be found by:

$$Nr_{pw} = \sum_{i=1}^n N_i - N_w \quad (15)$$

Re_{pw} is expressed as follows:

$$Re_{pw} = Nr_{pw} c_r \alpha_r \quad (16)$$

2.6. The Expected Revenue Re_{we} from the Extended Warranty

The extended warranty is sold to the lessee. Its price is based on the length of the extended coverage considering a cost per time unit c_{we} . The expected revenue Re_{we} from the extended warranty is given by:

$$Re_{we} = c_{we} \sum_{i=1}^n we_i \quad (17)$$

2.7. The Expected Total Profit of the Lessor

The expected total profit PR across the equipment lifecycle can be expressed as follows using the previously developed equations:

$$PR = 12 \left(P(A_0)T_1 + P(A_1)T_2 + \sum_{i=2}^{n-1} P(A_i)T_{i+1} \right) + Nr_{pw} c_r \alpha_r + c_{we} \sum_{i=1}^n we_i - c_r N_w - \sum_{j=1}^{n-1} X_{k,j} c_{m_k} - \sum_{i=1}^n C_{Pi} - C_a \quad (18)$$

Our objective is to simultaneously find the optimal efficiency levels of PM to be carried out between consecutive lease periods and the optimal extended warranty periods to be offered to lessees, thereby maximizing the expected total profit for the lessor.

In order to solve the problem and to determine the optimal scenario among the possible K^{n-1} scenarios along with the extended warranty periods, a numerical approach is adopted. It consists in an iterative procedure that considers varying the proportionality factor x_e for the extended warranty period ($we_i = x_e (T_i - w_{bi})$). The procedure is initialized with $x_e = 0$ (i.e., no extended warranty is offered). At each iteration with a value of x_e (x_e is incremented by Δx_e until it reaches 1), the procedure computes the expected total profit given by Equation (18) for each of the K^{n-1} PM scenarios. The value of x_e and the PM scenario that yields the greatest expected total profit is selected as the optimal solution.

This iterative procedure has been tested with multiple examples of combinations of input parameters. In the following section, one of these examples is presented to prove the use of the proposed model.

3. Numerical Example

We examine a repairable piece of equipment leased for $n = 5$ periods of time during its lifecycle $L = 15$ years. It is a three-phase separator leased to oil companies in Tunisia and North Africa by a Tunisian service company that leases and maintains equipment such as separators, generators, gas scrubbers, oil tanks, etc. Preventive maintenance is performed at the end of lease periods i ($i = 1, 2, \dots, 4$) at discrete time instants τ_1, τ_2, τ_3 and τ_4 . As no PM is carried out at the end of the equipment lifecycle, the end of the fifth period is not included.

The time to failure of the equipment follows a Weibull probability distribution with a shape parameter $\alpha = 2$ and a scale parameter $\lambda = 1.5$. The relating mean time to failure (MTTF) is 1.33 years.

The failure rate function is given by:

$$r(t) = \left(\frac{\alpha}{\lambda}\right) \left(\frac{t}{\lambda}\right)^{\alpha-1}$$

For illustration purposes, the following input data are considered (Table 1). These data are not exactly the real data of the industry concerned. Some modifications have been made for confidentiality reasons.

Table 1. Input data.

$C_a = \text{USD } 18,000$; $c_r = \text{USD } 200$; $c_p = \text{USD } 60$; $c_{we} = 420$; $k_0 = 0.95$; $d_L = 0.4$; $\beta = 0.09$
$K = 6$; $M = 5$; $N_{Tmax} = 2$ failures; $\alpha_r = 0.4$; $x = 0.25$ (warranty period is set to the fourth of each lease period T_i); $\Delta x_e = 0.05$
$T_1 = 4$ years, $T_2 = 2$ years, $T_3 = 3$ years, $T_4 = 4$ years and $T_5 = 2$ years

Table 2 shows the possible PM packages ($k = 1, 2, \dots, 6$) including their respective PM levels m_k , improvement factors $\delta(m_k)$ and costs c_{m_k} .

Table 2. PM levels with their corresponding improvement factors $\delta(m_k)$ and costs c_{m_k} .

k	PM Level (m_k)	$\delta(m_k)$	c_{m_k} (USD)
1	0	1 (No age reduction)	0
2	1	0.74	150
3	2	0.41	300
4	3	0.2	550
5	4	0.09	720
6	5	0.04	900

Using the input data presented above, the iterative procedure has been run considering the 1296 (6^4) possible combinations of the six PM levels that can be performed at instants τ_j ($j = 1, 2, 3, 4$).

The obtained optimal combination that produces the highest expected profit over the equipment lifecycle is as follows:

$$\begin{matrix}
 k & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\
 \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
 \end{matrix}$$

$$(k, \tau_j) = \begin{cases} 1 : \text{means that PM package } k \text{ (PM level } m_k) \text{ is adopted at the end of the } j\text{th lease period} \\ 0 : \text{otherwise} \end{cases}$$

It is interesting to note that the optimal solution does not suggest implementing the same maintenance efficiency level for all PM actions. In fact, the best PM strategy suggests initiating with a high PM level and then gradually adopting smaller PM levels as the lease periods follow each other.

Simultaneously, the optimal values of the extended warranty periods have been obtained. They correspond to the optimal factor of proportionality $x_e^* = 0.1$. They are given in Table 3 below.

Table 3. The optimal extended warranty periods (years).

w_{e1}	w_{e2}	w_{e3}	w_{e4}	w_{e5}
0.3	0.15	0.225	0.3	0.15

Table 4 below presents the profit PR^* and its components.

Table 4. The different components of the optimal expected profit (USD).

PR^*	4806.6
Re_L	25,674.7
Re_{pw}	1113.9
Re_{w_e}	472.5
C_a	18,000
C_{rep}	845.4
C_{PM}	2890
C_P	719.1

Below is a sensitivity analysis showing how changing a few key input parameters affects the optimal strategy.

3.1. Effect of the Variation of the Base Warranty Factor of Proportionality x_b on the Optimal Strategy

As shown in Table 5, if the lessor offers longer base warranty periods (higher factor of proportionality x_b), the expected total profit is reduced. This can be attributed to two factors: the increase in the expected total cost of repairs and the decrease in the expected revenue generated during the post-warranty periods. One can also observe that with longer base warranty durations, the best strategy recommends adopting higher maintenance levels earlier in the equipment lifecycle and offering shorter extended warranty durations. With a base warranty of half the lease term, the optimal strategy is to not offer an extended warranty in all terms.

3.2. Effect of the Scale Parameter λ of the Time to Failure Probability Distribution

Keeping all other parameters at their initial values, we examined the impact of varying λ (more or less reliable equipment). Looking at Table 6, in case λ increases (more reliable equipment), the best policy proposes adopting lower maintenance levels earlier in the equipment lifecycle (there is less need for high PM levels). Also, the expected total penalty cost decreases to zero for an increase of MTTF from 0.89 years to 3.1 years, which causes an increase in the expected total profit. It is also clear that, as expected, an increase in the

equipment's reliability engenders an increase in the extended warranty durations, which significantly increases the expected income from the sale of extended warranties.

Table 5. Effect of the factor of proportionality x_b on the optimal strategy.

	$x_b = 0$ (No Warranty)	$x_b = 0.25$	$x_b = 0.5$
PR^* (USD)	6379.1	4806.6	3182.1
Re_L (USD)	25,674.7	25,674.7	25,994.7
Re_{pw} (USD)	1185.6	1113.9	576.6
Re_{w_e} (USD)	1890	472.5	0
C_{rep} (USD)	747.6	845.4	1441.5
C_{PM} (USD)	2890	2890	3420
C_P (USD)	733.6	719.1	527.7
$w_{e_i} [i = 1, \dots, n]$ (years)	[1.2, 0.6, 0.9, 1.2, 0.6]	[0.3, 0.15, 0.225, 0.3, 0.15]	[0, 0, 0, 0, 0]
Corresponding PM strategy	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Table 6. Effect of λ on the optimal strategy.

	$\lambda = 1$ (MTTF = 0.89 Years)	$\lambda = 1.5$ (MTTF = 1.33 Years)	$\lambda = 3.5$ (MTTF = 3.1 Years)
PR^* (USD)	4173	4806.6	8492
Re_L (USD)	25,994.7	25,674.7	25,478.8
Re_{pw} (USD)	2681	1113.9	0
Re_{w_e} (USD)	0	472.5	4725
C_{rep} (USD)	1009.2	845.4	1071.8
C_{PM} (USD)	3420	2890	2640
C_P (USD)	2073.5	719.1	0
$w_{e_i} [i = 1, \dots, n]$ (years)	[0, 0, 0, 0, 0]	[0.3, 0.15, 0.225, 0.3, 0.15]	[3, 1.5, 2.25, 3, 1.5]
Corresponding PM strategy	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

3.3. Effect of the Variation of the Price (per Month) c_{we} of the Extended Warranty on the Optimal Solution

Table 7 shows that as the price per month of the extended warranty increases, the duration of that warranty also increases significantly. As a result, both the expected income from the sale of the extended warranty and the expected total cost of repair actions over the warranty period increase. Thus, logically, there is necessarily a decrease in the expected revenue from post-warranty repairs. We can also see that the lessor's best strategy is not to offer extended warranties ($w_{ei} = 0$) if the warranty extension is offered at a low price (case of USD 210 per month).

Table 7. Effect of c_{we} on the optimal strategy.

	$c_{we} = \text{USD } 210\text{Month}$	$c_{we} = \text{USD } 420\text{Month}$	$c_{we} = \text{USD } 840\text{Month}$
PR^* (USD)	4783.4	4806.6	8045.7
Re_L (USD)	25,674.7	25,674.7	25,994.7
Re_{pw} (USD)	1335.7	1113.9	0
Re_{w_e} (USD)	0	472.5	9450
C_{rep} (USD)	568.6	845.4	5060.8
C_{PM} (USD)	2890	2890	3420
C_P (USD)	768.4	719.1	918.2
$we_i [i = 1, \dots, n]$ (years)	[0, 0, 0, 0, 0]	[0.3, 0.15, 0.225, 0.3, 0.15]	[3, 1.5, 2.25, 3, 1.5]
Corresponding PM strategy	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

3.4. Effect of the Variation of the Profit Margin α_r of Each Repair Action Executed throughout the Post-Warranty Period

Table 8 illustrates how the length of the warranty substantially reduces as the profit margin increases. Both the expected total cost of repairs within the warranty period and the expected revenue from the sale of the extended warranty decrease as a result of this. Consequently, the expected income generated during the post-warranty periods must increase. It is clear that in cases where the profit margin is high, the lessor's optimal strategy would be to adopt lower maintenance levels early in the equipment lifecycle and not to propose extended warranties.

Table 8. Effect of α_r on the optimal strategy.

	$\alpha_r = 0.2$	$\alpha_r = 0.4$	$\alpha_r = 0.8$
PR^* (USD)	4470.9	4806.6	6187.8
Re_L (USD)	25,994.7	25,674.7	25,478.8
Re_{pw} (USD)	275.8	1113.9	2847.9
Re_{w_e} (USD)	1653.7	472.5	0
C_{rep} (USD)	1505.4	845.4	642.2
C_{PM} (USD)	3420	2890	2640
C_P (USD)	527.9	719.1	856.7
$we_i [i = 1, \dots, n]$ (years)	[1.05, 0.525, 0.7875, 1.05, 0.525]	[0.3, 0.15, 0.225, 0.3, 0.15]	[0, 0, 0, 0, 0]
Corresponding PM strategy	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

3.5. Effect of the Variation of the Penalty Cost per Unit c_p on the Optimal Solution

The obtained results in Table 9 reveal that when the contract specifies larger penalties c_p , the optimal approach for the lessor is to implement increased maintenance levels earlier in the equipment lifecycle. This helps prevent surpassing the maximum allowable failures

and incurring substantial penalties. Additionally, it is observed that the lessor's optimal choice is to refrain from providing extended warranties if the penalty cost is reduced. As a result, this leads to increased expected income generated throughout the post-warranty periods.

Table 9. Effect of the variation of c_p on the optimal strategy.

	$c_p = \text{USD } 10$	$c_p = \text{USD } 60$	$c_p = \text{USD } 100$
PR^* (USD)	5477.8	4806.6	4434.1
Re_L (USD)	25,478.8	25,674.7	25,994.7
Re_{pw} (USD)	1423.9	1113.9	709.9
Re_{w_e} (USD)	0	472.5	1181.2
C_{rep} (USD)	642.2	845.4	1142.2
C_{PM} (USD)	2640	2890	3420
C_P (USD)	142.7	719.1	889.5
$w_{e_i} [i = 1, \dots, n]$ (years)	[0, 0, 0, 0, 0]	[0.3, 0.15, 0.225, 0.3, 0.15]	[0.75, 0.375, 0.5625, 0.75, 0.375]
Corresponding PM strategy	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

4. Conclusions

In this work, we considered a piece of equipment leased multiple times over its life cycle with a base warranty and a probable extended warranty offered for each lease period. The warranty periods are not uniform. A mathematical model has been established to find the optimal extended warranty durations and the optimal levels of PM to be executed by the lessor in between consecutive lease periods at the same time throughout the equipment lifecycle. The rental periods have different durations, and they comprise variable warranty periods. A numerical example and a sensitivity study have been offered and discussed.

The obtained results illustrated the applicability and the consistency of our model. The sensitivity analysis results highlighted how variations in specific input parameters impact the optimal strategy for the lessor throughout the equipment's lifecycle. We concluded that the optimal PM strategy will generally follow this pattern: start with a high PM level and then gradually move to lower PM levels as lease periods progress toward the end of the equipment lifecycle. This conclusion is based on the presented example relative to the three-phase separator, but also on several other pieces of equipment for which we applied the model. As expected, we found that the condition where the factor of proportionality $x_b = 0$ (no base warranty is offered) yields the highest profit. However, the lessor is obliged to provide a warranty for obvious marketing reasons.

The proposed mathematical model is generic. It is suitable not only for equipment related to the oil and gas industry which inspired this work but also for a large variety of equipment including but not limited to vehicles, trucks, ships and different kinds of machinery and production systems leased by manufacturing and service companies.

This work can be expanded for further investigation to include different types of warranty policies, such as pro-rata or free replacement. More practical settings, including PM and repairs with non-negligible durations can be considered for the maintenance policy. A valuable enhancement to this study would involve considering the lessee's income generated from the equipment.

Author Contributions: Conceptualization, A.B.M. and A.C.; Methodology, M.S.A. and S.D.; Software, A.B.M. and M.S.A.; Validation, A.B.M. and A.C.; Formal analysis, A.B.M. and S.D.; Writing – original draft, A.B.M. and A.C.; Writing – review & editing, A.B.M. and A.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author/s.

Conflicts of Interest: The authors declare no conflict of interest.

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