

## Article

# An Extended EWMA Control Chart with Multiple Dependent State Sampling for COM–Poisson Processes and Its Application to Air Quality Index Monitoring Data

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## Abstract

Traditional attribute control charts for defect counts are commonly developed under the assumption that count data follow a homogeneous Poisson distribution. However, this assumption is often violated in practical applications. To overcome this limitation, a two-parameter Poisson distribution, the Conway–Maxwell–Poisson (CMP or COM–Poisson) distribution, has been widely used to construct control charts capable of effectively monitoring count data exhibiting over- or under-dispersion. Furthermore, the multiple dependent state (MDS) sampling scheme evaluates the current process status not only based on the present sample but also by incorporating information from previous samples, thereby achieving higher detection efficiency than single sampling schemes. This study integrates the COM–Poisson distribution with the MDS sampling strategy to develop an attribute control chart based on the extended exponentially weighted moving average statistic. The average run length is obtained under various shift magnitudes using probability-based computations. The simulation results demonstrate that the proposed chart substantially outperforms existing approaches in the prompt detection of out-of-control conditions. A real-world air quality index (AQI) monitoring study showed that the proposed chart effectively detected increases in weekly AQI counts and provided earlier warnings of potential air quality deterioration.

**Keywords:** average run length; count data; COM–Poisson distribution; extended EWMA control chart; multiple dependent state sampling

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## 1. Introduction

Control charts are the core tools of statistical process control. They are designed to monitor process conditions in real time through statistical methods, reduce variability in quality characteristics, and thus improve process stability. According to the quality characteristic type, Shewhart classified control charts into two major categories: variable and attribute charts. Variable charts deal with continuous data, such as length, weight, and time, with typical examples including  $\bar{X}$ - $R$  and  $\bar{X}$ - $S$  charts. Attribute charts handle categorical or count data, such as the number of conforming and nonconforming items.

Common charts of this type include the  $p$ ,  $np$ ,  $c$ , and  $u$  charts, which are based on binomial or Poisson distributions. However, conventional Shewhart charts rely primarily on a single sample at each sampling epoch, which limits their sensitivity to small and moderate process shifts and does not fully exploit historical information. Roberts [1] proposed the exponentially weighted moving average (EWMA) chart, which combines current and past observations through a recursive weighting scheme to enhance the detection of small shifts. Its extension, the extended EWMA (EEWMA) chart, can further reduce the average run length (ARL) under out-of-control conditions by appropriately selecting smoothing parameters, thereby improving process monitoring responsiveness and efficiency. Subsequently, Naveed et al. [2] proposed an EEWMA control chart that, when appropriate smoothing parameters are chosen, can reduce the out-of-control ARL, thereby enhancing process monitoring responsiveness and efficiency.

Traditional attribute control charts for count data are constructed under the Poisson assumption that the mean is equal to the variance, which is often violated in practice because of over- or under-dispersion. Conway and Maxwell [3] addressed this issue by proposing the two-parameter Conway–Maxwell–Poisson (COM–Poisson) distribution, which can accommodate both over- and under-dispersed data and includes Geometric, Poisson, and Bernoulli distributions as special cases. Building on this framework, Saghir and Lin [4] developed the Generalized EWMA (GEWMA) control chart for monitoring COM–Poisson count data and empirically demonstrated that it yields superior ARL performance compared to the Sellers [5] control chart. Aslam et al. [6] applied a repetitive sampling scheme to the GEWMA control chart for the COM–Poisson distribution and found that the proposed chart can outperform the conventional GEWMA chart by yielding smaller ARL values. Aslam et al. [7] proposed a modified EWMA control chart for processes that follow a COM–Poisson distribution and demonstrated that the proposed chart detects out-of-control conditions more rapidly than existing control charts, as evidenced by improved ARL performance. Alevizakos and Koukouvinos [8] introduced a double EWMA control chart for monitoring COM–Poisson attribute data, referred to as the CMP-DEWMA chart, and showed that this chart outperforms the standard EWMA chart in detecting downward shifts in the process mean. Adeoti et al. [9] proposed a homogeneously weighted moving average (HWMA) control chart for COM–Poisson-distributed count data, called the CMP-HWMA chart. Their findings indicated that the proposed chart performs competitively with existing control charts in detecting shifts in both location and dispersion parameters. Alevizakos and Koukouvinos [10] employed the progressive mean (PM) statistic to develop a CMP-PM chart for COM–Poisson count data. Simulation results indicated that the CMP-PM chart outperforms the Sellers, GEWMA, and  $\mu$ -CUSUM charts across nearly all shift magnitudes for over- and under-dispersed data.

Furthermore, the classical single sampling scheme constructs the control chart statistic using only the current sample and determines whether a process is out of control without utilizing information from previous samples. Achieving earlier detection through the adoption of alternative sampling schemes, including repetitive, ranked set, sequential, and multiple dependent state (MDS) sampling, can also improve the effectiveness of control charts. Wortham and Baker [11] proposed the MDS sampling plan that incorporates information from previous samples into the decision rule, making it more efficient than the single sampling plan. Several subsequent studies have integrated the MDS sampling scheme into various distributions and control chart designs. Balamurali and Jun [12] and Aslam et al. [13–15] have provided additional studies demonstrating the application of MDS. Aslam et al. [13] introduced the application of the MDS sampling concept to control chart development, proposing a new  $\bar{X}$  chart with double control limits based on the MDS sampling procedure. The proposed chart demonstrated superior performance to the

traditional  $\bar{X}$  chart in terms of ARLs. Aslam et al. [16] proposed a control chart for exponentially distributed quality characteristics using the MDS sampling scheme, which features double control limits with two control limit constants and is derived from a normal approximation achieved via transformation. Aslam et al. [17] proposed an MDS-based control chart for gamma distribution. Aldosari et al. [18] used successive sampling along with MDS repetitive sampling to develop a control chart for monitoring the process mean. Arshad et al. [19] designed an MDS-based control chart to track the changes in process variance. Naveed et al. [20] presented an EEWMA control chart based on MDS sampling to pinpoint slight variation in the process mean, which demonstrated substantially superior performance in rapidly detecting small shifts in the process mean.

Regarding the design of control charts for count data combined with the MDS sampling plan, Aslam et al. [21] proposed an attribute control chart based on the MDS technique, in which current and previous sample information are utilized to enhance performance. Aldosari et al. [22] combined repetitive sampling and the MDS sampling plan to develop a control chart for count data. Zhou et al. [23] employed the MDS sampling plan to construct a joint-adaptive  $np$  control chart. Aslam et al. [14] developed an attribute EWMA control chart to monitor nonconformities following the COM–Poisson distribution using the MDS scheme. Their ARL performance analysis demonstrated that the proposed chart can detect out-of-control conditions more quickly than the GEWMA chart proposed by Saghir and Lin [4].

Based on the above literature, memory-type control charts have evolved from the traditional EWMA framework to EEWMA, MEWMA, HWMA, and DEWMA structures to improve small-shift detection, while the COM–Poisson distribution has become a flexible model for monitoring equi-dispersed, over-dispersed, and under-dispersed count data. Although several COM–Poisson based control charts have been developed, existing studies have mostly considered the COM–Poisson distribution, memory-type statistics, and dependent-state sampling strategies separately. Therefore, a unified monitoring framework that integrates the COM–Poisson distribution, the EEWMA statistic, and the MDS sampling mechanism remains insufficiently explored. To address this methodological gap, this study first employs the EEWMA statistic to develop a CMP-EEWMA control chart for count data following the COM–Poisson distribution and then incorporates the MDS sampling scheme to propose the MDS-CMP-EEWMA chart. Simulation studies are conducted in the R program to compare ARL performance and small-shift detection capability under different choices for the additional smoothing parameter and various settings of the location parameter.

The remainder of this paper is organized as follows. Section 2 presents the design of the proposed MDS-CMP-EEWMA chart. Section 3 describes the computation of the ARL. Section 4 describes the performance of the proposed chart and compares it with existing control charts. Section 5 presents a comparative study. Section 6 proposes a case study regarding air quality index monitoring. Finally, Section 7 provides concluding remarks.

## 2. Designing the Proposed Control Charts

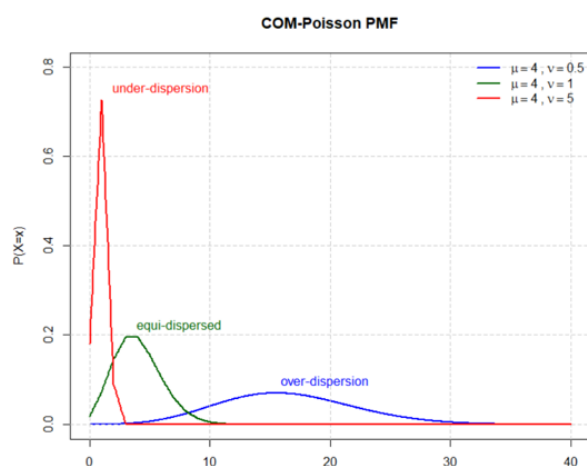
### 2.1. The COM–Poisson Distribution

Conway and Maxwell [3] introduced the COM–Poisson distribution as a flexible discrete probability distribution that generalizes the Poisson distribution by incorporating an additional dispersion parameter. This formulation effectively addresses the issues of over- and under-dispersion, which are commonly observed in count data. Let  $X$  be a random variable following the COM–Poisson distribution; its probability mass function is defined as:

$$P(x; \mu, v) = \frac{\mu^x}{(x!)^v} \cdot \frac{1}{z(\mu, v)}, \text{ for } x = 0, 1, 2, \dots, \quad (1)$$

where  $\mu (>0)$  and  $v (>0)$  denote the location and dispersion parameters, respectively. Figure 1 illustrates the data patterns for three different values of the dispersion parameter. When  $v < 1$ , the distribution exhibits over-dispersion; when  $v = 1$ , it reduces to the equi-dispersed Poisson distribution; and when  $v > 1$ , it represents under-dispersion. The nor-

malizing constant  $z(\mu, v) = \sum_{j=0}^{\infty} \frac{\mu^j}{(j!)^v}$  ensures that the probability mass function sums to one. Table 1 lists several well-known probability distributions that appear as special cases of the COM-Poisson distribution.



**Figure 1.** Three probability distributions associated with the COM-Poisson distribution when same  $\mu$  and different  $v$ .

According to Conway and Maxwell [3], if  $X$  is a random variable following the COM-Poisson distribution, its mean and variance can be expressed as:

$$E(X) = \mu \frac{\partial \log(z(\mu, v))}{\partial \mu} \approx \mu^{1/v} - \frac{v-1}{2v} \quad (2)$$

$$Var(X) = \frac{\partial E(X)}{\partial \log \mu} \approx \frac{1}{v} \cdot \mu^{1/v} \quad (3)$$

When  $v \leq 1$  or  $\mu > 10^v$ , Equations (2) and (3) provide good approximations of the mean and standard deviation (Shmueli et al., [24]). For the given values of  $\mu$  and  $v$ , the mean and variance of the COM-Poisson distribution can be computed using the *compoisson* package in R. However, when the parameters  $\mu$  and  $v$  are unknown, maximum likelihood estimation can be used for the mean and variance (Sellers, [5]).

**Table 1.** Three probability distributions associated with the COM-Poisson distribution.

| Conditions             | $z(\mu, v)$                                      | $P(x; \mu, v)$              | Distribution |
|------------------------|--------------------------------------------------|-----------------------------|--------------|
| $\mu < 1, v = 0$       | $\sum_{j=0}^{\infty} \mu^j = \frac{1}{1-\mu}$    | $\mu^x (1-\mu)$             | Geometric    |
| $v = 1$                | $\sum_{j=0}^{\infty} \frac{\mu^j}{j!} = e^{\mu}$ | $\frac{e^{-\mu} \mu^x}{x!}$ | Poisson      |
| $v \rightarrow \infty$ | $1 + \mu$                                        | $\frac{\mu}{1+\mu}$         | Bernoulli    |

## 2.2. The CMP-EEWMA Chart

Naveed et al. [2] first proposed the EEWMA control chart to monitor shifts in the process mean. Motivated by their work, this study extends the EEWMA framework to the COM-Poisson distribution and develops the CMP-EEWMA control chart. At time  $t$ , the corresponding statistic  $E_t$  is defined as:

$$E_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1 - \lambda_1 + \lambda_2) E_{t-1}, \quad t = 1, 2, 3, \dots, \quad (4)$$

the general expression of  $E_t$  is given by:

$$E_t = \lambda_1 X_t + b X_{t-1} + ab X_{t-2} + a^2 b X_{t-3} + \dots + a^{t-2} b X_1 - a^{t-1} \lambda_2 X_0 + a^t E_0, \quad (5)$$

where  $a = (1 - \lambda_1 + \lambda_2)$ ,  $b = a\lambda_1 - \lambda_2$ , and  $X_t$  denotes the observation from COM-Poisson distribution at time  $t$ . The derivation of Equation (5) is presented in Appendix A.  $\lambda_1$  and  $\lambda_2$  are smoothing parameters that satisfy  $0 < \lambda_1 \leq 1$  and  $0 \leq \lambda_2 < \lambda_1$ . The location parameter of the COM-Poisson distribution is assumed to be  $\mu_0$  when the process is in

control. The initial  $X_0$  and  $E_0$  values are set to  $\mu_0^{1/v} - \frac{v-1}{2v}$ . When  $\lambda_2 = 0$ , the CMP-EEWMA chart reduces to the classical GEWMA chart proposed by Saghir and Lin [4]. For consistency, we subsequently refer to the GEWMA chart as the CMP-EWMA chart. The expected value and variance of Equation (5) can be expressed as:

$$E(E_t) = \mu_0^{1/v} - \frac{v-1}{2v}, \quad (6)$$

and

$$\text{Var}(E_t) = \left\{ \left( \lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2 \right) \left( \frac{1 - (1 - \lambda_1 + \lambda_2)^{2(t-1)}}{1 - (1 - \lambda_1 + \lambda_2)^2} \right) \right\} \cdot \frac{\mu_0^{1/v}}{v} \quad (7)$$

The derivation of Equations (6) and (7) are presented in Appendix B. The upper and lower control limits as well as the center line of the CMP-EEWMA chart are given by:

$$\begin{aligned} UCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} + k_1 \sqrt{\left( \lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2 \right) \left( \frac{1 - (1 - \lambda_1 + \lambda_2)^{2(t-1)}}{1 - (1 - \lambda_1 + \lambda_2)^2} \right) \cdot \frac{\mu_0^{1/v}}{v}} \\ CL &= \mu_0^{1/v} - \frac{v-1}{2v} \\ LCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} - k_1 \sqrt{\left( \lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2 \right) \left( \frac{1 - (1 - \lambda_1 + \lambda_2)^{2(t-1)}}{1 - (1 - \lambda_1 + \lambda_2)^2} \right) \cdot \frac{\mu_0^{1/v}}{v}} \end{aligned} \quad (8)$$

As time  $t \rightarrow \infty$ , the control limits converge to steady-state values, which are given by:

$$\begin{aligned} UCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} + k_1 \sqrt{\left( \lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2 \right) \cdot \frac{\mu_0^{1/v}}{v}} \\ CL &= \mu_0^{1/v} - \frac{v-1}{2v} \\ LCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} - k_1 \sqrt{\left( \lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2 \right) \cdot \frac{\mu_0^{1/v}}{v}} \end{aligned} \quad (9)$$

where  $k_1$  indicates a control limit constant of the CMP-EEWMA chart. The CMP-EEWMA statistic  $E_t$  exceeding  $UCL_1$  or falling below the  $LCL_1$  indicates an out-of-control process, and appropriate corrective actions must be taken to return the process to a stable state.

### 2.3. The MDS-CMP-EEWMA Chart

Aslam et al. [14] proposed an attribute EWMA control chart based on the MDS scheme for monitoring nonconformities modeled by the COM–Poisson distribution (hereafter referred to as the MDS-CMP-EWMA chart). The COM–Poisson distribution can characterize different types of count data through its dispersion parameter. For over-dispersed data, observations exhibit greater natural process variation, so small process shifts may be masked by random fluctuations. In contrast, for under-dispersed data, observations are more concentrated, and small shifts may not immediately exceed the control limits but may instead appear as gradual deviations from the center line over successive samples. Therefore, relying only on the current monitoring statistic may reduce the sensitivity of a control chart to small and persistent shifts. By incorporating the MDS sampling mechanism into the COM–Poisson monitoring framework, the states of preceding samples can be included in the current decision rule, allowing the chart to accumulate weak abnormal signals from recent observations and improve the detection of small shifts in COM–Poisson count processes. Accordingly, this study employs the EEWMA statistic to develop the MDS-CMP-EEWMA chart to enhance the small-shift detection ability of the existing MDS-CMP-EWMA chart.

To incorporate the sampling concept of the MDS scheme, the upper and lower control limits of the CMP-EEWMA chart (Equation (9)) are taken as the outer control limits of the MDS-CMP-EEWMA chart. An additional set of inner control limits is defined as:

$$\begin{aligned} UCL_2 &= \mu_0^{1/v} - \frac{v-1}{2v} + k_2 \sqrt{\left( \frac{\lambda_1^2 - 2(1-\lambda_1+\lambda_2)\lambda_1\lambda_2 + \lambda_2^2}{1-(1-\lambda_1+\lambda_2)^2} \right) \cdot \frac{\mu_0^{1/v}}{v}} \\ CL &= \mu_0^{1/v} - \frac{v-1}{2v} \\ LCL_2 &= \mu_0^{1/v} - \frac{v-1}{2v} - k_2 \sqrt{\left( \frac{\lambda_1^2 - 2(1-\lambda_1+\lambda_2)\lambda_1\lambda_2 + \lambda_2^2}{1-(1-\lambda_1+\lambda_2)^2} \right) \cdot \frac{\mu_0^{1/v}}{v}} \end{aligned} \quad (10)$$

where  $k_2$  is another control limit constant for the MDS-CMP-EEWMA chart related to the MDS scheme. If the CMP-EEWMA statistic  $E_t$  exceeds the outer control limits  $(LCL_1, UCL_1)$ , an out-of-control signal is generated. Otherwise, the process is deemed to be in control only when the preceding  $i$  samples lie within the inner control limits  $(LCL_2, UCL_2)$ ; if this condition is not satisfied, the process is considered out of control. The proposed MDS-CMP-EEWMA chart reduces to Aslam et al.'s [14] MDS-CMP-EWMA chart when  $\lambda_2 = 0$ . Furthermore, it simplifies the CMP-EWMA chart proposed by Saghir and Lin [4] when  $i = 0$  and  $k_1 = k_2$ .

### 3. Average Run Length Computation

Saleh et al. [25] recently noted out that Formula (3) adopted by Aslam et al. [26] for evaluating the ARL of the generalized MDS sampling (GMDSS)- $\bar{X}$  control chart may be deficient in its derivation or state-space modeling, which can lead to biased ARL estimation. To clarify this issue, Saleh et al. [25] developed a Markov-chain framework to recompute the ARL under different initial states and conducted cross-validation via Monte Carlo simulations. Their results showed that the Markov-chain-based ARLs were highly consistent with the simulation outcomes, whereas the ARLs obtained from the formula of Aslam et al. [26] differed substantially from both, supporting the view that the existing formula may result in misleading conclusions regarding detection performance.

Motivated by the above methodological concerns, this study further examines the validity of ARL evaluation for the MDS-CMP-EEWMA chart. The comparative results are summarized in Table 2, using the Monte Carlo simulation results as the benchmark.

As shown in Table 2, the ARLs computed from Aslam et al.'s [14] formula exhibit considerable discrepancies from those obtained in our simulations across most parameter combinations and shift scenarios, indicating that the existing computational formula may lead to erroneous ARL assessments. Accordingly, in terms of methodology, this study adopts Monte Carlo simulation as the primary basis for ARL evaluation to ensure the reliability and accuracy of the results. Subsequently, parameter adjustment and performance comparisons are conducted on this basis.

The ARL, defined as the expected number of samples taken before an out-of-control signal is triggered, is one of the most important and widely used performance measures for evaluating control charts. When a process is in control, the  $ARL_0$  should be sufficiently large to avoid frequent false alarms. Conversely, when the process is out of control, a smaller  $ARL_1$  is desirable to enable quicker detection of process changes.

Therefore, this study employs a Monte Carlo simulation in R to evaluate the monitoring performance of the proposed MDS-CMP-EEWMA chart. During the simulation, the run length (RL) is recorded for each replication, and the ARL is estimated by adjusting the control limit constants  $k_1$  and  $k_2$ . For the shift design, a multiplicative shift is adopted to describe changes in the process mean such that the location parameter under the shifted state is  $\mu_1 = c\mu_0$ , where  $c = 1$  corresponds to the in-control state,  $c > 1$  indicates an upward process mean shift, and  $0 < c < 1$  indicates a downward process mean shift. The simulation procedure is summarized below.

**Table 2.** ARL values of the MDS-CMP-EWMA charts using simulation and Aslam et al.'s [14] formula when  $\lambda_1 = 0.05, 0.2$ , and  $i = 2, 3$ .

| $k_1 = 2.9123; k_2 = 2.1887$ and $i = 2$ |                    |                             |                    |                   | $k_1 = 3.3910; k_2 = 2.0644$ and $i = 3$ |                   |                        |                   |
|------------------------------------------|--------------------|-----------------------------|--------------------|-------------------|------------------------------------------|-------------------|------------------------|-------------------|
|                                          |                    | Aslam et al.'s [14] Formula |                    |                   |                                          |                   | Monte Carlo Simulation |                   |
| $c$                                      | $\lambda_1 = 0.05$ | $\lambda_1 = 0.2$           | $\lambda_1 = 0.05$ | $\lambda_1 = 0.2$ | $\lambda_1 = 0.05$                       | $\lambda_1 = 0.2$ | $\lambda_1 = 0.05$     | $\lambda_1 = 0.2$ |
| 0.8750                                   | 1.01               | 3.15                        | 16.97              | 15.71             | 1.01                                     | 2.48              | 15.73                  | 13.72             |
| 0.9250                                   | 1.63               | 15.22                       | 32.72              | 37.21             | 1.46                                     | 11.38             | 29.52                  | 30.16             |
| 0.9375                                   | 2.47               | 25.16                       | 42.11              | 49.70             | 2.06                                     | 19.49             | 37.45                  | 39.35             |
| 0.9500                                   | 4.68               | 42.82                       | 57.80              | 68.53             | 3.63                                     | 35.17             | 50.65                  | 52.57             |
| 0.9625                                   | 11.18              | 73.79                       | 86.00              | 95.97             | 8.54                                     | 65.33             | 73.01                  | 71.39             |
| 0.9750                                   | 32.92              | 123.82                      | 140.01             | 130.29            | 27.03                                    | 117.89            | 113.60                 | 94.37             |
| 0.9875                                   | 106.31             | 183.39                      | 226.06             | 157.31            | 100.46                                   | 182.20            | 173.19                 | 113.48            |
| 1.0000                                   | 200.01             | 200.01                      | 270.05             | 153.07            | 200.00                                   | 200.00            | 202.29                 | 111.65            |
| 1.0125                                   | 87.22              | 146.03                      | 192.72             | 121.85            | 82.07                                    | 143.72            | 153.18                 | 93.23             |
| 1.0250                                   | 24.72              | 82.94                       | 116.08             | 88.45             | 20.61                                    | 78.21             | 97.60                  | 70.64             |
| 1.0375                                   | 8.34               | 43.92                       | 73.44              | 62.54             | 6.62                                     | 39.09             | 63.44                  | 51.99             |
| 1.0500                                   | 3.67               | 23.70                       | 50.13              | 45.37             | 3.00                                     | 20.03             | 44.71                  | 38.45             |
| 1.0625                                   | 2.10               | 13.47                       | 37.07              | 33.74             | 1.83                                     | 11.02             | 33.53                  | 29.12             |
| 1.0750                                   | 1.48               | 8.17                        | 28.88              | 25.87             | 1.37                                     | 6.60              | 26.29                  | 22.85             |
| 1.1250                                   | 1.01               | 2.14                        | 14.72              | 11.79             | 1.01                                     | 1.87              | 13.78                  | 11.02             |

Step 1: Set the target in-control  $ARL_0$ , which serves as the design criterion for selecting chart parameters and adjusting the control limits.

Step 2: Specify the Monte Carlo replications. For each replication, generate a random sample  $X_t$  from the COM-Poisson distribution with the in-control location parameter  $\mu_0$  and dispersion parameter  $v$ , indicating the number of nonconformities at time  $t$ . Then, compute the CMP-EEWMA statistic  $E_t$  (Equation (4)) using the predefined smoothing parameters  $\lambda_1$  and  $\lambda_2$ .

Step 3: The process is considered in control if  $E_t$  lies within the inner control limits. If  $E_t$  falls outside the outer control limits, the process is considered out-of-control. Otherwise, the process is considered in control only if the previous  $i$  samples fall within the inner control limits; otherwise, it is considered out of control. Record the RL for each replication and adjust control limit constants  $k_1$  and  $k_2$  to satisfy the desired in-control  $ARL_0$ .

Step 4: When a process shift occurs such that  $\mu_1 = c\mu_0$ , compute the corresponding out-of-control  $ARL_1$  for each shift magnitude.

#### 4. Performance of the Proposed Control Chart

Based on the above procedure, numerical tables of the ARL values for the proposed MDS-CMP-EEWMA control chart are constructed under various parameter combinations, following Aslam et al.'s [14] proposed MDS-CMP-EWMA control chart framework.

Tables 3–5 present out-of-control  $ARL_1$  values for different shift magnitudes of 0.875, 0.9625, 0.9750, 0.9875, 1.000, 1.0125, 1.0250, 1.0375, and 1.125. Note that when the shift magnitude is  $c=1.000$ , the corresponding value represents the in-control  $ARL_0 \approx 200$ . These tables report the ARL values of the MDS-CMP-EEWMA charts with  $\mu_0 = 4$  under various smoothing parameter combinations:  $\lambda_1 = 0.05$  with  $\lambda_2 = 0.01, 0.02, 0.03, \text{ and } 0.04$ , and  $\lambda_1 = 0.2$  with  $\lambda_2 = 0.04, 0.08, 0.12, \text{ and } 0.16$ . Additionally, the results are evaluated under different MDS sampling strategies ( $i=0, 2, 3, 4$ ) and dispersion parameters  $v=0.5, 1.0, 5.0$  for Tables 3, 4 and 5, respectively. Specifically, the MDS-CMP-EEWMA control chart reduces to the CMP-EEWMA control chart when  $i=0$ .

Tables 3–5 present a consistent pattern in which  $ARL_1$  values decrease as the shift magnitude increases, indicating improved detection ability for larger process mean shifts. The numerical results show that the proposed MDS-CMP-EEWMA chart provides more efficient and robust detection, particularly for small shifts and under various dispersion conditions. The important findings are summarized below.

- (i) For a downward shift ( $c < 1$ ), the MDS-CMP-EEWMA chart generally produces lower  $ARL_1$  values than the CMP-EEWMA chart, demonstrating enhanced sensitivity in early detection capability for processing small downward shifts. The effect of the MDS scheme is evident because increasing  $i$  (i.e., incorporating more previous samples) improves detection performance, particularly for small and moderate shifts.
- (ii) As the dispersion parameter increases from  $v=0.5$  to  $v=5.0$ , the  $ARL_1$  values increase overall, indicating that higher variability makes shift detection more challenging. However, the relative advantage of the MDS-CMP-EEWMA chart remains consistent. Moreover, for downward shifts among the smoothing parameter combina-

tions considered, setting  $\lambda_1 = 0.05$  and  $\lambda_2 = 0.03$  yields relatively small  $ARL_1$  values, indicating superior sensitivity in detecting process shifts. For shifts near the in-control state ( $c = 1.000$ ), all charts maintain ARL values close to the nominal level of 200, confirming proper in-control performance.

- (iii) For an upward shift ( $c > 1$ ), the results show that the conventional CMP-EEWMA chart without the MDS strategy attains smaller  $ARL_1$  values in most scenarios. This finding implies that, in the upward shift setting, the traditional single sample decision mechanism is more sensitive to increases in the monitoring statistic induced by the shift. The relatively limited performance of the MDS-CMP-EEWMA chart for upward shifts may be related to both the distributional characteristics of the COM-Poisson model and the decision structure of the MDS mechanism. Under different dispersion parameters, the COM-Poisson distribution may exhibit different skewness and tail behaviors. When an upward shift occurs, relatively large observations may cause the CMP-EEWMA statistic to approach or exceed the upper control limit more directly.

**Table 3.** The  $ARL$  values of the MDS-CMP-EEWMA charts when  $\mu_0 = 4$ ,  $v = 0.5$ , and  $ARL_0 \approx 200$ .

| Scheme             |                    |        | Shift $c$ |        |        |        |        |        |        |        |       |       |
|--------------------|--------------------|--------|-----------|--------|--------|--------|--------|--------|--------|--------|-------|-------|
| $i$                | $k_1$              | $k_2$  | 0.875     | 0.9625 | 0.9750 | 0.9875 | 1.000  | 1.0125 | 1.0250 | 1.0375 | 1.125 |       |
| $\lambda_1 = 0.05$ | $\lambda_2 = 0.01$ |        |           |        |        |        |        |        |        |        |       |       |
|                    | 0                  | 2.1852 | -         | 15.80  | 70.91  | 109.88 | 171.22 | 200.05 | 148.60 | 93.17  | 60.95 | 13.45 |
|                    | 2                  | 2.6222 | 1.9929    | 15.94  | 69.87  | 108.26 | 168.10 | 200.00 | 150.53 | 95.43  | 62.07 | 13.93 |
|                    | 3                  | 3.0592 | 2.0037    | 15.96  | 69.90  | 108.18 | 168.52 | 200.01 | 150.77 | 95.49  | 62.06 | 14.02 |
|                    | 4                  | 3.4963 | 2.0102    | 15.98  | 69.85  | 108.44 | 169.13 | 200.00 | 150.62 | 95.11  | 61.98 | 14.04 |
|                    | $\lambda_2 = 0.02$ |        |           |        |        |        |        |        |        |        |       |       |
|                    | 0                  | 2.1584 | -         | 16.54  | 70.01  | 108.01 | 169.21 | 200.06 | 145.38 | 90.57  | 59.71 | 13.86 |
|                    | 2                  | 2.5900 | 1.9197    | 16.40  | 67.71  | 104.00 | 164.80 | 200.00 | 148.85 | 93.33  | 61.12 | 14.31 |
|                    | 3                  | 3.0217 | 1.9352    | 16.42  | 67.64  | 104.09 | 164.46 | 200.05 | 149.03 | 93.34  | 61.10 | 14.42 |
|                    | 4                  | 3.4534 | 1.9453    | 16.46  | 67.72  | 104.22 | 164.60 | 200.01 | 148.74 | 93.10  | 60.97 | 14.44 |
|                    | $\lambda_2 = 0.03$ |        |           |        |        |        |        |        |        |        |       |       |
|                    | 0                  | 2.1416 | -         | 17.92  | 70.65  | 107.55 | 169.25 | 200.01 | 141.37 | 87.42  | 58.34 | 14.40 |
|                    | 2                  | 2.5699 | 1.8263    | 17.28  | 66.78  | 101.66 | 162.04 | 200.01 | 147.02 | 91.39  | 60.35 | 14.85 |
|                    | 3                  | 2.9982 | 1.8502    | 17.33  | 66.66  | 101.26 | 161.65 | 200.02 | 147.43 | 91.58  | 60.43 | 15.01 |
|                    | 4                  | 3.4265 | 1.8680    | 17.40  | 66.93  | 101.83 | 162.38 | 200.03 | 147.02 | 91.23  | 60.39 | 15.04 |
|                    | $\lambda_2 = 0.04$ |        |           |        |        |        |        |        |        |        |       |       |
|                    | 0                  | 2.1871 | -         | 21.61  | 77.53  | 114.42 | 175.47 | 200.00 | 136.61 | 84.02  | 57.33 | 15.15 |
|                    | 2                  | 2.6245 | 1.7175    | 19.38  | 69.72  | 103.61 | 162.38 | 200.02 | 144.65 | 89.34  | 60.20 | 15.47 |
|                    | 3                  | 3.0619 | 1.7458    | 19.30  | 69.06  | 102.56 | 161.32 | 200.02 | 144.86 | 89.91  | 60.58 | 15.73 |
|                    | 4                  | 3.4993 | 1.7725    | 19.40  | 69.25  | 102.81 | 161.77 | 200.04 | 143.93 | 89.41  | 60.23 | 15.80 |
| $\lambda_1 = 0.2$  | $\lambda_2 = 0.04$ |        |           |        |        |        |        |        |        |        |       |       |
|                    | 0                  | 2.6438 | -         | 18.14  | 132.44 | 182.92 | 217.99 | 200.01 | 147.63 | 100.27 | 68.90 | 12.34 |
|                    | 2                  | 3.1726 | 2.2467    | 16.67  | 111.26 | 160.75 | 204.12 | 200.02 | 153.82 | 105.84 | 72.68 | 12.66 |
|                    | 3                  | 3.7013 | 2.2562    | 16.51  | 109.96 | 158.36 | 201.75 | 200.02 | 153.96 | 105.81 | 72.54 | 12.82 |
|                    | 4                  | 4.2301 | 2.2671    | 16.52  | 110.16 | 159.10 | 202.37 | 200.03 | 153.54 | 105.48 | 72.33 | 12.96 |
|                    | $\lambda_2 = 0.08$ |        |           |        |        |        |        |        |        |        |       |       |
|                    | 0                  | 2.6631 | -         | 18.03  | 127.39 | 183.70 | 222.05 | 200.00 | 144.14 | 95.75  | 65.20 | 12.28 |
|                    | 2                  | 3.1957 | 2.1990    | 16.22  | 102.98 | 152.46 | 200.81 | 200.00 | 151.49 | 102.46 | 69.01 | 12.57 |
|                    | 3                  | 3.7283 | 2.2169    | 16.06  | 101.07 | 150.26 | 199.26 | 200.00 | 152.42 | 103.30 | 69.43 | 12.75 |

|                    |        |        |       |        |        |        |        |        |        |       |       |
|--------------------|--------|--------|-------|--------|--------|--------|--------|--------|--------|-------|-------|
| 4                  | 4.2610 | 2.2338 | 16.10 | 101.24 | 150.40 | 199.97 | 200.04 | 151.55 | 102.36 | 69.03 | 12.88 |
| $\lambda_2 = 0.12$ |        |        |       |        |        |        |        |        |        |       |       |
| 0                  | 2.6899 | -      | 18.99 | 124.77 | 184.88 | 227.00 | 200.02 | 137.56 | 90.02  | 61.58 | 12.50 |
| 2                  | 3.2279 | 2.1268 | 16.41 | 94.54  | 143.62 | 195.10 | 200.01 | 147.56 | 96.79  | 64.95 | 12.68 |
| 3                  | 3.7659 | 2.1523 | 16.21 | 92.18  | 140.03 | 192.47 | 200.01 | 148.30 | 97.48  | 65.31 | 12.95 |
| 4                  | 4.3038 | 2.1797 | 16.30 | 92.81  | 140.80 | 195.06 | 200.00 | 147.84 | 96.66  | 64.68 | 13.09 |
| $\lambda_2 = 0.16$ |        |        |       |        |        |        |        |        |        |       |       |
| 0                  | 2.7305 | -      | 23.18 | 130.22 | 194.86 | 240.31 | 200.01 | 130.29 | 84.02  | 57.90 | 13.51 |
| 2                  | 3.2766 | 2.0175 | 18.41 | 90.00  | 136.86 | 193.72 | 200.03 | 141.87 | 90.63  | 61.30 | 13.43 |
| 3                  | 3.8227 | 2.0523 | 18.11 | 86.98  | 132.88 | 189.82 | 200.03 | 143.96 | 92.61  | 52.16 | 13.80 |
| 4                  | 4.3688 | 2.2898 | 18.22 | 87.43  | 133.33 | 191.44 | 200.01 | 142.96 | 91.61  | 61.49 | 14.01 |

Note:  $i=0$  indicates the CMP-EEWMA chart;  $i=2,3,4$  indicates the MDS-CMP-EEWMA charts.

**Table 4.** The  $ARL$  values of the MDS-CMP-EEWMA charts when  $\mu_0 = 4$ ,  $v = 1.0$ , and  $ARL_0 \approx 200$ .

| Scheme             |        |        | Shift $c$ |        |        |        |        |        |        |        |       |  |
|--------------------|--------|--------|-----------|--------|--------|--------|--------|--------|--------|--------|-------|--|
| $i$                | $k_1$  | $k_2$  | 0.875     | 0.9625 | 0.9750 | 0.9875 | 1.000  | 1.0125 | 1.0250 | 1.0375 | 1.125 |  |
| $\lambda_1 = 0.05$ |        |        |           |        |        |        |        |        |        |        |       |  |
| $\lambda_2 = 0.01$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.1820 | -      | 56.64     | 168.79 | 187.85 | 200.04 | 200.04 | 186.30 | 165.04 | 142.74 | 51.00 |  |
| 2                  | 2.6184 | 1.9910 | 55.30     | 163.83 | 184.64 | 198.53 | 200.00 | 187.94 | 168.64 | 146.25 | 52.39 |  |
| 3                  | 3.0548 | 2.0003 | 55.29     | 163.60 | 184.52 | 197.96 | 200.01 | 187.12 | 168.23 | 145.97 | 52.29 |  |
| 4                  | 3.4912 | 2.0068 | 55.36     | 164.02 | 184.97 | 198.56 | 200.04 | 186.84 | 167.95 | 145.71 | 52.26 |  |
| $\lambda_2 = 0.02$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.1587 | -      | 56.55     | 167.48 | 188.56 | 201.19 | 200.00 | 185.45 | 163.20 | 140.54 | 50.13 |  |
| 2                  | 2.5904 | 1.9178 | 54.06     | 159.71 | 182.21 | 196.92 | 200.03 | 188.52 | 167.70 | 144.79 | 51.90 |  |
| 3                  | 3.0221 | 1.9342 | 54.09     | 159.99 | 182.91 | 197.33 | 200.00 | 188.78 | 167.84 | 145.10 | 51.93 |  |
| 4                  | 3.4539 | 1.9440 | 54.19     | 160.18 | 183.11 | 196.73 | 200.02 | 188.13 | 167.49 | 144.56 | 51.75 |  |
| $\lambda_2 = 0.03$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.1434 | -      | 58.10     | 168.40 | 190.16 | 203.38 | 200.01 | 182.50 | 159.27 | 135.19 | 48.81 |  |
| 2                  | 2.5904 | 1.8282 | 53.96     | 157.16 | 180.09 | 196.71 | 200.00 | 188.11 | 166.69 | 143.37 | 51.40 |  |
| 3                  | 3.0221 | 1.8502 | 53.96     | 156.91 | 180.17 | 196.31 | 200.01 | 188.24 | 166.67 | 143.48 | 51.66 |  |
| 4                  | 3.4539 | 1.8666 | 54.06     | 157.23 | 180.91 | 196.16 | 200.01 | 187.97 | 165.31 | 142.71 | 51.46 |  |
| $\lambda_2 = 0.04$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.1885 | -      | 66.49     | 178.17 | 198.98 | 208.69 | 200.04 | 178.77 | 152.10 | 127.24 | 47.60 |  |
| 2                  | 2.6262 | 1.7217 | 57.81     | 158.52 | 182.04 | 197.74 | 200.01 | 186.19 | 162.30 | 138.31 | 50.97 |  |
| 3                  | 3.0639 | 1.7438 | 57.11     | 156.63 | 181.01 | 197.15 | 200.02 | 187.26 | 163.20 | 139.42 | 51.37 |  |
| 4                  | 3.5016 | 1.7711 | 57.32     | 157.50 | 180.89 | 197.76 | 200.00 | 186.51 | 162.25 | 138.65 | 51.33 |  |
| $\lambda_1 = 0.2$  |        |        |           |        |        |        |        |        |        |        |       |  |
| $\lambda_2 = 0.04$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.6514 | -      | 128.13    | 240.94 | 234.34 | 219.02 | 200.04 | 178.84 | 157.09 | 137.71 | 55.23 |  |
| 2                  | 3.1817 | 2.2510 | 98.14     | 216.64 | 218.11 | 213.09 | 200.03 | 183.73 | 165.01 | 146.13 | 58.91 |  |
| 3                  | 3.7120 | 2.2581 | 95.78     | 214.66 | 216.82 | 211.77 | 200.02 | 183.91 | 165.28 | 146.37 | 58.79 |  |
| 4                  | 4.2422 | 2.2692 | 96.03     | 216.42 | 217.81 | 212.30 | 200.01 | 184.44 | 165.33 | 146.08 | 58.54 |  |
| $\lambda_2 = 0.08$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.6713 | -      | 122.22    | 246.64 | 237.45 | 221.07 | 200.05 | 176.79 | 152.94 | 132.64 | 52.36 |  |
| 2                  | 3.2056 | 2.2052 | 87.88     | 209.72 | 214.94 | 211.29 | 200.01 | 183.93 | 164.20 | 144.12 | 56.27 |  |
| 3                  | 3.7398 | 2.1835 | 83.99     | 204.59 | 211.74 | 209.75 | 200.03 | 185.32 | 166.78 | 146.38 | 57.39 |  |
| 4                  | 4.2741 | 2.2336 | 85.66     | 208.75 | 214.22 | 210.94 | 200.01 | 183.42 | 164.01 | 144.03 | 55.92 |  |
| $\lambda_2 = 0.12$ |        |        |           |        |        |        |        |        |        |        |       |  |
| 0                  | 2.7024 | -      | 121.64    | 258.98 | 247.05 | 225.85 | 200.01 | 172.78 | 147.59 | 125.84 | 49.14 |  |
| 2                  | 3.2429 | 2.1352 | 79.52     | 204.40 | 212.43 | 210.62 | 200.03 | 182.98 | 161.51 | 139.81 | 52.99 |  |

|                    |        |        |        |        |        |        |        |        |        |        |       |
|--------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------|
| 3                  | 3.7834 | 2.1533 | 76.48  | 200.42 | 209.54 | 210.16 | 200.01 | 182.19 | 161.66 | 140.94 | 53.50 |
| 4                  | 4.3238 | 2.1783 | 77.30  | 202.06 | 210.85 | 210.15 | 200.01 | 181.70 | 160.30 | 139.62 | 53.06 |
| $\lambda_2 = 0.16$ |        |        |        |        |        |        |        |        |        |        |       |
| 0                  | 2.7553 | -      | 136.25 | 293.93 | 268.67 | 235.34 | 200.04 | 167.59 | 140.75 | 119.00 | 46.88 |
| 2                  | 3.3064 | 2.0247 | 75.81  | 203.94 | 213.68 | 213.17 | 200.04 | 179.96 | 155.51 | 133.17 | 50.09 |
| 3                  | 3.8574 | 2.0501 | 72.93  | 197.84 | 210.21 | 211.00 | 200.03 | 180.04 | 157.08 | 134.62 | 50.75 |
| 4                  | 4.4085 | 2.0867 | 73.58  | 200.01 | 212.39 | 212.09 | 200.01 | 179.30 | 156.89 | 133.22 | 50.54 |

Note:  $i=0$  indicates the CMP-EEWMA chart;  $i=2,3,4$  indicates the MDS-CMP-EEWMA charts.

**Table 5.** The  $ARL$  values of the MDS-CMP-EEWMA charts when  $\mu_0 = 4$ ,  $v = 5.0$ , and  $ARL_0 \approx 200$ .

| Scheme             |                   |                    | Shift $C$          |        |        |        |        |        |        |        |        |        |        |
|--------------------|-------------------|--------------------|--------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|                    | $i$               | $k_1$              | $k_2$              | 0.875  | 0.9625 | 0.9750 | 0.9875 | 1.000  | 1.0125 | 1.0250 | 1.0375 | 1.125  |        |
| $\lambda_1 = 0.05$ |                   | $\lambda_2 = 0.01$ |                    |        |        |        |        |        |        |        |        |        |        |
|                    | 0                 | 2.2600             | -                  | 148.10 | 189.42 | 193.95 | 197.60 | 200.04 | 201.12 | 201.39 | 202.40 | 182.05 |        |
|                    | 2                 | 2.7120             | 1.9944             | 150.32 | 190.57 | 194.66 | 198.01 | 200.04 | 200.63 | 200.77 | 200.74 | 178.78 |        |
|                    | 3                 | 3.1640             | 2.0081             | 149.87 | 190.53 | 194.55 | 198.09 | 200.01 | 200.47 | 201.02 | 201.25 | 179.86 |        |
|                    | 4                 | 3.6160             | 2.0235             | 148.73 | 190.04 | 194.14 | 198.01 | 200.02 | 200.82 | 201.29 | 201.88 | 181.18 |        |
|                    |                   | $\lambda_2 = 0.02$ |                    |        |        |        |        |        |        |        |        |        |        |
|                    | 0                 | 2.2475             | -                  | 146.72 | 189.72 | 193.56 | 197.30 | 200.00 | 201.24 | 201.39 | 201.94 | 178.10 |        |
|                    | 2                 | 2.6970             | 1.8870             | 145.23 | 189.48 | 193.40 | 196.90 | 200.01 | 201.52 | 202.27 | 202.93 | 180.59 |        |
|                    | 3                 | 3.1465             | 1.9219             | 144.16 | 188.22 | 192.66 | 197.00 | 200.01 | 202.12 | 203.16 | 203.60 | 183.06 |        |
|                    | 4                 | 3.5960             | 1.9469             | 142.57 | 187.91 | 192.78 | 196.81 | 200.01 | 202.31 | 203.64 | 204.27 | 183.50 |        |
|                    |                   | $\lambda_2 = 0.03$ |                    |        |        |        |        |        |        |        |        |        |        |
|                    | 0                 | 2.2475             | -                  | 146.10 | 190.64 | 194.29 | 197.93 | 200.02 | 201.54 | 201.01 | 200.77 | 173.23 |        |
|                    | 2                 | 2.6970             | 1.8870             | 138.77 | 186.54 | 191.84 | 196.02 | 200.02 | 202.88 | 204.28 | 205.35 | 186.05 |        |
|                    | 3                 | 3.1465             | 1.9219             | 137.45 | 185.86 | 191.03 | 196.29 | 200.03 | 202.42 | 203.76 | 205.40 | 186.55 |        |
|                    | 4                 | 3.5960             | 1.9469             | 137.58 | 186.24 | 191.09 | 196.48 | 200.00 | 202.54 | 203.58 | 205.16 | 185.37 |        |
|                    |                   | $\lambda_2 = 0.04$ |                    |        |        |        |        |        |        |        |        |        |        |
|                    | 0                 | 2.2098             | -                  | 151.00 | 191.66 | 195.03 | 198.01 | 200.02 | 200.69 | 199.23 | 197.34 | 165.74 |        |
|                    | 2                 | 2.6518             | 1.8737             | 140.03 | 187.13 | 192.58 | 197.12 | 200.03 | 202.44 | 202.91 | 203.44 | 181.94 |        |
|                    | 3                 | 3.0937             | 1.9072             | 139.16 | 186.76 | 192.02 | 196.08 | 200.00 | 202.31 | 203.31 | 203.95 | 181.00 |        |
|                    | 4                 | 3.5357             | 1.9377             | 140.58 | 187.75 | 192.44 | 196.77 | 200.02 | 202.99 | 203.87 | 204.19 | 179.48 |        |
|                    | $\lambda_1 = 0.2$ |                    | $\lambda_2 = 0.04$ |        |        |        |        |        |        |        |        |        |        |
|                    |                   | 0                  | 2.7786             | -      | 173.20 | 196.84 | 198.77 | 199.23 | 200.00 | 200.33 | 200.26 | 199.58 | 182.33 |
|                    |                   | 2                  | 3.3343             | 2.2554 | 158.19 | 190.71 | 194.72 | 197.14 | 200.05 | 202.62 | 204.70 | 205.99 | 201.22 |
|                    |                   | 3                  | 3.8900             | 2.3090 | 158.52 | 191.06 | 194.78 | 197.27 | 200.00 | 202.28 | 203.76 | 205.00 | 200.09 |
| 4                  |                   | 4.4458             | 2.3282             | 161.60 | 192.23 | 195.61 | 197.74 | 200.02 | 201.70 | 202.66 | 203.39 | 195.38 |        |
|                    |                   | $\lambda_2 = 0.08$ |                    |        |        |        |        |        |        |        |        |        |        |
| 0                  |                   | 2.7625             | -                  | 174.37 | 197.82 | 199.00 | 199.15 | 200.06 | 199.58 | 198.58 | 197.51 | 176.54 |        |
| 2                  |                   | 3.3150             | 2.2652             | 154.64 | 189.99 | 193.92 | 196.58 | 200.04 | 202.31 | 204.18 | 205.30 | 200.47 |        |
| 3                  |                   | 3.8675             | 2.3241             | 153.50 | 190.19 | 194.30 | 196.80 | 200.01 | 202.17 | 204.29 | 205.07 | 199.64 |        |
| 4                  |                   | 4.4200             | 2.3481             | 155.94 | 191.10 | 195.17 | 197.17 | 200.03 | 201.71 | 203.32 | 204.29 | 195.67 |        |
|                    |                   | $\lambda_2 = 0.12$ |                    |        |        |        |        |        |        |        |        |        |        |
| 0                  |                   | 2.7204             | -                  | 178.98 | 201.27 | 200.84 | 200.54 | 200.00 | 199.12 | 196.74 | 194.65 | 167.24 |        |
| 2                  |                   | 3.2645             | 2.2908             | 154.94 | 191.67 | 195.32 | 197.60 | 200.04 | 201.68 | 201.63 | 201.69 | 189.31 |        |
| 3                  |                   | 3.8086             | 2.3333             | 153.99 | 191.21 | 194.86 | 197.39 | 200.02 | 200.89 | 201.40 | 202.05 | 188.66 |        |
| 4                  |                   | 4.3526             | 2.3611             | 154.55 | 190.97 | 194.80 | 197.42 | 200.00 | 200.94 | 201.65 | 202.30 | 187.27 |        |
|                    |                   | $\lambda_2 = 0.16$ |                    |        |        |        |        |        |        |        |        |        |        |
| 0                  |                   | 2.5910             | -                  | 183.72 | 204.92 | 204.98 | 203.63 | 200.02 | 197.68 | 193.44 | 190.06 | 154.37 |        |
| 2                  |                   | 3.1092             | 2.2955             | 160.77 | 194.19 | 197.04 | 198.34 | 200.01 | 200.02 | 199.07 | 198.22 | 174.18 |        |

|   |        |        |        |        |        |        |        |        |        |        |        |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 3 | 3.6274 | 2.3058 | 159.86 | 193.96 | 197.05 | 198.70 | 200.05 | 199.14 | 198.15 | 197.02 | 170.46 |
| 4 | 4.1456 | 2.3167 | 157.68 | 193.77 | 196.83 | 198.53 | 200.02 | 199.25 | 198.33 | 196.78 | 169.79 |

Note:  $i=0$  indicates the CMP-EEWMA chart;  $i=2,3,4$  indicates the MDS-CMP-EEWMA charts.

## 5. Comparative Study

### 5.1. Comparison Among the MDS-CMP-EEWMA, MDS-CMP-EWMA, CMP-EEWMA, and CMP-EWMA Charts

Saghir and Lin [4] proposed the GEWMA control chart, herein referred to as the CMP-EWMA chart, which can efficiently monitor both under- and over-dispersed count data. Following Saghir and Lin [4], the CMP-EWMA statistic  $W_t$  is defined as:

$$W_t = \lambda_1 X_t - (1 - \lambda_1) W_{t-1}, \quad t = 1, 2, 3, \dots, \quad (11)$$

where  $\lambda_1$  ( $0 < \lambda_1 \leq 1$ ) is the weight assigned to the most recent observation. Based on the mean and variance of the COM-Poisson distribution given in Equations (2) and (3), as time  $t \rightarrow \infty$ , the control limits of the CMP-EWMA control chart are given by:

$$\begin{aligned} UCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} + k_1 \sqrt{\left( \frac{\lambda_1}{2-\lambda_1} \right) \cdot \frac{\mu_0^{1/v}}{v}} \\ CL &= \mu_0^{1/v} - \frac{v-1}{2v} \\ LCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} - k_1 \sqrt{\left( \frac{\lambda_1}{2-\lambda_1} \right) \cdot \frac{\mu_0^{1/v}}{v}} \end{aligned} \quad (12)$$

where the control limit constant  $k_1$  is used to determine the in-control ARL of the CMP-EWMA control chart. Interested readers may refer to Saghir and Lin [4] for further details.

To incorporate the sampling concept of the MDS scheme, Aslam et al. [14] developed the MDS-CMP-EWMA control chart for monitoring nonconformities. The upper and lower control limits of the CMP-EWMA chart (Equation (12)) are adopted as the outer control limits of the MDS-CMP-EWMA chart. An additional set of inner control limits is defined as:

$$\begin{aligned} UCL_2 &= \mu_0^{1/v} - \frac{v-1}{2v} + k_2 \sqrt{\left( \frac{\lambda_1}{2-\lambda_1} \right) \cdot \frac{\mu_0^{1/v}}{v}} \\ CL &= \mu_0^{1/v} - \frac{v-1}{2v} \\ LCL_2 &= \mu_0^{1/v} - \frac{v-1}{2v} - k_2 \sqrt{\left( \frac{\lambda_1}{2-\lambda_1} \right) \cdot \frac{\mu_0^{1/v}}{v}} \end{aligned} \quad (13)$$

where  $k_2$  is another control limit constant and satisfies  $0 < k_2 \leq k_1$ . The MDS-CMP-EWMA chart reduces Saghir and Lin's [4] CMP-EWMA chart when previous samples  $i=0$  and  $k_1=k_2$ . For further details, readers may refer to Aslam et al. [14].

A comprehensive comparison is conducted between the proposed MDS-CMP-EEWMA and CMP-EEWMA charts and existing MDS-CMP-EWMA and CMP-EWMA charts for detecting small shifts in the process mean. Saghir and Lin [4] have recommended a weight within the range  $0.05 \leq \lambda_1 \leq 0.25$  to monitor small process shifts effectively. Table 6 presents the  $ARL_1$  values of these four control charts with  $\lambda_1 = 0.05$  and  $\lambda_2 = \{0, 0.03\}$ , considering  $\mu_0 = 4$ ,  $v = \{0.5, 1.0, 5.0\}$  and an MDS scheme with  $i = \{0, 2, 3, 4\}$ , at an in-control ARL of approximately 200. Specifically, the MDS-CMP-EEWMA and MDS-CMP-EWMA charts reduce to the CMP-EEWMA and CMP-EWMA

charts, respectively, when  $i = 0$ . Moreover, the MDS-CMP-EEWMA and CMP-EEWMA charts reduce to the MDS-CMP-EWMA and CMP-EWMA charts, respectively, when  $\lambda_2 = 0$ .

Table 6 shows that EEWMA-based charts consistently outperform the EWMA-based charts by yielding lower  $ARL_1$  values, indicating faster detection of small process mean shifts. The MDS-CMP-EEWMA chart demonstrates the best overall performance across most scenarios. Including the second smoothing parameter ( $\lambda_2 = 0.03$ ) further enhances sensitivity, particularly for small-to-moderate shifts. Incorporating the MDS scheme ( $i > 0$ ) improves detection efficiency because the MDS-CMP-EEWMA chart generally performs better than the CMP-EEWMA chart. As the dispersion parameter  $v$  increases, all charts show higher  $ARL_1$  values, reflecting increased detection difficulty; however, the relative superiority of the MDS-CMP-EEWMA chart remains. The performance differences across charts are most noticeable for small downward shifts and become less significant as the shift magnitude increases. Overall, the findings suggest that the proposed MDS-CMP-EEWMA chart provides the most efficient and reliable performance for monitoring small process mean shifts.

**Table 6.** Comparisons among the MDS-CMP-EEWMA, MDS-CMP-EWMA, CMP-EEWMA, and CMP-EWMA charts when  $\mu_0 = 4$  at  $\lambda_1 = 0.05$  and  $\lambda_2 = \{0, 0.03\}$ .

| Scheme                               |        |        |        | Shift $c$ |        |        |        |        |        |        |        |       |
|--------------------------------------|--------|--------|--------|-----------|--------|--------|--------|--------|--------|--------|--------|-------|
| $i$                                  | $k_1$  | $k_2$  |        | 0.875     | 0.9625 | 0.9750 | 0.9875 | 1.000  | 1.0125 | 1.0250 | 1.0375 | 1.125 |
| $v = 0.5$                            |        |        |        |           |        |        |        |        |        |        |        |       |
| $\lambda_1 = 0.05, \lambda_2 = 0$    |        |        |        |           |        |        |        |        |        |        |        |       |
| 0(CMP-EWMA)                          | 2.2177 | -      | 15.41  | 73.24     | 113.75 | 172.78 | 200.03 | 151.02 | 95.59  | 62.55  | 13.22  |       |
| 2(MDS-CMP-EWMA)                      | 2.6612 | 2.0517 | 15.67  | 72.60     | 112.72 | 170.85 | 200.01 | 151.90 | 96.83  | 63.09  | 13.66  |       |
| 3(MDS-CMP-EWMA)                      | 3.1048 | 2.0593 | 15.68  | 72.60     | 112.70 | 171.33 | 200.00 | 151.78 | 96.88  | 63.12  | 13.75  |       |
| 4(MDS-CMP-EWMA)                      | 3.5483 | 2.0632 | 15.68  | 72.57     | 112.66 | 171.51 | 200.01 | 151.68 | 96.78  | 63.04  | 13.76  |       |
| $\lambda_1 = 0.05, \lambda_2 = 0.03$ |        |        |        |           |        |        |        |        |        |        |        |       |
| 0(CMP-EEWMA)                         | 2.1416 | -      | 17.92  | 70.65     | 107.55 | 169.25 | 200.01 | 141.37 | 87.42  | 58.34  | 14.40  |       |
| 2(MDS-CMP-EEWMA)                     | 2.5699 | 1.8263 | 17.28  | 66.78     | 101.66 | 162.04 | 200.01 | 147.02 | 91.39  | 60.35  | 14.85  |       |
| 3(MDS-CMP-EEWMA)                     | 2.9982 | 1.8502 | 17.33  | 66.66     | 101.26 | 161.65 | 200.02 | 147.43 | 91.58  | 60.43  | 15.01  |       |
| 4(MDS-CMP-EEWMA)                     | 3.4265 | 1.8680 | 17.40  | 66.93     | 101.83 | 162.38 | 200.03 | 147.02 | 91.23  | 60.39  | 15.04  |       |
| $v = 1.0$                            |        |        |        |           |        |        |        |        |        |        |        |       |
| $\lambda_1 = 0.05, \lambda_2 = 0$    |        |        |        |           |        |        |        |        |        |        |        |       |
| 0(CMP-EWMA)                          | 2.2135 | -      | 58.17  | 170.99    | 189.94 | 201.50 | 200.02 | 186.14 | 167.01 | 144.95 | 52.00  |       |
| 2(MDS-CMP-EWMA)                      | 2.6562 | 2.0506 | 57.43  | 169.52    | 189.13 | 200.55 | 200.03 | 188.21 | 169.05 | 147.56 | 52.98  |       |
| 3(MDS-CMP-EWMA)                      | 3.0989 | 2.0573 | 57.33  | 168.89    | 188.94 | 200.19 | 200.00 | 187.55 | 168.76 | 147.21 | 52.91  |       |
| 4(MDS-CMP-EWMA)                      | 3.5416 | 2.0606 | 57.25  | 168.70    | 188.78 | 200.04 | 200.00 | 187.31 | 168.71 | 146.96 | 52.83  |       |
| $\lambda_1 = 0.05, \lambda_2 = 0.03$ |        |        |        |           |        |        |        |        |        |        |        |       |
| 0(CMP-EEWMA)                         | 2.1434 | -      | 58.10  | 168.40    | 190.16 | 203.38 | 200.01 | 182.50 | 159.27 | 135.19 | 48.81  |       |
| 2(MDS-CMP-EEWMA)                     | 2.5904 | 1.8282 | 53.96  | 157.16    | 180.09 | 196.71 | 200.00 | 188.11 | 166.69 | 143.37 | 51.40  |       |
| 3(MDS-CMP-EEWMA)                     | 3.0221 | 1.8502 | 53.96  | 156.91    | 180.17 | 196.31 | 200.01 | 188.24 | 166.67 | 143.48 | 51.66  |       |
| 4(MDS-CMP-EEWMA)                     | 3.4539 | 1.8666 | 54.06  | 157.23    | 180.91 | 196.16 | 200.01 | 187.97 | 165.31 | 142.71 | 51.46  |       |
| $v = 5.0$                            |        |        |        |           |        |        |        |        |        |        |        |       |
| $\lambda_1 = 0.05, \lambda_2 = 0$    |        |        |        |           |        |        |        |        |        |        |        |       |
| 0(CMP-EWMA)                          | 2.2637 | -      | 148.47 | 189.45    | 193.56 | 196.86 | 200.01 | 201.07 | 201.47 | 202.79 | 185.55 |       |
| 2(MDS-CMP-EWMA)                      | 2.7164 | 2.1004 | 152.85 | 191.46    | 194.84 | 197.88 | 200.00 | 200.89 | 201.57 | 201.73 | 180.75 |       |
| 3(MDS-CMP-EWMA)                      | 3.1692 | 2.1034 | 152.73 | 191.53    | 194.78 | 197.63 | 200.01 | 200.74 | 201.46 | 201.90 | 180.87 |       |
| 4(MDS-CMP-EWMA)                      | 3.6219 | 2.1116 | 151.42 | 191.00    | 194.25 | 197.26 | 200.01 | 200.84 | 201.35 | 201.81 | 181.65 |       |

$$\lambda_1 = 0.05, \lambda_2 = 0.03$$

|                  |        |        |        |        |        |        |        |        |        |        |        |
|------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0(CMP-EEWMA)     | 2.2282 | -      | 146.10 | 190.64 | 194.29 | 197.93 | 200.02 | 201.54 | 201.01 | 200.77 | 173.23 |
| 2(MDS-CMP-EEWMA) | 2.6738 | 1.8432 | 138.77 | 186.54 | 191.84 | 196.02 | 200.02 | 202.88 | 204.28 | 205.35 | 186.05 |
| 3(MDS-CMP-EEWMA) | 3.1195 | 1.8921 | 137.45 | 185.86 | 191.03 | 196.29 | 200.03 | 202.42 | 203.76 | 205.40 | 186.55 |
| 4(MDS-CMP-EEWMA) | 3.5651 | 1.9262 | 137.58 | 186.24 | 191.09 | 196.48 | 200.00 | 202.54 | 203.58 | 205.16 | 185.37 |

### 5.2. Comparison with Existing COM–Poisson Memory-Type Control Charts

This section compares three representative COM–Poisson memory-type control charts reported in the literature: the CMP-EEWMA chart, which serves as the baseline model in this study; the COM–Poisson Double Exponentially Weighted Moving Average (CMP-DEWMA) chart proposed by Alevizakos et al. [8]; and the COM–Poisson homogeneously weighted moving average (CMP-HWMA) chart proposed by Adeoti et al. [9]. These charts were selected because they are all designed for monitoring COM–Poisson count processes and employ different memory-type structures to enhance the detection of small-to-moderate process shifts.

Alevizakos et al. [8] proposed the CMP-DEWMA control chart by extending the CMP-EWMA chart with a double-memory structure to improve the detection of small process shifts. The CMP-DEWMA charting statistics are defined as

$$\begin{aligned} Y_t &= \lambda_1 R_t + (1 - \lambda_1) Y_{t-1}, \\ Z_t &= \lambda_1 Y_t + (1 - \lambda_1) R_{t-1} \end{aligned} \quad (14)$$

where  $\lambda_1$  is the smoothing factor,  $0 < \lambda_1 \leq 1$ . The statistic  $Z_t$  can be written as

$$Z_t = \lambda_1^2 \sum_{i=1}^t (t-i+1)(1-\lambda_1)^{t-i} R_i + t\lambda_1(1-\lambda_1)^t Z_0 + (1-\lambda_1)^t Z_0 \quad (15)$$

Adeoti et al. [9] proposed the CMP-HWMA control chart for monitoring count data with equi-dispersion, over-dispersion, and under-dispersion. The CMP-HWMA chart assigns a specific weight to the current observation and distributes the remaining weight equally among the previous observations. Its charting statistic is defined as

$$H_i = \omega J_i + (1 - \omega) \bar{J}_{i-1} \quad (16)$$

where  $\omega$  is the smoothing constant satisfying  $0 < \omega \leq 1$ , and  $\bar{J}_{i-1}$  denotes the mean of the previous observations, given by

$$\bar{J}_{i-1} = \frac{1}{i-1} \sum_{t=1}^{i-1} J_t \quad (17)$$

To ensure a fair comparison, all control charts were evaluated under the same process parameter settings and shift conditions, considering two smoothing parameter settings,  $\lambda_1 = 0.05$  and 0.25. The resulting  $ARL_1$  values are summarized in Table 7. The symbol “–” indicates that the corresponding  $ARL_1$  value for that shift magnitude was not reported in the original study. Table 7 shows that, under upward process shifts, the CMP-EEWMA chart yields smaller  $ARL_1$  values than the CMP-DEWMA chart in most small-shift scenarios, indicating that the CMP-EEWMA chart can generally detect upward changes in the process mean more rapidly. In contrast, because the original CMP-HWMA study did not report the corresponding  $ARL_1$  results for several small-shift scenarios, a complete and direct comparison between the CMP-EEWMA and CMP-HWMA charts under small shifts cannot currently be made. However, for the larger-shift scenarios reported in the original study, the CMP-EEWMA and CMP-HWMA charts exhibit similar  $ARL_1$  values, suggesting comparable detection performance for larger process shifts. Overall, the available results indicate that the CMP-EEWMA chart is more sensitive than the CMP-DEWMA chart in detecting small upward shifts, while it also demonstrates competitive

performance relative to the CMP-HWMA chart; nevertheless, further numerical evidence is needed to fully assess their relative performance under small-shift conditions.

**Table 7.** Comparisons among the CMP-EEWMA, CMP-DEWMA, and CMP-HWMA charts under different shift magnitudes.

|       | CMP-EEWMA          | CMP-DEWMA<br>(Alevizakos et al. [8]) | CMP-HWMA<br>(Adeoti et al. [9]) | CMP-EEWMA          | CMP-DEWMA<br>(Alevizakos et al. [8]) | CMP-HWMA<br>(Adeoti et al. [9]) |
|-------|--------------------|--------------------------------------|---------------------------------|--------------------|--------------------------------------|---------------------------------|
|       | $\lambda_1 = 0.05$ | $\lambda_1 = 0.05$                   | $\lambda_1 = 0.05$              | $\lambda_1 = 0.25$ | $\lambda_1 = 0.25$                   | $\lambda_1 = 0.25$              |
| $c$   | $\lambda_2 = 0.03$ | -                                    | -                               | $\lambda_2 = 0.16$ | -                                    | -                               |
| 0.700 | 7.39               | -                                    | 3.45                            | 6.34               | -                                    | 4.47                            |
| 0.800 | 11.00              | -                                    | 5.69                            | 10.70              | -                                    | 8.15                            |
| 0.875 | 17.92              | 20.18                                | -                               | 21.41              | 15.14                                | -                               |
| 0.900 | 22.89              | -                                    | 15.53                           | 31.19              | -                                    | 27.84                           |
| 0.925 | 31.58              | 31.04                                | -                               | 52.22              | 35.08                                | -                               |
| 0.950 | 50.64              | 47.01                                | -                               | 106.03             | 65.95                                | -                               |
| 0.975 | 107.55             | 96.04                                | -                               | 219.02             | 140.33                               | -                               |
| 1.000 | 200.01             | 200.49                               | 199.72                          | 200.01             | 200.86                               | 200.68                          |
| 1.025 | 87.42              | 100.99                               | -                               | 90.67              | 115.11                               | -                               |
| 1.050 | 42.13              | 47.19                                | -                               | 45.03              | 52.27                                | -                               |
| 1.075 | 26.35              | 30.32                                | -                               | 26.35              | 28.48                                | -                               |
| 1.100 | 18.77              | -                                    | 12.78                           | 17.57              | -                                    | 16.63                           |
| 1.125 | 14.40              | 18.94                                | -                               | 12.72              | 12.77                                | -                               |
| 1.200 | 8.16               | -                                    | 4.47                            | 6.45               | 5.26                                 | 5.60                            |
| 1.300 | 4.83               | -                                    | 2.56                            | 3.53               | 3.55                                 | 3.02                            |

## 6. Case Study: Air Quality Index Monitoring

To demonstrate the applicability and monitoring performance of the proposed control chart, air quality data are considered as a case study, given the growing importance of air quality monitoring in sustainability issues. An air quality index (AQI) value ranging from 0 to 50 is classified as good, indicating that air quality is satisfactory and poses minimal health concerns. AQI values above 50 may adversely affect sensitive populations, with the severity of health impacts increasing as the AQI rises. Therefore, an AQI value greater than 50 was used as the threshold to define an air-quality exceedance event in this study, because it represents the transition from “Good” air quality to a level that may begin to affect sensitive populations.

The AQI data used in this case study were obtained from the Taiwan Air Quality Monitoring Network and the Environmental Open Data Platform of the Ministry of Environment, Taiwan (<https://airtw.moenv.gov.tw/ENG/default.aspx>, (accessed on 10 June 2026)). This study collected hourly AQI observations from the Zhongli air quality monitoring station in Taoyuan City in 2025. The Zhongli station is located in Zhongli District, Taoyuan City, and is classified as a traffic air quality monitoring station. Since traffic monitoring stations are generally established in areas with heavy traffic flow or areas strongly influenced by traffic emissions, the Zhongli station can represent an air quality monitoring scenario affected by urban traffic emissions.

The AQI is calculated based on the sub-indices of major air pollutants, including  $PM_{2.5}$ ,  $PM_{10}$ ,  $O_3$ ,  $CO$ ,  $SO_2$ , and  $NO_2$ . The maximum sub-index value is reported as the AQI value at each time point, and the corresponding pollutant is identified as the primary pollutant. Before the analysis, the completeness of the hourly AQI data was examined, and missing or invalid observations were excluded. The valid AQI data were then converted into binary indicators; an hourly AQI value greater than 50 was assigned

a value of 1, and 0 otherwise. These binary indicators were subsequently aggregated on a weekly basis, resulting in 52 weekly observations, as shown in Table 8.

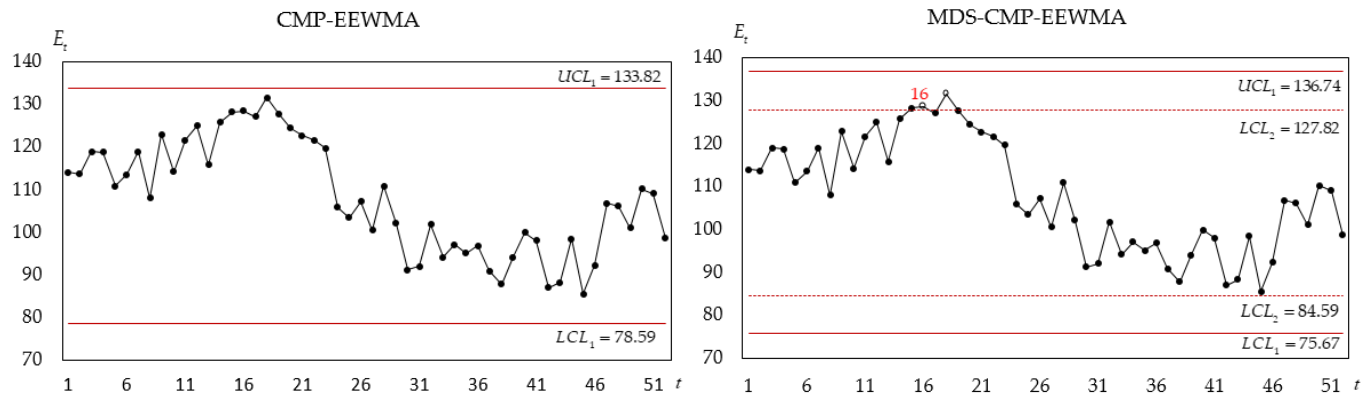
In addition, because the Zhongli station is located in an urban area with frequent traffic activities, AQI variations may be jointly affected by local traffic emissions, urban activity intensity, and seasonal meteorological conditions. In particular, seasonal differences in wind fields, rainfall, boundary layer height, and pollutant dispersion conditions may lead to periodic variations in AQI exceedance counts. Therefore, this study uses the weekly number of hours with AQI values greater than 50 as the monitoring variable. This transformation not only converts hourly air quality conditions into count data but also enables the proposed control chart to be evaluated in terms of its ability to detect potential air quality deterioration in a real environmental monitoring context.

Before implementing the MDS-CMP-EEWMA control chart, the COM-Poisson distribution was fitted to the weekly count data using the maximum likelihood estimation method. The estimated parameter values were  $\hat{\mu} = 1.2633$  and  $\hat{v} = 0.0511$ . Since  $v < 1$ , the data exhibit an over-dispersed pattern, indicating that the COM-Poisson distribution is more suitable than the ordinary Poisson distribution for modeling this dataset.

**Table 8.** The CMP-EEWMA and MDS-CMP-EEWMA charts for monitoring weekly AQI count data for Zhongli in 2025.

| Week | $X_t$ | $E_t$    | Week | $X_t$ | $E_t$    | Week | $X_t$ | $E_t$    | Week | $X_t$ | $E_t$    |
|------|-------|----------|------|-------|----------|------|-------|----------|------|-------|----------|
| 1    | 145   | 113.9665 | 14   | 159   | 125.6912 | 27   | 44    | 100.4842 | 40   | 119   | 99.8442  |
| 2    | 137   | 113.6078 | 15   | 165   | 128.2236 | 28   | 107   | 110.8248 | 41   | 106   | 98.0105  |
| 3    | 159   | 118.9435 | 16   | 159   | 128.4946 | 29   | 64    | 102.0718 | 42   | 49    | 86.9300  |
| 4    | 150   | 118.7458 | 17   | 146   | 127.1148 | 30   | 17    | 91.1489  | 43   | 63    | 88.2128  |
| 5    | 104   | 110.7959 | 18   | 164   | 131.4703 | 31   | 36    | 91.9830  | 44   | 119   | 98.4043  |
| 6    | 119   | 113.5241 | 19   | 138   | 127.5714 | 32   | 96    | 101.7437 | 45   | 50    | 85.4281  |
| 7    | 145   | 118.9431 | 20   | 120   | 124.3886 | 33   | 59    | 94.1139  | 46   | 91    | 92.2110  |
| 8    | 85    | 107.9854 | 21   | 112   | 122.6130 | 34   | 81    | 97.1094  | 47   | 164   | 106.7626 |
| 9    | 164   | 122.8660 | 22   | 109   | 121.5885 | 35   | 74    | 95.0650  | 48   | 149   | 106.0521 |
| 10   | 112   | 114.1114 | 23   | 102   | 119.6850 | 36   | 87    | 96.8224  | 49   | 115   | 100.9700 |
| 11   | 150   | 121.6269 | 24   | 36    | 105.7776 | 37   | 59    | 90.8295  | 50   | 158   | 110.1312 |
| 12   | 161   | 124.9618 | 25   | 38    | 103.3865 | 38   | 50    | 87.7563  | 51   | 143   | 109.0459 |
| 13   | 108   | 115.8034 | 26   | 70    | 107.1710 | 39   | 89    | 94.0461  | 52   | 84    | 98.6041  |

To examine the goodness-of-fit of the COM-Poisson distribution, a chi-square goodness-of-fit test was conducted. The test statistic was 2.1507 with 2 degrees of freedom, yielding a  $p$ -value of 0.3412. Since the  $p$ -value exceeds 0.05, the null hypothesis that the weekly AQI count data follow a COM-Poisson distribution cannot be rejected at the 5% significance level. Therefore, the weekly AQI count data for Zhongli in 2025 can be reasonably modeled using the COM-Poisson distribution. Accordingly, the dataset is used to assess the monitoring performance of the proposed CMP-EEWMA and MDS-CMP-EEWMA charts. For the MDS sampling framework, the previous three samples ( $i = 3$ ) are incorporated into the monitoring procedure. The performance of the proposed MDS-CMP-EEWMA chart is then compared with that of the conventional CMP-EEWMA chart without the MDS scheme, as illustrated in Figure 2.



**Figure 2.** The AQI results of the CMP-EEWMA and MDS-CMP-EEWMA charts.

Both the CMP-EEWMA and MDS-CMP-EEWMA charts are constructed using the smoothing parameters  $\lambda_1=0.2$  and  $\lambda_2=0.16$ . Since both charts employ the same monitoring statistic, denoted by  $E_t$ , the corresponding values are reported in Table 6. For the CMP-EEWMA chart, the control limit constant is  $k_1=2.7526$ , which yields an upper control limit of  $UCL_1=133.82$  and a lower control limit of  $LCL_1=78.59$ . For the MDS-CMP-EEWMA chart, the previous three samples ( $i=3$ ) are incorporated into the sampling procedure. The resulting control limit constants are  $k_1=2.9326$  and  $k_2=2.3139$ , corresponding to outer control limits of  $UCL_1=136.74$  and  $LCL_1=75.67$ , respectively. In addition, the inner control limits are  $UCL_2=127.82$  and  $LCL_2=84.59$ . These limits define the decision regions used by the MDS-CMP-EEWMA chart to enhance monitoring sensitivity while maintaining the desired in-control performance.

For monitoring weekly AQI count data, an earlier signal corresponds to a shorter detection time and greater sensitivity to shifts in weekly AQI counts. As shown in Table 8 and Figure 2, all monitoring statistics remain within the control limits of the CMP-EEWMA chart, indicating that no out-of-control condition is detected. In contrast, the proposed MDS-CMP-EEWMA chart generates its first out-of-control signal at week 16. This indicates that the signal around week 16 reflects a sustained period of elevated AQI conditions rather than a single isolated hourly event. This result demonstrates that incorporating information from previous samples through the MDS sampling scheme enhances the chart's ability to detect process shifts that remain undetected by the conventional CMP-EEWMA chart. Consequently, the MDS-CMP-EEWMA chart is more sensitive to increases in weekly AQI count data, thereby providing earlier warnings of potential air quality deterioration.

## 7. Conclusions

This study examines COM-Poisson count data. First, a CMP-EEWMA control chart is proposed. Subsequently, an MDS sampling scheme is incorporated to develop the MDS-CMP-EEWMA chart, thereby enhancing shift detection performance. Methodologically, a Monte Carlo-based design procedure is adopted to determine control limit constants  $k_1$  and  $k_2$  to construct the corresponding outer and inner control limits under a pre-specified in-control requirement. The numerical results indicate that for downward process shifts, parameter settings  $\lambda_1=0.05$  and  $\lambda_2=0.03$  yield smaller  $ARL_1$  values across most shift magnitudes, implying faster signaling, particularly for small shifts. Moreover, the proposed MDS-CMP-EEWMA chart produces substantially smaller  $ARL_1$  values than

the CMP-EEWMA chart without the MDS sampling strategy, demonstrating that incorporating dependent-state decision rules can effectively reduce detection delay. In addition, the simulation results show that the proposed MDS-CMP-EEWMA chart provides earlier signals than the CMP-EEWMA, existing MDS-CMP-EWMA, and CMP-EWMA charts in many small-shift scenarios, particularly for downward shifts. However, the proposed chart is not uniformly superior under all conditions; in some upward shift scenarios, the conventional CMP-EEWMA chart may perform better. Therefore, the selection of an appropriate control chart should consider the shift direction, dispersion condition, and parameter settings.

Application to weekly AQI count data revealed that the proposed MDS-CMP-EEWMA chart can detect abnormal increases in weekly AQI exceedance counts earlier than the conventional CMP-EEWMA chart. Thus, it can serve as an early-warning tool for environmental monitoring and support timely air quality management and sustainability-related decision-making.

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**Conflicts of Interest:** The authors declare no conflicts of interest.

## Appendix A

The statistic  $E_t$  of the CMP-EEWMA chart in Equation (4) is as follows:

$$E_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1 - \lambda_1 + \lambda_2) E_{t-1}. \quad (\text{A1})$$

To derive the general form of the statistic at time point  $t$ , we first let  $t=1$ , which yields:

$$E_1 = \lambda_1 X_1 - \lambda_2 X_0 + (1 - \lambda_1 + \lambda_2) E_0. \quad (\text{A2})$$

When  $t=2$ , it follows that

$$\begin{aligned} E_2 &= \lambda_1 X_2 - \lambda_2 X_1 + (1 - \lambda_1 + \lambda_2) E_1 \\ &= \lambda_1 X_2 - \lambda_2 X_1 + (1 - \lambda_1 + \lambda_2) [\lambda_1 X_1 - \lambda_2 X_0 + (1 - \lambda_1 + \lambda_2) E_0] \\ &= \lambda_1 X_2 - [\lambda_2 - \lambda_1(1 - \lambda_1 + \lambda_2)] X_1 - \lambda_2(1 - \lambda_1 + \lambda_2) X_0 + (1 - \lambda_1 + \lambda_2)^2 E_0. \end{aligned} \quad (\text{A3})$$

When  $t=3$ , it follows that

$$\begin{aligned} E_3 &= \lambda_1 X_3 - \lambda_2 X_2 + (1 - \lambda_1 + \lambda_2) E_2 \\ &= \lambda_1 X_3 - \lambda_2 X_2 + (1 - \lambda_1 + \lambda_2) \\ &\quad \times \{ \lambda_1 X_2 - [\lambda_2 - \lambda_1(1 - \lambda_1 + \lambda_2)] X_1 - \lambda_2(1 - \lambda_1 + \lambda_2) X_0 + (1 - \lambda_1 + \lambda_2)^2 E_0 \} \\ &= \lambda_1 X_3 - [\lambda_2 - \lambda_1(1 - \lambda_1 + \lambda_2)] X_2 \\ &\quad - [\lambda_2(1 - \lambda_1 + \lambda_2) - \lambda_1(1 - \lambda_1 + \lambda_2)^2] X_1 - \lambda_2(1 - \lambda_1 + \lambda_2)^2 X_0 + (1 - \lambda_1 + \lambda_2)^3 E_0. \end{aligned} \quad (\text{A4})$$

From (A2)–(A4), the general expression of  $E_t$  is given by:

$$E_t = \lambda_1 X_t + bX_{t-1} + abX_{t-2} + a^2bX_{t-3} + \dots + a^{t-2}bX_1 - a^{t-1}\lambda_2 X_0 + a^t E_0, \quad (A5)$$

where  $a = (1 - \lambda_1 + \lambda_2)$  and  $b = a\lambda_1 - \lambda_2$ . The general expression is completely derived.

## Appendix B

Assume that  $X_t$  is an independent random sample following a COM-Poisson distribution with mean  $\mu_0^{1/v} - \frac{v-1}{2v}$  and variance  $\frac{\mu_0^{1/v}}{v}$  at time  $t$ . The expected value of the CMP-EEWMA statistic  $E_t$  in Equation (A5) is:

$$\begin{aligned} E(E_t) &= E(\lambda_1 X_t + bX_{t-1} + abX_{t-2} + a^2bX_{t-3} + \dots + a^{t-2}bX_1 - a^{t-1}\lambda_2 X_0 + a^t E_0) \\ &= (\lambda_1 + b + ab + \dots + a^{t-2}b - a^{t-1}\lambda_2 + a^t) \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \\ &= (\lambda_1 + b[1 + a + a^2 + \dots + a^{t-2}] - a^{t-1}\lambda_2 + a^t) \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \\ &= \left( \lambda_1 + b \left[ \frac{1-a^{t-1}}{1-a} \right] - a^{t-1}\lambda_2 + a^t \right) \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \end{aligned}$$

Substitute  $b = a\lambda_1 - \lambda_2$  then,

$$\begin{aligned} E(E_t) &= \left\{ \lambda_1 + [a\lambda_1 - \lambda_2] \left[ \frac{1-a^{t-1}}{1-a} \right] - a^{t-1}\lambda_2 + a^t \right\} \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \\ &= \left\{ \lambda_1 + \frac{a\lambda_1(1-a^{t-1})}{1-a} - \left[ \frac{\lambda_2(1-a^{t-1})}{1-a} + a^{t-1}\lambda_2 \right] + a^t \right\} \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \\ &= \left\{ \lambda_1 \left[ \frac{1-a^t}{1-a} \right] - \lambda_2 \left[ \frac{1-a^t}{1-a} \right] + a^t \right\} \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \\ &= \left\{ (\lambda_1 - \lambda_2) \left[ \frac{1-a^t}{1-a} \right] + a^t \right\} \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \end{aligned}$$

Substitute  $a = (1 - \lambda_1 + \lambda_2)$  then,

$$\begin{aligned} E(E_t) &= \left\{ (\lambda_1 - \lambda_2) \left[ \frac{1-(1-\lambda_1+\lambda_2)^t}{1-(1-\lambda_1+\lambda_2)} \right] + (1-\lambda_1+\lambda_2)^t \right\} \left( \mu_0^{1/v} - \frac{v-1}{2v} \right) \\ &= \mu_0^{1/v} - \frac{v-1}{2v} \end{aligned} \quad (A6)$$

The variance of the CMP-EEWMA statistic  $E_t$  in Equation (A5) is:

$$\begin{aligned} Var(E_t) &= Var(\lambda_1 X_t + bX_{t-1} + abX_{t-2} + a^2bX_{t-3} + \dots + a^{t-2}bX_1 - a^{t-1}\lambda_2 X_0 + a^t E_0) \\ &= (\lambda_1^2 + b^2 + a^2b^2 + \dots + a^{2(t-2)}b^2) \cdot \frac{\mu_0^{1/v}}{v} \\ &= (\lambda_1^2 + b^2(1 + a^2 + \dots + a^{2(t-2)})) \cdot \frac{\mu_0^{1/v}}{v} \\ &= \left( \lambda_1^2 + b^2 \left( \frac{1-a^{2(t-1)}}{1-a^2} \right) \right) \cdot \frac{\mu_0^{1/v}}{v} \end{aligned}$$

Since  $b = a\lambda_1 - \lambda_2$ , we get,

$$\begin{aligned}
 \text{Var}(E_t) &= \left\{ \lambda_1^2 + [a^2 \lambda_1^2 - 2a\lambda_1\lambda_2 + \lambda_2^2] \left( \frac{1-a^{2(t-1)}}{1-a^2} \right) \right\} \cdot \frac{\mu_0^{1/v}}{v} \\
 &= \left\{ \lambda_1^2 + [a\lambda_1 - \lambda_2]^2 \left( \frac{1-a^{2(t-1)}}{1-a^2} \right) \right\} \cdot \frac{\mu_0^{1/v}}{v} \\
 &= \left\{ (\lambda_1^2 - 2a\lambda_1\lambda_2 + \lambda_2^2) \left( \frac{1-a^{2(t-1)}}{1-a^2} \right) \right\} \cdot \frac{\mu_0^{1/v}}{v}
 \end{aligned}$$

Substitute  $a = (1 - \lambda_1 + \lambda_2)$ , we get,

$$\text{Var}(E_t) = \left\{ (\lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2) \left( \frac{1 - (1 - \lambda_1 + \lambda_2)^{2(t-1)}}{1 - (1 - \lambda_1 + \lambda_2)^2} \right) \right\} \cdot \frac{\mu_0^{1/v}}{v} \quad (\text{A7})$$

When  $t \rightarrow \infty$ , the CMP-EEWMA chart is approximated as:

$$\begin{aligned}
 UCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} + k_1 \sqrt{\left( \frac{\lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2}{1 - (1 - \lambda_1 + \lambda_2)^2} \right) \cdot \frac{\mu_0^{1/v}}{v}} \\
 CL &= \mu_0^{1/v} - \frac{v-1}{2v} \\
 LCL_1 &= \mu_0^{1/v} - \frac{v-1}{2v} - k_1 \sqrt{\left( \frac{\lambda_1^2 - 2(1 - \lambda_1 + \lambda_2)\lambda_1\lambda_2 + \lambda_2^2}{1 - (1 - \lambda_1 + \lambda_2)^2} \right) \cdot \frac{\mu_0^{1/v}}{v}}
 \end{aligned} \quad (\text{A8})$$

Equations (6) and (7) are complete derivation in context.

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