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Dual Pairs of Holomorphic Representations of Lie Groups from a Vector-Coherent-State Perspective

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Abstract: It is shown that, for both compact and non-compact Lie groups, vector-coherent-state methods provide straightforward derivations of holomorphic representations on symmetric spaces. Complementary vector-coherent-state methods are introduced to derive pairs of holomorphic representations which are bi-orthogonal duals of each other with respect to a simple Bargmann inner product. It is then shown that the dual of a standard holomorphic representation has an integral expression for its inner product, with a Bargmann measure and a simply-defined kernel, which is not restricted to discrete-series representations. Dual pairs of holomorphic representations also provide practical ways to construct orthonormal bases for unitary irreps which bypass the need for evaluating the integral expressions for their inner products. This leads to practical algorithms for the application of holomorphic representations to model problems with dynamical symmetries in physics.

Keywords: coherent-state representations; vector-coherent-state representations; dual holomorphic representations; representations of simple Lie groups

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1. Introduction

Pairs of groups that have dual representations on a Hilbert space, such as those given by the Schur-Weyl duality theorem [1,2] and by Howe's dual reductive pairs [3], have wide applications in both mathematics [4] and physics [5,6]. They were shown [7], for example, to expose an intimate relationship between symmetry groups and dynamical groups in the quantum mechanics of many-particle systems. However, this contribution is concerned with a different kind of duality relationship that emerges in vector-coherent-state theory [8,9].

VCS theory is a physics version of the mathematical theory of induced representations [10,11] which focuses on the construction of irreducible representations. Vector coherent states were introduced [8] for the purpose of deriving explicit realisations of the holomorphic representations of the non-compact symplectic groups and their Lie algebras, as needed in applications of a symplectic model in nuclear collective dynamics [12,13], and were subsequently found to have many other applications [11,14–17], *i.e.*, induced representations of a more general type [18–22]. Partially-coherent state representations, related to VCS representations, were also introduced for this purpose by Deenen and Quesne [23]. A recent review of the applications of VCS theory in nuclear physics can be found in [24]. Classes of vector coherent states, for which there is a resolution of the identity, have been considered more recently by Ali and others [25–27] but not, as far as we are aware, for the construction of Lie group or Lie algebra representations.

Holomorphic discrete-series representations of connected non-compact simple Lie groups were defined by Harish-Chandra [28–30] and further explored by Godement [31], Gelbart [32] and others [33].

However, for computational applications in physics, the evaluation of the integral expressions for their inner products posed problems. Moreover, they applied only to holomorphic representations which are in the discrete series, even though there are others that are sometimes needed. (Discrete series representations of a Lie group are subrepresentations of its regular representation.) Thus, complementary coherent-state methods were developed [8,9] for deriving the matrix elements of the infinitesimal generators of many Lie groups in terms of a VCS generalisation of standard coherent-state representations [34–36]. Essential ingredients of VCS theory, which enabled it to provide practical computational algorithms, were the so-called K-matrix methods [8,37] for calculating inner products.

VCS theory was shown [38] to reproduce the Harish-Chandra/Godement expressions of the $\mathrm{Sp}(N, \mathbb{R})$ holomorphic discrete-series representations. It was also used to derive explicit representations of numerous Lie algebras, including a graded (super) Lie algebra [17], as outlined in several reviews [11,39,40]. In this paper, a refined version of K-matrix theory is developed and its underlying structure is exposed in terms of complementary VCS irreps on simply-defined extensions of Bargmann Hilbert spaces. These irreps are not unitary but are equivalent to unitary irreps. They are biorthogonal duals of each other relative to the extended Bargmann inner products and in combination, a complementary pair of irreps define an orthonormal basis for an irreducible unitary representation. This approach provides systematic procedures for calculating the explicit matrices of irreducible Lie algebra representations as needed in applications of dynamical symmetry in physics. Moreover, they are not restricted to discrete-series representations and apply to both compact and non-compact Lie groups.

For the purpose of constructing unitary irreps of a Lie group G_0, coherent states are most usefully defined [36,41] as elements of a minimum-dimensional orbit of G_0 within the Hilbert space of an irreducible unitary representation. For simplicity, we here restrict consideration to connected semi-simple Lie groups and irreps with extremal (highest- or lowest-weight) states. A minimum-dimensional orbit for such an irrep is one that contains an extremal state. The Lie algebra of G, the complex extension of G_0, is then of the form $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-$, where $\mathfrak{k}$ is the complex extension of the Lie algebra of the isotropy subgroup $K_0 \subset G_0$ that leaves the extremal state invariant; $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are, respectively, subalgebras of raising and lowering operators.

Harish-Chandra showed that the quotient $K_0 \backslash G_0$ can be regarded in a natural way as a bounded open subset of the complex vector space $\mathfrak{n}_+$. This is shown in the VCS context in Section 2.3. It gives a natural complex structure on the quotient space and makes it possible to discuss holomorphic functions. This underlies the construction of holomorphic discrete series [28–30].

Construction of a holomorphic unitary irrep of G_0 by coherent-state methods is then straightforward when $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are Abelian. However, this condition imposes an unacceptable restriction on the set of holomorphic representations that can be constructed by coherent state methods. Fortunately, the construction can be extended by considering a larger group $K_0 \subset G_0$ such that $\mathfrak{g}$ continues to be of the form $\mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-$ for some subalgebras $\mathfrak{n}_\pm$ which are Abelian. Consider an irrep of G_0 which contains an irrep of K_0 that is extremal in the sense that it is killed by $\mathfrak{n}_-$ (or by $\mathfrak{n}_+$). Then, instead of the orbit of a single extremal state, we consider the orbit of this K_0-irrep. With this extension, it becomes straightforward to construct all the holomorphic discrete series irreps, considered by Harish-Chandra, plus others that are limits of the discrete series.

It is known [42,43] that coherent-state orbits are diffeomorphic to coadjoint orbits. It is also known [44,45] that coadjoint orbits are classical phase spaces and carry classical representations of the dynamics of the model for which G_0 is a dynamical group. Thus, the construction of the unitary irreps of a Lie group G_0 by VCS methods is, in effect, a partial realisation of Kirillov's objectives of constructing unitary irreps from coadjoint orbits. As shown by Bartlett *et al.* [46], the VCS construction is also a powerful tool in realising the objectives of geometric quantisation.

This paper starts with consideration of a unitary irrep with lowest weight κ of a connected non-compact semi-simple Lie group G_0 that has a Lie algebra $\mathfrak{g}_0$ with complex extension $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-$ for which $\mathfrak{k}$ is the complex extension of a compact subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$ and $\mathfrak{n}_\pm$ are, respectively, subalgebras of Abelian raising and lowering operators. Vector coherent states are then defined for this irrep based on a set of lowest-grade states $\{|\kappa\alpha\rangle\}$, *i.e.*, states that are annihilated by the lowering operators of $\mathfrak{n}_-$ and are a basis for an irrep $\hat{\sigma}_\kappa$ of $\mathfrak{k}_0$.

We adopt conventions standard in physics, in which $\hat{X}^\dagger$ denotes the Hermitian adjoint of an operator $\hat{X}$, $\tilde{M}$ denotes the transpose of a matrix M, and $*$ denotes complex conjugation. Bases of raising and lowering operators will be denoted, respectively, by $\{A_i, i = 1, \ldots, d\}$ and $\{B_i, i = 1, \ldots, d\}$, and the representation of any element $X \in \mathfrak{g}$ in the representation with lowest weight κ will be denoted by $\hat{X}$. The raising and lowering operators will be defined such that, in a unitary irrep, $\hat{B}_i = \hat{A}_i^\dagger$.

Following a derivation in Section 2 and 3 of the Harish-Chandra holomorphic discrete-series representations from a VCS perspective, the following sections develop the dual VCS theory of holomorphic representation. An early version of the theory was initiated by the authors [47] in terms of coherent-state triplets and led to many analytical results for scalar coherent-state representations. The present VCS developments are new and apply to a much wider class of representations.

If $\mathbb{H}^\kappa$ denotes the Hilbert space of a unitary irrep of the Lie algebra $\mathfrak{g}_0$ with lowest weight κ, a state $|\psi_\nu\rangle \in \mathbb{H}^\kappa$ is observed to have two naturally-defined VCS wave functions: one, Φ_ν, defined by the expansion of the state $|\psi_\nu\rangle$ as a combination of vector coherent states

$$\begin{aligned} |\psi_\nu\rangle &= \sum_\alpha \int e^{\sum_i z_i^* \hat{A}_i} |\kappa\alpha\rangle \Phi_{\nu\alpha}(z)\, dv(z), \\ &= \int e^{\sum_i z_i^* \hat{A}_i} \Phi_\nu(z)\, dv(z), \end{aligned} \tag{1}$$

where $dv(z)$ is any conveniently-chosen measure; the other, Ψ_ν, defined by its overlaps with vector coherent states

$$\Psi_\nu(z) = \sum_\alpha |\kappa\alpha\rangle \langle\kappa\alpha| e^{\sum_i z_i \hat{B}_i} |\psi_\nu\rangle. \tag{2}$$

(Note that a state in a Hilbert space can be represented by a variety of wave functions. For example, a given eigenstate of a particle in a harmonic oscillator potential can be represented by a function of its position coordinates and by a function of its momentum coordinates. A remarkable property of a dual pair of VCS wave functions for a state is that they are different functions of a common set of variables.)

If one chooses dv to be the Bargmann measure [48], for which

$$\int e^{\sum_i x_i y_i^*} \phi(y)\, dv(y) = \phi(x), \tag{3}$$

one obtains from these equations the powerful results

$$\langle\psi_\mu|\psi_\nu\rangle = (\Phi_\mu, \Psi_\nu) \tag{4}$$

where

$$(\Phi_\mu, \Psi_\nu) = \sum_\alpha \int \Phi_{\mu\alpha}^*(z) \Psi_{\nu\alpha}(z)\, dv(z) \tag{5}$$

and

$$\Psi_\nu(x) = \hat{S}\Phi_\nu(x) = \int \hat{S}(x, y^*) \Phi_\nu(y)\, dv(y), \tag{6}$$

with

$$\hat{S}(x, y^*) = \sum_{\alpha,\beta} |\kappa\alpha\rangle \langle\kappa\alpha| e^{\sum_i x_i \hat{B}_i} e^{\sum_i y_i^* \hat{A}_i} |\kappa\beta\rangle \langle\kappa\beta|. \tag{7}$$

Systematic procedures are given in the text for constructing the unitary VCS irreps with these dual VCS wave functions. Similar constructions are also given for compact Lie groups defined by highest-weights.

2. VCS Construction of Holomorphic Discrete-series Representations

It is shown in this section that VCS theory reproduces the standard results for holomorphic-discrete-series representations.

Let G_0 be a real, simple and connected Lie group with Lie algebra $\mathfrak{g}_0$ which, if non-compact, has a faithful finite-dimensional representation. (Extension to a reductive Lie group is straightforward but will not be considered.) Without loss of generality, we regard G_0 as a matrix subgroup of $\mathrm{GL}(n,\mathbb{C})$. Let G with Lie algebra $\mathfrak{g}$ denote the complex extension of G_0. Then, G_0 has unitary holomorphic representations if its Lie algebra $\mathfrak{g}_0$ has a compact subalgebra $\mathfrak{k}_0$ that contains a Cartan subalgebra for G_0 and $\mathfrak{g}$ is expressible as a direct sum

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-, \tag{8}$$

in which $\mathfrak{k}$ is the complex extension of $\mathfrak{k}_0$, and $\mathfrak{n}_\pm$ are, respectively, Abelian subalgebras of raising and of lowering operators for which $[\mathfrak{k},\mathfrak{n}_\pm] \in \mathfrak{n}_\pm$ and $[\mathfrak{n}_-,\mathfrak{n}_+] \in \mathfrak{k}$.

The conditions on G_0 are such that, if $K_0 \subset G_0$ is a compact subgroup with Lie algebra $\mathfrak{k}_0$, G_0/K_0 and $K_0\backslash G_0$ are symmetric spaces, and G_0 and K_0 have defining representations of the block-matrix form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0, \tag{9}$$

$$k = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in K_0, \tag{10}$$

with $a, e \in M_{pp}(\mathbb{C})$, $d, f \in M_{qq}(\mathbb{C})$, $b \in M_{pq}(\mathbb{C})$ and $c \in M_{qp}(\mathbb{C})$. Likewise, the subgroups of G generated by the subalgebras $\mathfrak{n}_\pm$ can be defined by matrix representations of the form

$$\begin{pmatrix} I_p & x \\ 0 & I_q \end{pmatrix} = e^{X(x)} \quad \text{with} \quad X(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}_+, \tag{11}$$

$$\begin{pmatrix} I_p & 0 \\ z & I_q \end{pmatrix} = e^{Z(z)} \quad \text{with} \quad Z(z) = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \in \mathfrak{n}_-, \tag{12}$$

in which $I_p \in M_{pp}$ and $I_q \in M_{qq}$ are identity matrices.

Substantial use will be made in the following of the Gauss factorisation

$$g(a,b,c,d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I_p & bd^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} I_p & 0 \\ d^{-1}c & I_q \end{pmatrix}, \tag{13}$$

where a, b, c, and d are real or complex matrices, which applies to any matrix of the given block matrix form, provided $\det(d) \neq 0$.

2.1. Holomorphic VCS Representations of the Group G_0

Let $\hat{\mathcal{U}}^\kappa$ denote an irreducible unitary representation of G_0 and its Lie algebra $\mathfrak{g}_0$ on a Hilbert space $\mathbb{H}^\kappa$ and let $\hat{T}^\kappa$ denote the extension of $\hat{\mathcal{U}}^\kappa$ to the complex Lie algebra $\mathfrak{g}$. We also let $\hat{T}^\kappa$ denote the extension of $\hat{\mathcal{U}}^\kappa$ from the compact subgroup $K_0 \subset G_0$ to the corresponding complex group $K \subset G$. The matrix

$$c_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \tag{14}$$

then has the property that the subalgebra $\mathfrak{k}$ is an eigenspace of $\mathrm{ad}(c_0)$ with zero eigenvalue, *i.e.*,

$$[c_0, X] = 0, \quad \text{for } X \in \mathfrak{k}, \tag{15}$$

and the subalgebras $\mathfrak{n}_\pm$ are eigenspaces of $\mathrm{ad}(c_0)$ with respective eigenvalues ± 2, given by

$$[c_0, X(x)] = +2\,X(x), \quad [c_0, Z(z)] = -2\,Z(z). \tag{16}$$

We consider holomorphic representations for which the spectrum of the self-adjoint operator $\hat{c}_0 = \hat{T}^\kappa(c_0)$ on $\mathbb{H}^\kappa$ is bounded from below (if the required spectrum is bounded from above, replace c_0 with -c_0). The eigenspace belonging to the least $\hat{c}_0$ eigenvalue is then called the lowest-grade subspace. It is the subspace $\mathbb{H}_0^\kappa$ of states in $\mathbb{H}^\kappa$ that are annihilated by the lowering operators representing elements of $\mathfrak{n}_-$.

Each eigenspace of $\hat{c}_0$ is a K_0-invariant subspace of $\mathbb{H}^\kappa$ and is a direct sum of subspaces for unitary irreps of K_0. In particular, the lowest-grade subspace, $\mathbb{H}_0^\kappa$, carries a single irrep of $\mathfrak{k}$, which we denote by $\hat{\sigma}^\kappa$. As will be seen, this irrep $\hat{\sigma}^\kappa$ uniquely determines, up to unitary equivalence, the irrep $\hat{\mathcal{U}}^\kappa$ of G_0. Thus, the irrep $\hat{\mathcal{U}}^\kappa$ and its Hilbert space $\mathbb{H}^\kappa$ are appropriately labelled by the highest weight κ of the irrep $\hat{\sigma}^\kappa$. Thus, if $\hat{\Pi}^\kappa$ denotes the projection operator

$$\hat{\Pi}^\kappa : \mathbb{H}^\kappa \to \mathbb{H}_0^\kappa \tag{17}$$

to the lowest-grade subspace of the Hilbert space $\mathbb{H}^\kappa$ that we wish to construct, the representation $\hat{\sigma}^\kappa$ is related to the representation $\hat{T}^\kappa$ by the intertwining relationship

$$\hat{\sigma}^\kappa(k)\hat{\Pi}^\kappa = \hat{\Pi}^\kappa \hat{T}^\kappa(k), \quad \forall\, k \in K \subset G. \tag{18}$$

The objective is now to induce the representation $\hat{\mathcal{U}}^\kappa$ of G_0 from the irrep $\hat{\sigma}^\kappa$ of its subgroup K_0 by VCS methods. If $\hat{Z}(z)$ represents an element $Z(z) \in \mathfrak{n}_-$, its Hermitian adjoint $\hat{Z}^\dagger(z)$ represents an element of $\mathfrak{n}_+$ and is a raising operator for the representation under construction. A set of coherent states $\{e^{\hat{Z}^\dagger(z)}|\alpha\rangle, Z(z) \in \mathfrak{n}_-\}$ is then defined for every vector $|\alpha\rangle$ in the space of lowest-grade states.

Holomorphic wave functions for states of the Hilbert space $\mathbb{H}^\kappa$ can now be defined. Observe that an element $Z(z) \in \mathfrak{n}_-$ can be expanded $Z(z) = \sum_i z_i B_i$, in a basis $\{B_i, i = 1, \ldots, d\}$ for the d-dimensional subalgebra $\mathfrak{n}_-$. A vector z, with components $\{z_i\}$, is then an element of a complex vector space $\mathcal{Z}$ that is isomorphic, as a vector space, to $\mathfrak{n}_-$, and the Hilbert space $\mathbb{H}^\kappa$ is spanned by a subset of the coherent states $\{e^{\hat{Z}^\dagger(z)}|\alpha\rangle, |\alpha\rangle \in \mathbb{H}_0^\kappa, z \in D\}$, where D is an open neighbourhood of $z = 0$ in $\mathcal{Z}$. Thus, in this paper, the same symbol z is used, without ambiguity, to denote both an element of $\mathcal{Z}$ and the corresponding matrix representing an element of $\mathfrak{n}_-$ in Equation (12). It follows that a state vector $|\psi\rangle \in \mathbb{H}^\kappa$ can be represented by a VCS wave function Ψ which is a holomorphic function of the complex variables $\{z_i\}$ that takes values in $\mathbb{H}_0^\kappa$, given by

$$\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}|\psi\rangle, \quad z \in D. \tag{19}$$

The corresponding holomorphic VCS representation $\hat{\Gamma}$, isomorphic to the desired irrep $\hat{\mathcal{U}}^\kappa$, is then defined by

$$\hat{\Gamma}(g)\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}\hat{\mathcal{U}}^\kappa(g)|\psi\rangle, \quad \forall\, g \in G_0. \tag{20}$$

In fact, because the VCS wave functions are holomorphic functions of the variables $\{z_i\}$, their values are defined at all $z \in \mathcal{Z}$ by their values in any open neighbourhood D of $z = 0$ in $\mathcal{Z}$. Let $\mathcal{F}^\kappa(D)$ denote the space of holomorphic functions on D with values in $\mathbb{H}_0^\kappa$, defined by Equation (19). The VCS linear mapping

$$\mathrm{VCS} : \mathbb{H}^\kappa \to \mathcal{F}^\kappa(D);\ |\psi\rangle \mapsto \Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}|\psi\rangle \tag{21}$$

then intertwines the actions $\hat{\mathcal{U}}^\kappa(g)$ and $\hat{\Gamma}(g)$ for all $g \in G_0$, i.e.,

$$\hat{\Gamma}(g)\hat{\Pi}^\kappa e^{\hat{Z}(z)} = \hat{\Pi}^\kappa e^{\hat{Z}(z)}\hat{\mathcal{U}}^\kappa(g). \tag{22}$$

Consider the action $\hat{\Gamma}(g)$ defined by Equation (20) of a group element $g = g(a,b,c,d) \in G_0$ starting from the observation that the product $e^{\hat{Z}(z)}\hat{\mathcal{U}}^{\kappa}(g)$ is a representation of the element

$$e^{Z(z)}g = \begin{pmatrix} I_p & 0 \\ z & I_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c+za & d+zb \end{pmatrix} \in \mathrm{G}. \tag{23}$$

Thus, provided $\det(d+zb) \neq 0$, the product $e^{Z(z)}g$ has the Gauss factorisation given, according to Equation (13), by

$$\begin{aligned} e^{Z(z)}g(a,b,c,d) = &\begin{pmatrix} I_p & b(d+zb)^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} a-b(d+zb)^{-1}(c+za) & 0 \\ 0 & d+zb \end{pmatrix} \\ &\times \begin{pmatrix} I_p & 0 \\ (d+zb)^{-1}(c+za) & I_q \end{pmatrix}. \end{aligned} \tag{24}$$

It follows that, for $g \in G_0$ and $z \in D$,

$$\hat{\Gamma}(g)\Psi(z) = \hat{\Pi}\Gamma^{\kappa} e^{\hat{Z}(z)}\hat{\mathcal{U}}^{\kappa}(g)|\psi\rangle = \hat{\sigma}^{\kappa}(k)\Psi\big((d+zb)^{-1}(c+za)\big), \tag{25}$$

where $k \in K$ is the element given in the defining representation by

$$k = \begin{pmatrix} a-b(d+zb)^{-1}(c+za) & 0 \\ 0 & d+zb \end{pmatrix}. \tag{26}$$

The domain D on which the VCS representation $\hat{\Gamma}$ acts can now be defined as a convenient subset of $\mathcal{Z}$ that is invariant under the transformations

$$z \to (d+zb)^{-1}(c+za) \tag{27}$$

for all the group elements $g(a,b,c,d) \in G_0$. The domain D must also exclude any z for which $\det(d+zb)$ could vanish, or be such that all such points are of zero measure. Such a domain is identified with a subset of $K_0\backslash G_0$ as follows. Let $P \subset G$ denote the parabolic subgroup with Lie algebra

$$\mathfrak{p} = \mathfrak{n}_+ \oplus \mathfrak{k}. \tag{28}$$

The action given by Equation (27) allows us to canonically identify $K_0\backslash G_0$ with an open submanifold of $P\backslash G$ [29,49,50], and by regarding $\exp\big(Z(z)\big)$ as a representative of a coset $P\exp\big(Z(z)\big) \in P\backslash G$, the complex numbers $z = \{z_i\}$ become coordinates for $P\backslash G$. This is discussed further in Section 2.3.

Note that the above results have not been restricted to representations of the discrete series.

2.2. Holomorphic Representation of a Lie Algebra $\mathfrak{g}$

The VCS derived representation $\hat{\Gamma}(X)$ of an element X in the Lie algebra $\mathfrak{g}$ is defined as for the Lie group G_0 by

$$\hat{\Gamma}(X)\Psi(z) = \hat{\Pi}\Gamma^{\kappa} e^{\hat{Z}(z)}\hat{X}|\psi\rangle, \quad \hat{Z}(z) = z\cdot\hat{B} = \sum_i z_i\hat{B}_i, \tag{29}$$

where $\{B_i\}$ is a basis for $\mathfrak{n}_-$. Explicit expressions are obtained from the expansion

$$e^{\hat{Z}}\hat{X} = \big(\hat{X} + [\hat{Z},\hat{X}] + \tfrac{1}{2!}[\hat{Z},[\hat{Z},\hat{X}]]\big)e^{\hat{Z}} \tag{30}$$

and the identities

$$\hat{\Pi}^\kappa \hat{B}_i e^{\hat{Z}(z)}|\psi\rangle = \partial_i \Psi(z) \equiv \partial\Psi(z)/\partial z_i, \tag{31}$$

$$\hat{\Pi}^\kappa \hat{C}_p e^{\hat{Z}(z)}|\psi\rangle = \hat{\mathfrak{C}}_p \Psi(z), \quad \forall\, C_p \in \mathfrak{k}, \tag{32}$$

$$\hat{\Pi}^\kappa \hat{A}_i e^{\hat{Z}(z)}|\psi\rangle = 0, \tag{33}$$

where $\hat{A}_i$ is the Hermitian adjoint of $B_i^\dagger$, $\hat{\mathfrak{C}}_p = \hat{\sigma}^\kappa(C_p)$ is defined, in accordance with Equation (18), by

$$\hat{\sigma}^\kappa(C_p)\hat{\Pi}^\kappa = \hat{\Pi}^\kappa \hat{C}_p, \quad \forall\, C_p \in \mathfrak{k}, \tag{34}$$

and $\{C_p\}$ is a basis for $\mathfrak{k}$, the Lie algebra of K. From these definitions, it follows that

$$\hat{\Gamma}(B_i) = \partial_i, \tag{35}$$

$$\hat{\Gamma}(C_p) = \hat{\mathfrak{C}}_p + \sum_{ij} \hat{z}_i C_p^{ij} \partial_j, \tag{36}$$

$$\hat{\Gamma}(A_j) = \sum_{ip} \hat{z}_i D_p^{ij} \Big(\hat{\mathfrak{C}}_p + \tfrac{1}{2}\sum_{kl} \hat{z}_k C_p^{kl} \partial_l\Big), \tag{37}$$

where $\hat{z}_i$ is the multiplicative operator for which $\hat{z}_i\Psi(z) = z_i\Psi(z)$. The first of these equations is obtained immediately. The second is obtained from the expansion $[B_i, C_p] = \sum_j C_p^{ij} B_j$ and

$$e^{Z(z)} C_p = \Big(C_p + [Z(z), C_p]\Big) e^{Z(z)} = \Big(C_p + \sum_j z_i C_p^{ij} B_j\Big) e^{Z(z)}. \tag{38}$$

The third is obtained from the expansion $[B_i, A_j] = \sum_p D_p^{ij} C_p$ followed by Equation (38).

2.3. The Inner Product for Discrete-series Representations

This section confirms that, when the VCS construction is applied to a discrete-series representation, it reproduces the familiar expression for the inner products of its wave functions as obtained by the standard methods [28–31,51].

Let $\{|\kappa\alpha\rangle\}$ denote an orthonormal basis for the lowest-grade subspace $\mathbb{H}_0^\kappa$ of the Hilbert space $\mathbb{H}^\kappa$ for an irreducible discrete-series representation $\hat{\mathcal{U}}^\kappa$ of G_0. The inner product for the Hilbert space of standard coherent-state wave functions for this representation is then obtained from the resolution of the identity on $\mathbb{H}^\kappa$ [34,35],

$$\hat{I} = \int_{G_0} \hat{\mathcal{U}}^{\kappa\dagger}(g)|\kappa\alpha\rangle\langle\kappa\alpha|\hat{\mathcal{U}}^\kappa(g)\, dv(g), \tag{39}$$

where $dv(g)$ is the suitably normalised G_0-invariant measure. If the integral in this expression is restricted to the compact subgroup $K_0 \subset G_0$, then the modified operator continues to be the identity operator on the subspace $\mathbb{H}_0^\kappa \subset \mathbb{H}^\kappa$ but on $\mathbb{H}^\kappa$ it becomes the projection operator

$$\hat{\Pi}^\kappa = \int_{K_0} \hat{\mathcal{U}}^{\kappa\dagger}(k)|\kappa\alpha\rangle\langle\kappa\alpha|\hat{\mathcal{U}}^\kappa(k)\, dv(k) \;:\; \mathbb{H}^\kappa \to \mathbb{H}_0^\kappa. \tag{40}$$

This implies that the identity operator (39) can be expressed

$$\hat{I} = \int_{G_0} \hat{\mathcal{U}}^{\kappa\dagger}(g)\hat{\Pi}^\kappa\hat{\mathcal{U}}^\kappa(g)\, \frac{1}{\dim\hat{\sigma}^\kappa}\, dv(g), \tag{41}$$

where $\dim\hat{\sigma}^\kappa$ is the dimension of the irrep κ of K_0. However, because of the K_0 invariance of $\hat{\Pi}^\kappa$,

$$\hat{\mathcal{U}}^{\kappa\dagger}(k)\hat{\Pi}^\kappa\hat{\mathcal{U}}^\kappa(k) = \hat{\Pi}^\kappa, \quad \text{for } k \in K_0, \tag{42}$$

the integrand $\hat{\mathcal{U}}^{\kappa\dagger}(g)\hat{\Pi}^{\kappa}\hat{\mathcal{U}}^{\kappa}(g)$ in (41) is unchanged if g is replaced by any g' in the $K_0\backslash G_0$ coset $K_0 g$. Thus, the integral in Equation (41) effectively reduces to an integral over the symmetric space $K_0\backslash G_0$.

To derive the Harish-Chandra expression for the inner product of a discrete series holomorphic irrep from this resolution of the identity for the irrep, we need to express the identity operator of Equation (41) as an integral over $K_0\backslash G_0$ coset representatives.

The Gauss factorisation, given by Equation (13), defines a $G \to \mathcal{Z}$ map in which

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto z(g) = d^{-1}c. \tag{43}$$

The set of elements of G that map to a single $z \in \mathcal{Z}$ is then a $P\backslash G$ coset, of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} I_p & 0 \\ z & I_q \end{pmatrix} = \begin{pmatrix} a+bz & b \\ dz & d \end{pmatrix} \in G. \tag{44}$$

It has been shown by Harish-Chandra [29] that, for z in an open subset $D \subset \mathcal{Z}$, the subset of elements of the corresponding $P\backslash G$ coset that lie in G_0 form a $K_0\backslash G_0$ coset with a representative element that can be expressed in the form

$$g(z) = e^{X(x(z))} h(z) e^{Z(z)} \in G_0. \tag{45}$$

The expression (41) of the identity $\hat{I}$ can now be replaced by

$$\hat{I} = \int_D e^{\hat{Z}^{\dagger}(z)} \hat{T}^{\dagger}(h(z)) \hat{\Pi}^{\kappa} \hat{T}(h(z)) e^{Z(z)}\, d\mu(z), \tag{46}$$

where $d\mu(z)$ is the G_0-invariant measure on the domain D diffeomorphic to $K_0\backslash G_0$. The inner product of state vectors $|\psi\rangle$ and $|\psi'\rangle$ in $\mathbb{H}^{\kappa}$ is then expressed in terms of their VCS wave functions, Ψ and Ψ' by

$$\langle\psi|\hat{I}|\psi'\rangle = \int_D \langle\hat{\sigma}^{\kappa}(h(z))\Psi(z)|\hat{\sigma}^{\kappa}(h(z))\Psi'(z)\rangle\, d\mu(z), \tag{47}$$

where $\langle\hat{\sigma}^{\kappa}(h(z))\Psi(z)|\hat{\sigma}^{\kappa}(h(z))\Psi'(z)\rangle$ is an inner product of $\hat{\sigma}^{\kappa}(h(z))\Psi(z)$ and $\hat{\sigma}^{\kappa}(h(z))\Psi'(z)$ as vectors in $\mathbb{H}^{\kappa}_0$.

3. A $G_0 = \mathbf{Sp}(N, \mathbb{R})$ Example

In quantum mechanics the position and momentum coordinates $(x_1, \ldots, x_N, p_1, \ldots, p_N)$ of a particle are mapped to linear operators $(\hat{x}_1, \ldots, \hat{x}_N, \hat{p}_1, \ldots, \hat{p}_N)$ on a Hilbert space that satisfy the Heisenberg commutation relations

$$[\hat{x}_j, \hat{x}_k] = [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{p}_k] = \mathrm{i}\hbar\delta_{j,k}. \tag{48}$$

The real symplectic group $G_0 = \mathrm{Sp}(N, \mathbb{R})$ is then defined as the group of linear transformations of such position and momentum coordinates that preserve these commutation relations. However, for present purposes, $\mathrm{Sp}(N, \mathbb{R})$ is equivalently expressed as the group of complex linear transformations that preserve the commutation relations

$$[c_j, c_k] = [c_j^{\dagger}, c_k^{\dagger}] = 0, \quad [c_j, c_k^{\dagger}] = \delta_{j,k}. \tag{49}$$

of the harmonic-oscillator raising and lowering operators, which are related to the position and momentum coordinates by

$$\hat{x}_j = \frac{1}{\sqrt{2}\,a}(c_j^{\dagger} + c_j), \quad \hat{p}_j = \mathrm{i}\hbar\frac{a}{\sqrt{2}}(c_j^{\dagger} - c_j), \tag{50}$$

where a is a unit of inverse length. Defining $\mathrm{Sp}(N,\mathbb{R})$ as a subgroup of linear transformations of harmonic-oscillator raising and lowering operators makes use of the isomorphism

$$\mathrm{Sp}(N,\mathbb{R}) \simeq \mathrm{Sp}(N,\mathbb{C}) \cap \mathrm{U}(N,N). \tag{51}$$

3.1. A Defining Representation of Sp$(N,\mathbb{R})$

An $\mathrm{Sp}(N,\mathbb{C})$ matrix satisfies the condition

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} & \tilde{\gamma} \\ \tilde{\beta} & \tilde{\delta} \end{pmatrix} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \tag{52}$$

where α, β, γ, δ are $N \times N$ matrices and $\tilde{\alpha}$ is the transpose of α. This implies that

$$\alpha\tilde{\beta} = \beta\tilde{\alpha}, \quad \gamma\tilde{\delta} = \delta\tilde{\gamma}, \quad \alpha\tilde{\delta} - \beta\tilde{\gamma} = I_N. \tag{53}$$

Restriction to the subgroup $\mathrm{Sp}(N,\mathbb{R}) \simeq \mathrm{Sp}(N,\mathbb{C}) \cap \mathrm{U}(N,N)$ further requires that $\delta = \alpha^*$ and $\gamma = \beta^*$. Thus, $\mathrm{Sp}(N,\mathbb{R})$ is the group of $2N \times 2N$ complex matrices

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} ;\ \alpha\alpha^\dagger - \beta\beta^\dagger = I_N,\ \alpha\tilde{\beta} = \beta\tilde{\alpha} \right\}. \tag{54}$$

In this realisation, the group $\mathrm{U}(N)$ is embedded in $\mathrm{Sp}(N,\mathbb{R})$ as the subgroup of matrices of the form

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} ;\ \alpha\alpha^\dagger = I_N \right\}. \tag{55}$$

The group $\mathrm{Sp}(N,\mathbb{C})$ has Gauss factorisation as a product of three matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} I_N & \beta\delta^{-1} \\ 0 & I_N \end{pmatrix} \begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} I_N & 0 \\ \delta^{-1}\gamma & I_N \end{pmatrix}. \tag{56}$$

However, the third equation of (53) implies that

$$\begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \tilde{\delta}^{-1} & 0 \\ 0 & \delta \end{pmatrix}. \tag{57}$$

It follows that $\mathrm{GL}(N,\mathbb{C})$, the complex extension of $\mathrm{U}(N)$, is embedded in $\mathrm{Sp}(N,\mathbb{C})$ as the subgroup

$$\left\{ \begin{pmatrix} \delta & 0 \\ 0 & \tilde{\delta}^{-1} \end{pmatrix} ;\ \delta \in \mathrm{GL}(N,\mathbb{C}) \right\}. \tag{58}$$

A basis for a realisation of the $\mathfrak{sp}(N,\mathbb{C})$ Lie algebra is given by the operators

$$\hat{C}_{ij} = c_i^\dagger c_j + \tfrac{1}{2}, \quad \hat{A}_{ij} = c_i^\dagger c_j^\dagger, \quad \hat{B}_{ij} = c_i c_j, \tag{59}$$

which satisfy the commutation relations

$$[\hat{C}_{ij}, c_k^\dagger] = \delta_{j,k} c_i^\dagger, \quad [\hat{C}_{ij}, c_k] = -\delta_{i,k} c_j, \tag{60}$$

$$[\hat{A}_{ij}, c_k^\dagger] = 0, \quad [\hat{A}_{ij}, c_k] = -\delta_{j,k} c_i^\dagger - \delta_{i,k} c_j^\dagger, \tag{61}$$

$$[\hat{B}_{ij}, c_k^\dagger] = \delta_{j,k} c_i + \delta_{i,k} c_j, \quad [\hat{B}_{ij}, c_k] = 0. \tag{62}$$

It follows that, in the above defining representation,

$$C_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}, \tag{63}$$

$$A_{ij} = \begin{pmatrix} 0 & -E_{ij} - E_{ji} \\ 0 & 0 \end{pmatrix}, \tag{64}$$

$$B_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}, \tag{65}$$

where E_{ij} is the $N \times N$ matrix with elements

$$(E_{ij})_{lk} = \delta_{i,l}\delta_{j,k}. \tag{66}$$

Subgroups of $\mathrm{Sp}(N,\mathbb{C})$ generated by these basis elements then comprise matrices of the form

$$\exp\Big(\sum_{ij} y_{ij} C_{ij}\Big) = \begin{pmatrix} e^{y} & 0 \\ 0 & e^{-\tilde{y}} \end{pmatrix}, \tag{67}$$

$$\exp\Big(-\tfrac{1}{2}\sum_{ij} x_{ij} A_{ij}\Big) = \begin{pmatrix} I_N & x \\ 0 & I_N \end{pmatrix}, \tag{68}$$

$$\exp\Big(\tfrac{1}{2}\sum_{ij} z_{ij} B_{ij}\Big) = \begin{pmatrix} I_N & 0 \\ z & I_N \end{pmatrix}, \tag{69}$$

in which x and z are symmetric. An $\mathrm{Sp}(N,\mathbb{C})$ element

$$g(y,z) = \exp\Big(\sum_{ij} \big[y_{ij}C_{ij} + \tfrac{1}{2} z_{ij} B_{ij} - \tfrac{1}{2} z^*_{ij} A_{ij}\big]\Big) \tag{70}$$

is an element of $\mathrm{Sp}(N,\mathbb{R})$ when y is skew-Hermitian.

3.2. VCS Representations of the $Sp(N,\mathbb{R})$ Lie Group

We consider an irrep $\hat{\mathcal{U}}^\kappa$ of $\mathrm{Sp}(N,\mathbb{R})$ with a lowest grade irrep $\hat{\sigma}^\kappa$ of the subgroup $\mathrm{U}(N)$ of highest weight κ. Thus, with the embeddings of $K_0 = \mathrm{U}(N)$ in its complexification $K = \mathrm{GL}(N,\mathbb{C})$ and $\mathrm{Sp}(N,\mathbb{R})$ in $\mathrm{Sp}(N,\mathbb{C})$, in accord with the equations in Section 3.1, and with $\hat{\Pi}^\kappa : \mathbb{H}^\kappa \to \mathbb{H}^\kappa_0$ being the projection operator of the Hilbert space for the representation $\hat{\mathcal{U}}^\kappa$ to its lowest-grade subspace, it follows that

$$\hat{\sigma}^\kappa(\alpha)\,\hat{\Pi}^\kappa = \hat{\Pi}^\kappa\,\hat{\mathcal{U}}^\kappa \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}, \quad \text{for } \alpha \in \mathrm{U}(N), \tag{71}$$

$$\hat{\sigma}^\kappa(\delta)\,\hat{\Pi}^\kappa = \hat{\Pi}^\kappa\,\hat{T}^\kappa \begin{pmatrix} \delta & 0 \\ 0 & \tilde{\delta}^{-1} \end{pmatrix}, \quad \text{for } \delta \in \mathrm{GL}(N,\mathbb{C}). \tag{72}$$

A state $|\psi\rangle \in \mathbb{H}^\kappa$ has the VCS wave function

$$\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)} |\psi\rangle, \tag{73}$$

where $\hat{Z}(z) = \frac{1}{2}\sum_{ij} z_{ij}\hat{B}_{ij}$. The Hilbert space $\mathcal{H}^\kappa$ of such wave functions is then a module for a VCS irrep $\hat{\Gamma}$ for which

$$\hat{\Gamma}(g)\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)} \hat{\mathcal{U}}^\kappa(g)|\psi\rangle, \quad \forall\, g \in \mathrm{Sp}(N,\mathbb{R}). \tag{74}$$

For g in the defining $\mathrm{Sp}(N,\mathbb{R})$ matrix representation,

$$e^{Z(z)}g = \begin{pmatrix} I_N & 0 \\ z & I_N \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ z\alpha+\beta^* & z\beta+\alpha^* \end{pmatrix} \tag{75}$$

is an $\mathrm{Sp}(N,\mathbb{C})$ matrix which, by Equations (56) and (57), has the factorised form

$$\begin{aligned} e^{Z(z)}g = {} & \begin{pmatrix} I_N & \beta(z\beta+\alpha^*)^{-1} \\ 0 & I_N \end{pmatrix}\begin{pmatrix} (\tilde{\beta}z+\alpha^\dagger)^{-1} & 0 \\ 0 & z\beta+\alpha^* \end{pmatrix} \\ & \times \begin{pmatrix} I_N & 0 \\ (z\beta+\alpha^*)^{-1}(z\alpha+\beta^*) & I_N \end{pmatrix}. \end{aligned} \tag{76}$$

It follows that

$$\hat{\Gamma}(g)\Psi(z) = \hat{\sigma}^\kappa((\tilde{\beta}z+\alpha^\dagger)^{-1})\Psi((z\beta+\alpha^*)^{-1}(z\alpha+\beta^*)). \tag{77}$$

3.3. Representations of the Sp(N, ℝ) Lie Algebra

The above $\mathfrak{sp}(N,\mathbb{R})$ matrices satisfy the commutation relations

$$[B_{ij}, A_{kl}] = \delta_{ik}C_{lj} + \delta_{il}C_{kj} + \delta_{jk}C_{li} + \delta_{jl}C_{ki}, \tag{78}$$
$$[C_{ij}, A_{kl}] = \delta_{jk}A_{il} + \delta_{jl}A_{ik}, \tag{79}$$
$$[C_{ij}, B_{kl}] = -\delta_{ik}B_{lj} - \delta_{il}B_{kj}, \tag{80}$$

and, with $\hat{Z}(z) = \frac{1}{2}\sum_{ij} z_{ij}\hat{B}_{ij}$, the VCS representation of an element $X \in \mathfrak{sp}(N,\mathbb{C})$ is defined by

$$\hat{\Gamma}(X)\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}\hat{X}|\psi\rangle. \tag{81}$$

Thus, the VCS operators representing elements of the $\mathfrak{sp}(N,\mathbb{C})$ Lie algebra are determined by use of the expansion

$$e^{\hat{Z}}\hat{X} = \big(\hat{X} + [\hat{Z},\hat{X}] + \tfrac{1}{2!}[\hat{Z},[\hat{Z},\hat{X}]]\big)e^{\hat{Z}}, \tag{82}$$

and the identities

$$\hat{\Pi}^\kappa\hat{C}_{ij}e^{\hat{Z}(z)}|\psi\rangle = \hat{\mathfrak{C}}_{ij}\Psi(z), \tag{83}$$
$$\hat{\Pi}^\kappa\hat{B}_{ij}e^{\hat{Z}(z)}|\psi\rangle = \nabla_{ij}\Psi(z), \tag{84}$$
$$\hat{\Pi}^\kappa\hat{A}_{ij}e^{\hat{Z}(z)}|\psi\rangle = 0, \tag{85}$$

where $\hat{\mathfrak{C}}_{ij} = \hat{\sigma}^\kappa(C_{ij})$, in accordance with Equation (18), and

$$\nabla_{ij}\Psi(z) = (1+\delta_{i,j})\frac{\partial\Psi(z)}{\partial z_{ij}}, \tag{86}$$

with ∇_{ij} defined such that

$$[\nabla_{ij}, z_{kl}] = \delta_{j,k}\delta_{i,l} + \delta_{i,k}\delta_{j,l}. \tag{87}$$

Then, because $\hat{Z}(z)$ is a sum of $\mathfrak{sp}(N,\mathbb{R})$ lowering operators, the expansions of Equation (82) terminate at or before the third term and we obtain the VCS representation [8,9]

$$\hat{\Gamma}(C_{ij}) = \hat{\mathfrak{C}}_{ij} + (z\nabla)_{ij}, \tag{88}$$
$$\hat{\Gamma}(B_{ij}) = \nabla_{ij}, \tag{89}$$
$$\hat{\Gamma}(A_{ij}) = (\hat{\mathfrak{C}}z)_{ij} + (\hat{\mathfrak{C}}z)_{ji} + \tfrac{1}{2}(z\nabla z)_{ij} + \tfrac{1}{2}(z\nabla z)_{ji} - 4z_{ij}, \tag{90}$$

in which, for example, $\hat{z}\nabla$ is an $N \times N$ matrix with components $(\hat{z}\nabla)_{ij} = \sum_k \hat{z}_{ik}\nabla_{kj}$.

3.4. Inner Products for the Holomorphic Discrete Series Representations of Sp$(N,\mathbb{R})$

To derive the inner product for a holomorphic discrete series representations of $\mathrm{Sp}(N,\mathbb{R})$, we need an expression of the identity resolution as an integral in the form given by Equation (46). The required integral is obtained by use of the following claim.

Claim: The expansion

$$g(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} I_N & \beta(\alpha^*)^{-1} \\ 0 & I_N \end{pmatrix} \begin{pmatrix} (\alpha^\dagger)^{-1} & 0 \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} I_N & 0 \\ (\alpha^{-1}\beta)^* & I_N \end{pmatrix} \tag{91}$$

of an element $g(\alpha,\beta) \in \mathrm{Sp}(N,\mathbb{R})$ defines an isomorphism between the $\mathrm{U}(N)\backslash\mathrm{Sp}(N,\mathbb{R})$ cosets and a subset of vectors in $D \subset \mathcal{Z}$.

Proof: First observe that the map

$$\mathrm{Sp}(N,\mathbb{R}) \to \mathcal{Z} : g(\alpha,\beta) \mapsto z(\alpha,\beta) = (\alpha^{-1}\beta)^*, \tag{92}$$

is $\mathrm{U}(N)$ invariant. This follows because, for $g(a,0) \in \mathrm{U}(N)$,

$$g(a,0)g(\alpha,\beta) = g(a\alpha, a\beta) \quad \text{and} \quad z(a\alpha, a\beta) = z(\alpha,\beta). \tag{93}$$

Conversely, if $z(\alpha',\beta') = z(\alpha,\beta)$ it follows that $\alpha' = a\alpha$ and $\beta' = a\beta$, for some a and that

$$g(\alpha',\beta') = g(a,0)g(\alpha,\beta). \tag{94}$$

□

In accordance with Section 2.3, a representative of a $K_0\backslash G_0$ coset can then be identified with an element $z \in D \subseteq \mathcal{Z}$ by an inverse map

$$D \to \mathrm{Sp}(N,\mathbb{R}) : z \mapsto g(\alpha, \alpha z^*), \tag{95}$$

for a suitable α. For $g(\alpha,\alpha z^*)$ to be an element of $\mathrm{Sp}(N,\mathbb{R})$, α and z must satisfy the constraint

$$|\alpha|^2 - \alpha z^\dagger z \alpha^\dagger = \alpha(I_N - z^\dagger z)\alpha^\dagger = I_N. \tag{96}$$

Thus, we can choose the coset representative $g(z) = g(\alpha,\alpha z^*)$, as in [38], for any z in the image of the map, given in Equation (92), by setting

$$\alpha = \alpha^\dagger = (I_N - z^\dagger z)^{-\frac{1}{2}}. \tag{97}$$

The resolution of the identity, given in general by Equation (46), is then expressed for $G_0 = \mathrm{Sp}(N,\mathbb{R})$ by

$$\hat{I} = \int_D e^{\hat{Z}^\dagger(z)} \hat{T}^\dagger(h(z)) \hat{\Pi}^\kappa \hat{T}(h(z)) e^{\hat{Z}(z)}\, d\mu((z), \tag{98}$$

with

$$h(z) = (I_N - z^\dagger z)^{\frac{1}{2}}, \tag{99}$$

where $d\mu(z)$ is the $\mathrm{U}(N)$-invariant measure on the domain $D \subset \mathcal{Z}$, diffeomorphic to $\mathrm{U}(N)\backslash\mathrm{Sp}(N,\mathbb{R})$, and hereby identified as the multi-dimensional unit disk of vectors $z \in \mathcal{Z}$ of length $|z| < 1$.

The inner product of VCS wave functions, expressed generally by Equation (47), is now given for $\mathrm{Sp}(N,\mathbb{R})$ by

$$\langle \Psi|\Psi'\rangle = \int_D \langle \hat{\sigma}^\kappa((I-z^*z)^{\frac{1}{2}})\Psi(z)|\hat{\sigma}^\kappa((I-z^*z)^{\frac{1}{2}})\Psi'(z)\rangle \, d\mu(z), \tag{100}$$

where $\langle \hat{\sigma}^\kappa((I-z^*z)^{\frac{1}{2}})\Psi(z)|\hat{\sigma}^\kappa((I-z^*z)^{\frac{1}{2}})\Psi'(z)\rangle$ is the inner product of $\hat{\sigma}^\kappa((I-z^*z)^{\frac{1}{2}}))\Psi(z)$ and $\hat{\sigma}^\kappa((I-z^*z)^{\frac{1}{2}})\Psi'(z)$ as vectors in the Hilbert space for the U(N) irrep $\hat{\sigma}^\kappa$. The measure $d\mu(z)$ is determined by VCS methods [38], consistent with previously known results [52,53], to be given to within a normalization factor by

$$d\mu(z) = (I-z^*z)^{-(N+1)} \prod_{i\le j} dx_{ij}dy_{ij}, \tag{101}$$

where $x_{ij} = \mathrm{Re}\, z_{ij}$ and $y_{ij} = \mathrm{Im}\, z_{ij}$.

4. Dual VCS Holomorphic Representations

Sections 2 and 3 have shown that VCS methods can be used to derive the holomorphic discrete-series representations of many simple Lie groups. This section proceeds to show that holomorphic representations, in general, have dual pairs of non-unitary VCS representations which, in combination, provide simple algebraic procedures for constructing orthonormal bases for irreducible unitary representations. It also shows that dual representations are not restricted to discrete-series representations and, in fact, neither are they restricted to holomorphic representations [47].

The notations are the same as in Section 2: G_0 is a real connected simple Lie group with a faithful finite-dimensional representation and a compact subgroup K_0; G and K are the complex extensions of G_0 and K_0; the Lie algebra $\mathfrak{g}$ of G is a sum

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-, \tag{102}$$

in which $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{n}_\pm$ are, respectively, Abelian Lie algebras of raising and lowering operators having the property that $[\mathfrak{k},\mathfrak{n}_\pm] \in \mathfrak{n}_\pm$ and $[\mathfrak{n}_-,\mathfrak{n}_+] \in \mathfrak{k}$; $\mathbb{H}^\kappa$ is the Hilbert space for a unitary irrep $\hat{\mathcal{U}}^\kappa$ of G_0 with lowest-weight κ; $\mathbb{H}^\kappa_0 \subset \mathbb{H}^\kappa$ is the lowest-grade subspace of states in $\mathbb{H}^\kappa$ that are annihilated by the lowering operators of $\mathfrak{n}_-$ and is the Hilbert space for a unitary irrep of K_0 of highest-weight κ given by the restriction of $\hat{\mathcal{U}}^\kappa$ to $K_0 \subset G_0$; $\hat{T}^\kappa$ denotes the extension of $\hat{\mathcal{U}}^\kappa$ to the complex Lie algebra $\mathfrak{g}$ and also to K; $\hat{X}$ denotes the representation $\hat{T}^\kappa(X)$ of an element $X \in \mathfrak{g}$.

4.1. Dual VCS Wave Functions

Dual VCS wave functions for the states of $\mathbb{H}^\kappa$ are constructed as follows. Define $\mathcal{F}^\kappa$ to be the linear space of entire-analytic vector-valued functions of a set of complex variables $\{z_i, i = 1,\ldots,\dim(\mathfrak{n}_+)\}$ of the form

$$\phi(z) = \sum_\alpha \phi_\alpha(z)|\kappa\alpha\rangle, \tag{103}$$

in which $\{|\kappa\alpha\rangle\}$ is an orthonormal basis for the lowest-grade subspace $\mathbb{H}^\kappa_0 \subset \mathbb{H}^\kappa$ and the functions $\{\phi_\alpha\}$ are polynomials. The space $\mathcal{H}^\kappa$ of VCS wave functions is then defined as the image of the vector-space homomorphism

$$\mathbb{H}^\kappa \to \mathcal{F}^\kappa : |\psi\rangle \mapsto \Psi \tag{104}$$

for which

$$\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}|\psi\rangle = \sum_\alpha |\kappa\alpha\rangle\langle\kappa\alpha|e^{\hat{Z}(z)}|\psi\rangle, \quad \forall\, |\psi\rangle \in \mathbb{H}^\kappa, \tag{105}$$

where $\hat{Z}(z) = \sum_i z_i \hat{B}_i$ and $\{B_i\}$ is a basis for $\mathfrak{n}_-$.

Conversely, a space $\mathcal{H}^{\kappa *}$ of dual VCS* wave functions is defined as follows. For each $i = 1, \ldots, \dim(\mathfrak{n}_+)$, choose $A_i \in \mathfrak{n}_+$ and $B_i \in \mathfrak{n}_-$ so that $\hat{A}_i$ is the Hermitian adjoint of $\hat{B}_i$. Then $\mathcal{H}^{\kappa *}$ is identified with the factor space $\mathcal{F}^\kappa / \mathcal{F}_0^\kappa$, where $\mathcal{F}_0^\kappa$ is the kernel of the map

$$\mathcal{F}^\kappa \to \mathbb{H}^\kappa : \phi \mapsto |\psi\rangle = \sum_\alpha \phi_\alpha(\hat{A}) |\kappa\alpha\rangle, \tag{106}$$

in which each variable z_i of a function ϕ_α is replaced by the corresponding element $\hat{A}_i$ in the representation $\hat{T}^\kappa(\mathfrak{n}_+)$. Because $\mathcal{F}^\kappa$ is a vector space, $\mathcal{H}^{\kappa *} = \mathcal{F}^\kappa / \mathcal{F}_0^\kappa$ is isomorphic, as a vector space, to a subspace of $\mathcal{F}^\kappa$.

Consider the maps defined in Equations (106) and (104):

$$\mathcal{F}^\kappa \to \mathbb{H}^\kappa \to \mathcal{F}^\kappa. \tag{107}$$

Neither of them is an isomorphism in general. Thus, we insert two more spaces to obtain the following sequence of maps:

$$\mathcal{F}^\kappa \to \mathcal{H}^{\kappa *} \to \mathbb{H}^\kappa \to \mathcal{H}^\kappa \hookrightarrow \mathcal{F}^\kappa \;:\; \phi \mapsto \Phi \mapsto |\psi\rangle \mapsto \Psi \mapsto \Psi, \tag{108}$$

where $\mathcal{H}^\kappa \hookrightarrow \mathcal{F}^\kappa$ is a simple embedding. The first space inserted, $\mathcal{H}^{\kappa *}$, is the quotient of $\mathcal{F}^\kappa$ by the kernel of the first map in Equation (107). This makes the second map in Equation (108) an isomorphism. The second insertion, $\mathcal{H}^\kappa$, is the image of the second map in Equation (107). This makes the third map in Equation (108) an isomorphism. In other words, we have vector-space isomorphisms $\mathcal{H}^{\kappa *} \to \mathbb{H}^\kappa \to \mathcal{H}^\kappa$ involving a space of VCS wave functions and its dual. We shall see that these isomorphisms can be exploited to simplify various calculations.

To determine inner products and the completion of $\mathcal{F}$, $\mathcal{H}^\kappa$, and $\mathcal{H}^{\kappa *}$ to Hilbert spaces, it is convenient to start with the inner product

$$(\phi, \phi') = \sum_\alpha \int \phi_\alpha^*(z) \phi'_\alpha(z)\, dv(z) \tag{109}$$

for $\mathcal{F}^\kappa$, in which $dv(z)$ is the Bargmann measure [48]. This is the inner product for a coherent-state representation of a multi-dimensional harmonic oscillator on a space of entire analytic wave functions such that, if $n = \{n_1, n_2, \ldots, n_{\dim(\mathfrak{n}_+)}\}$ is a set of non-negative integers and a wave function ϕ_α is expanded

$$\phi_\alpha(z) = \sum_n \phi_{\alpha n} \prod_{i \in n} \frac{z_i^{n_i}}{\sqrt{n_i!}}, \tag{110}$$

then

$$(\phi, \phi') = \sum_{n\alpha} \phi_{\alpha n}^* \phi'_{\alpha n}. \tag{111}$$

This inner product (ϕ, ϕ') and the corresponding measure $dv(z)$ have the useful property that

$$\int e^{\hat{Z}^\dagger(z)} \phi_\alpha(z)\, dv(z) = \int e^{\sum_i z_i^* \hat{A}_i} \phi_\alpha(z)\, dv(z) = \phi_\alpha(\hat{A}). \tag{112}$$

As shown in [54], the calculation of such inner products is facilitated by use of the Capelli identities [2,55–58] when the $\{z_i\}$ variables are elements of a matrix $\{z_{jk}\}$, as is generally the case in the VCS construction.

4.2. Inner Products and Dual VCS Representations

In this Section, inner products are defined for $\mathcal{H}^\kappa$, $\mathbb{H}^\kappa$ and $\mathcal{H}^{\kappa *}$ such that they are isomorphic as Hilbert spaces and carry a representation of K and a unitary irrep of G_0. The constructions start with the Bargmann inner product (ϕ, ϕ') for $\mathcal{F}^\kappa$, and show that, with the definitions and notations of

Equation (108), the VCS wave functions satisfy the relationship $(\Phi_\mu, \Psi_\nu) = (\Psi_\mu, \Phi_\nu) = \langle\psi_\mu|\psi_\nu\rangle$, where $\Phi_\nu \in \mathcal{H}^{\kappa*}$ and $\Psi_\nu \in \mathcal{H}^\kappa$ are dual VCS wave functions for a state vector $|\psi_\nu\rangle \in \mathbb{H}^\kappa$. Thus, the dual VCS wave functions have orthogonal bases that are mutually biorthogonal relative to the Bargmann inner product and, in combination, define an orthonormal basis for the Hilbert space $\mathbb{H}^\kappa$.

By use of the identity (112), the map in Equation (106) is expressed in terms of the Bargmann integral of Equation (109) by

$$\phi \mapsto |\psi\rangle = \sum_\alpha \phi_\alpha(\hat{A})|\kappa\alpha\rangle = \sum_\alpha \int e^{\hat{Z}^\dagger(z)}|\kappa\alpha\rangle\phi_\alpha(z)\, dv(z), \tag{113}$$

where $Z^\dagger(z) = \sum_i z_i^* \hat{B}_i^\dagger = \sum_i z_i^* \hat{A}_i$. Equations (113) and (105) then show that the sequence of maps $\mathcal{F}^\kappa \to \mathbb{H}^\kappa \to \mathcal{H}^\kappa$ is given by

$$\phi \mapsto |\psi\rangle = \int e^{\hat{Z}^\dagger(y)}\phi(y)\, dv(y) \mapsto \Psi(x) = \Pi^\kappa e^{\hat{Z}(x)}|\psi\rangle = \int \Pi^\kappa e^{\hat{Z}(x)} e^{\hat{Z}^\dagger(y)}\phi(y)\, dv(y). \tag{114}$$

Because $\mathcal{H}^{\kappa*} = \mathcal{F}^\kappa / \mathcal{F}_0^\kappa$, where $\mathcal{F}_0^\kappa$ is the kernel of the map $\mathcal{F}^\kappa \to \mathbb{H}^\kappa$, the sequence $\mathcal{H}^{\kappa*} \to \mathbb{H}^\kappa \to \mathcal{H}^\kappa$ is similarly given by

$$\Phi \mapsto |\psi\rangle = \int e^{\hat{Z}^\dagger(y)}\Phi(y)\, dv(y) \mapsto \Psi(x) = \Pi^\kappa e^{\hat{Z}(x)}|\psi\rangle = \int \Pi^\kappa e^{\hat{Z}(x)} e^{\hat{Z}^\dagger(y)}\Phi(y)\, dv(y). \tag{115}$$

Thus, the relationship between a wave function $\Phi \in \mathcal{H}^{\kappa*}$ and its counterpart $\Psi \in \mathcal{H}^\kappa$, both of which have vector values in $\mathbb{H}_0^\kappa$, is given by the equation

$$\Psi(x) = \hat{S}\Phi(x) = \int \hat{S}(x, y^*)\Phi(y)\, dv(y) \tag{116}$$

with

$$\hat{S}(x, y^*) = \hat{\Pi}^\kappa e^{\hat{Z}(x)} e^{\hat{Z}^\dagger(y)} \hat{\Pi}^\kappa. \tag{117}$$

The inner product $\langle\psi|\psi'\rangle$ of $\mathbb{H}^\kappa$ and a corresponding inner product for $\mathcal{H}^{\kappa*}$ are then expressed by

$$(\Phi, \hat{S}\Phi') = \langle\psi|\psi'\rangle = \int \Phi^\dagger(x)\hat{S}(x, y^*)\Phi'(y)\, dv(x)\, dv(y). \tag{118}$$

Thus, $\mathcal{H}^{\kappa*}$ and $\mathcal{H}^\kappa$ are, respectively, the Hilbert spaces

$$\mathcal{H}^{\kappa*} = \{\Phi \in \mathcal{F}^\kappa \mid (\Phi, \hat{S}\Phi) < \infty\}. \tag{119}$$

and

$$\mathcal{H}^\kappa = \{\Psi = \hat{S}\Phi \mid \Phi \in \mathcal{H}^{\kappa*}\}. \tag{120}$$

It follows that an orthonormal basis $\{\Phi_\nu\}$ for $\mathcal{H}^{\kappa*}$ for which

$$(\Phi_\mu, \hat{S}\Phi_\nu) = \delta_{\mu,\nu} \tag{121}$$

defines an orthonormal basis for $\{\Psi_\nu = \hat{S}\Phi_\nu\}$ for $\mathcal{H}^\kappa$ such that together they form biorthogonal bases for which

$$(\Phi_\mu, \Psi_\nu) = \delta_{\mu,\nu}. \tag{122}$$

The function $\hat{S}(x, y^*)$ in Equation (118) appears as a weight function for the inner product of the Hilbert space $\mathcal{H}^{\kappa*}$ relative to the Bargmann measure. Thus, it serves a parallel role to the Bergman kernel [52] for the inner product of $\mathcal{H}^\kappa$ in terms of a Euclidean measure. Moreover, the inner product of Equation (118) has the advantage over the inner product in terms of the $K_0\backslash G_0$ invariant measure in that it is not restricted to discrete-series representations by the convergence constraint on the resolution of the identity given by Equation (47).

The relationships between the triplet of Hilbert spaces $\mathcal{H}^{\kappa *}$, $\mathbb{H}^{\kappa}$, and $\mathcal{H}^{\kappa}$ relate the VCS representation $\hat{\Gamma}$ of G_0 on $\mathcal{H}^{\kappa}$ to a dual VCS* representation $\hat{\Theta}$ on $\mathcal{H}^{\kappa *}$ and relate both of these VCS representations to the equivalent unitary representation $\hat{\mathcal{U}}^{\kappa}$ on $\mathbb{H}^{\kappa}$ in accordance with the commuting diagram

$$\begin{array}{ccccc} \mathcal{H}^{\kappa *} & \longrightarrow & \mathbb{H}^{\kappa} & \longrightarrow & \mathcal{H}^{\kappa} \\ \downarrow \hat{\Theta} & & \downarrow \hat{\mathcal{U}}^{\kappa} & & \downarrow \hat{\Gamma} \\ \mathcal{H}^{\kappa *} & \longrightarrow & \mathbb{H}^{\kappa} & \longrightarrow & \mathcal{H}^{\kappa} \end{array} . \tag{123}$$

The relationships are expressed by the equations

$$\langle \psi | \hat{\mathcal{U}}^{\kappa}(g) | \psi' \rangle = \big(\Phi, \hat{\Gamma}(g)\Psi'\big) = \big(\Psi, \hat{\Theta}(g)\Phi'\big), \quad \forall\, g \in G_0, \tag{124}$$

$$\langle \psi | \hat{X} | \psi' \rangle = \big(\Phi, \hat{\Gamma}(X)\Psi'\big) = \big(\Psi, \hat{\Theta}(X)\Phi'\big), \quad \forall\, X \in \mathfrak{g}. \tag{125}$$

Together with Equation (117) and the map

$$\hat{S} : \mathcal{H}^{\kappa *} \to \mathcal{H}^{\kappa} \; ; \; \Phi \mapsto \Psi = \hat{S}\Phi, \tag{126}$$

they show that the dual VCS representations, $\hat{\Gamma}$ and $\hat{\Theta}$, are intertwined by $\hat{S}$, i.e.,

$$\hat{\Gamma}(g)\hat{S} = \hat{S}\hat{\Theta}(g), \quad \forall\, g \in G_0, \tag{127}$$

$$\hat{\Gamma}(X)\hat{S} = \hat{S}\hat{\Theta}(X), \quad \forall\, X \in \mathfrak{g}. \tag{128}$$

An interpretation of $\hat{S}(x, y^*)$ is obtained by inserting the identity $\hat{I} = \sum_\nu |\nu\rangle\langle\nu|$, where $\{|\nu\rangle\}$ is an orthonormal basis for $\mathbb{H}^{\kappa}$, into Equation (117) to obtain

$$\hat{S}(x, y^*) = \sum_\nu \hat{\Pi}^{\kappa} e^{\hat{Z}(x)} |\nu\rangle\langle\nu| e^{\hat{Z}^\dagger(y)}\, \hat{\Pi}^{\kappa} = \sum_\nu \Psi_\nu(x)\Psi^\dagger_\nu(y), \tag{129}$$

in which it is noted that

$$\Psi_\nu(x) = \sum_\alpha |\kappa\alpha\rangle\langle\kappa\alpha| e^{\hat{Z}(x)} |\nu\rangle = \sum_\alpha |\kappa\alpha\rangle \Psi_{\nu\alpha}(x) \tag{130}$$

implies that

$$\Psi^\dagger_\nu(y) = \sum_\alpha \langle\nu| e^{\hat{Z}^\dagger(y)} |\kappa\alpha\rangle\langle\kappa\alpha| = \sum_\alpha \Psi^*_{\nu\alpha}(y)\langle\kappa\alpha|. \tag{131}$$

This expression indicates that for an infinite-dimensional irrep of a non-compact group, $\hat{S}(x, y^*)$ is not, in general, a well-defined function for unrestricted values of x and y. However, like the Dirac delta function in the inner product

$$\int \psi^*(x)\delta(x, y)\phi(y)\, dx\, dy = \int \psi^*(x)\phi(x)\, dx, \tag{132}$$

it is well-defined as a distribution on dual VCS wave functions with integration over the entire multi-dimensional complex plane and, as a consequence, the VCS representations are not restricted to those of the discrete series.

An explicit expression for the inner product of Equation (118) is obtained from the observation that the operator $e^{\hat{Z}(x)} e^{\hat{Z}^\dagger(y)}$ in Equation (117) is a representation of the group product

$$e^{Z(x)} e^{Z^\dagger(y)} = \begin{pmatrix} I_p & 0 \\ x & I_q \end{pmatrix} \begin{pmatrix} I_p & \epsilon y^\dagger \\ 0 & I_q \end{pmatrix} = \begin{pmatrix} I_p & \epsilon y^\dagger \\ x & I_q + \epsilon x y^\dagger \end{pmatrix} \tag{133}$$

with $\epsilon = \pm 1$ according as G_0 is, respectively, compact or non-compact. Gauss factorization then gives

$$e^{Z(x)}e^{Z^\dagger(y)} = \begin{pmatrix} I_p & \epsilon y^\dagger(I_q+\epsilon xy^\dagger)^{-1} \\ 0 & I_q \end{pmatrix} \times \begin{pmatrix} I_p - \epsilon y^\dagger(I_q+\epsilon xy^\dagger)^{-1}x & 0 \\ 0 & I_q+\epsilon xy^\dagger \end{pmatrix} \begin{pmatrix} I_p & 0 \\ (I_q+\epsilon xy^\dagger)^{-1}x & I_q \end{pmatrix} \tag{134}$$

from which it follows that

$$\hat{S}(x, y^*) = \hat{\sigma}^\kappa(h(x, \epsilon y^*)), \tag{135}$$

where $h(x, \epsilon y^*)$ is the $K \subset G$ matrix

$$h(x, \epsilon y^*) = \begin{pmatrix} I_p - \epsilon y^\dagger(I_q+\epsilon xy^\dagger)^{-1}x & 0 \\ 0 & I_q+\epsilon xy^\dagger \end{pmatrix}. \tag{136}$$

It also follows that

$$\langle\psi|\psi'\rangle = (\Phi, \hat{S}\Phi') = \int \Phi^\dagger(x)\hat{\sigma}^\kappa(h(x,\epsilon y^*))\Phi'(y)\, dv(x)\, dv(y). \tag{137}$$

This expression of the inner product is particularly useful for scalar-valued VCS wave functions (i.e., when the representation $\hat{\sigma}^\kappa$ is one-dimensional) [47]. However, as now shown, the construction of orthonormal bases and the calculation of matrix elements of Lie algebra observables, as needed in quantum mechanical applications, is also achieved by alternative and, for generic vector-valued representations, more practical algebraic methods.

4.3. Dual representations of the Lie algebra $\mathfrak{g}$

The VCS representation $\hat{\Gamma}(X)$ of an element $X \in \mathfrak{g}$, defined by

$$\hat{\Gamma}(X)\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}\hat{X}|\psi\rangle \tag{138}$$

and given explicitly by Equations (35)–(37), has the more useful expression

$$\hat{\Gamma}(B_i) = \partial_i, \tag{139}$$

$$\hat{\Gamma}(C_p) = \hat{\mathfrak{C}}_p + \sum_{ij} \hat{z}_i C_p^{ij}\partial_j, \tag{140}$$

$$\hat{\Gamma}(A_i) = [\hat{\Lambda}, \hat{z}_i], \tag{141}$$

where

$$\hat{\Lambda} = \sum_{ijp} \hat{z}_i D_p^{ij}\left(\hat{\mathfrak{C}}_p\partial_j + \tfrac{1}{4}\sum_{kl}\hat{z}_k C_p^{kl}\partial_l\partial_j\right) \tag{142}$$

and the coefficients C_p^{ij} and D_p^{ij} are defined, in Section 2.2, by the expansions

$$[B_i, C_p] = \sum_j C_p^{ij} B_j, \quad [B_i, A_j] = \sum_p D_p^{ij} C_p. \tag{143}$$

Equation (141) is shown to reproduce Equation (37) by use of the symmetry relation

$$\sum_p D_p^{ij} C_p^{kl} = \sum_p D_p^{kj} C_p^{il} \tag{144}$$

which follows from the Jacobi identity

$$[B_k[B_i, A_j]] = [[B_k, B_i], A_j] + [B_i, [B_k, A_j]] = [B_i, [B_k, A_j]]. \tag{145}$$

It is more useful than Equation (37), when $\hat{\Lambda}$ is K_0 invariant and $\sum_i [\hat{C}_p, \Lambda] = 0$, because $\hat{\Lambda}$ is then diagonal in a K_0-coupled basis for the Hilbert space of the desired representation and the action of the operator $\hat{\Gamma}(A_i)$ is much simplified.

Claim: The operator $\hat{\Lambda}$ is K_0 invariant if the elements $\{A_i\}$ and $\{B_i\}$ of $\mathfrak{n}_\pm$ are chosen such that the sum $\sum_i A_i B_i$ is K_0-invariant.

Proof: K_0-invariance of $\sum_i A_i B_i$ implies that $\sum_i [C_p, A_i B_i] = 0$ and

$$\sum_i [C_p, A_i] B_i + \sum_i A_i [C_p, B_i] = 0. \tag{146}$$

From the expansion $[B_i, C_p] = \sum_j C_p^{ij} B_j$, it then follows that

$$[C_p, A_j] = \sum_i A_i C_p^{ij}, \tag{147}$$

and, hence, that

$$[\hat{\Gamma}(C_p), \hat{\Gamma}(A_j)] = [\hat{\Gamma}(C_p), [\hat{\Lambda}, \hat{z}_j]] = \sum_i [\hat{\Lambda}, \hat{z}_i] C_p^{ij}. \tag{148}$$

It also follows from Equation (140) that

$$[\hat{\Gamma}(C_p), \hat{z}_j] = \sum_i \hat{z}_i C_p^{ij}. \tag{149}$$

Thus, from the identity

$$\begin{aligned} [\hat{\Gamma}(C_p), [\hat{\Lambda}, \hat{z}_j]] &= [[\hat{\Gamma}(C_p), \hat{\Lambda}], \hat{z}_j] + [\hat{\Lambda}, [\hat{\Gamma}(C_p), \hat{z}_j]] \\ &= [[\hat{\Gamma}(C_p), \hat{\Lambda}], \hat{z}_j] + \sum_i [\hat{\Lambda}, \hat{z}_j] C_p^{ij}, \end{aligned} \tag{150}$$

it follows that

$$[[\hat{\Gamma}(C_p), \hat{\Lambda}], \hat{z}_i] = 0. \tag{151}$$

Now, because $\hat{\Gamma}(C_p)$ and $\hat{\Lambda}$ are both sums of terms that are of the same order in the variables $\{z_i\}$ as in the $\{\partial_i\}$ derivative operators, the only way that Equation (151) can be satisfied is if $[\hat{\Gamma}(C_p), \hat{\Lambda}]$ is independent of any z_i. However, it is indeed ascertained that $[\hat{\Gamma}(C_p), \hat{\Lambda}]$ has no z_i-independent term for any z_i. Therefore $[\hat{\Gamma}(C_p), \hat{\Lambda}] = 0$. □

The dual VCS* representation $\hat{\Theta}$ of $\mathfrak{g}$ is now determined starting from the definition of Equation (113), which asserts that a state $|\psi\rangle = \sum_\alpha \Phi_\alpha(\hat{A})|\kappa\alpha\rangle$ has a VCS* wave function $\Phi(z) = \sum_\alpha |\kappa\alpha\rangle\, \Phi_\alpha(z)$ and implies that $\hat{\Theta}(A_i) = \hat{z}_i$. The other components of the $\hat{\Theta}$ representation are generated by requiring that they respect the commutation relations of Equations (143) and (147) and have the property that when restricted to the space of lowest-grade states, an operator $\Theta(C_p)$ restricts to $\hat{\mathfrak{C}}_p$. Equation (147) implies that

$$[\hat{\Theta}(C_p), \hat{z}_j] = \sum_i \hat{z}_i C_p^{ij} \tag{152}$$

and, hence, that

$$\hat{\Theta}(C_p) = \hat{\mathfrak{C}}_p + \sum_i \hat{z}_i C_p^{ij} \partial_j = \hat{\Gamma}(C_p). \tag{153}$$

Then, because $[B_i, A_j]$ is an element of $\mathfrak{k}$, it follows that

$$\begin{aligned}[\hat{\Theta}(B_i), \hat{\Theta}(A_j)] &= [\hat{\Gamma}(B_i), \hat{\Gamma}(A_j)] \\ &= [\partial_i, [\hat{\Lambda}, \hat{z}_j]] = [[\partial_i, \hat{\Lambda}], \hat{z}_j]\end{aligned} \tag{154}$$

and that $\hat{\Theta}(B_i) = [\partial_i, \hat{\Lambda}]$. Thus, we obtain the dual VCS* representation of the Lie algebra $\mathfrak{g}$:

$$\hat{\Theta}(A_i) = \hat{z}_i, \tag{155}$$

$$\hat{\Theta}(C_p) = \hat{\Gamma}(C_p) = \hat{\mathfrak{C}}_p + \sum_{ij} \hat{z}_i C_p^{ij} \partial_j, \tag{156}$$

$$\hat{\Theta}(B_i) = [\partial_i, \hat{\Lambda}]. \tag{157}$$

4.4. Non-unitary Representations on $\mathcal{F}^\kappa$

In a unitary representation, the elements of the Lie algebra $\mathfrak{g}$ satisfy the Hermiticity relations

$$\hat{A}_i^\dagger = \hat{B}_i \quad \text{and} \quad \hat{C}_p^\dagger = \hat{C}_p. \tag{158}$$

However, when acting on the Hilbert space $\mathcal{F}^\kappa$ with Bargmann inner product given by Equation (109) for which ∂_i is the Hermitian adjoint of $\hat{z}_i$, the representations $\hat{\Gamma}$ and $\hat{\Theta}$ satisfy the Hermiticity relations

$$\begin{aligned}&\hat{\Gamma}^\dagger(A_i) = \hat{\Theta}(B_i), \quad \hat{\Gamma}^\dagger(B_i) = \hat{\Theta}(A_i), \\ &\hat{\Gamma}^\dagger(C_p) = \hat{\Gamma}(C_p) = \hat{\Theta}^\dagger(C_p) = \hat{\Theta}(C_p).\end{aligned} \tag{159}$$

An important observation is that the representations $\hat{\Gamma}$ and $\hat{\Theta}$ are both expressed in terms of the elements of the lowest-grade representation $\hat{\sigma}^\kappa$ of the compact subalgebra $\mathfrak{k}_0$, for which $\mathfrak{C}_p = \hat{\sigma}^\kappa(C_p)$, and a commuting Heisenberg-Weyl algebra with commutation relations

$$[\hat{\mathfrak{C}}_p, \hat{\mathfrak{C}}_q] = \textstyle\sum_r C_{pq}^r \mathfrak{C}_r, \tag{160}$$

$$[\partial_i, \hat{z}_j] = \delta_{i,j} \hat{I}, \tag{161}$$

$$[\hat{\mathfrak{C}}_p, \hat{z}_i] = [\hat{\mathfrak{C}}_p, \delta_i] = 0, \tag{162}$$

where $\{C_{pq}^r\}$ are structure constants for $\mathfrak{k}_0$. The direct sum of these two Lie algebras, which is obtained as a contraction limit of the Lie algebra $\mathfrak{g}$ [54], has an irreducible unitary representation on $\mathcal{F}^\kappa$ with actions defined simply by

$$\hat{\mathfrak{C}}_p \sum_\alpha |\kappa\alpha\rangle \phi_\alpha(z) = \sum_\alpha \hat{\sigma}^\kappa(C_p) |\kappa\alpha\rangle \phi_\alpha(z), \tag{163}$$

$$\partial_i \sum_\alpha |\kappa\alpha\rangle \phi_\alpha(z) = \sum_\alpha |\kappa\alpha\rangle \partial_i \phi_\alpha(z), \quad \hat{z}_i \sum_\alpha |\kappa\alpha\rangle \phi_\alpha(z) = \sum_\alpha |\kappa\alpha\rangle z_i \phi_\alpha(z). \tag{164}$$

This representation then defines dual representations of the Lie algebra $\mathfrak{g}$ on $\mathcal{F}^\kappa$ by the operators $\hat{\Gamma}(X)$ and $\hat{\Theta}(X)$ for $X \in \mathfrak{g}$. In general, these representations of $\mathfrak{g}$ are neither unitary nor irreducible. However, they are easily constructed and, as now shown, lead to a practical construction of an orthonormal basis for the desired irreducible unitary representation.

The commutation relations, Equation (149), for a simple Lie algebra, show that the $\{\hat{z}_i\}$ operators transform as a basis for an irreducible finite-dimensional K_0 representation which, for a suitable choice of the $\{A_i\}$ basis, is unitary. Thus, it is possible to classify a basis of polynomials in the variables $\{z_i\}$ by the labels of irreducible unitary K_0 representations. It is also possible to couple these polynomials to the states of the lowest-grade irrep $\hat{\sigma}^{\kappa}$ to form an orthonormal basis for $\mathcal{F}^{\kappa}$. Let $\{\phi_{swr}\}$ denote such a basis of vector-valued functions that are eigenfunctions of $\hat{\Lambda}$,

$$\hat{\Lambda}\phi_{swr} = \Omega_{sw}\phi_{swr}, \tag{165}$$

and are orthonormal with respect to the Bargmann inner product

$$(\phi_{swr}, \phi_{tw'r'}) = \delta_{s,t}\delta_{w,w'}\delta_{r,r'}, \tag{166}$$

where w and r label basis states for an irreducible K_0-invariant subspace of $\mathcal{F}^{\kappa}$ of highest weight w, and s is a multiplicity index to distinguish irreducible K_0 subspaces of common w. Thus, in both the $\hat{\Gamma}$ and $\hat{\Theta}$ representations on $\mathcal{F}^{\kappa}$, the subalgebra $\mathfrak{k}_0 \subset \mathfrak{g}_0$ is represented as a sum of irreducible unitary representations of highest weights labelled by w. Matrix elements of the raising and lowering operators of the Lie algebra $\mathfrak{g}$ are then given in the $\hat{\Gamma}$ representation by

$$(\phi_{s'w'r'}, \hat{\Gamma}(A_i)\phi_{swr}) = (\Omega_{s'w'} - \Omega_{sw})(\phi_{s'w'r}, z_i\phi_{swr}), \tag{167}$$
$$(\phi_{swr}, \hat{\Gamma}(B_i)\phi_{s'w'r'}) = (\phi_{swr}, \partial_i\phi_{s'w'r}) = (\phi_{s'w'r'}, z_i\phi_{swr})^*, \tag{168}$$

and in the dual $\hat{\Theta}$ representation by

$$(\phi_{s'w'r'}, \hat{\Theta}(A_i)\phi_{swr}) = (\phi_{s'w'r'}, z_i\phi_{swr}), \tag{169}$$
$$(\phi_{swr}, \hat{\Theta}(B_i)\phi_{s'w'r'}) = (\phi_{s'w'r'}, z_i\phi_{swr})^*(\Omega_{s'w'} - \Omega_{sw}). \tag{170}$$

Neither of the dual representations $\hat{\Gamma}$ and $\hat{\Theta}$ of $\mathfrak{g}_0$ is unitary on $\mathcal{F}^{\kappa}$. Nor, in general, are they irreducible. However, with orthonormal bases $\{\Phi_\nu\}$ and $\{\Psi_\nu\}$ for the respective VCS Hilbert spaces $\mathcal{H}^{\kappa}$ and $\mathcal{H}^{\kappa *}$, defined in terms of an operator $\hat{S}$ by Equations (121) and (122), they have irreducible unitary representations on these spaces with matrix elements

$$\langle\mu|\hat{X}|\nu\rangle = (\Phi_\mu, \hat{\Gamma}(X)\Psi_\nu) = (\Psi_\mu, \hat{\Theta}(X)\Phi_\nu), \quad \forall\, X \in \mathfrak{g}. \tag{171}$$

Thus, given that the bases $\{\Phi_\nu\}$ and $\{\Psi_\nu = \hat{S}\Phi_\nu\}$ are defined by the operator $\hat{S}$ and the equation $(\Phi_\mu, \hat{S}\Phi_\nu) = \delta_{\mu,\nu}$, the matrix elements of Equation (171) are determined simply by those of $\hat{S}$.

4.5. Orthonormal VCS Wave Functions

Recall that the operator $\hat{S}$ has an integral expression $\hat{S}\Phi(x) = \int \hat{S}(x,y^*)\Phi(y)\,dv(y)$ in which $\hat{S}(x,y^*)$ is given by Equation (129) as a sum $\sum_{\nu wr} \Psi_{\nu wr}(x)\Psi^{\dagger}_{\nu wr}(y)$ in which $\{\Psi_{\nu wr}\}$ is an orthonormal K_0-coupled basis for $\mathcal{H}^{\kappa}$. It follows that the operator $\hat{S}$ conserves the K_0 quantum numbers and has matrix elements

$$(\phi_{swr}, \hat{S}\phi_{tw'r'}) = \delta_{w,w'}\delta_{r,r'}S^{w}_{st} \tag{172}$$

in the basis $\{\phi_{swr}\}$ for $\mathcal{F}^{\kappa}$ given by

$$S^{w}_{st} = \sum_{\nu}(\phi_{swr}, \Psi_{\nu wr})(\Psi_{\nu wr}, \phi_{twr}). \tag{173}$$

Thus, if the $\{\Psi_{\nu wr}\}$ basis wave functions are expanded

$$\Psi_{\nu wr} = \sum_{s}\phi_{swr}\mathcal{K}^{w}_{s\nu}, \tag{174}$$

the S^{w} matrices are given by

$$S^{w}_{st} = \sum_{\nu}(\phi_{swr}, \Psi_{\nu wr})(\Psi_{\nu wr}, \phi_{twr}) = \sum_{\nu}\mathcal{K}^{w}_{s\nu}\mathcal{K}^{w*}_{t\nu}. \tag{175}$$

Particularly useful for computational purposes is the observation that the S^w submatrices are finite dimensional. Moreover, a recursion relation for these matrices is obtained from the equation

$$\sum_i \hat{\Gamma}(A_i)\hat{S}\hat{\Gamma}(B_i) = \hat{S}\sum_i \hat{z}_i \partial_i = \hat{S}\hat{N}, \tag{176}$$

where $\hat{N} = \sum_i \hat{z}_i \partial_i$ is an operator that measures the degree in the z variables of any vector in $\mathcal{F}^\kappa$ on which it operates. Equation (176) is obtained from the identities $\hat{\Gamma}(A_i)\hat{S} = \hat{S}\Theta(A_i) = \hat{S}\hat{z}_i$ (cf. Equation (128)) and $\hat{\Gamma}(B_i) = \partial_i$. With the expression $\hat{\Gamma}(A_i) = [\hat{\Lambda}, \hat{z}_i]$, it becomes

$$\hat{S}\hat{N} = \sum_i [\hat{\Lambda}, \hat{z}_i]\hat{S}\partial_i. \tag{177}$$

Taking matrix elements of both sides of (177) gives

$$\left(\phi_{sw'r'}, \hat{S}^{w'}\hat{N}\phi_{tw'r'}\right) = \sum_i \left(\phi_{sw'r'}, [\hat{\Lambda}, z_i]\hat{S}\partial_i \phi_{tw'r'}\right) \tag{178}$$

and the recursion relation

$$S^{w'}_{st} = \frac{1}{N_{w'}} \sum_{iuvwr} \left(\Omega_{sw'} - \Omega_{uw}\right)\left(\phi_{sw'r'}, \hat{z}_i \phi_{uwr}\right)\left(\phi_{tw'r'}, \hat{z}_i \phi_{vwr}\right)^* S^w_{uv}\,, \tag{179}$$

where N_w, defined by $\hat{N}\phi_{swr} = N_w \phi_{swr}$, is the degree in the $\{z_i\}$ variables of ϕ_{swr} and use has been made of the Hermiticity relation

$$\left(\phi_{vwr}, \partial_i \phi_{tw'r'}\right) = \left(\phi_{tw'r'}, \hat{z}_i \phi_{vwr}\right)^*. \tag{180}$$

Equation (179) is much simplified by use of the Wigner-Eckart theorem [59]

$$\left(\phi_{sw'r'}, \hat{z}_i \phi_{uwr}\right) = \sum_\rho (wr, \eta i|\rho w'r')\left(\phi_{sw'}\|\hat{z}\|\phi_{uw}\right)_\rho, \tag{181}$$

in which it is understood that the vector $\hat{z}$ transforms according to an irrep η of K_0. The Wigner-Eckart theorem expresses the many matrix elements $\left(\phi_{sw'r'}, \hat{z}_i \phi_{uwr}\right)$, for given values of w and w', in terms of a few so-called reduced matrix elements $\left(\phi_{sw'}\|\hat{z}\|\phi_{uw}\right)_\rho$ and the Clebsch-Gordan coefficients $(wr, \eta i|w'r')_\rho$ for the decomposition of a tensor product $\eta \otimes w$ of irreducible K_0 representations into a sum of irreps; ρ indexes the multiplicity of an irrep w' in this decomposition. With the sum rule for K_0 Clebsch-Gordan coefficients

$$\sum_{ri} (wr, \eta i|\rho w'r')(wr, \eta i|\rho' w'r')^* = \delta_{\rho,\rho'}, \tag{182}$$

the recursion relation (179) simplifies to

$$S^{w'}_{st} = \frac{1}{N_{w'}} \sum_{uv\rho w} \left(\Omega_{sw'} - \Omega_{uw}\right)\left(\phi_{sw'}\|\hat{z}\|\phi_{uw}\right)_\rho \left(\phi_{tw'}\|\hat{z}\|\phi_{vw}\right)^*_\rho S^w_{uv}. \tag{183}$$

Setting $S^\kappa = 1$ for the multiplicity-free lowest-grade multiplicity-free K_0 irrep, for which $w = \kappa$ and $N_\kappa = 0$, this equation sequentially determines the S^w matrices for which $N_w = 1, 2, 3, \ldots$.

Now, because the non-zero S^w matrices are positive and Hermitian, they can be diagonalized and expressed at each step of the recursive process in terms of their eigenvalues, $(k^w_\nu)^2$, by

$$S^w_{st} = \sum_\nu U^w_{s\nu}(k^w_\nu)^2 U^{w*}_{t\nu}, \tag{184}$$

where U^w is a unitary matrix. Equation (183) then becomes

$$S^{w'}_{st} = \frac{1}{N_{w'}} \sum_{uv\rho wv} \big(\Omega_{sw'} - \Omega_{uw}\big) \big(\phi_{sw'} \| \hat{z} \| \phi_{uw}\big)_\rho \big(\phi_{tw'} \| \hat{z} \| \phi_{vw}\big)^*_\rho U^w_{uv} (k^w_\nu)^2 U^{w*}_{v\nu}. \tag{185}$$

Note that some of the eigenvalues $(k^w_\nu)^2$ vanish, in general, in accordance with the $G_0 \downarrow K_0$ branching rules.

From Equation (184) and the identity $S^w_{st} = \sum_\nu \mathcal{K}^w_{s\nu} \mathcal{K}^{w*}_{t\nu}$, an orthonormal basis of VCS wave functions $\{\Psi\nu wr\}$ is now given by

$$\Psi_{\nu wr} = \sum_s \phi_{swr} \mathcal{K}^w_{s\nu} = \sum_s \phi_{swr} U^w_{s\nu} k^w_\nu, \tag{186}$$

and, from the duality relation $\big(\Phi_{\mu wr}, \Psi_{\nu wr}\big) = \delta_{\mu,\nu}$, a dual VCS basis is given by

$$\Phi_{\mu wr} = \sum_t \phi_{twr} \frac{1}{k^w_\mu} U^{w*}_{t\mu}, \tag{187}$$

with μ restricted to values for which $k^w_\mu \neq 0$. From Equation (106), an orthonormal basis $\{|\mu wr\rangle\}$ for $\mathbb{H}^\kappa$ is obtained, in terms of polynomials in the $\{\hat{A}_i\}$ raising operators acting on the lowest-grade states $\{|\kappa\alpha\rangle\}$, in the form

$$|\mu wr\rangle = \sum_t \frac{1}{k^w_\mu} U^{w*}_{\mu t} \phi_{twr}(\hat{A}). \tag{188}$$

(Recall that a wave function ϕ is a vector-valued function of z variables for which $\phi(z) = \sum_\alpha \phi_\alpha(z)|\kappa\alpha\rangle$, where $\{\kappa\alpha\}$ are basis vectors for the lowest-grade subspace $\mathbb{H}^\kappa_0 \subset \mathbb{H}^\kappa$. Thus, in accord with Equation (106),

$$\phi(\hat{A}) = \sum_\alpha \phi_\alpha(\hat{A})|\kappa\alpha\rangle \tag{189}$$

is a state vector in $\mathbb{H}^\kappa$.)

4.6. Matrix Elements of the Group $\mathbf{G_0}$ and the Lie Algebra $\mathfrak{g}$

From the expression (77) and Equations (186) and (187), the matrix elements of a holomorphic representation of lowest weight κ are given, relative to an orthonormal basis, by

$$\begin{aligned} \langle \mu wr|\hat{\mathcal{U}}^\kappa(g)|\nu w'r'\rangle &= \big(\Phi_{\mu wr}, \hat{\Gamma}(g)\Psi_{\nu w'r'}\big) \\ &= \sum_{st} \frac{1}{k^w_\mu} U^{w*}_{s\mu} \big(\phi_{swr}, \hat{\Gamma}(g)\phi_{tw'r'}\big) U^{w'}_{t\nu} k^{w'}_\nu. \end{aligned} \tag{190}$$

Similarly, for the Lie algebra, matrix elements of the operator $\hat{X}$ representing an element $X \in \mathfrak{g}$ are evaluated from either of the expressions

$$\langle \mu wr|\hat{X}|\nu w'r\rangle = \sum_{st} \frac{1}{k^w_\mu} U^{w*}_{s\mu} \big(\phi_{swr}, \hat{\Gamma}(X)\phi_{tw'r'}\big) U^{w'}_{t\nu} k^{w'}_\nu, \tag{191}$$

$$= \sum_{st} k^w_\mu U^{w*}_{s\mu} \big(\phi_{swr}, \hat{\Theta}(X)\phi_{tw'r'}\big) U^{w'}_{t\nu} \frac{1}{k^{w'}_\nu}. \tag{192}$$

Thus, for an element $C_p \in \mathfrak{k}_0$, for which

$$\hat{\Gamma}(C_p) = \hat{\Theta}(C_p) \tag{193}$$

and the observation that the matrix elements of a Lie algebra cannot connect its different irreps and are identical within equivalent irreps, we obtain

$$\langle \mu w r|\hat{C}_p|\nu w' r'\rangle = \delta_{\mu,\nu}\delta_{w,w'}(\phi_{swr}, \hat{\Gamma}(C_p)\phi_{swr'}), \tag{194}$$

independent of the multiplicity indices. For a lowering operator, for which $\hat{\Gamma}(B_i) = \partial_i$, and a raising operator, for which $\hat{\Theta}(A_i) = \hat{z}_i$, we obtain expressions

$$\langle \mu w r|\hat{B}_i|\nu w' r'\rangle = \sum_{st} \frac{1}{k^w_\mu} U^{w*}_{s\mu}(\phi_{swr}, \partial_i \phi_{tw'r'}) U^{w'}_{t\nu} k^{w'}_\nu, \tag{195}$$

$$\langle \nu w' r'|\hat{A}_i|\mu w r\rangle = \sum_{st} k^{w'}_\nu U^{w'*}_{t\nu}(\phi_{tw'r'}, \hat{z}_i \phi_{swr}) U^{w}_{s\mu} \frac{1}{k^w_\mu}, \tag{196}$$

which satisfy the Hermiticity relationship

$$\langle \nu w' r'|\hat{A}_i|\mu w r\rangle = \langle \mu w r|\hat{B}_i|\nu w' r'\rangle^*, \tag{197}$$

as required for a unitary irrep.

The above expressions simplify dramatically for the states $\{|\nu w r\rangle\}$ of a multiplicity-free irrep w for which the label ν takes a single value. The S^w matrices are then one-dimensional and given by single real numbers, $S^w = (k^w)^2 \geq 0$. Matrix elements between such multiplicity-free states are then obtained directly from the Hermiticity requirement of Equation (197) as follows. First observe that

$$\langle w' r'|\hat{A}_i|w r\rangle = (\Phi_{w'r'}, [\hat{\Lambda}, \hat{z}_i]\Psi_{wr}) = \frac{1}{k^{w'}}(\Omega_{w'} - \Omega_w)(\phi_{w'r'}, \hat{z}_i \phi_{wr}) k^w. \tag{198}$$

Equating this expression to its Hermitian adjoint

$$\langle w r|\hat{B}_i|w' r'\rangle^* = \frac{1}{k^w}(\phi_{wr}, \partial_i \phi_{w'r'})^* k^{w'} = \frac{1}{k^w}(\phi_{w'r'}, \hat{z}_i \phi_{wr}) k^{w'}, \tag{199}$$

then gives

$$\left(\frac{k^{w'}}{k^w}\right)^2 = \Omega_{w'} - \Omega_w \tag{200}$$

and

$$\langle w' r'|\hat{A}_i|w r\rangle = (\Omega_{w'} - \Omega_w)^{\frac{1}{2}}(\phi_{w'r'} \hat{z}_i \phi_{wr}), \tag{201}$$

$$\langle w r|\hat{B}_i|w' r'\rangle = (\Omega_{w'} - \Omega_w)^{\frac{1}{2}}(\phi_{w'r'}, \hat{z}_i \phi_{wr})^*. \tag{202}$$

Similarly, for a group element $g \in G_0$, the matrix elements of a multiplicity-free representation simplify to

$$\langle w' r'|\hat{\mathcal{U}}^\kappa(g)|w r\rangle = (\Omega_{w'} - \Omega_w)^{-\frac{1}{2}}(\phi_{w'r'}, \hat{\Gamma}(g)\phi_{wr}). \tag{203}$$

5. Dual VCS Representations of the $\mathfrak{sp}(N, \mathbb{R})$ Lie Algebras

As an example of dual VCS representations, we consider the holomorphic representation with lowest weight of an $\mathfrak{sp}(N, \mathbb{R})$ Lie algebra. An example for a compact Lie algebra is considered in the following section.

All unitary representations of $\mathfrak{sp}(N, \mathbb{R})$ with lowest- or highest-weight states, including the double-valued (metaplectic) projective representations, have holomorphic VCS realisations. The construction of these VCS representations provides orthonormal basis wave functions for the corresponding holomorphic representations of the $\mathrm{Sp}(N, \mathbb{R})$ Lie groups and expressions for the matrix elements of their Lie algebras in these bases in terms of $\mathrm{U}(N)$ Clebsch-Gordan (also called Wigner) coefficients and Racah coefficients [60–62]. The many holomorphic irreps of $\mathrm{Sp}(3, \mathbb{R})$ with lowest weights are of particular importance in nuclear physics as they arise in the microscopic theory of nuclear collective dynamics.

5.1. Dual Representations of the $\mathfrak{sp}(N, \mathbb{R})$ Lie Algebra on $\mathcal{F}^\kappa$

For the symplectic groups, the submatrix z of Equation (69) is symmetric with elements $z_{ij} = z_{ji}$ for $i, j = 1, \ldots, N$. With $z \cdot \hat{B} = \frac{1}{2}\sum_{ij} z_{ij}\hat{B}_{ij}$, the VCS representations of the $\mathfrak{sp}(N, \mathbb{R})$ Lie algebra, obtained from Equations (139)–(142) and (155)–(157), are given by

$$\begin{aligned} \hat{\Gamma}(B_{ij}) &= \nabla_{ij}, & \hat{\Theta}(B_{ij}) &= [\nabla_{ij}, \hat{\Lambda}], \\ \hat{\Gamma}(C_{ij}) &= \hat{\mathfrak{C}}_{ij} + (\hat{z}\nabla)_{ij}, & \hat{\Theta}(C_{ij}) &= \hat{\mathfrak{C}}_{ij} + (\hat{z}\nabla)_{ij}, \\ \hat{\Gamma}(A_{ij}) &= [\hat{\Lambda}, \hat{z}_{ij}], & \hat{\Theta}(A_{ij}) &= \hat{z}_{ij}, \end{aligned} \tag{204}$$

where $\nabla_{ij} = (1 + \delta_{i,j})\frac{\partial}{\partial z_{ij}}$ and $\hat{\Lambda}$ is the U(N)-invariant operator

$$\hat{\Lambda} = \tfrac{1}{2}(\hat{\mathfrak{C}} + \hat{z}\nabla)\cdot(\hat{\mathfrak{C}} + \hat{z}\nabla) - \tfrac{1}{4}\hat{z}\nabla\cdot\hat{z}\nabla - \tfrac{1}{4}(N+1)\hat{z}\cdot\nabla, \tag{205}$$

with $(\hat{z}\nabla)_{ij} = \sum_k \hat{z}_{ik}\nabla_{kj}$, and $\hat{z}\cdot\nabla = \sum_i (\hat{z}\nabla)_{ii}$.

As in any VCS representation, the $\hat{\Gamma}$ and $\hat{\Theta}$ operators are expressed in terms of a simpler Lie algebra with elements

$$\hat{\mathcal{C}}_{ij} = \hat{\mathfrak{C}}_{ij} + (\hat{z}\nabla)_{ij}, \quad \hat{z}_{ij}, \quad \nabla_{ij}, \quad i, j = 1, \ldots, N. \tag{206}$$

Such a Lie algebra, known in nuclear physics as a U(N)-boson algebra [63], is a semi-direct sum of a $\mathfrak{u}(N)$ algebra and a Heisenberg-Weyl algebra with raising and lowering operators given, respectively, by $\{\hat{z}_{ij}\}$ and $\{\nabla_{ij}\}$ and with commutation relations

$$[\nabla_{ij}, \hat{z}_{kl}] = (\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k})\hat{I}, \tag{207}$$

$$[\hat{\mathcal{C}}_{ij}, \hat{z}_{kl}] = [(\hat{z}\nabla)_{ij}, \hat{z}_{kl}] = \delta_{j,k}\hat{z}_{il} + \delta_{j,l}\hat{z}_{ik}, \tag{208}$$

where $\hat{I}$ is the identity operator.

It follows that the Hilbert space $\mathcal{F}^\kappa$ for an irreducible unitary representation of a U(N)-boson algebra is also a Hilbert space for dual VCS $\hat{\Gamma}$ and $\hat{\Theta}$ representations of $\mathfrak{sp}(N, \mathbb{R})$ from which unitary irreps of $\mathfrak{sp}(N, \mathbb{R})$ can be constructed.

5.2. Representations of the $U(N)$-boson Algebra

A Heisenberg-Weyl algebra has a unique unitary representation which can be combined with those of a $\mathfrak{u}(N)$ algebra to construct unitary representations of the U(N)-boson algebra given by their semi-direct sum. Unitary irreps of the U(3)-boson algebra in an orthonormal U(3)-coupled basis, were determined [64,65] for use in nuclear collective models. The representations of boson algebras with symmetric matrices of raising and lowering operators $\{\hat{z}_{ij}, i, j = 1, \ldots, N\}$ and $\{\nabla_{ij}, i, j = 1, \ldots, N\}$ with arbitrary N and with matrices of other symmetries, as needed for construction of holomorphic representations of other Lie algebras, were subsequently constructed [54,66], by use of Capelli identities [2,55–58], and expressed in terms of the so-called U(N)-reduced matrix elements of the generalised Wigner-Eckart theorem [60,61].

For example, the 6 linearly-independent variables

$$\{\bar{z}_{ij} = z_{ij}/\sqrt{1 + \delta_{i,j}},\ 0 \le i \le j \le 3\}, \tag{209}$$

which transform under U(3) as components of a U(3) $\{2, 0, 0\}$ tensor, are regarded as the boson raising operators for a Bargmann coherent-state representation of a 6-dimensional harmonic oscillator. Thus, an orthonormal basis for the 6-dimensional harmonic oscillator is given by polynomials $\{\chi_{nq}(z)\}$, that separate into U(3)-invariant subsets, each of which is characterised by a highest weight $n \equiv \{n_1, n_2, n_3\}$ given by a triplet of even integers with

$$n_1 \geq n_2 \geq n_3 \geq 0. \tag{210}$$

The U(3) irreps spanned by these subsets are those of Littlewood's D_3 series [67].

The U(3)-reduced matrix elements of the boson raising operators in this basis have been determined [64,65] and are given explicitly by

$$\begin{aligned}(m\|\hat{z}\|n) = &\left(\frac{(n_1+4)(n_1-n_2+2)(n_1-n_3+3)}{2(n_1-n_2+3)(n_1-n_3+4)}\right)^{\frac{1}{2}} \delta_{m_1,n_1+2}\delta_{m_2,n_2}\delta_{m_3,n_3} \\ &+ \left(\frac{(n_2+3)(n_1-n_2)(n_2-n_3+2)}{2(n_1-n_2-1)(n_2-n_3+3)}\right)^{\frac{1}{2}} \delta_{m_1,n_1}\delta_{m_2,n_2+2}\delta_{m_3,n_3} \\ &+ \left(\frac{(n_3+2)(n_2-n_3)(n_1-n_3+1)}{2(n_1-n_3)(n_2-n_3-1)}\right)^{\frac{1}{2}} \delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3+2}\,.\end{aligned} \tag{211}$$

Now, if the lowest-grade U(3) states of an Sp$(3,\mathbb{R})$ irrep belong to a U(3) irrep $\{\kappa\}$, an orthonormal basis for the corresponding irrep of the U(3)-boson algebra is given by the U(3)-coupled product states

$$\phi_{\kappa n\rho w r}(z) = \left[\chi_n(z) \otimes |\kappa\rangle\right]_{\rho w r}, \quad n \in D_3, \tag{212}$$

where ρ indexes the multiple occurences of a U(N) irrep w in the tensor product $n \otimes \kappa$. The corresponding reduced matrix elements of the boson raising operators in this (round bracket) basis are then given [65] by

$$(\phi_{\kappa m\rho' w'}\|\hat{z}\|\phi_{\kappa n\rho w}) = U(\kappa\, n\, w'\Delta_{11}; w\,\rho\, m\,\rho')\,(m\|\hat{z}\|n), \tag{213}$$

where $U(\kappa\, n\, w'\Delta_{11}; w\,\rho\, m\,\rho')$ is a known U(3) Racah recoupling coefficient [62,68]. The corresponding reduced matrix elements for other U(N)-boson representations are similarly expressed in term of the reduced boson matrix elements given in [66].

The matrix elements of the VCS $\hat{\Gamma}$ and $\hat{\Theta}$ representations given by the U(N)-boson expansions of Equation (204) are now obtained immediately from the observation that the U(N)-invariant operator $\hat{\Lambda}$ is diagonal in the $\{\phi_{\kappa n\rho w r}\}$ basis given in Equation (212) with eigenvalues given by $\hat{\Lambda}\phi_{\kappa n\rho w r} = \Omega_{nw}\phi_{\kappa n\rho w r}$, where

$$\Omega_{nw} = \frac{1}{4}\sum_{i}^{N}\Big[2w_i^2 - n_i(n_i+N+1) + \sum_{j>i}^{N}(2w_i - 2w_j - n_i + n_j)\Big]. \tag{214}$$

Thus, we obtain the matrix elements

$$\begin{aligned}(\phi_{\kappa m\rho' w' r}, \hat{\Gamma}(A_{ij})\phi_{\kappa n\rho w r}) &= (\Omega_{mw'} - \Omega_{nw})(\phi_{\kappa m\rho' w' r}, \hat{z}_{ij}\phi_{\kappa n\rho w r}), \\ (\phi_{\kappa n\rho w}, \hat{\Gamma}(B_{ij})\phi_{\kappa m\rho' w' r'}) &= (\phi_{\kappa m\rho' w' r}, \hat{z}_{ij}\phi_{\kappa n\rho w r})^*,\end{aligned} \tag{215}$$

$$\begin{aligned}(\phi_{\kappa m\rho' w' r}, \hat{\Theta}(A_{ij})\phi_{\kappa n\rho w r}) &= (\phi_{\kappa m\rho' w' r}, \hat{z}_{ij}\phi_{\kappa n\rho w r}), \\ (\phi_{\kappa n\rho w}, \hat{\Theta}(B_{ij})\phi_{\kappa m\rho' w' r'}) &= (\Omega_{mw'} - \Omega_{nw})(\phi_{\kappa m\rho' w' r}, \hat{z}_{ij}\phi_{\kappa n\rho w r})^*,\end{aligned} \tag{216}$$

of the $\mathfrak{sp}(N,\mathbb{R})$ raising and lowering operators in the non-unitary representations given by the actions of $\hat{\Gamma}$ and $\hat{\Theta}$ on $\mathcal{F}^\kappa$ in the above-defined U(N)-boson basis.

5.3. Unitary Irreps of the $sp(N,\mathbb{R})$ Lie Algebra

Matrix elements of a unitary irrep between states of multiplicity-free $\mathrm{U}(N)$ states, i.e., states for which the multiplicity indices ρ and ρ' are redundant, are obtained immediately from Equations (215) and (216) and given by

$$\langle \kappa m w' r|\hat{A}_{ij}|\kappa n w r\rangle = (\Omega_{mw'} - \Omega_{nw})^{\frac{1}{2}} \langle \kappa m w' r|\hat{z}_{ij}|\kappa n w r\rangle, \tag{217}$$

$$\langle \kappa n w|\hat{B}_{ij}|\kappa m w' r'\rangle = (\Omega_{mw'} - \Omega_{nw})^{\frac{1}{2}} \langle \kappa m w' r|\hat{z}_{ij}|\kappa n w r\rangle^*. \tag{218}$$

More generally, they are given, in terms of the dual VCS representations by

$$\langle \nu w' r'|\hat{A}_{ij}|\mu w r\rangle = (\Psi_{\nu w' r'}, \Theta(A_{ij})\Phi_{\mu w r}) = (\Psi_{\nu w' r'}, \hat{z}_{ij}\Phi_{\mu w r}), \tag{219}$$

$$\langle \mu w r|\hat{B}_{ij}|\nu w' r'\rangle = (\Phi_{\mu w r}, \Gamma(B_{ij})\Psi_{\nu w' r'}) = (\Phi_{\nu w r}, \nabla_{ij}\Psi_{\nu w' r'}). \tag{220}$$

However, to make use of these matrix elements, one must determine the VCS wave functions

$$\Psi_{\nu w r} = \sum_s \phi_{swr} U^w_{s\nu} k^w_\nu, \tag{221}$$

$$\Phi_{\mu w r} = \sum_t \phi_{twr} \frac{1}{k^w_\mu} U^{w*}_{t\mu}, \tag{222}$$

of Equations (186) and (187). This is achieved by solving the recursion relation (185) for the S^w matrices and expressing them in the form

$$S^w_{st} = \sum_\nu U^w_{s\nu}(k^w_\nu)^2 U^{w*}_{t\nu}. \tag{223}$$

The matrix elements of a unitary irrep are then given by

$$\langle \nu w' r'|\hat{A}_{ij}|\mu w r\rangle = \sum_{st} k^{w'}_\nu U^{w'*}_{t\nu} (\phi_{tw'r'}, \hat{z}_{ij}\phi_{swr})\, U^w_{s\mu} \frac{1}{k^w_\mu}, \tag{224}$$

$$\langle \mu w r|\hat{B}_{ij}|\nu w' r'\rangle = \langle \nu w' r'|\hat{A}_{ij}|\mu w r\rangle^*. \tag{225}$$

6. Holomorphic VCS Representations of a Compact Lie Group

The above presentation of VCS theory has focussed on holomorphic representations with lowest weights of simple Lie groups and their Lie algebras. The construction applies to both compact and non-compact Lie groups. For a non-compact Lie group G_0, a holomorphic representation is commonly induced from a representation of a maximal compact subgroup K_0. However, when G_0 is compact, the subgroup K_0 is automatically compact; it is then only required that the homogeneous space $K_0\backslash G_0$ be symmetric.

In fact, the Lie groups that have holomorphic representations frequently come in pairs, one of which is compact and the other non-compact. The holomorphic representations of such pairs are then induced from common compact subgroups. For example, a compact $\mathrm{Sp}(N)$ and a non-compact $\mathrm{Sp}(N,\mathbb{R})$ group both have holomorphic representations induced from the same irrep of a common $\mathrm{U}(N)$ subgroup. Similarly, $\mathrm{SU}(p+q)$ and $\mathrm{SU}(p,q)$ have holomorphic representations induced from a representation of their common $\mathrm{S}[\mathrm{U}(p)\times\mathrm{U}(q)]$ subgroup. However, when G_0 is compact, it is generally useful to induce its holomorphic representations from a highest-grade irrep of $K_0 \subset G_0$.

Outlining the essential steps of the construction of a holomorphic representation with highest weight of a Lie group and its Lie algebra is useful at this stage because it provides a concise summary of the essential methods employed in the VCS approach.

6.1. The Basic Construction

The notations parallel those of Section 2: G_0 is now a simple connected compact Lie group with a subgroup K_0; G and K are the complex extensions of G_0 and K_0; the Lie algebra $\mathfrak{g}$ of G is a sum

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-, \tag{226}$$

in which $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{n}_\pm$ are, respectively, Abelian Lie algebras of raising and lowering operators having the property that $[\mathfrak{k}, \mathfrak{n}_\pm] \in \mathfrak{n}_\pm$ and $[\mathfrak{n}_-, \mathfrak{n}_+] \in \mathfrak{k}$; $\mathbb{H}^\kappa$ is the Hilbert space for a unitary irrep $\hat{\mathcal{U}}^\kappa$ of G_0 with highest-weight κ; $\mathbb{H}_0^\kappa \subset \mathbb{H}^\kappa$ is the highest-grade subspace of states in $\mathbb{H}^\kappa$ that are annihilated by the raising operators of $\mathfrak{n}_+$ and is the Hilbert space for a unitary irrep of K_0 of highest-weight κ given by the restriction of $\hat{\mathcal{U}}^\kappa$ to $K_0 \subset G_0$; $\hat{T}^\kappa$ denotes the extension to K of the irrep $\hat{\mathcal{U}}^\kappa$ of K_0; $\hat{X}$ denotes the representation $\hat{\mathcal{U}}^\kappa$ of an element $X \in \mathfrak{g}_0$ or by extension of $X \in \mathfrak{g}$. Thus the factor space $K_0 \backslash G_0$ is again a symmetric space. Also, in parallel with Equations (9) and (10), elements of G_0 and K_0 are expressed in the form

$$g(a,b,c,d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0, \quad k(e,f) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in K_0, \tag{227}$$

and, as in Equations (11) and (12), elements of the subgroups of G generated by the subalgebras $\mathfrak{n}_\pm$ are expressed in the form

$$\begin{pmatrix} I_p & z \\ 0 & I_q \end{pmatrix} = e^{Z(z)}, \quad \text{with} \quad Z(z) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}_+, \tag{228}$$

$$\begin{pmatrix} I_p & 0 \\ x & I_q \end{pmatrix} = e^{X(x)}, \quad \text{with} \quad X(x) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \in \mathfrak{n}_-, \tag{229}$$

where I_p and I_q are, respectively, M_{pp} and M_{qq} identity matrices.

The plan is now to construct $\hat{\mathcal{U}}^\kappa$ as a holomorphic representation induced from a representation $\hat{\sigma}^\kappa$ of the subgroup K_0 on the highest-grade subspace $\mathbb{H}_0^\kappa \subset \mathbb{H}^\kappa$. A projection operator $\hat{\Pi}^\kappa : \mathbb{H}^\kappa \to \mathbb{H}_0^\kappa$ is defined in terms of an orthonormal basis $\{|\kappa\alpha\rangle\}$ for $\mathbb{H}_0^\kappa$ by $\hat{\Gamma}^\kappa = \sum_\alpha |\kappa\alpha\rangle\langle\kappa\alpha|$. Almost every element $g(a,b,c,d) \in G_0$ (elements with $\det(a) \neq 0$) then has the Gauss factorization

$$g(a,b,c,d) = \begin{pmatrix} I_p & 0 \\ x & I_q \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d-cz \end{pmatrix} \begin{pmatrix} I_p & z \\ 0 & I_q \end{pmatrix}, \tag{230}$$

with $x = ca^{-1}$ and $z = a^{-1}b$. The VCS wave function for a state $|\psi\rangle \in \mathbb{H}^\kappa$ and a VCS representation $\hat{\Gamma}$ isomorphic to $\hat{\mathcal{U}}^\kappa$ are then defined by

$$\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}|\psi\rangle, \tag{231}$$

$$\hat{\Gamma}(g)\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}\hat{\mathcal{U}}^\kappa(g)|\psi\rangle, \quad \forall g \in G_0. \tag{232}$$

In the defining representation, Gauss factorisation gives

$$\begin{pmatrix} I_p & z \\ 0 & I_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ x & I_q \end{pmatrix} \begin{pmatrix} a+zc & 0 \\ 0 & d-cy \end{pmatrix} \begin{pmatrix} I_p & y \\ 0 & I_q \end{pmatrix} \tag{233}$$

with $x = c(a+zc)^{-1}$ and $y = (a+zc)^{-1}(b+zd)$. It then follows, with $g = g(a,b,c,d)$, that

$$\hat{\Gamma}(g)\Psi(z) = \hat{\Pi}^\kappa e^{\hat{Z}(z)}\hat{\mathcal{U}}^\kappa(g)|\psi\rangle = \hat{\sigma}^\kappa(k)\Psi(y), \tag{234}$$

with $k = k(a + zc, d - cy)$, where Ψ is a polynomial in the elements of the matrix z in Equation (233) restricted to a subset D that is invariant under the map

$$g(a,b,c,d) : D \to D; z \mapsto (a+zc)^{-1}(b+zd), \quad \forall\, g(a,b,c,d) \in G_0. \tag{235}$$

6.2. Dual VCS Representations of the Lie Algebra $\mathfrak{g}$ with Highest Weight

As shown in Section 4.3, a Lie algebra with holomorphic representations and complex extension $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{k} \oplus \mathfrak{n}_-$ has a VCS expression in terms of the structure constants C_p^{ij} and D_p^{ij}, in the commutation relations

$$[B_i, C_p] = \sum_j C_p^{ij} B_j, \quad [C_p, A_j] = \sum_j A_i C_p^{ij}, \quad [B_i, A_j] = \sum_p D_p^{ij} C_p, \tag{236}$$

where $\{A_i\}$ and $\{B_i\}$ are related bases for the respective raising and lowering operator subalgebras $\mathfrak{n}_\pm$, such that the sum $\sum_i A_i B_i$ is K_0 invariant, and $\{C_p\}$ is a basis for $\mathfrak{k}$. The VCS representation $\hat{\Gamma}(X)$ of an element $X \in \mathfrak{g}$, defined by

$$\hat{\Gamma}(X)\hat{\Pi}^\kappa e^{\sum_i z_i \hat{A}_i} = \hat{\Pi}^\kappa e^{\sum_i z_i \hat{A}_i} \hat{X}, \tag{237}$$

is then given by

$$\hat{\Gamma}(A_i) = \partial_i, \tag{238}$$

$$\hat{\Gamma}(C_p) = \hat{\mathfrak{C}}_p - \sum_{ij} C_p^{ij} \hat{z}_j \partial_i, \tag{239}$$

$$\hat{\Gamma}(B_i) = -\sum_j D_p^{ij} \hat{z}_j (\hat{\mathfrak{C}}_p - \tfrac{1}{2} \sum_{kl} C_p^{lk} \hat{z}_k \partial_l) = [\hat{\Lambda}, \hat{z}_i], \tag{240}$$

where $\hat{\mathfrak{C}}_p = \hat{\sigma}^\kappa(C_p)$ is defined, in accordance with Equation (18), by $\hat{\sigma}^\kappa(C_p)\hat{\Pi}^\kappa = \hat{\Pi}^\kappa \hat{C}_p$, and $\hat{\Lambda}$ is the K-invariant operator

$$\hat{\Lambda} = -\sum_j D_p^{ij} \hat{z}_j (\hat{\mathfrak{C}}_p \partial_i - \tfrac{1}{4} \sum_{kl} C_p^{lk} \hat{z}_k \partial_l \partial_i). \tag{241}$$

Let $\{|\kappa\alpha\rangle\}$ denote an orthonormal basis for the representation $\hat{\sigma}^\kappa$ of the subalgebra $\mathfrak{k}_0 \subset \mathfrak{g}_0$ on the highest-grade subspace $\mathbb{H}_0^\kappa \subset \mathbb{H}^\kappa$. And let $\{\phi_n\}$ denote a Bargmann basis of wave functions $\phi_n(z) = \prod_i^{\dim \mathfrak{n}_-} z_i^{n_i} / \sqrt{n_i!}$. Then the product vector-valued wave functions

$$\phi_{\kappa\alpha n}(z) = |\kappa\alpha\rangle \phi_n(z) \tag{242}$$

are an orthonormal basis for the Hilbert space $\mathcal{F}^\kappa$ for a representation of the direct sum of $\mathfrak{k}_0$ and the Heisenberg-Weyl algebra generated by $\{z_i\}$ and $\{\partial_i\}$. However, to take advantage of the K_0-invariance of $\hat{\Lambda}$, it is preferable to start with a K_0-coupled basis for this representation. Such a basis can be constructed because the variables $\{z_i, i = 1, \ldots, \dim \mathfrak{n}_-\}$ transform as a basis for an irrep of K_0 of dimension $\dim \mathfrak{n}_-$. Let $\{\phi_{swr}\}$ denote such K_0-coupled basis functions that are orthonormal with respect to the Bargmann inner product

$$(\phi_{swr}, \phi_{tw'r'}) = \delta_{st}\delta_{ww'}\delta_{rr'}, \tag{243}$$

where w and r label basis states for an irreducible K_0-invariant subspace of $\mathcal{F}^\kappa$ highest weight w, and s is a multiplicity index to distinguish irreducible K_0 subspaces of common w. These wave functions are naturally constructed as eigenfunctions of the $\hat{\Lambda}$ operator;

$$\hat{\Lambda}\phi_{swr} = \Omega_{sw}\phi_{swr} \tag{244}$$

and, because $\hat{\Lambda}$ is K_0 invariant, its eigenvalues are independent of r.

We next seek VCS wave functions $\{\Psi_{\nu w r}\}$ for an orthonormal basis $\{|\nu w r\rangle\}$ for the Hilbert space $\mathbb{H}^\kappa$ of the unitary irrep $\hat{\mathcal{U}}^\kappa$ as linear combinations

$$\Psi_{\nu w r} = \sum_s \phi_{swr} \mathcal{K}^w_{s\nu} \tag{245}$$

such that the expansion

$$\hat{\Gamma}(X)\Psi_{\nu w r} = \sum_{\mu w' s} \Psi_{\mu w' s} \langle \mu w' s | \hat{X} | \nu w r \rangle, \quad \forall\, X \in \mathfrak{g}, \tag{246}$$

gives matrix elements that satisfy the Hermiticity relationship

$$\langle \mu w r | \hat{X}^\dagger | \nu w' s \rangle = \langle \nu w' s | \hat{X} | \mu w r \rangle^* \tag{247}$$

required of a unitary representation. The required matrix elements will then be expressible in the form

$$\langle \nu w' s | \hat{X} | \mu w r \rangle = (\Phi_{\nu w' s}, \hat{\Gamma}(X)\Psi_{\mu w r}), \tag{248}$$

where $\{\Phi_{\nu w' s}\}$ is a bi-orthogonal dual basis that satisfies the equation

$$(\Phi_{\mu w' s}, \Psi_{\nu w r}) = (\Phi_{\nu w r}, \Psi_{\mu w' s})^* = \delta_{\mu,\nu}\delta_{w',w}\delta_{s,r}. \tag{249}$$

A complementary dual VCS* representation $\hat{\Theta}$, defined by

$$(\Psi_{\mu w s}, \hat{\Theta}(X)\Phi_{\nu w' t}) = (\Phi_{\mu w s}, \hat{\Gamma}(X)\Psi_{\nu w' t}), \tag{250}$$

follows from the Hermiticity relation (247)

$$(\Psi_{\mu w r}, \hat{\Theta}(X^\dagger)\Phi_{\nu w' s}) = (\Phi_{\nu w' s}, \hat{\Gamma}(X)\Psi_{\mu w r})^* = (\Psi_{\mu w r}, \hat{\Gamma}^\dagger(X)\Phi_{\nu w' s}), \tag{251}$$

and implies that

$$\hat{\Theta}(X^\dagger) = \hat{\Gamma}^\dagger(X), \tag{252}$$

where the Hermitian adjoint $\hat{\Gamma}^\dagger$ is defined with respect to the Bargmann round-bracket inner product, and $X^\dagger$ is the element of $\mathfrak{g}$ represented by $\hat{X}^\dagger$ in a unitary representation. This simple relationship, together with the corresponding relationships $\hat{A}_i^\dagger = \hat{B}_i$, $\hat{C}_p^\dagger = \hat{C}_p$, leads directly to the dual VCS* representation

$$\hat{\Theta}(B_i) = z_i, \tag{253}$$

$$\hat{\Theta}(C_p) = \hat{\mathfrak{C}}_p - \sum_{ij} C_p^{ij} z_j \partial_i, \tag{254}$$

$$\hat{\Theta}(A_i) = [\partial_i, \hat{\Lambda}]. \tag{255}$$

The inner product for wave functions Φ and Φ' of the VCS* representation $\hat{\Theta}$ is expressed by $(\Phi, \hat{S}\Phi')$ in terms of an operator $\hat{S}$ for which $\Psi_{\nu w r} = \hat{S}\Phi_{\nu w r}$. Also, because the representations are constructed in a K_0-coupled basis, the matrix of the operator $\hat{S}$ in the Bargmann basis $\{\phi_{swr}\}$ is block diagonal with elements

$$(\phi_{swr}, \hat{S}\phi_{tw'r'}) = S^w_{st}\delta_{w,w'}\delta_{r,r'}. \tag{256}$$

The submatrices S^w are then obtained from the recursion relation,

$$\sum_i [\hat{\Lambda}, z_i]\hat{S}\partial_i = \hat{S}\sum_i z_i\partial_i, \tag{257}$$

derived from the intertwining relation

$$\sum_i \hat{\Gamma}(B_i)\hat{S}\hat{\Gamma}(A_i) = \hat{S}\hat{\Theta}(B_i)\hat{\Gamma}(A_i). \tag{258}$$

Once the S^w matrices have been determined, it is straightforward to derive the $\mathcal{K}^w$ matrices of Equation (245) and thereby express the VCS wave functions $\{\Psi_{\nu wr}\}$ as linear combinations of the Bargmann functions $\{\phi_{swr}\}$ as follows. Observe that the operation $\Phi_{\nu wr} \mapsto \hat{S}^w\Phi_{\nu wr}$ can be expressed in the integral form

$$\hat{S}^w\Phi_{\nu wr}(x) = \int S^w(x, z^*)\Phi_{\nu wr}(z)\, dv(z), \tag{259}$$

where $dv(z)$ is the Bargmann volume element. By setting

$$S^w(x, z^*) = \sum_{\mu r} \Psi_{\mu wr}(x)\Psi^\dagger_{\mu wr}(z^*), \tag{260}$$

we then obtain the desired result

$$\hat{S}^w\Phi_{\nu wr}(x) = \sum_\mu \Psi_{\mu wr}(x)(\Psi_{\mu wr}, \Phi_{\nu wr}) = \Psi_{\nu wr}(x). \tag{261}$$

The $\mathcal{K}^w$ matrices are then obtained from the matrix S^w with elements

$$S^w_{st} = (\phi_{swr}, \hat{S}^w\phi_{twr}) = \sum_\mu (\phi_{swr}, \Psi_{\mu wr})(\Psi_{\mu wr}, \phi_{twr}), \tag{262}$$

which, with Equation (245), gives

$$S^w_{st} = \sum_\mu \mathcal{K}^w_{s\mu}\mathcal{K}^{w*}_{t\mu}. \tag{263}$$

The S^w matrices are manifestly Hermitian. Thus, they can be diagonalised and brought to the form

$$S^w_{st} = \sum_\nu U^w_{s\nu}(k^w_\nu)^2 U^{w*}_{t\nu}, \tag{264}$$

where U^w is a unitary matrix and k^w_ν is real. The orthonormal VCS wave functions are then given for their respective Hilbert spaces by

$$\Psi_{\nu wr} = \sum_s \phi_{swr} U^w_{s\nu} k^w_\nu, \tag{265}$$

$$\Phi_{\nu wr} = \sum_s \phi_{swr} U^w_{s\nu} \frac{1}{k^w_\nu}, \tag{266}$$

and the matrix elements of the Lie algebra $\mathfrak{g}$ are given by

$$\langle \mu wr|\hat{X}|\nu w'r'\rangle = \sum_{st} \frac{1}{k^w_\mu} U^{w*}_{t\mu}(\phi_{twr}, \hat{\Gamma}(X)\phi_{swr})U^w_{s\nu}k^w_\nu. \tag{267}$$

6.3. Application to SU(3)

A simple application of the above construction is to induce an SU(3) irrep from an irrep of its subgroup $K_0 = \mathrm{S}[\mathrm{U}(1) \times \mathrm{U}(2)]$. This subgroup is isomorphic to U(2) and realised as matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \text{with } a \in \mathrm{U}(1),\ d \in \mathrm{U}(2),\ a = \det(d)^*. \tag{268}$$

Thus, if E_{ij} is a 3×3 matrix with entries

$$(E_{ij})_{kl} = \delta_{i,k}\delta_{j,l}, \tag{269}$$

the raising and lowering matrices for SU(3) are

$$A_1 = E_{12}, \quad A_2 = E_{13}, \quad B_1 = E_{21}, \quad B_2 = E_{31}. \tag{270}$$

For an SU(3) irrep, in which an element X in its Lie algebra is represented as an operator $\hat{X}$, a highest-weight state $|\lambda\mu\rangle$ is defined such that

$$\hat{E}_{12}|\lambda\mu\rangle = \hat{E}_{13}|\lambda\mu\rangle = \hat{E}_{23}|\lambda\mu\rangle = 0, \tag{271}$$

$$(\hat{E}_{11} - \hat{E}_{22})|\lambda\mu\rangle = \lambda|\lambda\mu\rangle, \tag{272}$$

$$(\hat{E}_{22} - \hat{E}_{33})|\lambda\mu\rangle = \mu|\lambda\mu\rangle. \tag{273}$$

A first step in the construction of an $\mathfrak{su}(3)$ irrep is to extend the highest-weight state to a set of highest-grade states for a $\mathfrak{u}(2)$ irrep. This is achieved by defining $\mathfrak{u}(2)$ operators

$$\hat{Q} = 2\hat{E}_{11} - \hat{E}_{22} - \hat{E}_{33} \tag{274}$$

$$\hat{S}_+ = \hat{E}_{23}, \quad \hat{S}_- = \hat{E}_{32}, \quad \hat{S}_0 = \tfrac{1}{2}(\hat{E}_{22} - \hat{E}_{33}) \tag{275}$$

and orthonormal highest-grade basis states $\{|sm\rangle\}$ that satisfy the equations

$$\hat{A}_i|sm\rangle = 0, \quad i = 1, 2, \tag{276}$$

$$\hat{Q}|sm\rangle = (2\lambda + \mu)|sm\rangle, \quad \hat{S}_0|sm\rangle = m|sm\rangle, \tag{277}$$

$$\hat{S}_\pm|sm\rangle = \sqrt{(s \mp m)(s \pm m + 1)}\,|s, m \pm 1\rangle, \tag{278}$$

with $s = \mu/2$.

Although it is not necessary, simplicity is gained by considering an $\mathrm{SU}(3) \subset \mathrm{U}(3)$ irrep. This has no impact on the results for SU(3) but it legitimises the expression of diagonal $\mathfrak{su}(3)$ matrices such as $E_{11} - E_{22}$ in terms of E_{11} and E_{22} as independent elements of the $\mathfrak{u}(3)$ Lie algebra. It also has the benefit, that it expresses representations of U(3) as induced from representations of its $\mathrm{U}(1)\times\mathrm{U}(2)$ subgroup. The highest-grade states $\{|sm\rangle\}$ are then also eigenstates of $\hat{E}_{11}$ and $\hat{E}_{22} + \hat{E}_{33}$ with eigenvalues given respectively by λ_1 and $\lambda_2 + \lambda_3$. The highest weight $(\lambda\mu)$ of an $\mathrm{SU}(3) \subset \mathrm{U}(3)$ irrep is likewise expressed in terms of the U(3) highest weight by

$$\lambda = \lambda_1 - \lambda_2, \quad \mu = \lambda_2 - \lambda_3. \tag{279}$$

With the above-defined highest-grade states $\{|sm\rangle\}$ and the projection operator $\Pi^{(\lambda\mu)} = \sum_m |sm\rangle\langle sm|$, a state $|\psi\rangle$ in the Hilbert space $\mathbb{H}^{(\lambda\mu)}$ for the corresponding SU(3) irrep has a VCS wave function defined by

$$\Psi(z) = \Pi^{(\lambda\mu)} e^{\hat{Z}(z)}|\psi\rangle, \quad \hat{Z}(z) = z_2\hat{E}_{12} + z_3\hat{E}_{13}. \tag{280}$$

The $\mathfrak{su}(3)$ Lie algebra then has a VCS representation defined in the usual way by

$$\hat{\Gamma}(X)\Psi(z) = \Pi^{(\lambda\mu)} e^{\hat{Z}(z)}\hat{X}|\psi\rangle, \quad X \in \mathfrak{su}(3), \tag{281}$$

and given by

$$\hat{\Gamma}(E_{1k}) = \partial_k, \quad k = 2,3, \tag{282}$$
$$\hat{\Gamma}(S_0) = \hat{\mathbb{S}}_0 + \tfrac{1}{2}(\hat{z}_2\partial_2 - \hat{z}_3\partial_3), \tag{283}$$
$$\hat{\Gamma}(S_+) = \hat{\mathbb{S}}_+ + \hat{z}_2\partial_3, \tag{284}$$
$$\hat{\Gamma}(S_-) = \hat{\mathbb{S}}_- + \hat{z}_3\partial_2, \tag{285}$$
$$\hat{\Gamma}(\mathcal{Q}) = (2\lambda + \mu) - 3(\hat{z}_2\partial_2 + \hat{z}_3\partial_3), \tag{286}$$
$$\hat{\Gamma}(E_{k1}) = [\hat{\Lambda}, \hat{z}_k], \quad k = 2,3, \tag{287}$$

where $\hat{\Lambda}$ is the U(2)-invariant operator

$$\hat{\Lambda} = \lambda_1 \sum_{k=2}^{3} \hat{z}_k\partial_k - \sum_{k,l=2}^{3} \hat{\mathbb{E}}_{kl}\hat{z}_l\partial_k - \tfrac{1}{2} \sum_{k,l=2}^{3} \hat{z}_k\hat{z}_l\partial_l\partial_k, \tag{288}$$

and $\hat{\mathbb{S}}_0$, $\hat{\mathbb{S}}_\pm$, $\hat{\mathbb{E}}_{11}$ and $\hat{\mathbb{E}}_{kl}$ denote the restrictions of the respective operators $\hat{S}_0$, $\hat{S}_\pm$, $\hat{E}_{11}$ and $\hat{E}_{kl}$ to highest-grade states. From Equation (252), the dual VCS* representation of $\mathfrak{su}(3)$ is given by

$$\hat{\Theta}(E_{k1}) = \hat{z}_k, \quad k = 2,3, \tag{289}$$
$$\hat{\Theta}(S_0) = \hat{\Gamma}(S_0), \quad \hat{\Theta}(S_\pm) = \hat{\Gamma}(S_\pm), \quad \hat{\Theta}(\mathcal{Q}) = \hat{\Gamma}(\mathcal{Q}), \tag{290}$$
$$\hat{\Theta}(E_{1k}) = [\partial_k, \hat{\Lambda}], \quad k = 2,3. \tag{291}$$

VCS wave functions, defined by Equation (280), are vector-valued functions of the variables $z = \{z_2, z_3\}$ of the form

$$\phi(z) = \sum_m |sm\rangle \phi_m(z), \tag{292}$$

and can be expanded in a basis of so-called U(2)-boson wave functions of this form as follows. Observe that z_2 and z_3 transform as spin-$\frac{1}{2}$ components of the 2-dimensional U(2) irrep $\{1,0\}$ and that an orthonormal Bargmann basis for a $\{2j,0\}$ irrep is given by the spin-j wave functions (with $2j$ a non-negative integer)

$$\chi_{jm}(z) = \frac{(z_2)^{j+m}(z_3)^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \quad m = -j, -j+1, \ldots, j. \tag{293}$$

VCS wave functions for a U(3) irrep are then expressed as finite linear combinations of the U(2)-coupled wave functions

$$\phi_{jSM}(z) = \left[\chi_j(z) \otimes |s\rangle\right]_{SM}. \tag{294}$$

To determine the eigenvalues of the operator $\hat{\Lambda}$ it is useful to express it in the more obviously U(2)-invariant form

$$\hat{\Lambda} = (\lambda_1 + 1)\sum_k \hat{z}_k\partial_k - \tfrac{1}{2}\sum_{kl}(\hat{\mathbb{E}}_{kl} + \hat{z}_k\partial_l)(\hat{\mathbb{E}}_{lk} + \hat{z}_l\partial_k) + \tfrac{1}{2}\sum_{kl}\hat{\mathbb{E}}_{kl}\hat{\mathbb{E}}_{lk}, \tag{295}$$

with k and l taking values 2 and 3. Then, by use of the identities

$$\tfrac{1}{2}\sum_{kl}(\hat{\mathbb{E}}_{kl} + \hat{z}_k\partial_l)(\hat{\mathbb{E}}_{lk} + \hat{z}_l\partial_k) = \hat{S}\cdot\hat{S} + \tfrac{1}{4}(\lambda_2 + \lambda_3 + \sum_k \hat{z}_k\partial_k)^2, \tag{296}$$
$$\tfrac{1}{2}\sum_{kl}\hat{\mathbb{E}}_{kl}\hat{\mathbb{E}}_{lk} = \hat{s}\cdot\hat{s} + \tfrac{1}{4}(\lambda_2 + \lambda_3)^2, \tag{297}$$

$\hat{\Lambda}$ is expressed

$$\hat{\Lambda} = (2\lambda + \mu + 1)\sum_k \hat{z}_k\partial_k - \tfrac{1}{4}\sum_{kl} z_k\partial_k\hat{z}_l\partial_l - \hat{S}\cdot\hat{S} + \hat{s}\cdot\hat{s}. \tag{298}$$

And, with the observation that

$$\sum_k \hat{z}_k \partial_k \phi_{jSM} = 2j\phi_{jSM}, \tag{299}$$

the U2)-boson wave functions of Equation (294) are eigenfunctions of $\hat{\Lambda}$ with M-independent eigenvalues

$$\Omega_{jS} = (2\lambda + \mu)j - S(S+1) + s(s+1) - j(j-2). \tag{300}$$

Matrix elements of the raising and lowering operators are now given in the U(2)-boson basis of Equation (294) for the VCS representation by

$$(\phi_{j'S'M'}, \hat{\Gamma}(E_{k1})\phi_{jSM}) = (\Omega_{j'S'} - \Omega_{jS})(\phi_{j'S'M'}, \hat{z}_k \phi_{jSM}). \tag{301}$$

$$(\phi_{jSM}, \hat{\Gamma}(E_{1k})\phi_{j'S'M'}) = (\phi_{j'S'M}, \hat{z}_k \phi_{jSM})^*, \tag{302}$$

and for the dual VCS* representation by

$$(\phi_{j'S'M'}, \hat{\Theta}(E_{k1})\phi_{jSM}) = (\phi_{j'S'M'}, \hat{z}_k \phi_{jSM}). \tag{303}$$

$$(\phi_{jSM}, \hat{\Theta}(E_{1k})\phi_{j'S'M'}) = (\Omega_{j'S'} - \Omega_{jS})(\phi_{j'S'M}, \hat{z}_k \phi_{jSM})^*. \tag{304}$$

It is evident that these matrix elements do not satisfy the Hermiticity relationships of a unitary representation. However, it is also observed that, in this SU(3) example, the states $\{\phi_{jSM}\}$ are uniquely labelled by jSM quantum numbers without multiplicity indices. Thus, the VCS wave functions $\{\Psi_{jSM}\}$ are uniquely defined, to within normalisation factors. Starting with the VCS wave functions $\{\Psi_{0sm} = \phi_{0sm}\}$ for the appropriately normalised highest-grade states for which $j = 0$, $S = s = \mu/2$, $M = m$, and $\Omega_{0s} = 1$, an orthonormal basis of VCS wave functions is then determined sequentially

$$\Psi_{jSM} = \mathcal{K}_{jS}\phi_{jSM}, \tag{305}$$

by setting

$$\frac{\mathcal{K}_{jS}}{\mathcal{K}_{j'S'}} = (\Omega_{j'S'} - \Omega_{jS})^{-\frac{1}{2}}., \quad \text{for } j' = j + \tfrac{1}{2} \text{ and } S' = S \pm \tfrac{1}{2} \tag{306}$$

for as long as $(\Omega_{j'S'} - \Omega_{jS}) > 0$. The VCS representation in this basis then satisfies the Hermiticity relations

$$\hat{\Gamma}(E_{k1})\Psi_{jSM} = \sum_{j'S'M'} (\Omega_{j'S'} - \Omega_{jS})^{\frac{1}{2}} \Psi_{j'S'M'}(\phi_{j'S'M'}, \hat{z}_k \phi_{jSM}), \tag{307}$$

$$\hat{\Gamma}(E_{1k})\Psi_{j'S'M'} = \sum_{jSM} (\Omega_{j'S'} - \Omega_{jS})^{\frac{1}{2}} \Psi_{jSM}(\phi_{jSM}, \hat{z}_k \phi_{jSM})^*, \tag{308}$$

of a unitary representation. The lowering sequence terminates and all $\{\phi_{j'S'M'}\}$ wave functions for which $(\Omega_{j'S'} - \Omega_{jS}) \leq 0$ in Equation (306) are discarded. Thus, a finite-dimensional SU(3) irrep is obtained with matrix elements

$$\langle j'S'M'|\hat{E}_{k1}|jSM\rangle = \langle jSM|\hat{E}_{1k}|j'S'M'\rangle^* = (\Omega_{j'S'} - \Omega_{jS})^{\frac{1}{2}}(\phi_{j'S'M'}, \hat{z}_k \phi_{jSM}). \tag{309}$$

In such a multiplicity-free case, the one-dimensional $\mathcal{K}_{jS}$ matrices are easily determined to be given by

$$\mathcal{K}_{j,S} = \sqrt{\frac{\lambda!(\lambda+\mu+1)!}{(\lambda + \frac{1}{2}\mu - j - S)!(\lambda + \frac{1}{2}\mu - j + S + 1)!}}. \tag{310}$$

Thus, with these $\mathcal{K}_{jS}$-matrix coefficients, the VCS wave functions have the explicit expressions

$$\Psi_{jSM}(z) = \mathcal{K}_{jS}\phi_{jSM}(z) = \mathcal{K}_{jS}[\phi_j(z) \otimes |s\rangle]_{SM}, \tag{311}$$

their dual VCS* partners are

$$\Phi_{jSM}(z) = \frac{1}{\mathcal{K}_{jS}}\phi_{jSM}(z) = \frac{1}{\mathcal{K}_{jS}}[\phi_j(z) \otimes |s\rangle]_{SM}, \tag{312}$$

and the corresponding states of the Hilbert space $\mathbb{H}^{(\lambda\mu)}$ are given by

$$|jSM\rangle = \frac{1}{\mathcal{K}_{jS}}[\phi_j(\hat{B}) \otimes |s\rangle]_{SM}. \tag{313}$$

7. Concluding remarks

The VCS methods described in this paper have been shown to successfully induce holomorphic representations with highest and/or lowest weights of any connected simple real Lie group G_0 and its Lie algebra from an irreducible unitary irrep of a compact subgroup $K_0 \subset G_0$ for which G_0/K_0 is a symmetric space. The VCS constructions are not restricted to discrete series representations and may be extended to reductive Lie groups. Examples of a group G_0 that does not have a subgroup K_0 for which G_0/K_0 is a symmetric space are given by the odd orthogonal groups $\mathrm{SO}(2N+1)$ for $N > 2$. However, even for these there are extensions of the VCS construction of holomorphic representations [16].

In spite of the above-mentioned generality of the VCS construction, it should be recognized that the explicit matrices of these representations are expressed not only in terms of the irreducible representation of the subgroup K_0 from which the representation of G_0 is induced but also in terms of the Clebsch-Gordan coupling and Racah decoupling coefficients for K_0. This, of course, is true of any inducing construction.

It should also be noted that, although the irreducible representations of all the Heisenberg-Weyl groups that feature in the VCS construction are uniquely defined, by the Stone-Von Neumann theorem [69,70], their construction in the needed K_0-coupled basis can be challenging. However, as indicated by LeBlanc [66], these representations can be inferred from the Capelli identities [2,55,56], when the subgroup K_0 is a unitary group. The derivation of these representations, when the variables $\{z_{ij}\}$ of the holomorphic inducing construction are either independent, symmetric $z_{ij} = z_{ji}$, or antisymmetric $z_{ij} = -z_{ji}$, has recently been further developed [54] and shown to correspond to the first [55], second [56], and third [57,58] Capelli identities, respectively. These representations are needed, for example, for the holomorphic representations of $\mathrm{U}(p,q)$ induced from an irrep of $\mathrm{U}(p) \times \mathrm{U}(q)$, of $\mathrm{Sp}(N,\mathbb{R})$ induced from a symmetric irrep of $\mathrm{U}N)$, and of $\mathrm{SO}^*(2N)$ induced from an anti-symmetric representation of $\mathrm{U}(N)$, respectively.

We also remark in closing that the VCS inducing construction of irreducible representations is by no means restricted to holomorphic representations. The essential requirement is the existence of two subgroups of G, the complex extension of G_0, such that a representation of one subgroup K_0 uniquely characterizes the desired representation of G_0 and the action of the other group extends the Hilbert space of this representation of K_0 to the Hilbert space of the irreducible representation of G_0. For example, irreps of SU(3) in an SO(3)-coupled basis have been induced from a highest-grade irrep of an SU(2) subgroup in which the Hilbert space of highest-grade states is extended to the Hilbert space of an irreducible SU(3) representation by the action of the SO(3) $\subset$ SU(3) subgroup [19,20,40]. The subgroups SU(2) and SO(3) have also been used to construct the irreducible representations of SO(5) in an SO(3) basis [22].

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