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Casorati Inequalities for Submanifolds in a Riemannian Manifold of Quasi-Constant Curvature with a Semi-Symmetric Metric Connection

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Abstract: By using new algebraic techniques, two Casorati inequalities are established for submanifolds in a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection, which generalize inequalities obtained by Lee *et al*. *J. Inequal. Appl.* **2014**, *2014*, 327.

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1. Introduction

In 1996, Chen initiated the following fundamental problem in his cornerstone work [1]: to establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of Riemannian submanifolds. The basic relationships discovered until now are inequalities and the study of this topic has attracted a lot of attention during the last two decades. We refer to [2] for a relatively recent survey on this topic.

On the other hand, Casorati introduced the Casorati curvature of an n-dimensional submanifold M of a Riemannian manifold, which is an extrinsic invariant defined as the normalized square of the length of the second fundamental form of the submanifold [3]. In 2007, Decu *et al*. introduced the normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ and established inequalities involving $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ for submanifolds in real space forms [4].

The proof of the inequalities in [4] is based on an optimization procedure by showing that the quadratic polynomial in the components of the second fundamental form is parabolic. And the above method was successfully applied to establish inequalities in terms of the Casorati curvatures for different submanifolds in various ambient spaces [5–10]. Recently, in [11–13], the authors obtained the corresponding Casorati inequalites by using Oprea's optimization methods on Riemannian submanifolds [14].

In [15,16], Mihai and Özgür established Chen inequalities for submanifolds of real, complex and Sasakian space forms endowed with semi-symmetric metric connections and in [17,18], Özgür and Murathan gave Chen inequalities for submanifolds of a locally conformal almost cosymplectic manifold and a cosymplectic space form endowed with semi-symmetric metric connections. On the other hand, Lee *et al.* proved inequalities involving the Casorati curvature of submanifolds in real, complex and Sasakian space forms endowed with a semi-symmetric metric connection in [7,8]. In an earlier paper, Özgür established Chen inequalities for submanifolds in a Riemannian manifold of quasi-constant curvature [19]. Just very recently, we obtained Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constatnt curvature with a semi-symmetric metric connection [20].

In this paper, by using new algebraic techniques, we establish inequalities involving the normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ for submanifolds in a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection, which generalize inequalities obtained in [7]. Our techniques can also be used to establish inequalities involving the generalized normalized δ-Casorati curvatures obtained in [5,9].

2. Preliminaries

To meet the requirements in the next sections, here, we briefly present the basic elements of the theory of a Riemannian manifold endowed with a semi-symmetric metric connection.

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold with the Riemannian connection $\hat{\overline{\nabla}}$, a linear connection $\overline{\nabla}$ and the Riemannian metric g. The torsion tensor field $\overline{T}$ of the linear connection $\overline{\nabla}$ is defined by

$$\overline{T}(\overline{X},\overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X},\overline{Y}]$$

for the vector fields $\overline{X},\overline{Y}$ on N^{n+p}.

The liner connection $\overline{\nabla}$ is said to be semi-symmetric if the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ satisfies the following relation

$$\overline{T}(\overline{X},\overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$$

for a 1-form ϕ on N^{n+p}. Further, if $\overline{\nabla}$ satisfies $\overline{\nabla}g = 0$, then $\overline{\nabla}$ is called a semi-symmetric metric connection [21]. In [21], Yano obtained a relation between the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$ which is given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \hat{\overline{\nabla}}_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X} - g(\overline{X},\overline{Y})P$$

where P is a vector field defined by

$$g(P,\overline{X}) = \phi(\overline{X}) \tag{1}$$

for any vector field $\overline{X}$ on N^{n+p}.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p} with the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$. On M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Riemannian connection denoted by $\hat{\nabla}$.

Let $\overline{R}$ be the curvature tensor of N^{n+p} with respect to $\overline{\nabla}$ and $\hat{\overline{R}}$ the curvature tensor of N^{n+p} with respect to $\hat{\overline{\nabla}}$. We also denote by R and $\hat{R}$ the curvature tensors associated to ∇ and $\hat{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\hat{\nabla}$, can be written as the following [22]

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \;\; \hat{\overline{\nabla}}_X Y = \hat{\nabla}_X Y + \hat{h}(X,Y)$$

for any vector fields X,Y on M, where h is a $(0,2)$ symmetric tensor on M^n and $\hat{h}$ is the second fundamental form associated to Riemaniann connection $\hat{\nabla}$. According to the formula (7) from [22] h is also symmetric.

The curvature tensor $\overline{R}$ with respect to the semi-symmetric metric connection $\overline{\nabla}$ on N^{n+p} can be written as [23]

$$\begin{aligned}\overline{R}(X,Y,Z,W) = {}& \hat{\overline{R}}(X,Y,Z,W) + \alpha(Y,Z)g(X,W) - \alpha(X,Z)g(Y,W) \\ & + \alpha(X,W)g(Y,Z) - \alpha(Y,W)g(X,Z)\end{aligned} \tag{2}$$

for any vector fields X, Y, Z, W on M^n, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = (\hat{\overline{\nabla}}_X \phi)Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y)$$

denote by λ the trace of α restricted on M^n.

The Gauss equation for the submanifold M^n into N^{n+p} with respect to the semi-symmetric metric connection is given by Nakao [22]

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \tag{3}$$

for any vector fields X, Y, Z, W on M^n.

According to the Equation (7) from [22] we have

Lemma 1. *If P given by Equation (1) is a tangent vector field on M^n, then $h = \hat{h}$.*

In N^{n+p} we choose a local orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$, such that, restricting to M^n, $e_1, \cdots, e_n$ are tangent to M^n.

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection. Then the scalar curvature τ with respect to the semi-symmetric metric connection is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ with respect to the semi-symmetric metric connection is defined by

$$\rho = \frac{2\tau}{n(n-1)}$$

We write

$$\hat{h}^r_{ij} = g(\hat{h}(e_i,e_j),e_r), \qquad h^r_{ij} = g(h(e_i,e_j),e_r)$$

Then we denote the Casorati curvature with respect to the semi-symmetric metric connection by

$$\mathcal{C} = \frac{1}{n} \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{n} (h^r_{ij})^2$$

Suppose now that L is an l-dimensional subspace of T_xM, $l \ge 2$, and $\{e_1, \cdots, e_l\}$ be an orthonormal basis of L. Then the Casorati curvature of the l-plane section L with respect to the semi-symmetric metric connection is given by

$$\mathcal{C}(L) = \frac{1}{l} \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{l} (h^r_{ij})^2$$

The normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ with respect to the semi-symmetric metric connection are given by [7]

$$[\delta_c(n-1)]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n}\inf\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_xM\}$$

and

$$[\hat{\delta}_c(n-1)]_x = 2\mathcal{C}_x - \frac{2n-1}{2n}\sup\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_xM\}$$

3. Main Results

Let (N, g, U) $(\dim N = m)$ be a Riemannian manifold with metric g and a unit vector field U. The structural group of this manifold is $O(m-1) \times 1$. $T_x N$ and $\mathfrak{X}N$ will stand for the tangent space to N at a point x and the algebra of smooth vector fields on N, respectively. The 1-form corresponding to the unit vector U is denoted by ξ, *i.e.*,

$$\xi(X) = g(U, X), \ \ X \in \mathfrak{X}N$$

The distribution of the 1-form ξ is denoted by Δ, *i.e.*, $\Delta(x) = \{X \in T_x N \mid \xi(X) = 0\}$.

Any section E in $T_x N$ determines an angle $\angle(E, U)$. A Riemannian manifold (N, g, U), $\dim N \geq 3$, is said to be of quasi-constant sectional curvature if for any arbitrary 2-plane E in $T_x N$ with $\angle(E, U) = \varphi$, the sectional curvature of E only depends on the point x and the angle φ.

We will consider an $(n+p)$-dimensional Riemanniann manifold N^{n+p} of quasi-constant curvature endowed with a semi-symmetric non-metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$.

From [24], the curvature tensor $\hat{\overline{R}}$ with respect to the Levi-Civita connection $\hat{\overline{\nabla}}$ on N^{n+p} is expressed by

$$\begin{aligned} \hat{\overline{R}}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = & \; a[g(\overline{X}, \overline{Z})g(\overline{Y}, \overline{W}) - g(\overline{Y}, \overline{Z})g(\overline{X}, \overline{W})] \\ & + b[g(\overline{X}, \overline{Z})\xi(\overline{Y})\xi(\overline{W}) - g(\overline{X}, \overline{W})\xi(\overline{Y})\xi(\overline{Z}) \\ & + g(\overline{Y}, \overline{W})\xi(\overline{X})\xi(\overline{Z}) - g(\overline{Y}, \overline{Z})\xi(\overline{X})\xi(\overline{W})] \end{aligned} \tag{4}$$

for any vector fields $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ on N^{n+p}, where a and b are scalar functions on N^{n+p}. If $b = 0$, it can be easily seen that the manifold reduces to a space of constant curvature.

We assume that an n-dimensional Riemannian manifold M^n is a submanifold in N^{n+p}. Decomposing the vector field U on M^n uniquely into its tangent and normal components $U^\top$ and $U^\perp$, respectively, we have

$$U = U^\top + U^\perp$$

Let us recall the following the definition from [25].

Definition 2. [25] Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. M^n is called invariantly quasi-umbilical if there exist p mutually orthogonal unit normal vectors $e_{n+1}, \cdots, e_{n+p}$ such that the shape operators with respect to all directions e_r have an eigenvalue of multiplicity $n-1$ and that for each e_r the distinguished eigendirection is the same.

The following is our main result.

Theorem 3. *Let M^n, $n \geq 3$, be an n-dimensional submanifold in a Riemanniann manifold N^{n+p} of quasi-constant curvature endowed with a semi-symmetric metric connection. Then we have:*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + a + \frac{2b}{n} \parallel U^\top \parallel^2 - \frac{2}{n}\lambda \tag{5}$$

Moreover, if P is tangent to M^n, the equality case of Equation (5) holds if and only if M^n is an invariantly quasi-umbilical submanifold in N^{n+p}, such that with respect to suitable orthonormal tangent frame $\{e_1, \cdots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \cdots, e_{n+p}\}$, the shape operators $A_r = A_{e_r}$, $r \in \{n+1, \cdots, n+p\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2a \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+p} = 0$$

(ii) *The normalized δ-Casorati curvature $\hat{\delta}_c(n-1)$ satisfies*

$$\rho \leq \hat{\delta}_c(n-1) + a + \frac{2b}{n} \parallel U^\top \parallel^2 - \frac{2}{n}\lambda \tag{6}$$

Moreover, if P is tangent to M^n, the equality case of Equation (6) holds if and only if M^n is an invariantly quasi-umbilical submanifold in N^{n+p}, such that with respect to suitable orthonormal tangent frame $\{e_1, \cdots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \cdots, e_{n+p}\}$, the shape operators $A_r = A_{e_r}$, $r \in \{n+1, \cdots, n+p\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+p} = 0$$

Remark 1. For $b = 0$, Theorem 3 is due to Theorem 1.1 in [7].

The proof of this theorem will be given after the following two lemmas.

Lemma 4. *Let $f(x_1, x_2, \cdots, x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \cdots, x_n) = n\sum_{i=1}^{n-1} x_i^2 + \frac{n-1}{2}x_n^2 - 2\sum_{1\leq i<j\leq n} x_i x_j$$

If $x_1 + x_2 + \cdots + x_n = \varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \geq 0,$$

where the equality holds if and only if

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{2}x_n = \frac{1}{n+1}\varepsilon$$

Proof. A simple calculation yields

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= n\sum_{i=1}^{n-1} x_i^2 + \frac{n-1}{2}x_n^2 - [(\sum_{i=1}^{n} x_i)^2 - (x_1^2 + x_2^2 + \cdots + x_n^2)] \\ &= (n+1)\sum_{i=1}^{n-1} x_i^2 + \frac{n+1}{2}x_n^2 - \varepsilon^2 \end{aligned} \tag{7}$$

On the other hand, using the Cauchy inequality we have

$$(\varepsilon - x_n)^2 = (\sum_{i=1}^{n-1} x_i)^2 \leq (n-1) \sum_{i=1}^{n-1} x_i^2 \tag{8}$$

where the equality holds if and only if $x_1 = x_2 = \cdots = x_{n-1}$.

Combining Equations (7) and (8), we have

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &\geq \frac{n+1}{n-1}(\varepsilon - x_n)^2 + \frac{n+1}{2}x_n^2 - \varepsilon^2 \\ &= \frac{1}{2(n-1)}[(n+1)x_n - 2\varepsilon]^2 \\ &\geq 0 \end{aligned}$$

which represents Lemma 4 to prove.

The following lemma can be proved in a similar way.

Lemma 5. *Let $f(x_1, x_2, \cdots, x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \cdots, x_n) = \frac{2n-3}{2} \sum_{i=1}^{n-1} x_i^2 + 2(n-1)x_n^2 - 2 \sum_{1 \leq i < j \leq n} x_i x_j$$

If $x_1 + x_2 + \cdots + x_n = \varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \geq 0$$

where the equality holds if and only if

$$x_1 = x_2 = \cdots = x_{n-1} = 2x_n = \frac{2}{2n-1}\varepsilon$$

Proof of Theorem 3 Now, we are ready to prove the theorem.

Let $x \in M^n$ and $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, \cdots, e_{n+p}\}$ be orthonormal bases of $T_x M^n$ and $T_x^{\perp} M$, respectively. From Equations (2)–(4) we have

$$\begin{aligned} R_{ijij} = a &+ b[g(U^{\top}, e_i)^2 + g(U^{\top}, e_j)^2] + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)) \\ &- \alpha(e_i, e_i) - \alpha(e_j, e_j) \end{aligned}$$

which implies

$$2\tau = n^2 H^2 - n\mathcal{C} + (n^2 - n)a + 2b(n-1) \parallel U^{\top} \parallel^2 - 2(n-1)\lambda \tag{9}$$

Consider the following function $\mathcal{P}$ which is a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = \frac{1}{2}n(n-1)\mathcal{C} + \frac{(n-1)(n+1)}{2}\mathcal{C}(L) - 2\tau + (n^2 - n)a + 2b(n-1) \parallel U^{\top} \parallel^2 - 2(n-1)\lambda \tag{10}$$

Assuming, without loss of generality, that L is spanned by $e_1, e_2, \cdots, e_{n-1}$, which together with Equation (9) gives

$$\begin{aligned}\mathcal{P} &= \sum_{r=n+1}^{n+p}\{n\sum_{i=1}^{n-1}(h_{ii}^r)^2 + \frac{n-1}{2}(h_{nn}^r)^2 + 2(n+1)\sum_{1\le i<j\le n-1}(h_{ij}^r)^2 \\ &\quad + (n+1)\sum_{i=1}^{n-1}(h_{in}^r)^2 - 2\sum_{1\le i<j\le n}h_{ii}^r h_{jj}^r\} \\ &\ge \sum_{r=n+1}^{n+p}\{n\sum_{i=1}^{n-1}(h_{ii}^r)^2 + \frac{n-1}{2}(h_{nn}^r)^2 - 2\sum_{1\le i<j\le n}h_{ii}^r h_{jj}^r\}\end{aligned}$$

On the other hand, we can set

$$h_{11}^r + \cdots + h_{nn}^r = k^r$$

where k^r are real constants. In fact, for a fixed normal vector e_r, k^r is the trace of the matrix (h_{ij}^r), which is invariant no matter how the entries h_{ij}^r change.

Then by Lemma 4,

$$\mathcal{P} \ge 0 \tag{11}$$

with the equality case holds if and only if

$$h_{11}^r = h_{22}^r = \cdots = h_{n-1,n-1}^r = \frac{1}{2}h_{nn}^r$$

Combining Equations (10) and (11) and the definition of $\delta_c(n-1)$, we can derive inequality Equation (5). The equality case of Equation (5) holds if and only if

$$h_{11}^r = h_{22}^r = \cdots = h_{n-1,n-1}^r = \frac{1}{2}h_{nn}^r;\ \ h_{ij}^r = 0, i \ne j$$

Moreover, if P given by Equation (1) is tangent to M, by using Lemma 1, we have

$$\hat{h}_{11}^r = \hat{h}_{22}^r = \cdots = \hat{h}_{n-1,n-1}^r = \frac{1}{2}\hat{h}_{nn}^r;\ \ \hat{h}_{ij}^r = 0, i \ne j$$

i.e., M^n is invariantly quasi-umbilical.

Considering the following quadratic polynomial in the components of the second fundamental form

$$\mathcal{Q} = 2n(n-1)\mathcal{C} + \frac{1}{2}(n-1)(1-2n)\mathcal{C}(L) - 2\tau + (n^2-n)a + 2b(n-1) \parallel U^\top \parallel^2 - 2(n-1)\lambda$$

and combining Equation (9), we have

$$\begin{aligned}\mathcal{Q} &= \sum_{r=n+1}^{n+p}\{\frac{2n-3}{2}\sum_{i=1}^{n-1}(h_{ii}^r)^2 + 2(n-1)(h_{nn}^r)^2 + (2n-1)\sum_{1\le i<j\le n-1}(h_{ij}^r)^2 \\ &\quad + 2(2n-1)\sum_{i=1}^{n-1}(h_{in}^r)^2 - 2\sum_{1\le i<j\le n}h_{ii}^r h_{jj}^r\} \\ &\ge \sum_{r=n+1}^{n+p}\{\frac{2n-3}{2}\sum_{i=1}^{n-1}(h_{ii}^r)^2 + 2(n-1)(h_{nn}^r)^2 - 2\sum_{1\le i<j\le n}h_{ii}^r h_{jj}^r\} \\ &\ge 0\end{aligned}$$

here we used Lemma 5. Then by the very definition of $\hat{\delta}_c(n-1)$, we can easily derive the inequality Equation (6). Also, the equality case can be easily verified.

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