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Reply

Reply to Frewer *et al*. Comments on Janocha *et al*. Lie Symmetry Analysis of the Hopf Functional-Differential Equation. *Symmetry* 2015, *7*, 1536–1566

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Abstract: We reply to the comment by Frewer and Khujadze regarding our contribution "Lie Symmetry Analysis of the Hopf Functional-Differential Equation" (Symmetry 2015, 7(3), 1536). The method developed by the present authors considered the Lie group analysis of the Hopf equations with functional derivatives in the equation, not the integro-differential equations in general. It was based on previous contributions (Oberlack and Wacławczyk, *Arch. Mech.* 2006, 58; Wacławczyk and Oberlack, *J. Math. Phys.* 2013, 54). In fact, three of the symmetries calculated in (Symmetry 2015, 7(3), 1536) break due to internal consistency constrains and conditions imposed on test functions, the same concerns the corresponding symmetries derived by Frewer and Khujadze and another, spurious symmetry, which was not discussed by Frewer and Khujadze. As a result, the same set of symmetries is obtained with both approaches.

Keywords: Lie symmetries; Hopf equation; Burgers equation; functional differential equations; turbulence

1. Introduction

We discuss objections of Frewer and Khujadze on the Lie group analysis applied to the functional Hopf-Burgers equation in [1]. We reject the criticism from Section 1 in their contribution. In Section 2 of this reply, we want to point to a fundamental mistake in Frewer and Khujadze as, in contrast to what is claimed by Frewer and Khujadze, we did not intend in [1] to develop and, in fact, did not develop an alternative approach to the previous approaches of Ibragimov [2], Fushchich [3] or Zawistowski [4] applicable to integro-differential equations. In fact, presently, a functional differential was investigated with respect to its symmetries, which, mathematically, is a very different object. The development of Lie symmetry methods for equations with functional derivatives started with the works [5] and [6]. Therein, the authors considered the Hopf–Burgers equation in the Fourier space. The method was first presented in [5]; next, solutions of the characteristic system of equations and invariant solutions for the functionals were derived in [6]. In [5,6], the transformation of the wavenumber space was not accounted for. The analysis proposed by Frewer and Khujadze is based on the method from [5,6].

Next, we refer to technical errors listed in Paragraphs E.1–E.3 and E.5 of the comment of Frewer and Khujadze. We reject the criticism from Paragraph E4. In Section 4, we discuss the main objection of Frewer and Khujadze. They correctly noted that internal consistency constraints break two symmetries derived in [1]; however, the corresponding transformations calculated by Frewer and Khujadze are also broken. Moreover, we expect that an additional, spurious symmetry follows from the modified method presented by Frewer and Khujadze. This symmetry was not discussed by Frewer and Khujadze; hence, first, they did not consider all consistency constrains, and second, they did not in fact perform full Lie symmetry analysis with their modified approach.

In fact, neither approach can derive symmetry transformations of the space variable x apart from the scaling and translation by a constant. The scaling of x was derived properly in [1], contrary to what Frewer and Khujadze discuss in their comment (Remark D2, Equations (50)–(54)). We show that Frewer and Khujadze introduced in this case the infinitesimals from their modified approach into formulas from [1], which, obviously, could not lead to the correct result. As we reject the critisism of Frewer and Khujadze from paragraph E4, as a consequence, the symmetry X_6 is formally admitted by the considered equation. This symmetry, however, breaks when the assumption of an asymptotic decay of the test functions is made.

As far as the scaling symmetry of Φ (X_7 in [1]) is concerned, we do not agree with Frewer and Khujadze. This symmetry is present also in other methods used to describe stochastic fields, namely the multipoint correlation function approach and the multipoint probability density function approach; see [7,8], where also its physical meaning was discussed.

2. Relation to Other Works

Frewer and Khujadze criticize [1] saying that: *"It is claimed that this new, third approach allows for a standard Lie-point symmetry analysis without having to directly transform the volume element of integration. Listed as the third item in [1, p. 1549], this approach is intended to avoid the 'more complicated' [1, p. 1549] approaches of Ibragimov et al. which rests on a Lie-Bäcklund analysis, and that of Fushchich and Zawistowski et al. which incorporates the transformation of the volume element (Jacobian) within a standard Lie-point analysis."*

All of the cited works concern the analysis of integro-differential equations, however, without functional derivatives in the equations. As it was pointed out in the first sentence in the abstract of [1] "we extend the classical Lie symmetry analysis from partial differential equations to integro-differential equations with functional derivatives". These type of functional equations were not considered in the works of Ibragimov, Fushchich or Zawistowski [2–4]. Moreover, it was never claimed in [1] to propose a method that is better (or simpler) than the approaches of Ibragimov *et al.*, Fushchich or Zawistowski *et al*. We referred to the approach from [4] as we wanted to account for the transformation of the x variable in the integral in our case of the Hopf-type functional equation; however, due to the mathematical difficulties, it was written in [1]: "One has to pay attention that analogous formula for the transformed integral should be used during the calculation of $\zeta_{;t}$, $\zeta_{;xy(x)y(x)}$ and $\zeta_{;xxy(x)}$,..., which, in our particular case of of the equation with functional derivatives, makes this approach more complicated".

In conclusion, integro-differential equations considered in the works of Ibragimov, Fushchich or Zawistowski [2–4] and equations of the Hopf-type (with functional derivatives), are fundamentally different. Instead, the extended Lie group method for equations with functional derivatives (however, without transformations of the space variable) was first proposed in [5,6].

3. Technical Errors

In [1], an extension of the n-point characteristic function:

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n, t) = \langle \mathrm{e}^{i\sum_{n=1}^{N} U(x_n,t)y_n} \rangle \tag{1}$$

towards a "continuum" limit $n \to \infty$:

$$\Phi([y(x)], t) = \langle \mathrm{e}^{i\int U(x,t)y(x)\mathrm{d}x} \rangle \tag{2}$$

was considered. We developed an approach to account for symmetry transformations of the functional equation with functional derivatives, e.g., the Hopf–Burgers equation:

$$\frac{\partial\phi([y(x)],t)}{\partial t} = \int_G y(x)\left(i\frac{\partial}{\partial x}\frac{\delta^2\phi([y(x)],t)}{\delta y(x)^2} + \nu\frac{\partial}{\partial x^2}\frac{\delta\phi([y(x)],t)}{\delta y(x)}\right)\mathrm{d}x \tag{3}$$

The variable $y(x)$ was presented as proposed by Klauder [9]:

$$y(x) = \sum_{n=1}^{\infty} h_n(x) y_n \tag{4}$$

to obtain:

$$\Phi_n([y(x)],t) = \langle \mathrm{e}^{i\sum_{n=1}^{\infty} y_n \int u(x,t)h_n(x)\mathrm{d}x} \rangle \tag{5}$$

The functional was further treated as a function of the infinite, but discrete set of variables y_n and t. In such a case, differentiating the functional Φ, e.g., with respect to t would lead to:

$$\frac{\mathcal{D}\overline{\phi}}{\mathcal{D}t} = \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{t}}\frac{\mathcal{D}\overline{t}}{\mathcal{D}t} + \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{x}}\frac{\mathcal{D}\overline{x}}{\mathcal{D}t} + \sum_n \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{y_n}}\frac{\mathcal{D}\overline{y_n}}{\mathcal{D}t} \tag{6}$$

This can be rewritten in terms of the functional derivatives as:

$$\frac{\mathcal{D}\overline{\phi}}{\mathcal{D}t} = \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{t}}\frac{\mathcal{D}\overline{t}}{\mathcal{D}t} + \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{x}}\frac{\mathcal{D}\overline{x}}{\mathcal{D}t} + \int_G \frac{\delta\overline{\phi}}{\delta\overline{y(x')}}\sum_{n=1}^{\infty}\frac{\mathcal{D}\overline{y_n}}{\mathcal{D}t}\overline{h_n}(\overline{x'})\mathrm{d}\overline{x'} \tag{7}$$

This was Equation (12) in [1] where h_n should be replaced by its correct form $\overline{h_n}$. If formula (5) is differentiated as described by Equation (6), it is found that Equation (6) is equivalent to Equation (7). So formulated, the problem would be in fact identical to the "discrete" case with $N \to \infty$.

In fact, such an approach is valid only for the transformations of y_n in Equation (4), or differently formulated, transformations $\overline{y}(\overline{x}) = \overline{y}(x)$ and and $\overline{h_n}(\overline{x}) = h_n(x)$. This condition was not considered in [1].

With the definition Equation (4), from Equation (7) the formula (15) in [1] was obtained. An alternative formula, without the decomposition Equation (4), (derived also in [10]), reads:

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\mathcal{D}\overline{t}}{\mathcal{D}t}\frac{\mathcal{D}}{\mathcal{D}\overline{t}} + \frac{\mathcal{D}\overline{x}}{\mathcal{D}t}\frac{\mathcal{D}}{\mathcal{D}\overline{x}} + \int_G \frac{\mathcal{D}\overline{y(x')\mathrm{d}x'}}{\mathcal{D}t}\frac{\delta}{\delta\overline{y(x')}} \tag{8}$$

This explains the objections raised up in Paragraphs E.1–E.3 in the comment of Frewer and Khujadze. The analysis without the use of the decomposition Equation (4) is given in the Appendix. However, as will be discussed below, symmetry transformations where $\partial\overline{x}/\partial t \neq 0$ and $\delta\overline{x}/\delta y(x) \neq 0$ must be also broken, and as a result, six symmetry transformations found also in [1] are left.

As is seen in the definition of the functional Equation (2), there are dual function spaces $U(x,t)$ and $y(x)$; however, Φ is presented as the functional of test functions $[y(x)]$ only. Instead of the transformation of $U(x,t)$, x, t and Φ, we accounted for the corresponding transformation of $y(x)\mathrm{d}x$, x, t and Φ. The question is how the functions in Equation (2) should transform and if it is possible to present the problem as such, that only the transformations of $y(x)$, x, t and Φ are accounted for. Let us consider a transformed time derivative of Equation (2). We should have the following equality:

$$\left\langle \frac{\partial}{\partial\overline{t}}\mathrm{e}^{i\int \overline{U}(\overline{x},\overline{t})\overline{y}(\overline{x})\mathrm{d}\overline{x}} \right\rangle = \left\langle \mathrm{e}^{i\int \overline{U}(\overline{x},\overline{t})\overline{y}(\overline{x})\mathrm{d}\overline{x}} i\int \frac{\partial\overline{U}(\overline{x},\overline{t})}{\partial\overline{t}}\overline{y}(\overline{x})\mathrm{d}\overline{x} \right\rangle \tag{9}$$

For the case of the Galilei transformation, this equality would be satisfied with: $\overline{y}(\overline{x}) = y(\overline{x}) = y(x+ct)$. In fact, due to a different argument of the test functions y the Galilei-invariance of the Hopf-Burgers equation is broken. The same concerns the projective symmetry X_4 from [1].

Let us consider the "Galilei-type" transformation derived by Frewer and Khujadze: $\overline{x} = x + ct$, $\overline{y}(\overline{x}) = y(x)$, $\overline{\Phi} = \Phi$, $\partial/\partial\overline{t} = \partial/\partial t - c\partial/\partial x$. With this, we obtain from Equation (9) an incorrect result:

$$\left\langle \mathrm{e}^{i\int U(x,t)y(x)\mathrm{d}x} i \int \frac{\partial U(x,t)}{\partial t} y(x)\mathrm{d}x \right\rangle \neq \left\langle \mathrm{e}^{i\int U(x,t)y(x)\mathrm{d}x} i \int \left[\frac{\partial U(x,t)}{\partial t} - c\frac{\partial U(x,t)}{\partial x}\right] y(x)\mathrm{d}x \right\rangle \tag{10}$$

Already, this inequality breaks in fact all of the obtained symmetries down to the case $\partial\overline{x}/\partial t = 0$. With the formula (7), Equation (9) is satisfied for the Galilei invariance X_5 derived in [1], however, $\overline{y}(\overline{x})$ transforms incorrectly to $y(x)$ instead of $y(\overline{x})$.

Note that there is an additional, spurious symmetry, which can be obtained with the proposal of Frewer and Khujadze and also the modified analysis given in the Appendix:

$$\overline{x} = x - iv(\mathrm{e}^{\epsilon} - 1)\int y(x)\mathrm{d}x, \quad \overline{y(x)} = \mathrm{e}^{\epsilon}y(x), \quad \overline{t} = \mathrm{e}^{\epsilon}t, \quad \overline{\Phi} = \Phi \tag{11}$$

(both approaches should give the same result as for this transformation $\mathrm{d}\overline{x} = \mathrm{d}x$). For such a case, the functional derivative reads:

$$\frac{\delta}{\delta\overline{y}(\overline{x})} = \mathrm{e}^{-\epsilon}\frac{\delta}{\delta y(x)} + iv(1 - \mathrm{e}^{-\epsilon})\frac{\partial}{\partial x}$$

This symmetry is also broken due to the condition:

$$\left\langle \frac{\delta}{\delta\overline{y}(\overline{x}')} \mathrm{e}^{i\int \overline{U}(\overline{x},\overline{t})\overline{y}(\overline{x})\mathrm{d}\overline{x}} \right\rangle = \left\langle \mathrm{e}^{i\int \overline{U}(\overline{x},\overline{t})\overline{y}(\overline{x})\mathrm{d}\overline{x}} i \int \left[\frac{\delta\overline{U}(\overline{x},\overline{t})}{\delta\overline{y}(\overline{x}')}\overline{y}(\overline{x}) + \frac{\delta\overline{y}(\overline{x})}{\delta\overline{y}(\overline{x}')}\overline{U}(\overline{x},\overline{t})\right] \mathrm{d}\overline{x} \right\rangle \tag{12}$$

Again, this equality is not satisfied by Equation (11), which means that the modification of the method proposed by Frewer and Khujadze still accounts for the transformations $\overline{y}(\overline{x}) = \overline{y}(x)$ only, similar to the method proposed in [1]. In fact, when the additional constrains are accounted for, Frewer and Khujadze obtain the same set of six symmetries, as derived with the analysis proposed in [1].

We do not agree with the objection raised by Frewer and Khujadze in Paragraph E.4. It was assumed in [1] that:

$$U^t(x = \pm\infty) = 0 \tag{13}$$

In such a case, not only the mean $\langle U^t(x = \pm\infty)\rangle$, but also any multipoint correlation function $\langle U^t(x_1) \cdot \dots \cdot U^t(x_n)\rangle$ is zero if, for any i, $x_i \to \infty$. This is not the case for the example presented by Frewer and Khujadze in Equation (77). In fact, the three-point correlation function computed from their formula (77) is:

$$\begin{aligned} \langle U^t(x)U^t(x')U^t(x'')\rangle &= \left.\frac{\delta^3\Phi}{\delta y(x)\delta y(x')\delta y(x'')}\right|_{y=0} \\ &= -i\Phi f^3(t)\mathrm{e}^{-\lambda(x^2+x'^2+x''^2)} + if(t)\Phi\left[2g(x,x') + 2g(x,x'') + 2g(x',x'')\right] \end{aligned} \tag{14}$$

In the limit $x \to \infty$, this formula equals $2if(t)\Phi g(x',x'')$, which clearly contradicts the assumption Equation (13).

Instead of giving rather specialized examples, we can consider the following expansion of the characteristic functional in the Taylor series (see [11]):

$$\begin{aligned} \Phi &= 1 + \int f_1(x,t)y(x)\mathrm{d}x + \iint f_2(x,x',t)y(x)y(x')\mathrm{d}x\mathrm{d}x' + \\ &+ \iiint f_3(x,x',x'',t)y(x)y(x')y(x'')\mathrm{d}x\mathrm{d}x'\mathrm{d}x'' + \dots \end{aligned} \tag{15}$$

where $f_2(x,x') = f_2(x',x)$, $f_3(x,x',x'') = f_3(x',x,x'') = f_3(x',x'',x)$, *etc.*, so that the multipoint correlations can be constructed from functions: $f_1, f_2, \ldots$ as

$$\langle U^t(x_1)\cdot\cdots\cdot U^t(x_n)\rangle = -i^n n! f_n(x_1,\ldots,x_n) \tag{16}$$

The first functional derivative of Φ reads:

$$\begin{aligned}\frac{\delta\Phi}{\delta y(x_i)} &= f_1(x_i,t) + \int 2f_2(x_i,x,t)y(x)\mathrm{d}x \\ &+ \iint 3f_3(x_i,x,x',t)y(x)y(x')\mathrm{d}x\mathrm{d}x'' + \ldots\end{aligned} \tag{17}$$

If all of the multipoint correlations tend to zero for $x_i \to \infty$, then also $f_1(x_i,t)$, and all functions f_n under the integrals are zero in this limit. Hence, with the assumption Equation (13), the functional derivatives of Φ are zero when a spatial variable tends to infinity.

In Paragraph E.5 of the Comment, Frewer and Khujadze wrote that one term was missing in Equation (A11) in [1]. This mistake did not have an impact on the further calculations in view of the further result given in Equation (49) in [1].

4. Choice of variables

The main objection raised by Frewer and Khujadze', is the problem of choosing $y(x)\mathrm{d}x$ as a variable. Instead, Frewer and Khujadze consider $y(x)$ and argue that their approach is correct, giving Examples (15)–(17) in their work. In none of the examples can $y(x)\mathrm{d}x$ be distinguished as a variable. Trivially, there are functionals where this is the case; however, we consider the characteristic functional of the form Equation (2) or the expansion Equation (15), where $y(x)$ is always accompanied by the element $\mathrm{d}x$. Hence, the existence of other Examples (15)–(17) does not prove that our analysis is erroneous. Moreover, the example provided by Frewer and Khujadze on page 11 in their comment (Equations (50)–(54)) does not prove anything either, as we also obtain a correct form of the scaling symmetry. We accounted for the infinitesimals of the transformed $y(x)\mathrm{d}x$, which we denoted by $\xi_{y(x)}\mathrm{d}x$. It is debatable if $\xi_{y(x)\mathrm{d}x}$ would be a better notation. This variable should not be confused with $\xi_{y(x)}$ in the comment of Frewer and Khujadze (we have chosen the notation $\xi_{y(x)}\mathrm{d}x$ for optical reasons, since the expressions like $\int \Phi_{,y(x)}\xi_{y(x)\mathrm{d}x}$ may look strange).

Related to the scaling symmetry, we find in our case:

$$\xi_t = 2t, \quad \xi_x = x, \quad \xi_{y(x)\mathrm{d}x} = y(x)\mathrm{d}x, \quad \eta = 0 \tag{18}$$

(see the result in Theorem 9 on page 23 in our paper with the constant a_1). If we substitute this result into the formula for $\zeta_{;y(x)}$ on page 1546, we also obtain the correct result:

$$\zeta_{;y(x)} = -\Phi_{,y(x)} \tag{19}$$

The error of Frewer and Khujadze becomes particularly apparent, as they introduce their result for $\xi_{y(x)} = 0$ into our formula for $\zeta_{;y(x)}$, where the infinitesimal $\xi_{y(x)\mathrm{d}x}$ of a variable $y(x)\mathrm{d}x$ is present.

Frewer and Khujadze derived the following relation between $\xi_{y(x)\mathrm{d}x}$ and the infinitesimal of the variable $y(x)$:

$$\xi_{y(x)\mathrm{d}x} = \left(\xi_{y(x)} + y\frac{\partial\xi_x}{\partial x}\right)\mathrm{d}x \tag{20}$$

(see formula (14) in their comment). It follows that, when $\xi_{y(x)\mathrm{d}x}$ and ξ_x are derived from the symmetry analysis, also the infinitesimal for the variable $y(x)$ can be derived. In [1], the whole bracketed term in Equation (20) was denoted by $\xi_{y(x)}$, which could, in fact, lead to confusion. In the following, we change the notation to:

$$\xi_{y(x)\mathrm{d}x} = \underbrace{\left(\xi_{y(x)} + y\frac{\partial \xi_x}{\partial x}\right)}_{\xi_{\gamma(x)}} \mathrm{d}x = \xi_{\gamma(x)}\mathrm{d}x \tag{21}$$

If we substitute relation (21) for $\xi_{\gamma(x)}$ into Equation (A4) we obtain the same formula for the prolonged infinitesimal $\zeta_{;y(x)}$, with 7 terms, as given in Equation (49) of the comment.

It should be noted that the choice of the variable to be transformed in [1] followed from the considerations of Hopf [11], who introduced $Y(\mathrm{d}x) = y(x)\mathrm{d}x$ and defined $Y(s)$ as the mass contained in the subset S of the considered interval. Hopf considered an example of a functional:

$$\Phi = \int_0^1 A(x)y(x)\mathrm{d}x \tag{22}$$

and defined the differential $\delta\Phi$ as:

$$\delta\Phi = \int_0^1 A(x)\delta Y(\mathrm{d}x) \tag{23}$$

For such a choice, the function $A(x)$ could be interpreted as a partial derivative of Φ with respect to its argument $Y(\mathrm{d}x)$.

The choice of $y(x)\mathrm{d}x$ as a variable might be a matter of discussion, and in particular, the notation chosen in [1] could be misleading. This latter issue is, however, a matter of notation and is not related to the mathematical content.

With the variable $y(x)\mathrm{d}x$ different results for the intermediate form of the projective symmetry are obtained. We rewrite Equation (8), which should account for possible transformations of $\overline{y}(\overline{x})$:

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\mathcal{D}\bar{t}}{\mathcal{D}t}\frac{\mathcal{D}}{\mathcal{D}\bar{t}} + \frac{\mathcal{D}\overline{x}}{\mathcal{D}t}\frac{\mathcal{D}}{\mathcal{D}\overline{x}} + \int_{\overline{G}} \frac{\mathcal{D}\overline{y(x')\mathrm{d}x'}}{\mathcal{D}t}\frac{\delta}{\overline{\delta y(x')}} \tag{24}$$

while the corresponding chain rule of Frewer and Khujadze reads:

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\mathcal{D}\bar{t}}{\mathcal{D}t}\frac{\mathcal{D}}{\mathcal{D}\bar{t}} + \frac{\mathcal{D}\overline{x}}{\mathcal{D}t}\frac{\mathcal{D}}{\mathcal{D}\overline{x}} + \int_{\overline{G}} \mathrm{d}\overline{x'}\frac{\mathcal{D}\overline{y(x')}}{\mathcal{D}t}\frac{\delta}{\overline{\delta y(x')}} \tag{25}$$

(this is given in Equation (A7) in the comment). Therefore, there is a difference in the last RHS term in formula (24) above and Equation (25) which leads to a different intermediate form of the projective symmetry.

Frewer and Khujadze wrote in their comment that their analysis led to the following form of the intermediate projective symmetry (see Equation (A27) in their comment; note that they considered a modified equation with a factor $1/2$ in the convective term):

$$\bar{t} = \frac{t}{1-t\epsilon}, \quad \overline{x} = \frac{x}{1-t\epsilon}, \quad \overline{y}(\overline{x}) = y(x), \quad \overline{\Phi} = \Phi \tag{26}$$

However, in such a case, when y remains invariant, $U(x,t)$ should transform, so we would have:

$$\bar{t} = \frac{t}{1-t\epsilon}, \quad \overline{x} = \frac{x}{1-t\epsilon}, \quad \overline{y}(\overline{x}) = y(x), \quad \overline{U} = (1-t\epsilon)U(x,t), \quad \overline{\Phi}([\overline{U(x,t)}], [\overline{y(x)}], t) = \Phi \tag{27}$$

As an immediate consequence, Φ is a functional of two function spaces U and y if both of them are to be transformed. This was not accounted for in the analysis of Frewer and Khujadze, as in such a case, the calculated derivative of a transformed variable $\bar{t}$ would have a different form.

To sum up, as it was argued, with both approaches applied to the Hopf-Burgers equation, the only transformations of x that are accounted for are the scaling and translation. Here, independently of the choice of $y(x)$ or $y(x)\mathrm{d}x$ as a variable the same final set of symmetries is obtained.

5. Breaking of Symmetries

Frewer and Khujadze correctly noticed that the condition $\partial y(x)/\partial t = 0$, which should hold also for the transformed variables $\partial \overline{y}(\overline{x})/\partial \overline{t} = 0$, finally breaks the Galilei invariance and the symmetry X_4. Moreover, with the method presented in the Appendix and, also, as we expect, with the analysis of Frewer and Khujadze, the following spurious symmetry is obtained:

$$\overline{x} = x - iv(\mathrm{e}^{\epsilon} - 1)\int y(x)\mathrm{d}x, \quad \overline{y(x)} = \mathrm{e}^{\epsilon} y(x), \quad \overline{t} = \mathrm{e}^{\epsilon} t, \quad \overline{\Phi} = \Phi$$

This symmetry also should be broken due to the condition Equation (12), or, more genarally, the requirement that $y(x)$ is an arbitrary function of space only both in the "old" and "new" variables.

As we rejected in Section 3 the criticism of Frewer and Khujadze given in paragraph E4, the transformation X_6 from [1] (translation of $y(x)$ by a constant) is formally admitted as a symmetry of the Hopf-Burgers equation. However, if we assume that the test functions $y(x)$ should decay asymptotically to 0 at infinity, symmetry X_6 breaks.

Finally, the Hopf-Burgers equation is invariant under the transformations given by X_1–X_3 and X_7–X_8 in [1].

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Appendix A.

Below, we present briefly the Lie group analysis procedure of the Hopf–Burgers Equation (3) without the use of the decomposition Equation (4). We only focus on these formulas and transformations, which are different from the corresponding formulas in [1].

Infinitesimals

By means of the differential operators presented in [1], we can calculate the infinitesimals $\zeta_{;\dots}$ as functionals of the infinitesimals $\xi_{y(z)\mathrm{d}z}, \xi_x, \xi_t, \eta_\phi$ or, using the relation Equation (21), infinitesimals of $\xi_{\gamma(z)}, \xi_x, \xi_t, \eta_\phi$. For the viscous Hopf–Burgers functional differential equation (FDE), we need the three infinitesimals $\zeta_{;t}$, $\zeta_{;xy(x)y(x)}$ and $\zeta_{;xxy(x)}$.

In order to calculate $\zeta_{;t}$, we differentiate the transformed Hopf functional $\overline{\phi}$ with respect to t, taking into account the fact that $\overline{\phi}$ does not depend explicitly on $\overline{x}$:

$$\frac{\mathcal{D}\overline{\phi}}{\mathcal{D}t} = \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{t}}\frac{\mathcal{D}\overline{t}}{\mathcal{D}t} + \int_{\overline{G}} \frac{\mathcal{D}\overline{\phi}}{\mathcal{D}\overline{y(x')\mathrm{d}x'}}\frac{\mathcal{D}\overline{y(x')\mathrm{d}x'}}{\mathcal{D}t} \tag{A1}$$

With the one-parameter Lie point transformations, Equation (A1) reads:

$$\frac{\mathcal{D}(\phi + \varepsilon\eta_\phi)}{\mathcal{D}t} = (\phi_{,t} + \varepsilon\zeta_{;t})\frac{\mathcal{D}(t + \varepsilon\xi_t)}{\mathcal{D}t} + \int_G (\phi_{,y(x')} + \varepsilon\zeta_{;y(x')})\frac{\mathcal{D}(y(x')\mathrm{d}x' + \varepsilon\xi_{\gamma(x')}\mathrm{d}x')}{\mathcal{D}t} \tag{A2}$$

Evaluating this equation in $\mathcal{O}(\varepsilon)$ and taking further steps as in [1] lead to an equation for $\zeta_{;t}$:

$$\zeta_{;t} = \frac{\partial\eta_\phi}{\partial t} + \phi_{,t}\left(\frac{\partial\eta_\phi}{\partial\phi} - \frac{\partial\xi_t}{\partial t}\right) - (\phi_{,t})^2\frac{\partial\xi_t}{\partial\phi} - \int_G \phi_{,y(x')}\frac{\partial\xi_{\gamma(x')}\mathrm{d}x'}{\partial t} - \int_G \phi_{,y(x')}\phi_{,t}\frac{\partial\xi_{\gamma(x')}\mathrm{d}x'}{\partial\phi} \tag{A3}$$

In order to calculate $\zeta_{;y(x)}$, we differentiate $\overline{\phi}$ with respect to $y(x)$. An analog calculation leads to:

$$\begin{aligned}\zeta_{;y(x)} \quad = \quad & \frac{\delta\eta_\phi}{\delta y(x)} + \phi_{,y(x)}\frac{\partial\eta_\phi}{\partial\phi} - \phi_{,t}\frac{\delta\xi_t}{\delta y(x)} - \phi_{,t}\phi_{,y(x)}\frac{\partial\xi_t}{\partial\phi} - \int_G \phi_{,y(x')}\frac{\delta\xi_{\gamma(x')}\mathrm{d}x'}{\delta y(x)} \\ & - \int_G \phi_{,y(x')}\phi_{,y(x)}\frac{\partial\xi_{\gamma(x')}\mathrm{d}x'}{\partial\phi}\end{aligned} \tag{A4}$$

Analogously, as was also done in [1], we derive the following forms of infinitesimals $\zeta_{;xy(x)y(x)}$ and $\zeta_{;xxy(x)}$:

$$\zeta_{;xy(x)y(x)} \quad = \quad \frac{\mathcal{D}\zeta_{;y(x)y(x)}}{\mathcal{D}x} - \phi_{,ty(x)y(x)}\frac{\mathcal{D}\xi_t}{\mathcal{D}x} - \int_G \phi_{,y(x')y(x)y(x)}\frac{\mathcal{D}\xi_{\gamma(x')}\mathrm{d}x'}{\mathcal{D}x} - \phi_{,xy(x)y(x)}\frac{\mathcal{D}\xi_x}{\mathcal{D}x} \tag{A5}$$

$$\zeta_{;xxy(x)} \quad = \quad \frac{\mathcal{D}\zeta_{;xy(x)}}{\mathcal{D}x} - \phi_{,txy(x)}\frac{\mathcal{D}\xi_t}{\mathcal{D}x} - \int_G \phi_{,y(x')xy(x)}\frac{\mathcal{D}\xi_{\gamma(x')}\mathrm{d}x'}{\mathcal{D}x} - \phi_{,xxy(x)}\frac{\mathcal{D}\xi_x}{\mathcal{D}x} \tag{A6}$$

Applying the differential operators introduced in [1], one can represent the infinitesimals Equations (A5) and (A6) as sums of partial and functional derivatives of $\xi_{\gamma(z)}\mathrm{d}z$, ξ_x, ξ_t, η_ϕ.

Determining the System of Equations for the Infinitesimals

In order to calculate the determining system of equations for the infinitesimals $\xi_{\gamma(z)}$, ξ_x, ξ_t, η_ϕ of the Hopf–Burgers Equation (3), we consider the equation:

$$\left[X^{(3)}F\right]_{F=0} = 0$$

The generator $X^{(3)}$ is given in [1]. The contributing summands of $X^{(3)}$ are given by:

$$\begin{aligned}X^{(3)}_{\text{modif}} = & \int_G \xi_{\gamma(x')}\mathrm{d}x'\frac{\delta}{\delta y(x')} + \zeta_{;t}\frac{\partial}{\partial\phi_{,t}} + \int_G\int_G \zeta_{;xy(x'')y(x')}\mathrm{d}x'\mathrm{d}x''\frac{\delta}{\delta\phi_{,xy(x'')y(x')}} \\ & + \int_G \zeta_{;xxy(x')}\mathrm{d}x'\frac{\delta}{\delta\phi_{,xxy(x')}}\end{aligned}$$

Applying $X^{(3)}_{\text{modif}}$ to F, we get the equation:

$$X^{(3)}_{\text{modif}}F = \zeta_{;t} - \int_G \xi_{\gamma(x)}\mathrm{d}x\left(i\phi_{,xy(x)y(x)} + \nu\phi_{,xxy(x)}\right) - \int_G y(x)\left(i\zeta_{;xy(x)y(x)}\mathrm{d}x + \nu\zeta_{;xxy(x)}\mathrm{d}x\right) = 0 \tag{A7}$$

In order to derive the final determining equation, we insert the infinitesimals $\zeta_{;t}$ (*cf.* Equation (A3)), $\zeta_{;xy(x)y(x)}$ and $\zeta_{;xxy(x)}$ and employ $F = 0$ in order to eliminate $\phi_{,t}$. With the same assumptions as taken in [1], the resulting Equation (A7) has the form:

$$\begin{aligned}0 = & \mathcal{A} + \int_G \phi_{,y(z)}\mathcal{B}\,\mathrm{d}z + \int_G\int_G \phi_{,y(x)y(z)}\mathcal{C}\,\mathrm{d}z\mathrm{d}x + \int_G\int_G \phi_{,y(x)}\phi_{,y(z)}\mathcal{D}\,\mathrm{d}z\mathrm{d}x \\ & + \int_G\int_G\int_G \phi_{,y(x)y(a)}\phi_{,y(z)}\mathcal{E}\,\mathrm{d}a\mathrm{d}z\mathrm{d}x + \int_G\int_G (\phi_{,y(x)})^2\phi_{,y(z)}\mathcal{F}\,\mathrm{d}z\mathrm{d}x \\ & + \int_G \phi_{,ty(x)}\mathcal{G}\,\mathrm{d}x + \int_G\int_G (\phi_{,y(x)})^2\phi_{,y(z)y(z)}\mathcal{H}\,\mathrm{d}z\mathrm{d}x + \int_G \phi_{,y(x)}\phi_{,ty(x)}\mathcal{I}\,\mathrm{d}x \\ & + \int_G \phi_{,ty(x)y(x)}\mathcal{J}\,\mathrm{d}x + \int_G\int_G \phi_{,y(z)y(x)y(x)}\mathcal{K}\,\mathrm{d}z\mathrm{d}x + \int_G\int_G \phi_{,y(x)y(x)}\phi_{,y(z)y(z)}\mathcal{L}\,\mathrm{d}z\mathrm{d}x\end{aligned}$$

Since the infinitesimals $\xi_{\gamma(z)}$, ξ_x, ξ_t, η_ϕ do not depend on derivatives of ϕ, all coefficients of all appearing derivatives of ϕ have to vanish:

$$\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{D} = \mathcal{E} = \mathcal{F} = \mathcal{G} = \mathcal{H} = \mathcal{I} = \mathcal{J} = \mathcal{K} = \mathcal{L} = 0$$

This leads to the system of linear FDE's for the infinitesimals ξ_γ, ξ_x, ξ_t, η_ϕ.

- $\mathcal{A} = 0$ reads:

$$\frac{\partial \eta_\phi}{\partial t} - \int_G y(x) \left(i \frac{\partial^3 \eta_\phi}{\partial x \partial (y(x)\mathrm{d}x)^2} + \nu \frac{\partial^3 \eta_\phi}{\partial x^2 \partial y(x)\mathrm{d}x} \right) \mathrm{d}x = 0 \tag{A8}$$

- $\mathcal{B} = 0$ reads:

$$\begin{aligned} 0 = {} & \nu \frac{\partial^2}{\partial z^2} \left(y(z) \frac{\partial \eta_\phi}{\partial \phi} \right) - \nu \frac{\partial^2}{\partial z^2} \left(y(z) \frac{\partial \xi_t}{\partial t} \right) - \frac{\partial \xi_{\gamma(z)}}{\partial t} - \nu \frac{\partial^2 \xi_{\gamma(z)}}{\partial z^2} \\ & - 2iy(z) \frac{\partial^2}{\partial z \partial \phi} \frac{\delta \eta_\phi}{\delta y(z)} + i\nu \int_G \frac{\partial^2}{\partial z^2} \left(y(x)y(z) \frac{\partial}{\partial x} \frac{\delta^2 \xi_t}{\delta y(x)^2} \right) \mathrm{d}x \\ & + i \int_G y(x) \frac{\partial}{\partial x} \frac{\delta^2 \xi_{\gamma(z)}}{\delta y(x)^2} \mathrm{d}x - i \frac{\partial}{\partial z} \left(y(z) \frac{\partial}{\partial z} \frac{\delta \xi_z}{\delta y(z)} \right) + 2i \frac{\partial}{\partial z} \left(y(z) \frac{\partial}{\partial \phi} \frac{\delta \eta_\phi}{\delta y(z)} \right) \\ & - i \int_G \frac{\partial}{\partial x} \left(y(x) \frac{\delta^2 \xi_{\gamma(z)}}{\delta y(x)^2} \right) \mathrm{d}x + i \frac{\partial^2}{\partial z^2} \left(y(z) \frac{\delta \xi_z}{\delta y(z)} \right) \\ & + \nu^2 \int_G \frac{\partial^2}{\partial z^2} \left(y(x)y(z) \frac{\partial^2}{\partial x^2} \frac{\delta \xi_t}{\delta y(x)} \right) \mathrm{d}x + \nu \int_G y(x) \frac{\partial^2}{\partial x^2} \frac{\delta \xi_{\gamma(z)}}{\delta y(x)} \mathrm{d}x \\ & - 2\nu \int_G \frac{\partial}{\partial x} \left(y(x) \frac{\partial}{\partial x} \frac{\delta \xi_{\gamma(z)}}{\delta y(x)} \right) \mathrm{d}x - \nu \frac{\partial}{\partial z} \left(y(z) \frac{\partial^2 \xi_z}{\partial z^2} \right) - \nu \frac{\partial^2}{\partial z^2} \left(y(z) \frac{\partial \eta_\phi}{\partial \phi} \right) \\ & + \nu \int_G \frac{\partial^2}{\partial x^2} \left(y(x) \frac{\delta \xi_{\gamma(z)}}{\delta y(x)} \right) \mathrm{d}x + 2\nu \frac{\partial^2}{\partial z^2} \left(y(z) \frac{\partial \xi_z}{\partial z} \right) \end{aligned} \tag{A9}$$

- $\mathcal{C} = 0$ reads:

$$\begin{aligned} 0 = {} & -i \frac{\partial}{\partial x} \left(y(x) \frac{\partial \eta_\phi}{\partial \phi} \right) \delta(x-z) + i \frac{\partial}{\partial x} \left(y(x) \frac{\partial \xi_t}{\partial t} \right) \delta(x-z) + i \frac{\partial \xi_{\gamma(x)}}{\partial x} \delta(x-z) \\ & + \int_G \frac{\partial}{\partial z} \left(y(a)y(z) \frac{\partial}{\partial a} \frac{\delta^2 \xi_t}{\delta y(a)^2} \right) \delta(x-z) \, \mathrm{d}a + 2iy(x) \frac{\partial}{\partial x} \frac{\delta \xi_{\gamma(z)}}{\delta y(x)} \\ & - 2i \frac{\partial}{\partial x} \left(y(x) \frac{\delta \xi_{\gamma(z)}}{\delta y(x)} \right) + i \frac{\partial}{\partial x} \left(y(x) \frac{\partial \eta_\phi}{\partial \phi} \right) \delta(x-z) - i \frac{\partial}{\partial x} \left(y(x) \frac{\partial \xi_x}{\partial x} \right) \delta(x-z) \\ & - i\nu \int_G \frac{\partial}{\partial z} \left(y(a)y(z) \frac{\partial^2}{\partial a^2} \frac{\delta \xi_t}{\delta y(a)} \right) \delta(x-z) \, \mathrm{d}a + \nu y(x) \frac{\partial^3 \xi_{\gamma(z)}}{\partial x^2 \partial \phi} + \nu y(x) \frac{\partial^2 \xi_{\gamma(z)}}{\partial x^2} \\ & - \nu \frac{\partial}{\partial x} \left(y(x) \frac{\partial \xi_{\gamma(z)}}{\partial x} \right) - \nu \frac{\partial}{\partial x} \left(y(x) \frac{\partial \xi_{\gamma(z)}}{\partial x} \right) \end{aligned} \tag{A10}$$

- $\mathcal{D} = 0$ reads:

$$
\begin{aligned}
0 = & -\nu^2 \frac{\partial^4}{\partial z^2 \partial x^2}\left(y(x)y(z)\frac{\partial \xi_t}{\partial \phi}\right) - \nu \frac{\partial^2}{\partial x^2}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right) + 2i\nu \frac{\partial^2}{\partial z^2}\left(y(x)y(z)\frac{\partial^2}{\partial x \partial \phi}\frac{\delta \xi_t}{\delta y(x)}\right) \\
& + 2iy(x)\frac{\partial^2}{\partial x \partial \phi}\frac{\delta \xi_{\gamma(z)}}{\delta y(x)} - i\frac{\partial}{\partial x}\left(y(x)\frac{\partial^2 \xi_x}{\partial x \partial \phi}\right)\delta(x-z) - 2i\nu \frac{\partial^3}{\partial z^2 \partial x}\left(y(x)y(z)\frac{\partial}{\partial \phi}\frac{\delta \xi_t}{\delta y(x)}\right) \\
& - 2i\frac{\partial}{\partial x}\left(y(x)\frac{\partial}{\partial \phi}\frac{\delta \xi_{\gamma(z)}}{\delta y(x)}\right)\delta(x-z) - 2i\frac{\partial}{\partial x}\left(y(x)\frac{\partial}{\partial \phi}\frac{\delta \xi_{\gamma(z)}}{\delta y(x)}\right) \\
& + 2i\frac{\partial}{\partial x}\left(y(x)\frac{\partial^2 \eta_\phi}{\partial \phi^2}\right)\delta(x-z) + 2i\frac{\partial^2}{\partial x^2}\left(y(x)\frac{\partial \xi_x}{\partial \phi}\right)\delta(x-z) \\
& - \nu \frac{\partial}{\partial x}\left(y(x)\frac{\partial^2 \xi_{\gamma(z)}}{\partial x \partial \phi}\right) - 2\nu \frac{\partial}{\partial x}\left(y(x)\frac{\partial^2 \xi_{\gamma(z)}}{\partial x \partial \phi}\right) - \nu \frac{\partial}{\partial x}\left(y(x)\frac{\partial^2 \xi_{\gamma(z)}}{\partial x \partial \phi}\right) \\
& + 2\nu \frac{\partial^2}{\partial x^2}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right) + \nu^2 \frac{\partial^4}{\partial z^2 \partial x^2}\left(y(x)y(z)\frac{\partial \xi_t}{\partial \phi}\right) + \nu \frac{\partial^2}{\partial x^2}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right) \\
& + \nu \frac{\partial^2}{\partial x^2}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right) - 2\nu^2 \frac{\partial^3}{\partial z^2 \partial x}\left(y(x)y(z)\frac{\partial^2 \xi_t}{\partial x \partial \phi}\right) + \nu^2 \frac{\partial^2}{\partial z^2}\left(y(x)y(z)\frac{\partial^3 \xi_t}{\partial x^2 \partial \phi}\right)
\end{aligned} \tag{A11}
$$

- $\mathcal{E} = 0$ reads:

$$
\begin{aligned}
0 = & \, 2i\nu \frac{\partial^3}{\partial z^2 \partial x}\left(y(x)y(z)\frac{\partial \xi_t}{\partial \phi}\right)\delta(x-a) + i\frac{\partial}{\partial x}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right)\delta(x-a) \\
& + 2\frac{\partial}{\partial x}\left(y(x)y(z)\frac{\partial^2}{\partial z \partial \phi}\frac{\delta \xi_t}{\delta y(z)}\right)\delta(x-a) + 2iy(z)\frac{\partial^2 \xi_{\gamma(x)}}{\partial z \partial \phi}\delta(a-z) + iy(x)\frac{\partial^2 \xi_{\gamma(z)}}{\partial x \partial \phi}\delta(x-a) \\
& - 2\frac{\partial^2}{\partial z \partial x}\left(y(x)y(z)\frac{\partial}{\partial \phi}\frac{\delta \xi_t}{\delta y(z)}\right)\delta(x-a) - 2i\frac{\partial}{\partial z}\left(y(z)\frac{\partial \xi_{\gamma(x)}}{\partial \phi}\right)\delta(a-z) \\
& - i\frac{\partial}{\partial x}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right)\delta(x-a) - 2i\frac{\partial}{\partial z}\left(y(z)\frac{\partial \xi_{\gamma(x)}}{\partial \phi}\right)\delta(a-z) \\
& - i\nu \frac{\partial^3}{\partial z^2 \partial x}\left(y(x)y(z)\frac{\partial \xi_t}{\partial \phi}\right)\delta(x-a) - i\frac{\partial}{\partial x}\left(y(x)\frac{\partial \xi_{\gamma(z)}}{\partial \phi}\right)\delta(x-a) \\
& - i\nu \frac{\partial^3}{\partial z^2 \partial x}\left(y(x)y(z)\frac{\partial \xi_t}{\partial \phi}\right)\delta(x-a) + i\nu \frac{\partial^2}{\partial z^2}\left(y(x)y(z)\frac{\partial^2 \xi_t}{\partial x \partial \phi}\right)\delta(x-a) \\
& + 2i\nu \frac{\partial^2}{\partial z \partial x}\left(y(x)y(z)\frac{\partial^2 \xi_t}{\partial z \partial \phi}\right)\delta(x-a) - 2i\nu \frac{\partial}{\partial x}\left(y(x)y(z)\frac{\partial^3 \xi_t}{\partial z^2 \partial \phi}\right)\delta(x-a)
\end{aligned} \tag{A12}
$$

- Equations $\mathcal{F} = 0, \mathcal{G} = 0, \mathcal{H} = 0, \mathcal{I} = 0, \mathcal{J} = 0, \mathcal{K} = 0$ and $\mathcal{L} = 0$ are the same as in [1].

Solution of the Determining System of Equations for the Infinitesimals

Similarly as in [1], equation $\mathcal{K} = 0$ leads to:

$$\frac{\partial \xi_{\gamma(z)}}{\partial x} = 0 \tag{A13}$$

and equations $\mathcal{J} = 0, \mathcal{I} = 0$ and $\mathcal{G} = 0$ to

$$\xi_t = \xi_t(t) \tag{A14}$$

With this, Equation (A12) leads to:

$$\frac{\partial \xi_{\gamma(z)}}{\partial \phi} = 0 \tag{A15}$$

Now, we take a look at the remaining four Equations (A8)–(A11). We start with Equation (A11). Considering Equations (A14) and (A15), Equation (A11) reads:

$$-i\frac{\partial}{\partial x}\left(y(x)\frac{\partial^2\xi_x}{\partial x\partial\phi}\right)\delta(x-z)+2i\frac{\partial}{\partial x}\left(y(x)\frac{\partial^2\eta_\phi}{\partial\phi^2}\right)\delta(x-z)+2i\frac{\partial^2}{\partial x^2}\left(y(x)\frac{\partial\xi_x}{\partial\phi}\right)\delta(x-z)=0$$

This equation has to hold for all choices of $y \in L^2(G,\mathbb{R})$, hence the coefficients of y'', y' and y have to vanish, which leads to:

$$\frac{\partial\xi_x}{\partial\phi}=0 \tag{A16}$$

$$2i\frac{\partial^2\eta_\phi}{\partial\phi^2}+3i\frac{\partial^2\xi_x}{\partial x\partial\phi}=0 \tag{A17}$$

$$2i\frac{\partial^3\eta_\phi}{\partial x\partial\phi^2}+i\frac{\partial^3\xi_x}{\partial x^2\partial\phi}=0 \tag{A18}$$

These equations mean that $\xi_x = \xi_x([y(x')], x, t)$ and that there are functionals f, g, such that:

$$\eta_\phi = f([y(x')],t)\phi + g([y(x')],t) \tag{A19}$$

For f, we choose the ansatz:

$$f([y(x')],t) = \int_G f_1(x',t)y(x')\mathrm{d}x' + f_2(t) \tag{A20}$$

The next equation we solve is Equation (A10). If we use Equations (A14) and (A13) and apply the product rule, Equation (A10) reads:

$$0=i\frac{\partial}{\partial x}\left(y(x)\frac{\partial\xi_t}{\partial t}\right)\delta(x-z)+i\frac{\partial\xi_{\gamma(x)}}{\partial x}\delta(x-z)-i\frac{\partial}{\partial x}\left(y(x)\frac{\partial\xi_x}{\partial x}\right)\delta(x-z)-2iy'(x)\frac{\delta\xi_{\gamma(z)}}{\delta y(x)} \tag{A21}$$

Considering the case $x \neq z$, we get:

$$\frac{\delta\xi_{\gamma(z)}}{\delta y(x)}=0 \tag{A22}$$

By virtue of Equations (A13) and (A15), we have $\xi_{\gamma(z)} = \xi_{\gamma(z)}([y(x')], z, t)$; we will use the following ansatz:

$$\xi_{\gamma(z)} = c(z,t) + c_0(z,t)y(z) \tag{A23}$$

Next, we want to consider Equation (A21) without the restriction $x \neq z$; hence, we integrate Equation (A21) with respect to $z \in G$. This leads to:

$$\frac{\partial}{\partial x}\left(y(x)\frac{\partial\xi_t}{\partial t}\right)+\frac{\partial\xi_{\gamma(x)}}{\partial x}-\frac{\partial}{\partial x}\left(y(x)\frac{\partial\xi_x}{\partial x}\right)-2\int_G y'(x)\frac{\delta\xi_{\gamma(z)}}{\delta y(x)}\mathrm{d}z=0$$

Now, we put in ansatz Equation (A23); make use of $\xi_t = \xi_t(t)$; and take into consideration that this equation has to hold for all choices of $y \in L^2(G,\mathbb{R})$; hence, the coefficients of $1, y, y'$ and $y'y'$ have to vanish:

$$\frac{\partial c(x,t)}{\partial x}=0 \Longrightarrow c=c(t) \tag{A24}$$

$$\frac{\partial c_0(x,t)}{\partial x}-\frac{\partial^2\xi_x}{\partial x^2}=0 \tag{A25}$$

$$\xi_t'(t)-\frac{\partial\xi_x}{\partial x}-c_0(x,t)=0 \tag{A26}$$

From Equations (A25) and (A26), it further follows that:

$$\frac{\partial^2 \xi_x}{\partial x^2} = 0 \tag{A27}$$

Considering Equations (A24) and (A25), ansatz Equation (A23) reads:

$$\xi_{\gamma(z)} = c(t) + \left(\xi_t'(t) - \frac{\partial \xi_z}{\partial z}\right) y(z) \tag{A28}$$

Now, we are ready to deal with Equation (A9). We use Equations (A14), (A20) (A27) and (A28) and the fact that the coefficients of 1, y, y' and y'' have to vanish. Then, Equation (A9) leads to:

$$-c'(t) = 0 \Longrightarrow c(t) = const. =: c \in \mathbb{R} \tag{A29}$$

$$-\xi_t''(t) + \frac{\partial^2 \xi_z}{\partial t \partial z} = 0 \tag{A30}$$

$$2if_1(z,t) + i\frac{\partial}{\partial z}\frac{\delta \xi_z}{\delta y(z)} = 0 \tag{A31}$$

$$2\nu\frac{\partial \xi_z}{\partial z} + i\frac{\delta \xi_z}{\delta y(z)} - \nu\xi_t'(t) = 0 \tag{A32}$$

If we calculate the derivative of Equation (A32) with respect to z and take into account Equation (A27), we obtain:

$$i\frac{\partial}{\partial z}\frac{\delta \xi_z}{\delta y(z)} = -2\nu\frac{\partial^2 \xi_z}{\partial z^2} = 0 \tag{A33}$$

which, substituted into Equation (A31), gives $f_1 = 0$. Hence, by ansatz Equation (A20):

$$f([y(x')], t) = f_2(t) = f(t) \tag{A34}$$

If we substitute Equation (A34) into Equation (A8), we get:

$$f'(t)\phi + \frac{\partial g}{\partial t} - \int_G y(x)\left(i\frac{\partial^3 g}{\partial x \partial (y(x)\mathrm{d}x)^2} + \nu\frac{\partial^3 g}{\partial x^2 \partial y(x)\mathrm{d}x}\right)\mathrm{d}x = 0 \tag{A35}$$

This equation has to hold for every ϕ; hence, the coefficient of ϕ has to vanish. This furnishes $f = const$. Equation (A35) further reads:

$$\frac{\partial g}{\partial t} - \int_G y(x)\left(i\frac{\partial^3 g}{\partial x \partial (y(x)\mathrm{d}x)^2} + \nu\frac{\partial^3 g}{\partial x^2 \partial y(x)\mathrm{d}x}\right)\mathrm{d}x = 0 \tag{A36}$$

An admissible ansatz for ξ_z is given by:

$$\xi_z = a_1(t) + a_2(t)z + \int_G a_3(x', t)y(x')\,\mathrm{d}x' \tag{A37}$$

In this ansatz, $a_1 \in \mathbb{R}$ is a constant and a_2 and a_3 are unknown functions. If we insert this ansatz into Equations (A30) and (A32), we get:

$$\xi_t = A_2(t) + a_4 t + a_5, \quad a_4, a_5 \in \mathbb{R} \tag{A38}$$

where $A_2'(t) = a_2(t)$ and

$$\xi_z = a_1(t) + a_2(t)z + i\nu\big(a_2(t) - a_4\big)\int_G y(x')\,\mathrm{d}x' \tag{A39}$$

We further insert Equations (A39), (A38) and (A29) into Equation (A28) to get an expression for $\xi_{\gamma(z)}$. Finally, we take into account the symmetry breaking restrictions concerning the functions f and g in Equations (A19) and (A20), as described in Section 3.3 in [1], and obtain the infinitesimals of the viscous Hopf–Burgers FDE:

$$\begin{aligned}
\xi_t &= A_2(t) + a_4 t + a_5 \\
\xi_x &= a_1(t) + a_2(t)x + iv\big(a_2(t) - a_4\big) \int_G y(x')\,\mathrm{d}x' \\
\xi_{y(x)\mathrm{d}x} &= \xi_{\gamma(x)}\mathrm{d}x = a_6 \mathrm{d}x + a_4 y(x)\mathrm{d}x \\
\eta_\phi &= a_7(\phi - 1) + g([y(x')], t)
\end{aligned}$$

where $a_4, a_5, a_6, a_7 \in \mathbb{R}$ are arbitrary constants, $a_1(t)$ and $a_2(t) = A_2'(t)$ are arbitrary functions and g is an arbitrary functional, which has to fulfill the viscous Hopf–Burgers FDE.

At the end, we consider the requirement that $y(x)$ is an arbitrary function of x only and $\overline{y(x)}$ should be a function of $\overline{x}$. As argued by Frewer and Khujadze, this would impose further constraints onto the symmetries, and we obtain $a_1(t) = a_1$, $a_2(t) = a_2$, $A_2 = a_2 t$. Moreover, due to the condition Equation (12) discussed in Section 3, also $a_4 = a_2$.

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