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Optimal Inequalities for the Casorati Curvatures of Submanifolds in Generalized Space Forms Endowed with Semi-Symmetric Non-Metric Connections

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Abstract: In this paper, we prove some optimal inequalities involving the intrinsic scalar curvature and the extrinsic Casorati curvature of submanifolds in a generalized complex space form with a semi-symmetric non-metric connection and a generalized Sasakian space form with a semi-symmetric non-metric connection. Moreover, we show that in both cases, the equalities hold if and only if submanifolds are invariantly quasi-umbilical.

Keywords: Casorati curvature; semi-symmetric non-metric connection; generalized complex space form; generalized Sasakian space form

1. Introduction

The theory of Chen invariants [1] is presently one of the most interesting research topics in differential geometry of submanifolds. He established some sharp inequalities, well-known as Chen's inequalities, for a submanifold in a real space form using the scalar curvature, the sectional curvature, Ricci curvature and the squared mean curvature. In other words, he gave simple relationships between the main intrinsic invariants and the extrinsic invariants of a submanifold in a real space form. It is well known that theorems which relate intrinsic and extrinsic curvatures of submanifolds always play an important role in differential geometry. So the study of this topic has attracted a lot of attention in the last two decades. Many Chen invariants and inequalities exist for the different classes of submanifolds in various ambient spaces; see [2–8] and reference therein.

On the other hand, Hayden [9] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. Yano [10] studied some properties of a Riemannian manifold with a semi-symmetric metric connection. Nakao [11] studied submanifolds in a Riemannian manifold with a semi-symmetric metric connection. Agashe and Chafle [12,13] introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold and studied submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection.

Mihai and Özgür [14,15] proved Chen's inequalities for submanifolds in a real space with a semi-symmetric metric connection, a complex space with a semi-symmetric metric connection and a Sasakian space form with a semi-symmetric metric connection. They also studied Chen's inequalities for submanifolds in a real space form endowed with a semi-symmetric non-metric connection [16]. By using two new algebraic lemmas Zhang et al. [17] obtained Chen's inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature endowed with a semi-symmetric non-metric connection.

Instead of the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form of the submanifold. The notion of Casorati curvature extends the concept of the principle direction of a hypersurface in a Riemannian manifold. Therefore, it is of great interest to obtain optimal inequalities for the Casorati curvatures of submanifolds in different manifolds. Decu et al. [18] obtained some optimal inequalities involving the scalar curvature and the Casorati curvature of a submanifold in a real space form. Some optimal inequalities involving Casorati curvatures were proved in [19–21] for slant submanifolds in quaternionic space forms. Recently, Lee et al. [22–24] proved optimal inequalities involving the Casorati curvature of submanifols in real and generalized space forms endowed with a semi-symmetric metric connection. Using a different algebra approach, Zhang et al. [25] established optimal inequalities involving the Casorati curvature of submanifols in a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection. But optimal inequalities involving the Casorati curvature of submanifolds in an ambient space with a semi-symmetric non-metric connection haven't been established.

In this paper, we will study some optimal inequalities involving the Casorati curvature of submanifols in a generalized space forms endowed with semi-symmetric non-metric connections.

2. Preliminaries

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold with a Riemannian metric g and a linear connection $\overline{\nabla}$ on N^{n+p}. If the torsion tensor $\overline{T}$ of $\overline{\nabla}$, defined by

$$\overline{T}(\overline{X},\overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X},\overline{Y}]$$

for any smooth vector fields $\overline{X}$ and $\overline{Y}$ on N^{n+p}, satisfies

$$\overline{T}(\overline{X},\overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$$

for a 1-form ϕ, then the linear connection $\overline{\nabla}$ is called a semi-symmetric connection. The vector field U is defined by $\phi(\overline{X}) = g(\overline{X},U)$ for any vector field $\overline{X}$ on N^{n+p}. If $\overline{\nabla}$ satisfies $\overline{\nabla}g = 0$, $\overline{\nabla}$ is called a semi-symmetric metric connection. If $\overline{\nabla}$ satisfies $\overline{\nabla}g \neq 0$, then $\overline{\nabla}$ is called a semi-symmetric non-metric connection.

Let $\overline{\nabla}'$ denote the Levi-Civita connection with respect to the Riemannian metric g on N^{n+p}. Agashe and Chafle [12] introduced a semi-symmetric non-metric connection $\overline{\nabla}$ which is given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}'_{\overline{X}}\bar{Y} + \phi(\overline{Y})\overline{X}, \tag{1}$$

for any smooth vector fields $\overline{X}$ and $\overline{Y}$ on N^{n+p}.

We will consider the Riemannian manifold N^{n+p} endowed with a semi-symmetric non-metric connection $\overline{\nabla}$ and the Levi-Civita connection $\overline{\nabla}'$. Let $\overline{R}$ and $\overline{R}'$ be curvature tensors of the Riemannian manifold N^{n+p} with respect to $\overline{\nabla}$ and $\overline{\nabla}'$, respectively. Then $\overline{R}$ can be written as [12]

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \overline{R}'(\overline{X},\overline{Y},\overline{Z},\overline{W}) + s(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) - s(\overline{Y},\overline{Z})g(\overline{X},\overline{W}) \tag{2}$$

for any smooth vector fields $\overline{X},\overline{Y},\overline{Z},\overline{W}$ on N^{n+p}, where $(0,2)$-tensor field s is given by

$$s(\overline{X},\overline{Y}) = (\overline{\nabla}'_{\overline{X}}\phi)\overline{Y} - \phi(\overline{X})\phi(\overline{Y}).$$

Denote by λ the trace of s.

Let M^n be an n-dimensional submanifold in the Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric non-metric connection denoted by ∇ and the induced Levi-Civita connection denoted by ∇'. We also denote by R and R' the curvature tensor on M^n with respect to ∇ and ∇', respectively.

The Gauss formulas with respect to $\overline{\nabla}$ and $\overline{\nabla}'$, respectively, can be written as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \overline{\nabla}'_X Y = \nabla'_X Y + h'(X,Y)$$

for any smooth vector fields X, Y on M^n, where h' is the second fundamental form of M^n in N^{n+p} and h is a (0,2)-tensor on M^n. From [13], we know

$$h = h'. \tag{3}$$

In [13], the Gauss equation for the submanifold M^n into N^{n+p} with respect to the semi-symmetric non-metric connection is

$$\begin{aligned}\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \\ + g(U,h(Y,Z))g(X,W) - g(U,h(X,Z))g(Y,W)\end{aligned} \tag{4}$$

for any smooth vector fields X, Y, Z, W on M^n.

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric non-metric connection ∇. For any orthonormal basis $\{e_1, \cdots, e_n\}$ of the tangent space $T_x M^n$ the scalar curvature τ at x with respect to the semi-symmetric non-metric connection is defined by

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ with respect to the semi-symmetric non-metric connection is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

Let $\{e_{n+1}, \cdots, e_{n+p}\}$ be an orthormal basis of the normal space $T_x^{\perp} M^n$. We denote by H the mean curvature vector of M^n with respect to the semi-symmetric non-metric connection, that is

$$H(x) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$$

We also set

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha), \ \ h'^{\alpha}_{ij} = g(h'(e_i, e_j), e_\alpha), \ \ i, j \in \{1, \cdots, n\}, \ \ \alpha \in \{n+1, \cdots, n+p\}.$$

Then the squared norm of h over dimension n is called the Casorati curvature of M^n with respect to the semi-symmetric non-metric connection, which is denoted by $\mathcal{C}$. That is,

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2.$$

Suppose that L is an l-dimensional subspace of $T_x M^n$, $l \geq 2$, and $\{e_1, \cdots, e_l\}$ is an orthonormal basis of L. Then the Casorati curvature of the l-plane section L with respect to the semi-symmetric non-metric connection is defined by

$$\mathcal{C}(L) = \frac{1}{l} \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{l} (h_{ij}^{\alpha})^2.$$

We define the normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ with respect to the semi-symmetric non-metric connection as the following:

$$[\delta_c(n-1)]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n}\inf\{\mathcal{C}(L) : L \text{ is a hyperplane of } T_xM\}$$

and

$$[\hat{\delta}_c(n-1)]_x = 2\mathcal{C}_x - \frac{2n-1}{2n}\sup\{\mathcal{C}(L) : L \text{ is a hyperplane of } T_xM\}.$$

The submanifold M^n is called invariantly quasi-umbilical if there exist p mutually orthogonal unit normal vectors $e_{n+1}, \cdots, e_{n+p}$ such that the shape operators with respect to all directions e_α have an eigenvalue of multiplicity $n-1$ and that for each e_α the distinguished eigendirection is the same [26].

Let us recall the following two lemmas in [25].

Lemma 1. *Let $f(x_1, x_2, \cdots, x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \cdots, x_n) = n\sum_{i=1}^{n-1} x_i^2 + \frac{n-1}{2}x_n^2 - 2\sum_{1\le i<j\le n} x_i x_j.$$

If $x_1 + x_2 + \cdots + x_n = \varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \ge 0,$$

with the equality holding if and only if

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{2}x_n = \frac{1}{n+1}\varepsilon.$$

Lemma 2. *Let $f(x_1, x_2, \cdots, x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \cdots, x_n) = \frac{2n-3}{2}\sum_{i=1}^{n-1} x_i^2 + 2(n-1)x_n^2 - 2\sum_{1\le i<j\le n} x_i x_j.$$

If $x_1 + x_2 + \cdots + x_n = \varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \ge 0,$$

with the equality holding if and only if

$$x_1 = x_2 = \cdots = x_{n-1} = 2x_n = \frac{2}{2n-1}\varepsilon.$$

3. Optimal Inequalities for the Casorati Curvatures of Submanifolds in a Generalized Complex Space form Endowed with a Semi-Symmetric Non-Metric Connection

A $2m$-dimensional almost Hermitian manifold (N, J, g) is said to be a generalized complex space form [27], if there exists two functions F_1 and F_2 on N such that

$$\begin{aligned}\overline{R}'(\overline{X},\overline{Y},\overline{Z},\overline{W}) = {} & F_1[g(\overline{Y},\overline{Z})g(\overline{X},\overline{W}) - g(\overline{X},\overline{Z})g(\overline{Y},\overline{W})] + F_2[g(\overline{X},J\overline{Z})g(J\overline{Y},\overline{W}) \\ & - g(\overline{Y},J\overline{Z})g(J\overline{X},\overline{W}) + 2g(\overline{X},J\overline{Y})g(J\overline{Z},\overline{W})]\end{aligned} \quad (5)$$

for any smooth vector fields $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ on N, where $\overline{R}'$ is the curvature tensor with respect to the Levi-Civita connection $\overline{\nabla}'$. In such a case, we will write $N(F_1, F_2)$.

We endow the generalized complex space form $N(F_1, F_2)$ with a semi-symmetric non-metric connection $\overline{\nabla}$. Let M^n be an n-dimensional submanifold of $N(F_1, F_2)$, $n \geq 3$. For any vector field X tangent to M, we decompose JX as

$$JX = PX + FX,$$

where PX and FX are tangential and normal components of JX, respectively. We also set

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

For submanifolds in the generalized complex space form with a semi-symmetric non-metric connection, we establish the following inequalities involving the normalized δ-curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$.

Theorem 1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold in a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ endowed with a semi-symmetric non-metric connection $\overline{\nabla}$. Then*

(i) The normalized δ-curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + F_1 + \frac{3}{n(n-1)} F_2 ||P||^2 - \frac{\lambda}{n} - \phi(H). \quad (6)$$

Moreover, the equality holds if and only if M^n is an invariantly quasi-umbilical submanifold.

(ii) The normalized δ-curvature $\hat{\delta}_c(n-1)$ satisfies

$$\rho \leq \hat{\delta}_c(n-1) + F_1 + \frac{3}{n(n-1)} F_2 ||P||^2 - \frac{\lambda}{n} - \phi(H). \quad (7)$$

Moreover, the equality holds if and only if M^n is an invariantly quasi-umbilical submanifold.

Proof. Let $e_1, \cdots, e_n$ and $e_{n+1}, \cdots, e_{2m}$ be orthonormal bases of $T_x M^n$ and $T_x^{\perp} M^n$, respectively, $x \in M^n$.

For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from (2), (4) and (5), we get

$$\begin{aligned} R_{ijji} = R(e_i, e_j, e_j, e_i) &= F_1 + 3F_2 g^2(Je_i, e_j) - s(e_j, e_j) \\ &+ g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)) - \phi(h(e_j, e_j)). \end{aligned}$$

By summation over $1 \leq i, \ \ j \leq n$, it follows that

$$2\tau(x) = n^2 H^2 - n\mathcal{C} + n(n-1)F_1 + 3F_2||P||^2 - (n-1)\lambda - n(n-1)\phi(H). \quad (8)$$

(i) Without loss of generality, we can assume that $L_0 = \text{span}\{e_1, \cdots, e_{n-1}\}$ satisfies

$$\mathcal{C}(L_0) = \inf\{\mathcal{C}(L) : L \text{ is a hyperplane of } T_x M\}.$$

We define the following function, denoted by $\mathcal{P}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned} \mathcal{P} = \frac{1}{2} n(n-1)\mathcal{C} + \frac{(n-1)(n+1)}{2}\mathcal{C}(L_0) - 2\tau \\ + n(n-1)F_1 + 3F_2||P||^2 - (n-1)\lambda - n(n-1)\phi(H). \end{aligned} \quad (9)$$

Using (8) we obtain

$$\begin{aligned}\mathcal{P} = &\sum_{\alpha=n+1}^{2m}[n\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + \frac{n-1}{2}(h_{nn}^{\alpha})^2 + 2(n+1)\sum_{1\leq i<j\leq n-1}(h_{ij}^{\alpha})^2 \\ &+ (n+1)\sum_{i=1}^{n-1}(h_{in}^{\alpha})^2 - 2\sum_{1\leq i<j\leq n-1}h_{ii}^{\alpha}h_{jj}^{\alpha}] \\ \geq &\sum_{\alpha=n+1}^{2m}[n\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + \frac{n-1}{2}(h_{nn}^{\alpha})^2 - 2\sum_{1\leq i<j\leq n-1}h_{ii}^{\alpha}h_{jj}^{\alpha}].\end{aligned}$$

Setting

$$f(h_{11}^{\alpha}, h_{22}^{\alpha}, \cdots, h_{nn}^{\alpha}) = n\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + \frac{n-1}{2}(h_{nn}^{\alpha})^2 - 2\sum_{1\leq i<j\leq n-1}h_{ii}^{\alpha}h_{jj}^{\alpha},$$

we consider the problem as following:

$$\min\{f(h_{11}^{\alpha}, h_{22}^{\alpha}, \cdots, h_{nn}^{\alpha}) : h_{11}^{\alpha} + h_{22}^{\alpha} + \cdots + h_{nn}^{\alpha} = k^{\alpha}, k^{\alpha} \text{ is some constant}\},$$

where $\alpha \in \{n+1, \cdots, 2m\}$.

By Lemma 1, we have

$$f(h_{11}^{\alpha}, h_{22}^{\alpha}, \cdots, h_{nn}^{\alpha}) \geq 0$$

with equality holding if and only if

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1,n-1}^{\alpha} = \frac{1}{2}h_{nn}^{\alpha}, \quad \forall\alpha \in \{n+1, \cdots, 2m\}.$$

Therefore, we have

$$\mathcal{P} \geq 0 \tag{10}$$

with equality holding if and only if

$$h_{ij}^{\alpha} = 0, \quad \forall i \neq j, \quad \forall\alpha \in \{n+1, \cdots, 2m\}$$

and

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1,n-1}^{\alpha} = \frac{1}{2}h_{nn}^{\alpha}, \quad \forall\alpha \in \{n+1, \cdots, 2m\}$$

From (9) and (10), we get

$$2\tau \leq \frac{1}{2}n(n-1)\mathcal{C} + \frac{(n-1)(n+1)}{2}\mathcal{C}(L) + n(n-1)F_1 + 3F_2||P||^2 - (n-1)\lambda - n(n-1)\phi(H).$$

Furthermore, we have

$$\rho \leq \frac{1}{2}\mathcal{C} + \frac{n+1}{2n}\mathcal{C}(L) + F_1 + \frac{3}{n(n-1)}F_2||P||^2 - \frac{\lambda}{n} - \phi(H).$$

By the definition of $\delta_c(n-1)$, we can obtain

$$\rho \leq \delta_c(n-1) + F_1 + \frac{3}{n(n-1)}F_2||P||^2 - \frac{\lambda}{n} - \phi(H).$$

And the equality holds if and only if

$$h'^{\alpha}_{11} = h'^{\alpha}_{22} = \cdots = h'^{\alpha}_{n-1,n-1} = \frac{1}{2}h'^{\alpha}_{nn}, \quad h'^{\alpha}_{ij} = 0, \quad \forall i \neq j, \quad \forall\alpha \in \{n+1, \cdots, 2m\}, \tag{11}$$

where we used the relation (3) of h and h'.

From (11), we know that M^n is invariantly quasi-umbilical.

(ii) Without loss of generality, we can also assume that $L_0 = \text{span}\{e_1, \cdots, e_{n-1}\}$ satisfies

$$\mathcal{C}(L_0) = \sup\{\mathcal{C}(L) : L \text{ is a hyperplane of } T_xM\}.$$

Considering the following quadratic polynomial in the components of the second fundamental form

$$\begin{aligned}\mathcal{Q} = & 2n(n-1)\mathcal{C} + \frac{1}{2}(n-1)(1-2n)\mathcal{C}(L) - 2\tau \\ & + n(n-1)F_1 + 3F_2||P||^2 - (n-1)\lambda - n(n-1)\phi(H).\end{aligned} \tag{12}$$

Using (8) we have

$$\begin{aligned}\mathcal{Q} = & \sum_{\alpha=n+1}^{2m}[\frac{2n-3}{2}\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + 2(n-1)(h_{nn}^{\alpha})^2 + (2n-1)\sum_{1\le i<j\le n-1}(h_{ij}^{\alpha})^2 \\ & + 2(2n-1)\sum_{i=1}^{n-1}(h_{in}^{\alpha})^2 - 2\sum_{1\le i<j\le n-1}h_{ii}^{\alpha}h_{jj}^{\alpha}] \\ \ge & \sum_{\alpha=n+1}^{2m}[\frac{2n-3}{2}\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + 2(n-1)(h_{nn}^{\alpha})^2 - 2\sum_{1\le i<j\le n-1}h_{ii}^{\alpha}h_{jj}^{\alpha}].\end{aligned}$$

Setting

$$f(h_{11}^{\alpha}, h_{22}^{\alpha}, \cdots, h_{nn}^{\alpha}) = \sum_{\alpha=n+1}^{2m}\frac{2n-3}{2}\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + 2(n-1)(h_{nn}^{\alpha})^2 - 2\sum_{1\le i<j\le n-1}h_{ii}^{\alpha}h_{jj}^{\alpha},$$

we consider the problem as following:

$$\min\{f(h_{11}^{\alpha}, h_{22}^{\alpha}, \cdots, h_{nn}^{\alpha}) : h_{11}^{\alpha} + h_{22}^{\alpha} + \cdots + h_{nn}^{\alpha} = k^{\alpha}, k^{\alpha} \text{ is some constant}\},$$

where $\alpha \in \{n+1, \cdots, 2m\}$.

By Lemma 2, we have

$$f(h_{11}^{\alpha}, h_{22}^{\alpha}, \cdots, h_{nn}^{\alpha}) \ge 0, \ \forall \alpha \in \{n+1, \cdots, 2m\}$$

with equality holding if and only if

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1,n-1}^{\alpha} = 2h_{nn}^{\alpha}.$$

Therefore, we have

$$\mathcal{Q} \ge 0 \tag{13}$$

with equality holding if and only if

$$h_{ij}^{\alpha} = 0, \quad \forall i \ne j, \quad \forall \alpha \in \{n+1, \cdots, 2m\}$$

and

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1,n-1}^{\alpha} = 2h_{nn}^{\alpha}, \quad \forall \alpha \in \{n+1, \cdots, 2m\}$$

Then by (12) and (13) and the definition of $\hat{\delta}_c(n-1)$, we can easily derive the inequality (7). And the equality can be also easily verified. □

Remark 1. *For $F_1 = F_2 = c$, where c is a constant, then from Theorem 1 we can get optimal inequalities for the Casorati curvatures of submanifolds in the complex space form $N^{2m}(4c)$ endowed with a semi-symmetric non-metric connection.*

4. Optimal Inequalities for the Casorati Curvatures of Submanifolds in a Generalized Sasakian Space form Endowed with a Semi-Symmetric Non-Metric Connection

Let N be a $(2m+1)$-dimensional almost contact metric manifold (see [28]) with an almost contact metric structure (φ, ξ, η, g) consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g on N satisfying

$$\varphi^2\overline{X} = -\overline{X} + \eta(\overline{X})\xi, \;\; \eta(\xi) = 1, \;\; \varphi\xi = 0, \;\; \eta \circ \varphi = 0,$$

$$g(\varphi\overline{X}, \varphi\overline{Y}) = g(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \;\; g(\overline{X}, \xi) = \eta(\overline{X}),$$

for all vector fields $\overline{X}, \overline{Y}$ on N. Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(\overline{X}, \overline{Y}) = g(\overline{X}, \varphi\overline{Y})$ is called the fundamental 2-form of N [28].

Given an almost contact metric manifold N with an almost contact metric structure (φ, ξ, η, g), N is called generalized Sasakian space form [29] if there exists three functions f_1, f_2 and f_3 on N such that

$$\begin{aligned}\overline{R}'(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= f_1[g(\overline{Y}, \overline{Z})g(\overline{X}, \overline{W}) - g(\overline{X}, \overline{Z})g(\overline{Y}, \overline{W})] + f_2[g(\overline{X}, \varphi\overline{Z})g(\varphi\overline{Y}, \overline{W}) \\ &- g(\overline{Y}, \varphi\overline{Z})g(\varphi\overline{X}, \overline{W}) + 2g(\overline{X}, \varphi\overline{Y})g(\varphi\overline{Z}, \overline{W})] + f_3[\eta(\overline{X})\eta(\overline{Z})g(\overline{Y}, \overline{W}) \\ &- \eta(\overline{Y})\eta(\overline{Z})g(\overline{X}, \overline{W}) + \eta(\overline{Y})\eta(\overline{W})g(\overline{X}, \overline{Z}) - \eta(\overline{X})\eta(\overline{W})g(\overline{Y}, \overline{Z})]\end{aligned} \tag{14}$$

for any smooth vector fields $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ on N, where $\overline{R}'$ is the curvature tensor with respect to the Levi-Civita connection $\overline{\nabla}'$. In such a case, we will write $N(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, where c is a constant, then N is a Sasakian space form.

Now we endow the generalized Sasakian space form $N(f_1, f_2, f_3)$ with a semi-symmetric non-metric connection $\overline{\nabla}$. Let M^n be an n-dimensional submanifold of $N(f_1, f_2, f_3)$, $n \geq 3$. We set

$$\varphi X = PX + FX$$

for any vector field X tangent to M^n, where PX and FX are tangential and normal components of φX, respectively. We also set

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j)$$

and decompose

$$\xi = \xi^\top + \xi^\perp,$$

where $\xi^\top$ and $\xi^\perp$ denote the tangential and normal components of ξ.

For submanifolds in a generalized Sasakian space form with the semi-symmetric non-metric connection, we establish the following inequalities involving the normalized δ-curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$.

Theorem 2. *Let $M^n, n \geq 3$, be an n-dimensional submanifold in a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ endowed with a semi-symmetric non-metric connection $\overline{\nabla}$. Then*
(i) The normalized δ-curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + f_1 + \frac{3}{n(n-1)} f_2 ||P||^2 - \frac{2}{n} f_3 ||\xi^\top||^2 - \frac{\lambda}{n} - \phi(H).$$

Moreover, the equality holds if and only if M^n is an invariantly quasi-umbilical submanifold.

(ii) The normalized δ-curvature $\hat{\delta}_c(n-1)$ satisfies

$$\rho \leq \hat{\delta}_c(n-1) + f_1 + \frac{3}{n(n-1)} f_2 ||P||^2 - \frac{2}{n} f_3 ||\xi^\top||^2 - \frac{\lambda}{n} - \phi(H).$$

Moreover, the equality holds if and only if M^n is an invariantly quasi-umbilical submanifold.

Proof. Let $e_1, \cdots, e_n$ and $e_{n+1}, \cdots, e_{2m+1}$ be orthonormal bases of $T_x M^n$ and $T_x^\perp M^n$, respectively, $x \in M^n$.

For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from (2), (4) and (14), we get

$$\begin{aligned} R_{ijji} = R(e_i, e_j, e_j, e_i) &= f_1 + 3 f_2 g^2(\varphi e_i, e_j) - f_3[\eta(e_i)^2 + \eta(e_j)^2] - s(e_j, e_j) \\ &+ g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)) - \phi(h(e_j, e_j)). \end{aligned}$$

By summation over $1 \leq i, \ \ j \leq n$, it follows that

$$\begin{aligned} 2\tau(x) = n^2 H^2 - n\mathcal{C} &+ n(n-1) f_1 + 3 f_2 ||P||^2 \\ &- 2(n-1) f_3 ||\xi^\top||^2 - (n-1)\lambda - n(n-1)\phi(H). \end{aligned}$$

The rest of the proof is the same as Theorem 1. So we will no longer describe here. □

Remark 2. *For $f_1 = \frac{c+3}{4}$,$f_2 = f_3 = \frac{c-1}{4}$, from Theorem 2 we can get the optimal inequalities for the Casorati curvatures of submanifolds in the Sasakian space form endowed with a semi-symmetric non-metric connection.*

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