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Invariant Subspaces of the Two-Dimensional Nonlinear Evolution Equations

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Abstract: In this paper, we develop the symmetry-related methods to study invariant subspaces of the two-dimensional nonlinear differential operators. The conditional Lie–Bäcklund symmetry and Lie point symmetry methods are used to construct invariant subspaces of two-dimensional differential operators. We first apply the multiple conditional Lie–Bäcklund symmetries to derive invariant subspaces of the two-dimensional operators. As an application, the invariant subspaces for a class of two-dimensional nonlinear quadratic operators are provided. Furthermore, the invariant subspace method in one-dimensional space combined with the Lie symmetry reduction method and the change of variables is used to obtain invariant subspaces of the two-dimensional nonlinear operators.

Keywords: symmetry group; invariant subspace; conditional Lie–Bäcklund symmetry; finite-dimensional dynamical system; nonlinear differential operator

MSC: 37K35; 37K25; 53A55

1. Introduction

The invariant subspace method is an effective one to perform reductions of nonlinear partial differential equations (PDEs) to finite-dimensional dynamical systems. In [1], Galaktionov and Svirshchevskii provide a systematic account of this approach and its various applications for a large variety of nonlinear PDEs. They also addressed some fundamental and open questions on the invariant subspaces of nonlinear PDEs. Many interesting results were obtained in this book. In [2–20], the extensions of the invariant subspace method and various applications to other nonlinear PDEs were also discussed. It is noticed that a large number of exact solutions, such as N-solitons of integrable equations, similarity solutions of nonlinear evolution equations and the generalized functional separable solutions to nonlinear PDEs, can be recovered by the invariant subspace methods [1,21–31]. In the one-dimensional space case, the invariant subspace method can be implemented by the conditional Lie–Bäcklund symmetry introduced independently by Zhdanov [32] and Fokas-Liu [33]. A key point for the invariant subspace approach is the estimate of maximal dimension of the invariant subspaces [1,5,6,15,16]. It was shown in [1,5] that for k-th order one-dimensional nonlinear operator of the form:

$$F[u] = F(x, u, u_x, \cdots, u^{(k)})$$

where $u^{(k)} = \partial^k u/\partial x^k$, the dimension of their invariant subspaces is bounded by $2k+1$. Such an estimate can be extended to the k-th order m-component nonlinear vector operators:

$$\vec{F}[\vec{u}] = \vec{F}(x, \vec{u}, \vec{u}_x, \cdots, \vec{u}^{(k)}). \tag{1}$$

In [15], we proved that the maximal dimension of the invariant subspaces for operator (1) is bounded by $2mk + 1$. This enables us to determine the maximal dimension preliminarily of the invariant subspaces of the nonlinear evolution equations. In contrast with the one-dimensional space case, only very limited results on the invariant subspaces of multi-dimensional PDEs were obtained. These results were obtained mostly by the ansatz-based method, and there are no systematic approaches to obtain these results. As mentioned in [1], the general problem of finding invariant subspaces for a wide class of nonlinear differential operators in the multi-dimensional case is not completely solved. A open question still remains: what is the maximal dimension of the two-dimensional k-th order scalar nonlinear operators of the form:

$$F[u] = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots, u^{(k)}),$$

where $u^{(k)} = \partial^{r+s}u/\partial x^r \partial y^s$, $r + s = k$ denotes all k-th order derivatives with respect to x and y?

It is of great interest to develop the invariant subspace method to study the multi-dimensional nonlinear evolution equations. Indeed, there are a number of examples whose exact solutions can be derived from the invariant subspace method; please refer to [1,2] for more examples on invariant subspaces of the $2 + 1$-dimensional nonlinear evolution equations. For instance, it is discovered that the operators:

$$J[u] = u\Delta_2 u - |\nabla u|^2, \quad (x, y) \in \mathbb{R}^2$$

and:

$$Q[u] = u\Delta_2^2 u - (\Delta_2 u)^2 + 2 \bigtriangledown u \bigtriangledown \Delta_2 u, \quad (x, y) \in \mathbb{R}^2$$

with $\Delta_2 = \partial_x^2 + \partial_y^2$ admit the following invariant subspaces:

$$\begin{aligned}
W_6 &= \mathcal{L}\{1, x, y, x^2, y^2, xy\}, \\
W_6 &= \mathcal{L}\{1, \cosh x, \cos y, \cosh(2x), \cos(2y), \cosh x \cos y\}, \\
W_{91} &= \mathcal{L}\{1, x, y, x^2 + y^2, xy, xr^2, yr^2, r^4\}, \quad r^2 = x^2 + y^2, \\
W_{92} &= \mathcal{L}\{1, \cosh(2x), \sinh(2x), \cos(2y), \sin(2y), \cosh x \cos y, \sinh x \cos y, \cosh x \sin y, \sinh x \sin y\}.
\end{aligned}$$

It was proven in [1] that the quadratic operator defined in $\mathbb{R}^N$:

$$K[u] = \alpha(\Delta_n u)^2 + \beta u \Delta_n u + \gamma |\nabla u|^2, \quad x \in \mathbb{R}^N$$

admits the invariant subspaces:

$$\begin{aligned}
W_2^r &= \mathcal{L}\{1, |x|^2, \}, \\
W_{N+1}^q &= \mathcal{L}\{1, x_1^2, x_2^2, \cdots, x_N^2\}, \\
W_n^q &= \mathcal{L}\{1, x_i x_j, 1 \le i, j \le N\}, \quad n = \frac{N(N+1)}{2} + 1, \\
W_N^{lin} &= \mathcal{L}\{x_1, x_2, \cdots, x_N\}
\end{aligned}$$

and the direct sum of subspaces:

$$W_N^{lin} \bigoplus W_n^q.$$

The purpose of this paper is to develop symmetry-related method to study invariant subspaces of nonlinear evolution equations in the two- or multi-dimensional case. The outline of this paper is as follows. In Section 2, we first give two direct extensions of the concept of invariant subspace in

$\mathbb{R}^2$. Then, the algorithm of this approach will be shown by looking for the invariant subspaces of the operator:

$$A[u] \equiv \alpha(\Delta_2 u)^2 + \gamma u \Delta_2 u + \delta |\nabla u|^2 + \varepsilon u^2 \quad \text{in} \quad \mathbb{R}^2,$$

where α, γ, δ and ε are constants, and $\alpha^2 + \gamma^2 + \delta^2 + \varepsilon^2 \neq 0$. In Section 3, the general description of the changes of variables for the two-dimensional invariant subspace method is given, which can be regarded as an extension to the invariant subspace method in the one-dimensional case. Since the two-dimensional nonlinear evolution equations can be reduced to one-dimensional equations by the Lie symmetry method, this fact combined with the invariant subspace method in the one-dimensional case will be used to obtain invariant subspaces of the corresponding two-dimensional nonlinear operators, which will be discussed in Section 4. As an example, we obtain many new invariant subspaces admitted by a quadratic differential operator $J[u]$. Section 5 is the concluding remarks on this work.

2. Direct Extensions of Invariant Subspaces

2.1. Direct Extensions in $\mathbb{R}^2$

Let us first give a brief account of the invariant subspace method as presented in [1]. Consider the general evolution equation:

$$u_t = F(x, u, u_x, u_{xx}, \cdots, u^{(k)}) \equiv F[u], \quad x \in \mathbb{R} \tag{2}$$

where F is a k-th-order ordinary differential operator with respect to the variable x and $F(\cdot)$ is a given sufficiently smooth function of the indicated variables. Let $\{f_i(x),\ i = 1, \cdots, n\}$ be a finite set of $n \geqslant 1$ linearly independent functions, and W_n^x denotes their linear span $W_n^x = \mathcal{L}\{f_1(x), \cdots, f_n(x)\}$. The subspace W_n^x is said to be invariant under the given operator F, if $F[W_n^x] \subseteq W_n^x$, and then operator F is said to preserve or admit W_n^x, which means:

$$F[\sum_{i=1}^{n} C_i f_i(x)] = \sum_{i=1}^{n} \Psi_i(C_1, \cdots, C_n) f_i(x)$$

for any $C(t) = (C_1(t), \cdots, C_n(t)) \in \mathbb{R}^n$, where Ψ_i are the expansion coefficients of $F[u] \in W_n^x$ in the basis $\{f_i\}$. It follows that if the linear subspace W_n^x is invariant with respect to F, then Equation (2) has solutions of the form:

$$u(t, x) = \sum_{i=1}^{n} C_i(t) f_i(x),$$

where $C_i(t)$ satisfy the n-dimensional dynamical system:

$$C_i' = \Psi_i(C_1, \cdots, C_n), \quad i = 1, \cdots, n.$$

Moreover, assume that the invariant subspace W_n^x is defined as the space of solutions of the linear n-th-order ODE:

$$L_x[v] \equiv \frac{d^n v}{dx^n} + a_{n-1}(x) \frac{d^{n-1} v}{dx^{n-1}} + \cdots + a_1(x) \frac{dv}{dx} + a_0(x) v = 0. \tag{3}$$

If the operator $F[u]$ admits the invariant subspace W_n^x, then the invariant condition with respect to F takes the form:

$$L_x[F[u]]|_{[H]} \equiv 0, \tag{4}$$

where $[H]$ denotes the equation $L_x[u] = 0$ and its differential consequences with respect to x. The invariant condition leads to the following theorem on the maximal dimension of an invariant subspace preserved by the operator F.

Theorem 1. *[1] If a linear subspace W_n^x determined by the space of solutions of linear Equation (3) is invariant under a nonlinear differential operator F of order k, then:*

$$n \leqslant 2k + 1.$$

It is inferred from Equation (4) and the invariant criteria for conditional Lie–Bäcklund symmetry [32,33] that Equation (2) admits the conditional Lie–Bäcklund symmetry:

$$\sigma = L_x[u].$$

To look for the exact solutions of the form:

$$u(t,x,y) = \sum_{i,j} C_{ij}(t) f_i(x) g_j(y) \tag{5}$$

of the two-dimensional nonlinear evolution equations:

$$u_t = F[u] \equiv F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots, u^{(k)}), \tag{6}$$

we now introduce the linear subspace:

$$\begin{aligned} W_{nm}^{xy} &= \mathcal{L}\{f_1(x)g_1(y), \cdots, f_n(x)g_1(y), \cdots, f_1(x)g_m(y), \cdots, f_n(x)g_m(y)\} \\ &\equiv \{\sum_{i,j} C_{ij} f_i(x) g_j(y),\ \forall (C_{11}, \cdots, C_{1m}, \cdots, C_{n1}, \cdots, C_{nm}) \in \mathbb{R}^{nm}\} \end{aligned}$$

as an extension to W_n^x. Assume that $F[u] = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots, u^{(k)})$ is a k-th-order differential operator with respect to the variables x and y, and $\{g_j(y),\ j = 1, \cdots, m\}$ is a finite set of $m \geqslant 1$ linearly independent functions of variable y. It is easy to see that the space $\{f_i(x)g_j(y),\ i = 1, \cdots, n,\ j = 1, \cdots, m\}$ is also a set of linearly independent functions. Let W_m^y denote the linear span of the set $\{g_j(y),\ j = 1, \cdots, m\}$, i.e., $W_m^y = \mathcal{L}\{g_1(y), \cdots, g_m(y)\}$. Similarly, the space W_m^y is defined as the space of solutions of the linear m-th-order ODE:

$$L_y[w] \equiv \frac{d^m w}{dy^m} + b_{m-1}(y)\frac{d^{m-1} w}{dy^{m-1}} + \cdots + b_1(y)\frac{dw}{dy} + b_0(y) w = 0. \tag{7}$$

If $u \in W_{nm}^{xy}$, then there exists a vector $(C_{11}(t), \cdots, C_{1m}(t), \cdots, C_{n1}(t), \cdots, C_{nm}(t)) \in \mathbb{R}^{nm}$, such that:

$$u = \sum_{i,j} C_{ij}(t) f_i(x) g_j(y). \tag{8}$$

We rewrite u as:

$$u = \sum_{i=1}^{n} (\sum_{j=1}^{m} (C_{ij}(t) g_j(y)) f_i(x) = \sum_{j=1}^{m} (\sum_{i=1}^{n} (C_{ij}(t) f_i(x)) g_j(y),$$

which means that:

$$L_x[u] = 0, \ \text{ and } \ L_y[u] = 0. \tag{9}$$

On the other hand, if the function $u = u(t,x,y)$ satisfies the condition (9), then u has the form (8). Indeed, $L_x[u] = 0$ means that there exists a vector function $(C_1(t,y), \cdots, C_n(t,y))$, such that:

$$u = \sum_{i=1}^{n} C_i(t,y) f_i(x),$$

while $L_y[u] = 0$ means that:

$$L_y[u] = L_y[\sum_{i=1}^{n} C_i(t,y) f_i(x)] = \sum_{i=1}^{n} f_i(x) L_y[C_i(t,y)] = 0.$$

Since $f_i(x)$ $(i = 1, \cdots, n)$ are linearly independent, the above equation leads to:

$$L_y[C_i(t,y)] = 0, \ i = 1, \cdots, n.$$

Hence, there exists a set of vectors $(C_{i1}(t), \cdots, C_{im}(t)) \in \mathbb{R}^m$, such that:

$$C_i(t,y) = \sum_{j=1}^{m} C_{ij}(t) g_j(y), \ i = 1, \cdots, n.$$

As above, we are able to obtain the invariance condition of the subspace W_{nm}^{xy} with respect to F, i.e., $F[W_{nm}^{xy}] \subseteq W_{nm}^{xy}$, which takes the form:

$$L_x[F[u]]|_{[H_x]\cap[H_y]} \equiv 0, \text{ and } L_y[F[u]]|_{[H_x]\cap[H_y]} \equiv 0, \tag{10}$$

where $[H_x] \cap [H_y]$ denotes $L_x[u] = 0$, $L_y[u] = 0$, and their differential consequences with respect to x and y. If $F[u]$ admits the invariant subspace W_{nm}^{xy}, then Equation (6) has solutions (5) and can be reduced to an nm-dimensional dynamic system.

We next consider a special case of the function (5). If $1 \in W_n^x \cap W_m^y$, then $a_0(x) = 0$ in (3) and $b_0(y) = 0$ in (7). Without loss of generality, we assume $f_1(x) = 1$ and $g_1(y) = 1$. Note that the function of the form:

$$u(t,x,y) = C_1(t) + \sum_{i=2}^{n} C_i(t) f_i(x) + \sum_{j=2}^{m} B_j(t) g_j(y) \tag{11}$$

is a special case of (5), which is a separable function with respect to spacial variables x and y. We denote:

$$\begin{aligned} W_{n+m-1}^{xy} &= \mathcal{L}\{1, f_2(x), \cdots, f_n(x), g_2(y), \cdots, g_m(y)\} \\ &\equiv \left\{ C_1(t) + \sum_{i=2}^{n} C_i(t) f_i(x) + \sum_{j=2}^{m} B_j(t) g_j(y) \right\}, \end{aligned}$$

which is a linear span of the set $\{1, f_i(x), g_j(y), i = 2, \cdots, n, j = 2, \cdots, m\}$. Clearly, if $u \in W_{n+m-1}^{xy}$, then:

$$L_x[u] = 0, \ \ L_y[u] = 0, \ \ \text{and } u_{xy} = 0. \tag{12}$$

On the other hand, if $u_{xy} = 0$, then the function u has the form:

$$u = f(t,x) + g(t,y).$$

From $L_x[u] = 0$ (notice that $a_0(x) = 0$), we obtain:

$$L_x[f(t,x) + g(t,y)] = L_x[f(t,x)] = 0,$$

which means that there exists a vector $(A_1(t), C_2(t), \cdots, C_n(t))$, such that:

$$f(t,x) = A_1(t) + \sum_{i=2}^{n} C_i(t) f_i(x).$$

Similarly, $L_y[u] = 0$ leads to:

$$g(t,y) = B_1(t) + \sum_{j=2}^{m} B_j(t) g_j(y),$$

where $B_j (j = 1, \cdots, m)$ are functions of t. We denote $C_1 = A_1 + B_1$. Hence, $u \in W^{xy}_{n+m-1}$ if and only if u satisfies the condition (12). Then, we can obtain the invariance condition of the subspace W^{xy}_{n+m-1} with respect to F, i.e., $F[W^{xy}_{n+m-1}] \subseteq W^{xy}_{n+m-1}$, which takes the form:

$$L_x[F[u]]|_{[H]} \equiv 0, \ \ L_y[F[u]]|_{[H]} \equiv 0, \ \ \text{and } (F[u])_{xy}|_{[H]} \equiv 0. \tag{13}$$

where $[H]$ denotes the set $\{L_x[u] = 0\} \cap \{L_y[u] = 0\} \cap \{u_{xy} = 0\}$, and their differential consequences with respect to x and y. In this case, Equation (6) has the solution of the form (11) and can be reduced to an $(n + m - 1)$-dimensional dynamic system.

Assume that the k-th-order differential operator $F[u]$, including the term $\partial^k u/\partial x^k$, admits the invariant subspace W^{xy}_{nm} (or W^{xy}_{n+m-1}), and note that the operator $F[u]$ can also be regarded as a differential operator only with respect to x; the first identity in the condition (10) (or (13)) leads to the estimate $n \leqslant 2k + 1$. The same estimate is also true for m.

Remark 1. *It is noted that the W^{xy}_{mn} and W^{xy}_{n+m-1} demonstrate two special forms of invariant subspaces of the operator $F[u]$. The general form can be introduced as below, which will be used in the following sections.*

Let $\{f_i(x,y),\ i = 1, \cdots, n\}$ be a finite set of $n \geqslant 1$ linearly independent functions, and W_n denote their linear span $W_n = \mathcal{L}\{f_1(x,y), \cdots, f_n(x,y)\}$. The subspace W_n is said to be invariant under the given operator $F[u]$, if $F[W_n] \subseteq W_n$, and then, operator $F[u]$ is said to preserve or admit W_n.

2.2. *Invariant Subspaces of a Quadratic Operator in $\mathbb{R}^2$*

Consider the quadratic operator:

$$A[u] \equiv \alpha(\Delta_2 u)^2 + \gamma u \Delta_2 u + \delta |\nabla u|^2 + \varepsilon u^2.$$

We will look for the invariant subspaces W^{xy}_{n+m-1} and W^{xy}_{nm} of $A[u]$. Note that the operator $A[u]$ is symmetric with respect to x and y; we assume that $n = m$. The cases of $n = 2,\ 3,\ 4,\ 5$ will be considered respectively. In the rest of this paper, the following notations will be used:

$$u_{r0} = \frac{\partial^r u}{\partial x^r},\ u_{0s} = \frac{\partial^s u}{\partial y^s},\ u_{rs} = \frac{\partial^{r+s} u}{\partial x^r \partial y^s},\ r, s = 1, 2, \cdots.$$

2.2.1. The Space W^{xy}_{n+n-1}

We first consider the case of $n = 3$. In this case, we look for the invariant subspaces W^{xy}_{3+3-1} of the operator $A[u]$, which are determined by the following ODEs:

$$L^3_x[v] \equiv \frac{d^3 v}{dx^3} + a_2 \frac{d^2 v}{dx^2} + a_1 \frac{dv}{dx} = 0,\ \ L^3_y[w] \equiv \frac{d^3 w}{dy^3} + b_2 \frac{d^2 w}{dy^2} + b_1 \frac{dw}{dy} = 0. \tag{14}$$

Here and hereafter, a_i, b_i are constants. The invariant conditions take the form:

$$G_1 = L_x^3[A[u]]|_{[H]} \equiv 0, \;\; G_2 = L_y^3[A[u]]|_{[H]} \equiv 0, \;\; \text{and } G_3 = (A[u])_{xy}|_{[H]} \equiv 0, \tag{15}$$

where $[H]$ denotes the set $\{L_x^3[u] = 0\} \cap \{L_y^3[u] = 0\} \cap \{u_{xy} = 0\}$ and their differential consequences with respect to x and y.

Substituting $A[u]$ into (15), we obtain:

$$\begin{aligned}
G_1 =&(-4a_2\delta - 4a_2^3\alpha - 3a_2\gamma + 6a_1\alpha a_2)u_{20}^2\\
&+ (6\varepsilon - 6a_1\gamma - 8a_2^2\alpha a_1 + a_2^2\gamma + 6a_1^2\alpha - 6a_1\delta)u_{10}u_{20} + (2a_2\varepsilon + a_2\gamma a_1 - 4a_2\alpha a_1^2)u_{10}^2,\\
G_2 =&(-4b_2\delta - 4b_2^3\alpha - 3b_2\gamma + 6b_1\alpha b_2)u_{02}^2\\
&+ (6\varepsilon - 6b_1\gamma - 8b_2^2\alpha b_1 + b_2^2\gamma + 6b_1^2\alpha - 6b_1\delta)u_{01}u_{02} + (2b_2\varepsilon - 4b_2\alpha b_1^2 + b_2\gamma b_1)u_{01}^2,\\
G_3 =&2\alpha b_2 a_2 u_{02}u_{20} + (2\alpha b_1 a_2 - \gamma a_2)u_{01}u_{20} + (-\gamma b_2 + 2\alpha b_2 a_1)u_{10}u_{02}\\
&+ (-\gamma b_1 - \gamma a_1 + 2\alpha b_1 a_1 + 2\varepsilon)u_{01}u_{10}.
\end{aligned}$$

In view of the coefficients in G_i $(i = 1,2,3)$, we deduce a system of $a_i, b_i, \alpha, \gamma, \delta$ and ε, which includes ten equations. Solving the resulting system, we arrive at the following results.

Proposition 1. *Assume that the subspaces W_{3+3-1}^{xy} are determined by the system (14). Then, the quadratic operators $A[u]$ in $\mathbb{R}^2$ preserving the invariant subspaces W_{3+3-1}^{xy} determined by $u_{xy} = 0$ and the following constraints are presented as below, where $\alpha, \gamma, \delta, \varepsilon, a_i, b_i (i = 1,2)$ are arbitrary constants.*

(1) $A[u] = \gamma[u\Delta_2 u - |\nabla u|^2]$, *with:*

$$L_x^3[v] = \frac{d^3v}{dx^3} - b_1\frac{dv}{dx} = 0, \quad L_y^3[w] = \frac{d^3w}{dy^3} + b_1\frac{dw}{dy} = 0;$$

(2) $A[u] = \alpha[(\Delta_2 u)^2 - b_2^2|\nabla u|^2]$, *with:*

$$L_x^3[v] = \frac{d^3v}{dx^3} = 0, \quad L_y^3[w] = \frac{d^3w}{dy^3} + b_2\frac{d^2w}{dy^2} = 0;$$

(3) $A[u] = \alpha[(\Delta_2 u)^2 - \frac{8}{9}b_2^2 u\Delta_2 u + \frac{16}{81}b_2^4 u^2]$, *with:*

$$L_x^3[v] = \frac{d^3v}{dx^3} - \frac{4}{9}b_2^2\frac{dv}{dx} = 0, \quad L_y^3[w] = \frac{d^3w}{dy^3} + b_2\frac{d^2w}{dy^2} + \frac{2}{9}b_2^2\frac{dw}{dy} = 0;$$

(4) $A[u] = \gamma[(a_1 + b_1)(\Delta_2 u)^2 + 4a_1 b_1 u\Delta_2 u + (a_1 - b_1)^2|\nabla u|^2 + a_1 b_1(a_1 + b_1)u^2]$, *with:*

$$L_x^3[v] = \frac{d^3v}{dx^3} + a_1\frac{dv}{dx} = 0, \quad L_y^3[w] = \frac{d^3w}{dy^3} + b_1\frac{dw}{dy} = 0;$$

(5) $A[u] = \alpha[(\Delta_2 u)^2 + b_1|\nabla u|^2]$, *with:*

$$L_x^3[v] = \frac{d^3v}{dx^3} = 0, \quad L_y^3[w] = \frac{d^3w}{dy^3} + b_1\frac{dw}{dy} = 0;$$

(6) $A[u] = \alpha(\Delta_2 u)^2 + \gamma u\Delta_2 u + (\gamma b_1 - \alpha b_1^2)u^2$, *with:*

$$L_x^3[v] = \frac{d^3v}{dx^3} + b_1\frac{dv}{dx} = 0, \; L_y^3[w] = \frac{d^3w}{dy^3} + b_1\frac{dw}{dy} = 0;$$

(7) $A[u] = \alpha(\Delta_2 u)^2 + \gamma u \Delta_2 u + \delta|\nabla u|^2$, *with:*

$$L_x^3[v] = \frac{d^3 v}{dx^3} = 0, \quad L_y^3[w] = \frac{d^3 w}{dy^3} = 0;$$

Solving the systems (14) yields the corresponding invariant subspaces. Here, we just present the invariant subspaces in the fourth case. The invariant subspaces for the other cases can be obtained in a similar manner. In the fourth case, we get the following invariant subspaces:

$$W_{3+3-1}^{xy} = \begin{cases} \mathcal{L}\{1, \cos(\sqrt{a_1}x), \sin(\sqrt{a_1}x), \cos(\sqrt{b_1}y), \sin(\sqrt{b_1}y)\}, & a_1 > 0, b_1 > 0, \\ \mathcal{L}\{1, \cos(\sqrt{a_1}x), \sin(\sqrt{a_1}x), \cosh(\sqrt{-b_1}y), \sinh(\sqrt{-b_1}y)\}, & a_1 > 0, b_1 < 0, \\ \mathcal{L}\{1, \cos(\sqrt{a_1}x), \sin(\sqrt{a_1}x), y, y^2)\}, & a_1 > 0, b_1 = 0, \\ \mathcal{L}\{1, \cosh(\sqrt{-a_1}x), \sinh(\sqrt{-a_1}x), \cosh(\sqrt{-b_1}y), \sinh(-\sqrt{-b_1}y)\}, & a_1 < 0, b_1 < 0, \\ \mathcal{L}\{1, \cosh(\sqrt{-a_1}x), \sinh(\sqrt{-a_1}x), y, y^2)\}, & a_1 < 0, b_1 = 0, \\ \mathcal{L}\{1, x, x^2, y, y^2\}, & a_1 = 0, b_1 = 0. \end{cases}$$

In the case of $n = 2$, we assume that the subspace W_{2+2-1}^{xy} is determined by the system:

$$L_x^2[v] \equiv \frac{d^2 v}{dx^2} + a_1 \frac{dv}{dx} = 0, \;\; L_y^2[w] \equiv \frac{d^2 w}{dy^2} + b_1 \frac{dw}{dy} = 0. \tag{16}$$

By the similar calculation, we obtain the following results.

Proposition 2. *Any operators $A[u]$ that admit the subspaces W_{2+2-1}^{xy} determined by the system* (16) *are presented as follows:*

(1) $A[u] = \gamma[(a_1^2 + b_1^2)(\Delta_2 u)^2 - 4a_1^2 b_1^2 u \Delta_2 u - (a_1^2 - b_1^2)^2 |\nabla u|^2 + a_1^2 b_1^2 (a_1^2 + b_1^2) u^2]$, *with:*

$$L_x^2[v] = \frac{d^2 v}{dx^2} + a_1 \frac{dv}{dx} = 0, \quad L_y^2[w] = \frac{d^2 w}{dy^2} + b_1 \frac{dw}{dy} = 0;$$

(2) $A[u] = \alpha[(\Delta_2 u)^2 - b_1^2 |\nabla u|^2]$, *with:*

$$L_x^2[v] = \frac{d^2 v}{dx^2} = 0, \quad L_y^2[w] = \frac{d^2 w}{dy^2} + b_1 \frac{dw}{dy} = 0;$$

(3) $A[u] = \alpha(\Delta_2 u)^2 + \gamma u \Delta_2 u - (\alpha b_1^2 + \gamma) b_1^2 u^2$, *with:*

$$L_x^2[v] = \frac{d^2 v}{dx^2} + b_1 \frac{dv}{dx} = 0, \quad L_y^2[w] = \frac{d^2 w}{dy^2} + b_1 \frac{dw}{dy} = 0;$$

(4) $A[u] = \alpha(\Delta_2 u)^2 + \gamma u \Delta_2 u - (\alpha b_1^2 + \gamma) b_1^2 u^2$, *with:*

$$L_x^2[v] = \frac{d^2 v}{dx^2} - b_1 \frac{dv}{dx} = 0, \quad L_y^2[w] = \frac{d^2 w}{dy^2} + b_1 \frac{dw}{dy} = 0;$$

(5) $A[u] = \alpha(\Delta_2 u)^2 + \gamma u \Delta_2 u + \delta|\nabla u|^2$, *with:*

$$L_x^2[v] = \frac{d^2 v}{dx^2} = 0, \quad L_y^2[w] = \frac{d^2 w}{dy^2} = 0;$$

In the case of $n = 4$, we consider the invariant subspaces W^{xy}_{4+4-1} admitted by the operator $A[u]$, which are determined by the following ODEs:

$$\begin{aligned} L_x^4[v] &\equiv \frac{d^4v}{dx^4} + a_3\frac{d^3v}{dx^3} + a_2\frac{d^2v}{dx^2} + a_1\frac{dv}{dx} = 0, \\ L_y^4[w] &\equiv \frac{d^4w}{dy^4} + b_3\frac{d^3w}{dy^3} + b_2\frac{d^2w}{dy^2} + b_1\frac{dw}{dy} = 0. \end{aligned} \tag{17}$$

By the similar calculation as that in the case of $n = 3$, the invariant condition:

$$(A[u])_{xy}|_{[H]} = 2\alpha u_{03}u_{30} + \gamma u_{10}u_{03} + \gamma u_{01}u_{30} + 2\varepsilon u_{10}u_{01} \equiv 0$$

leads to $\alpha = \gamma = \varepsilon = 0$, where $[H]$ denotes the set $\{L_x^4[u] = 0\} \cap \{L_y^4[u] = 0\} \cap \{u_{xy} = 0\}$, and their differential consequences with respect to x and y. The invariant condition:

$$L_x^4[A[u]]|_{[H]} \equiv 0, \;\; L_y^4[A[u]]|_{[H]} \equiv 0$$

yields $\delta = 0$, which shows that there are no operators $A[u]$ preserving the invariant subspaces determined by (17). Similarly, we are able to show that there are no operators $A[u]$ to preserve the subspace W^{xy}_{5+5-1} defined by the following ODEs:

$$\begin{aligned} L_x^5[v] &\equiv \frac{d^5v}{dx^5} + a_4\frac{d^4v}{dx^4} + a_3\frac{d^3v}{dx^3} + a_2\frac{d^2v}{dx^2} + a_1\frac{dv}{dx} = 0, \\ L_y^5[w] &\equiv \frac{d^5w}{dy^5} + b_4\frac{d^4w}{dy^4} + b_3\frac{d^3w}{dy^3} + b_2\frac{d^2w}{dy^2} + b_1\frac{dw}{dy} = 0. \end{aligned}$$

2.2.2. The Space W^{xy}_{nn}

From the invariant condition (10), a similar calculation as above leads to the following results.

Proposition 3. *There are no operators $A[u]$ admitting the invariant subspaces W^{xy}_{nn} determined by the system:*

$$\begin{aligned} L_x^n[v] &\equiv \frac{d^nv}{dx^n} + a_{n-1}\frac{d^{n-1}v}{dx^{n-1}} + \cdots + a_1\frac{dv}{dx} + a_0v = 0, \\ L_y^n[w] &\equiv \frac{d^nw}{dy^n} + b_{n-1}\frac{d^{n-1}w}{dy^{n-1}} + \cdots + b_1\frac{dw}{dy} + b_0w = 0, \end{aligned} \tag{18}$$

for $n = 3, 4, 5$. The operators $A[u]$, which preserve the invariant subspaces W^{xy}_{22} determined by the system (18) for $n = 2$, are given as follows:

(1) $A[u] = \alpha(\Delta_2u)^2 + \gamma u\Delta_2u - (a_0 + b_0)[\alpha(a_0 + b_0) - \gamma]u^2$, *with:*

$$L_x^2[v] = \frac{d^2v}{dx^2} + a_0v = 0, \;\; L_y^2[w] = \frac{d^2v}{dy^2} + b_0v = 0;$$

(2) $A[u] = \alpha[(\Delta_2u)^2 - b_1^2(2u\Delta_2u + |\nabla u|^2) + 2b_1^4u^2]$, *with:*

$$L_x^2[v] = \frac{d^2v}{dx^2} + b_1\frac{dv}{dx} = 0, \;\; L_y^2[w] = \frac{d^2v}{dy^2} + b_1\frac{dw}{dy} = 0;$$

(3) $A[u] = \alpha[(\Delta_2u)^2 - b_1^2(2u\Delta_2u + |\nabla u|^2) + 2b_1^4u^2]$, *with:*

$$L_x^2[v] = \frac{d^2v}{dx^2} - b_1\frac{dv}{dx} = 0, \;\; L_y^2[w] = \frac{d^2v}{dy^2} + b_1\frac{dw}{dy} = 0.$$

The invariant spaces of the following two nonlinear equations can be constructed in a similar manner.

Example 1. *Consider the Jacobian:*

$$J(u,\Delta u) = u_x\Delta_2 u_y - u_y\Delta_2 u_x \equiv u_x(u_{xxy}+u_{yyy}) - u_y(u_{xxx}+u_{xyy})$$

which is the nonlinear term in two-dimensional Rossby waves equation [34]:

$$\Delta u_t + J(u,\Delta u) + \beta u_x = 0.$$

It preserves the following invariant subspaces:

(1) W^{xy}_{2+2-1}, determined by the system:

$$L^2_x[v] = \frac{d^2v}{dx^2} + a_1\frac{dv}{dx} = 0, \quad L^2_y[w] = \frac{d^2w}{dy^2} + b_1\frac{dw}{dy} = 0, \quad \text{with } a_1b_1(a_1^2-b_1^2)=0;$$

(2) W^{xy}_{3+3-1}, determined by any of the following systems:

$$L^3_x[v] = \frac{d^3v}{dx^3} + a_1\frac{dv}{dx} = 0, \quad L^3_y[w] = \frac{d^3w}{dy^3} + a_1\frac{dw}{dy} = 0;$$
$$L^3_x[v] = \frac{d^3v}{dx^3} - b_2^2\frac{dv}{dx} = 0, \quad L^3_y[w] = \frac{d^3w}{dy^3} + b_2\frac{d^2w}{dy^2} = 0;$$
$$L^3_x[v] = \frac{d^3v}{dx^3} \pm a_2\frac{d^2v}{dx^2} = 0, \quad L^3_y[w] = \frac{d^3w}{dy^3} + a_2\frac{d^2w}{dy^2} = 0;$$

(3) W^{xy}_{4+4-1}, determined by the system:

$$L^4_x[v] = \frac{d^4v}{dx^4} + a_2\frac{d^2v}{dx^2} = 0, \quad L^4_y[w] = \frac{d^4w}{dy^4} + a_2\frac{d^2w}{dy^2} = 0;$$

(4) W^{xy}_{22}, determined by any of the following systems:

$$L^2_x[v] = \frac{d^2v}{dx^2} = 0, \quad L^2_y[w] = \frac{d^2w}{dy^2} + b_1\frac{dw}{dy} = 0;$$
$$L^2_x[v] = \frac{d^2v}{dx^2} + a_0v = 0, \quad L^2_y[w] = \frac{d^2w}{dy^2} + b_0w = 0.$$

Example 2. *The invariant subspaces $W_3 = \mathcal{L}\{1,x^2,y^2\}$ and $W_6 = \mathcal{L}\{1,x^2,y^2,x^2y^2,x^4,y^4\}$ admitted by Monge–Ampère operator $M[u] = u_{xx}u_{yy} - u_{xy}^2$ were given in [1]. Here, we are looking for more invariant subspaces of this operator. Indeed, it still admits the following invariant subspaces:*

(1) W^{xy}_{2+2-1}, *determined by the system:*

$$L^2_x[v] = \frac{d^2v}{dx^2} + a_1\frac{dv}{dx} = 0, \quad L^2_y[w] = \frac{d^2w}{dy^2} + b_1\frac{dw}{dy} = 0, \quad \textit{with } a_1b_1 = 0;$$

(2) W^{xy}_{3+3-1}, *determined by any of the following systems:*

$$L^3_x[v] = \frac{d^3v}{dx^3} = 0, \quad L^3_y[w] = \frac{d^3w}{dy^3} + b_2\frac{d^2w}{dy^2} + b_1\frac{dw}{dy} = 0;$$

(3) W_{22}^{xy}*, determined by any of the following systems:*

$$L_x^2[v] = \frac{d^2v}{dx^2} = 0, \quad L_y^2[w] = \frac{d^2w}{dy^2} + b_1\frac{dw}{dy} + b_0 w = 0, \quad \textit{with } b_0 b_1 = 0;$$
$$L_x^2[v] = \frac{d^2v}{dx^2} + a_0 v = 0, \quad L_y^2[w] = \frac{d^2w}{dy^2} + b_0 w = 0;$$

(4) $W_{33}^{xy} = \mathcal{L}\{1, x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2\}$*, determined by the system:*

$$L_x^3[v] = \frac{d^3v}{dx^3} = 0, \quad L_y^3[w] = \frac{d^3w}{dy^3} = 0.$$

3. Invariant Subspaces under the General Change of Variables

In King's papers [2,12], the formal solution of two-dimensional nonlinear diffusion equations:

$$C_1(t) + C_2(t)x + C_3(t)y + C_4(t)x^2 + C_5(t)xy + C_6(t)y^2 \tag{19}$$

was proposed as a non-group-invariant exact solution, which belongs to the subspace $W_6 = \mathcal{L}\{1, x, y, x^2, xy, y^2\}$. The solution:

$$\begin{aligned} U =& C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)y + C_5(t)y^2 + C_6(t)xy \\ &+ C_7(t)x(x^2+y^2) + C_8(t)y(x^2+y^2) + C_9(t)(x^2+y^2)^2 \end{aligned} \tag{20}$$

of the equation:

$$U_t = U\Delta_2 U - |\nabla U|^2 \equiv J[U], \quad (x,y) \in \mathbb{R}^2, \tag{21}$$

was presented as a generalization of solution (19). The derivation was based on the change of variables. King [2] discovered that Equation (21) was invariant under the following change of variables:

$$U^{(1)} = (x^2+y^2)^{-2}U, \;\; x^{(1)} = \frac{x}{x^2+y^2}, \;\; y^{(1)} = \frac{y}{x^2+y^2}, \;\; t^{(1)} = t, \tag{22}$$

which means that:

$$U_t = J[U] \quad \longrightarrow \quad U^{(1)}_{t^{(1)}} = J[U^{(1)}], \;\; \text{i.e.,} \;\; U_t = (x^2+y^2)^2 J[U^{(1)}].$$

Hence, $J[U] = (x^2+y^2)^2 J[U^{(1)}]$. On the other hand, since the operator $J[U^{(1)}]$ preserves the invariant subspace:

$$\begin{aligned} W_6^{(1)} &= \mathcal{L}\{1, x^{(1)}, (x^{(1)})^2, y^{(1)}, (y^{(1)})^2, x^{(1)}y^{(1)}\} \\ &\equiv \mathcal{L}\left\{1, \frac{x}{x^2+y^2}, \frac{x^2}{(x^2+y^2)^2}, \frac{y}{x^2+y^2}, \frac{y^2}{(x^2+y^2)^2}, \frac{xy}{(x^2+y^2)^2}\right\}, \end{aligned}$$

then the operator $J[U]$ preserves the corresponding subspace:

$$\widehat{W}_6 = \mathcal{L}\{x^2, y^2, xy, x(x^2+y^2), y(x^2+y^2), (x^2+y^2)^2\}.$$

In [1], Galaktionov and Svirshchevskii used the Lie symmetry of Equation (21) to give the invariant transformations of variables as (21). Then, they applied the invariant transformations and invariant subspaces of the corresponding one-dimensional equation of (21), i.e., $U_t = UU_{xx} - U_x^2$, to obtain the invariant subspaces W_{91} and W_{92}. In general, we have the following result.

Proposition 4. *Given a two-dimensional nonlinear differential operator $F[u]$ with respect to the variables x and y, if the nonlinear evolution Equation (6) is invariant under the transformation:*

$$u^{(1)} = r(x,y)u, \quad x^{(1)} = p(x,y), \quad y^{(1)} = q(x,y), \quad t^{(1)} = t, \tag{23}$$

and operator $F[u]$ admits the linear space $W_n = \mathcal{L}\{f_1(x,y), \cdots, f_n(x,y)\}$, then $F[u]$ also admits the linear space:

$$\widehat{W}_n = \mathcal{L}\{f_1(p(x,y), q(x,y))/r(x,y), \cdots, f_n(p(x,y), q(x,y))/r(x,y)\}.$$

Proof: Equation (6) is invariant under the transformation (23), which means $u^{(1)}_{t^{(1)}} = F[u^{(1)}]$. On the other hand, $u^{(1)}_{t^{(1)}} = r(x,y)u_t$. Hence, $F[u^{(1)}] = r(x,y)F[u]$. Assume that:

$$u^{(1)} = \sum_{i=1}^{n} C_i f_i(x^{(1)}, y^{(1)}),$$

where $C_i (i = 1, \cdots, n)$ are arbitrary functions of t. Correspondingly,

$$u = \frac{1}{r(x,y)} \sum_{i=1}^{n} C_i f_i(x^{(1)}, y^{(1)}).$$

$F[u^{(1)}]$ admits the subspace $W_n^{(1)} = \mathcal{L}\{f_1(x^{(1)}, y^{(1)}), \cdots, f_n(x^{(1)}, y^{(1)})\}$, which means that there exist functions $\Psi_i (i = 1, \cdots, n)$, such that:

$$F[u^{(1)}] = F[\sum_{i=1}^{n} C_i f_i(x^{(1)}, y^{(1)})] = \sum_{i=1}^{n} \Psi_i(C_1, \cdots, C_n) f_i(x^{(1)}, y^{(1)}),$$

i.e.,

$$r(x,y)F[u] = r(x,y)F[\frac{1}{r(x,y)} \sum_{i=1}^{n} C_i f_i(x^{(1)}, y^{(1)})] = \sum_{i=1}^{n} \Psi_i(C_1, \cdots, C_n) f_i(x^{(1)}, y^{(1)}).$$

Then, $F[u]$ admits the subspace:

$$\widehat{W}_n = \mathcal{L}\{f_1(p(x,y), q(x,y))/r(x,y), \cdots, f_n(p(x,y), q(x,y))/r(x,y)\}.$$

This completes the proof of the proposition. □

Example 3. *In Proposition 1, we find that the operator $J[U]$ admits the invariant subspaces:*

$$W_{51} = \mathcal{L}\{1, \cos(b_1 x), \sin(b_1 x), \cosh(b_1 y), \sinh(b_1 y)\} \text{ and.}$$

Hence, by the changes of variables (22), the following subspace:

$$\begin{aligned}\widehat{W}^{xy}_{3+3-1} = \mathcal{L}\Big\{&(x^2+y^2)^2, (x^2+y^2)^2\cos(b_1\frac{x}{x^2+y^2}), (x^2+y^2)^2\sin(b_1\frac{x}{x^2+y^2}),\\ &(x^2+y^2)^2\cosh(b_1\frac{y}{x^2+y^2}), (x^2+y^2)^2\sinh(b_1\frac{y}{x^2+y^2})\Big\}\end{aligned}$$

is invariant under $J[U]$.

Note that the transformation (22) is a special one, under which Equation (21) is invariant. We can introduce a general transformation. As for the one-dimensional case [1]; two two-dimensional operators $F[u]$ and $\widetilde{F}[\widetilde{u}]$ are said to be equivalent, if there exists the change of variables:

$$u = r(x,y)\widetilde{u}, \quad \widetilde{x} = p(x,y), \quad \widetilde{y} = q(x,y)$$

such that:

$$\widetilde{F}[\widetilde{u}] = F[u]/r(x,y).$$

It implies that if the operator $F[u]$ preserves the invariant subspace $W_n = \mathcal{L}\{f_1(x,y), \cdots, f_n(x,y)\}$, then the equivalent operator $\widetilde{F}[\widetilde{u}]$ preserves the invariant subspace $\widetilde{W}_n = \mathcal{L}\{\widetilde{f}_1(\widetilde{x},\widetilde{y}), \cdots, \widetilde{f}_n(\widetilde{x},\widetilde{y})\}$, where $\widetilde{f}_i(\widetilde{x},\widetilde{y}) = f_i(x(\widetilde{x},\widetilde{y}), y(\widetilde{x},\widetilde{y}))/r(x(\widetilde{x},\widetilde{y}), y(\widetilde{x},\widetilde{y}))(i = 1, \cdots, n)$.

4. Invariant Subspace in $\mathbb{R}$ and Lie's Classical Symmetries

The Lie theory of the symmetry group plays an important role for differential equations, which is a useful method to explore various properties and obtain exact solutions of nonlinear PDEs. The approach and its several extensions are illustrated in the books [35,36] and the papers [32,33,37,38]. One of the multiple applications of the Lie symmetry method is the similarity reduction of PDEs to ones with fewer variables. As usual, if an n-dimensional PDE admits one symmetry, then it can be reduced to an $n-1$-dimensional PDE equation and even to a ODE. It has been known that the invariant subspaces of one-dimensional differential operator were used to construct solutions of multi-dimensional nonlinear evolution equations of the radially symmetry form, which are one-dimensional evolution equations. For the two-dimensional case, the radially-symmetric solution can be regarded as the rotational-invariant solution. Accordingly, more invariant subspaces of two-dimensional operators can be obtained by combining the Lie symmetry method with the invariant subspaces of one-dimensional operators.

Example 4. *Consider the invariant subspaces preserved by the quadratic operator $J[U]$. The equation:*

$$u_t = \nabla \times (u^{-1}\nabla u) = (u^{-1}u_x)_x + (u^{-1}u_y)_y$$

can be changed into Equation (21) by the transformation $u = 1/U$. Indeed, for $u > 0$, the above equation can be rewritten as:

$$u_t = \triangle \ln u, \tag{24}$$

which is a well-known equation for describing the Ricci flow in a two-dimensional space [39]. Lie's classical symmetries of Equation (24) were computed in [40–45]. Indeed, Equation (24) admits the Lie group of symmetry with infinitesimal generator:

$$X = \xi\partial_x + \eta\partial_y + \tau\partial_t + \phi\partial_u,$$

where $\tau = k_1 + k_2 t, \xi = \xi(x,y), \eta = \eta(x,y)$ and ξ, η and ϕ satisfy the following constraints:

$$\phi = (2k_2 - 2\xi_x)u, \quad \xi_x - \eta_y = 0, \quad \eta_x + \xi_y = 0. \tag{25}$$

Clearly, the function $\xi = \xi(x,y)$ satisfies the two-dimensional Laplace equation:

$$\xi_{xx} + \xi_{yy} = 0.$$

Solving Equation (25), we obtain the following infinitesimal generators admitted by Equation (24):

$$\begin{aligned}
&X_1 = \partial_x + \partial_y, \quad X_2 = y\partial_x - x\partial_y, \quad X_3 = x\partial_x + y\partial_y - 2u\partial_u\\
&X_4 = xy\partial_x + \frac{1}{2}(y^2 - x^2)\partial_y - 2yu\partial_u, \quad X_5 = \frac{1}{2}(x^2 - y^2)\partial_x + xy\partial_y - 2xu\partial_u,\\
&X_6 = \sinh(ax)\sin(ay)\partial_x - \cosh(ax)\cos(ay)\partial_y - 2a\cosh(ax)\sin(ay)u\partial_u,\\
&X_7 = \sinh(ax)\cos(ay)\partial_x + \cosh(ax)\sin(ay)\partial_y - 2a\cosh(ax)\cos(ay)u\partial_u,\\
&X_8 = \sinh(ay)\sin(ax)\partial_x + \cosh(ay)\cos(ax)\partial_y - 2a\sinh(ay)\cos(ax)u\partial_u,\\
&X_9 = \sinh(ay)\cos(ax)\partial_x - \cosh(ay)\sin(ax)\partial_y + 2a\sinh(ay)\sin(ax)u\partial_u, \quad \text{etc.}
\end{aligned}$$

Here, a is a non-zero arbitrary constant. On the other hand, the corresponding infinitesimal generators admitted by the Equation (21) can be obtained by the transformation $u = 1/U$, i.e.,

$$u \longrightarrow \frac{1}{U}, \ \ \partial_u \longrightarrow -U^2\partial_U,$$

which reduce Equation (21) to one-dimensional equations. We denote them by $\widetilde{X}_i$ $(i = 1, \cdots, 9)$.

(1) $\widetilde{X}_1$. For $\widetilde{X}_1$, its invariants are $\widetilde{U} = U$ and $z = x + y$. The corresponding invariant solutions of (21) are $U = \widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies:

$$\widetilde{U}_t = 2(\widetilde{U}\widetilde{U}_{zz} - \widetilde{U}_z^2) \equiv \widetilde{J}^1[\widetilde{U}].$$

(2) $\widetilde{X}_2$. For $\widetilde{X}_2$, its invariants are $\widetilde{U} = U$ and $z = x^2 + y^2$. The corresponding invariant solutions of (21) are $v = \widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies:

$$\widetilde{U}_t = 4z\widetilde{U}\widetilde{U}_{zz} - 4z\widetilde{U}_z^2 + 4\widetilde{U}\widetilde{U}_z \equiv \widetilde{J}^2[\widetilde{U}].$$

(3) $\widetilde{X}_3$. For $\widetilde{X}_3$, its invariants are $\widetilde{U} = Ux^{-2}$ and $z = y/x$. The corresponding invariant solutions of (21) are $U = x^2\widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies:

$$\widetilde{U}_t = (1+z^2)\widetilde{U}\widetilde{U}_{zz} - (1+z^2)\widetilde{U}_z^2 + 2z\widetilde{U}\widetilde{U}_z - 2\widetilde{U}^2 \equiv \widetilde{J}^3[\widetilde{U}].$$

(4) $\widetilde{X}_4$. For $\widetilde{X}_4$, its invariants are $\widetilde{U} = vx^{-2}$ and $z = x + y^2/x$. The corresponding invariant solutions of (21) are $U = x^2\widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies:

$$\widetilde{U}_t = z^2\widetilde{U}\widetilde{U}_{zz} - z^2\widetilde{U}_z^2 + 2z\widetilde{U}\widetilde{U}_z - 2\widetilde{U}^2 \equiv \widetilde{J}^4[\widetilde{U}].$$

(5) $\widetilde{X}_5$. For $\widetilde{X}_5$, its invariants are $\widetilde{U} = y^{-2}U$ and $z = y + x^2/y$. The invariant solutions of (21) are $U = y^2\widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies $\widetilde{U}_t = \widetilde{J}^4[\widetilde{U}]$.

(6) $\widetilde{X}_6$. For $\widetilde{X}_6$, its invariants are $\widetilde{U} = \sinh^{-2}(ax)U$ and $z = \cos(ay)/\sinh(ax)$. The invariant solutions of (21) are $U = \sinh^2(ax)\widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies $\widetilde{U}_t = a^2\widetilde{J}^3[\widetilde{U}]$.

(7) $\widetilde{X}_7$. For $\widetilde{X}_7$, its invariants are $\widetilde{U} = \sinh^{-2}(ax)U$ and $z = \sin(ay)/\sinh(ax)$. The invariant solutions of (21) are $U = \sinh^2(ax)\widetilde{v}(z,t)$, where $\widetilde{U}(z,t)$ satisfies $\widetilde{U}_t = a^2\widetilde{J}^3[\widetilde{U}]$.

(8) $\widetilde{X}_8$. For $\widetilde{X}_8$, its invariants are $\widetilde{U} = \cosh^{-2}(ay)U$ and $z = \sin(ax)/\cosh(ay)$. The invariant solutions of (21) are $U = \cosh^2(ay)\widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies:

$$\widetilde{U}_t = a^2(1-z^2)\widetilde{U}\widetilde{U}_{zz} + a^2(z^2-1)\widetilde{U}_z^2 - 2a^2z\widetilde{U}\widetilde{U}_z + 2a^2\widetilde{U}^2 \equiv \widetilde{J}^5[\widetilde{U}].$$

(9) $\widetilde{X}_9$. For $\widetilde{X}_9$, its invariants are $\widetilde{U} = U\cosh^{-2}(ay)$ and $z = \cos(ax)/\cosh(ay)$. The invariant solutions of (21) are $U = \cosh^2(ay)\widetilde{U}(z,t)$, where $\widetilde{U}(z,t)$ satisfies $\widetilde{U}_t = \widetilde{J}^5[\widetilde{U}]$.

Using the invariant subspace method for the one-dimensional case, we find that the nonlinear operators $\widetilde{J}^i[\widetilde{U}] (i = 1, \cdots, 5)$ only admit two- and three-dimensional subspaces determined by spaces of solutions of linear ODEs as:

$$\frac{d^n w}{dz^n} + b_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + b_0(z)w = 0.$$

We concentrate on the three-dimensional invariant subspaces, which are listed as below:

(1) The operator $\widetilde{J}^1[\widetilde{U}]$ admits the invariant subspaces:

$$\widetilde{W}_3 = \begin{cases} \mathcal{L}\{1, z, z^2\}, & b = 0, \\ \mathcal{L}\{1, \cos(cz), \sin(cz)\}, & b = c^2, \\ \mathcal{L}\{1, \exp(cz), \exp(-cz)\}, & b = -c^2, \end{cases}$$

determined by the spaces of solutions of the ODE:

$$\frac{d^3w}{dz^3} + b\frac{dw}{dz} = 0.$$

(2) The operator $\widetilde{J}^2[\widetilde{U}]$ admits the invariant subspaces:

$$W_3 = \begin{cases} \mathcal{L}\{z, z\ln z, z(\ln z)^2\}, & b = -1, \\ \mathcal{L}\{z, z^{1-c}, z^{1+c}\}, & b = -1 + c^2, \\ \mathcal{L}\{z, z\sin(c\ln z), z\cos(c\ln z)\}, & b = -1 - c^2, \end{cases}$$

determined by the spaces of solutions of the ODE:

$$\frac{d^3w}{dz^3} + \frac{b}{z^2}\frac{dw}{dz} - \frac{b}{z^3}w = 0.$$

(3) The operator $\widetilde{J}^3[\widetilde{U}]$ admits the invariant subspaces:

$$\widetilde{W}_3 = \begin{cases} \mathcal{L}\{(1+z^2), (1+z^2)\arctan z, (1+z^2)(\arctan z)^2\}, & b = -4, \\ \mathcal{L}\{(1+z^2), (1+z^2)\sin(c\arctan z), (1+z^2)\cos(c\arctan z)\}, & b = -4 + c^2, \\ \mathcal{L}\{(1+z^2), (1+z^2)\exp(c\arctan z), (1+z^2)\exp(-c\arctan z)\}, & b = -4 - c^2, \\ \mathcal{L}\{1, z, z^2\}, & b = 0, \end{cases}$$

determined by the spaces of solutions of the ODE:

$$\frac{d^3w}{dz^3} + \frac{b}{(1+z^2)^2}\frac{dw}{dz} - \frac{2bz}{(1+z^2)^3}w = 0.$$

(4) The operator $\widetilde{J}^4[\widetilde{U}]$ admits the invariant subspaces:

$$\widetilde{W}_3 = \begin{cases} \mathcal{L}\{z^2, z^2\exp(-\frac{c}{z}), z^2\exp(\frac{c}{z})\}, & b = 2c^2, \\ \mathcal{L}\{z^2, z^2\sin(\frac{c}{z}), z^2\cos(\frac{c}{z})\}, & b = -2c^2, \\ \mathcal{L}\{1, z, z^2\}, & b = 0, \end{cases}$$

determined by the spaces of solutions of the ODE:

$$\frac{d^3w}{dz^3} - \frac{b}{2z^4}\frac{dw}{dz} + \frac{b}{z^5}w = 0.$$

(5) The operator $\widetilde{J}^5[\widetilde{U}]$ admits the invariant subspaces:

$$\widetilde{W}_3 = \begin{cases} \mathcal{L}\{(z^2-1), (z^2-1)\ln(\frac{z+1}{z-1}), (z^2-1)(\ln(\frac{z+1}{z-1}))^2\}, & b = 8, \\ \mathcal{L}\{(z^2-1), (z^2-1)\exp(c\mathrm{arctanhz}), (z^2-1)\exp(-c\mathrm{arctanhz})\}, & b = -8 + 8c^2, \\ \mathcal{L}\{(z^2-1), (z^2-1)\sin(c\mathrm{arctanhz}), (z^2-1)\cos(c\mathrm{arctanhz})\}, & b = -8 - 8c^2, \\ \mathcal{L}\{1, z, z^2\}, & b = 0, \end{cases}$$

determined by the spaces of solutions of the ODE:

$$\frac{d^3w}{dz^3} - \frac{b}{2(z^2-1)^2}\frac{dw}{dz} + \frac{bz}{(z^2-1)^3}w = 0,$$

where and hereafter b is an arbitrary constant, and c is a non-zero arbitrary constant.

Then, we can obtain the corresponding invariant subspaces preserved by $J[U]$, which are presented as below:

$$\begin{aligned}
W_3 &= \mathcal{L}\{1, x+y, (x+y)^2\},\\
W_3 &= \mathcal{L}\{1, \cos(c(x+y)), \sin(c(x+y))\},\\
W_3 &= \mathcal{L}\{1, \cosh(c(x+y)), \sinh(c(x+y))\},\\
W_3 &= \mathcal{L}\{x^2+y^2, (x^2+y^2)\ln(x^2+y^2), (x^2+y^2)(\ln(x^2+y^2))^2\},\\
W_3 &= \mathcal{L}\{x^2+y^2, (x^2+y^2)^{1-c}, (x^2+y^2)^{1+c}\},\\
W_3 &= \mathcal{L}\{x^2+y^2, (x^2+y^2)\sin(c\ln(x^2+y^2)), (x^2+y^2)\cos(c\ln(x^2+y^2))\},\\
W_3 &= \mathcal{L}\{x^2+y^2, (x^2+y^2)\arctan(\frac{y}{x}), (x^2+y^2)(\arctan(\frac{y}{x}))^2\},\\
W_3 &= \mathcal{L}\{x^2+y^2, (x^2+y^2)\sin(c\arctan(\frac{y}{x})), (x^2+y^2)\cos(c\arctan(\frac{y}{x}))\},\\
W_3 &= \mathcal{L}\{x^2+y^2, (x^2+y^2)\cosh(c\arctan(\frac{y}{x})), (x^2+y^2)\sinh(c\arctan(\frac{y}{x}))\},\\
W_3 &= \mathcal{L}\{x^2, xy, y^2\},\\
W_3 &= \mathcal{L}\{(x^2+y^2)^2, (x^2+y^2)^2\cosh(\frac{cx}{x^2+y^2}), (x^2+y^2)^2\sinh(\frac{cx}{x^2+y^2})\},\\
W_3 &= \mathcal{L}\{(x^2+y^2)^2, (x^2+y^2)^2\sin(\frac{cx}{x^2+y^2}), (x^2+y^2)^2\cos(\frac{cx}{x^2+y^2})\},\\
W_3 &= \mathcal{L}\{x^2, x(x^2+y^2), (x^2+y^2)^2\},\\
W_3 &= \mathcal{L}\{(\cos^2 ay+\sinh^2 ax), (\cos^2 ay+\sinh^2 ax)\arctan\frac{\cos ay}{\sinh ax},\\
&\qquad (\cos^2 ay+\sinh^2 ax)(\arctan\frac{\cos ay}{\sinh ax})^2\},\\
W_3 &= \mathcal{L}\{(\cos^2 ay+\sinh^2 ax), (\cos^2 ay+\sinh^2 ax)\sin(c\arctan\frac{\cos ay}{\sinh ax}),\\
&\qquad (\cos^2 ay+\sinh^2 ax)\cos(c\arctan\frac{\cos ay}{\sinh ax})\},\\
W_3 &= \mathcal{L}\{(\cos^2 ay+\sinh^2 ax), (\cos^2 ay+\sinh^2 ax)\cosh(c\arctan\frac{\cos ay}{\sinh ax}),\\
&\qquad (\cos^2 ay+\sinh^2 ax)\sinh(c\arctan\frac{\cos ay}{\sinh ax})\},\\
W_3 &= \mathcal{L}\{\sinh^2 ax, \cos ay\sinh ax, \cos^2 ay\},\\
W_3 &= \mathcal{L}\{(\sin^2 ax-\cosh^2 ay), (\sin^2 ax-\cosh^2 ay)\ln\frac{\sin ax+\cosh ay}{\sin ax-\cosh ay},\\
&\qquad (\sin^2 ax-\cosh^2 ay)(\ln\frac{\sin ax+\cosh ay}{\sin ax-\cosh ay})^2\},
\end{aligned}$$

$$W_3 = \mathcal{L}\{(\sin^2 ax - \cosh^2 ay), (\sin^2 ax - \cosh^2 ay)\sin(\mathrm{carctanh}\frac{\sin ax}{\cosh ay})$$
$$(\sin^2 ax - \cosh^2 ay)\cos(\mathrm{carctanh}\frac{\sin ax}{\cosh ay})\},$$
$$W_3 = \mathcal{L}\{(\sin^2 ax - \cosh^2 ay), (\sin^2 ax - \cosh^2 ay)\cosh(\mathrm{carctanh}\frac{\sin ax}{\cosh ay}),$$
$$(\sin^2 ax - \cosh^2 ay)\sinh(\mathrm{carctanh}\frac{\sin ax}{\cosh ay})\}.$$

Since the operator $J[U]$ is symmetric with respect to the variables x and y, the following invariant subspaces can also be obtained from the above invariant subspaces:

$$W_3 = \mathcal{L}\{\cos^2(ax), \sinh(ay)\cos(ax), \sinh^2(ay)\},$$
$$W_3 = \mathcal{L}\{\sin^2(ay), \cosh(ax)\sin(ay), \cosh^2(ax)\}.$$

Example 5. *Consider the two-dimensional porous medium equation:*

$$u_t = (u^p u_x)_x + (u^p u_y)_y, \quad p \neq 0, -1, \tag{26}$$

which can be changed into the equation:

$$U_t = U(U_{xx} + U_{yy}) + \frac{1}{p}(U_x^2 + U_y^2) \equiv J_p[U] \tag{27}$$

by the transformation $U = u^p$. Equation (27) admits the scaling invariance with the infinitesimal generator:

$$\widetilde{X}_3 = x\partial_x + y\partial_y + 2U\partial_U, \tag{28}$$

which possesses the invariants:

$$\widetilde{U} = \frac{1}{x^2}U(z,t), \quad z = \frac{y}{x}, \quad \widetilde{t} = t.$$

Under the Lie symmetry $\widetilde{X}_3$, this equation is reduced to:

$$\widetilde{U}_t = (1+z^2)\widetilde{U}\widetilde{U}_{zz} + \frac{1}{p}(1+z^2)\widetilde{U}_z^2 - \frac{2(p+2)}{p}z\widetilde{U}\widetilde{U}_z + \frac{2(p+2)}{p}\widetilde{U}^2 \equiv \widetilde{J}_p[\widetilde{U}].$$

The operator $\widetilde{J}_p[\widetilde{U}]$ admits invariant subspace $W_3 = \mathcal{L}\{1, z, z^2\}$ determined by ODE $d^3w/dz^3 = 0$. Hence, the operator $J_p[U]$ admits the invariant subspaces $W_3 = \mathcal{L}\{x^2, xy, y^2\}$. On the other hand, for $p = -4/3$, the operator $\widetilde{J}_p[\widetilde{U}]$ admits another invariant subspace $W_3 = \mathcal{L}\{1, z^2+1, \sqrt{z^2+1}\}$ determined by the ODE:

$$\frac{d^3w}{dz^3} - \frac{3}{z(z^2+1)}\frac{d^2w}{dz^2} + \frac{3}{z^2(1+z^2)}\frac{dw}{dz} = 0.$$

Therefore, the corresponding invariant subspace admitted by the operator $J_{-\frac{4}{3}}[U]$ is:

$$W_3 = \mathcal{L}\{x^2, x^2+y^2, x\sqrt{x^2+y^2}\} \equiv \mathcal{L}\{x^2, y^2, x\sqrt{x^2+y^2}\}.$$

Accordingly, some invariant subspaces of $J[U]$ can be obtained from the invariant subspace $\mathcal{L}\{1, z, z^2\}$ admitted by the operator $\widetilde{J}^i[\widetilde{U}]$, which is the polynomial subspace. The polynomial subspaces of nonlinear operators are studied in many papers, which were used to construct exact solutions of nonlinear evolution equations, including porous medium equations, thin film equations and Euler equations [1–3,12–14,28,29,46–49]. Using the Lie symmetry method, we may obtain polynomial invariant subspaces of some two-dimensional nonlinear operators. Note that in Examples 4 and 5, the invariant subspace $W_3 = \mathcal{L}\{x^2, xy, y^2\}$ can be obtained from the one-dimensional invariant subspace $\widetilde{W}_3 = \mathcal{L}\{1, z, z^2\}$ and the Lie group of symmetry (28). The subspace $\mathcal{L}\{1, z, z^2\}$ is determined by the space of solutions of linear ODE $\widetilde{U}_{zzz} = 0$, which can be explained by the conditional Lie–Bäcklund symmetry with character $\widetilde{U}_{zzz}$ [1,32,33]). Besides those, the nonlinear evolution equation $U_t = (UU_x)_y$ also admits the Lie group of transformation with the infinitesimal generator (28). By the similar calculations as above, we find that the operator $(UU_x)_y$ admits the invariant subspace $\mathcal{L}\{x^2, xy, y^2\}$. In [10], the operators preserving a given invariant subspace were discussed, for instance the space $\mathcal{M} = \{x^2, xy, y^2\}$, which was regarded as a "simple" problem for the affine annihilator.

Example 6. *Consider the evolution Monge–Ampère equation:*

$$u_t = u_{xx}u_{yy} - u_{xy}^2. \tag{29}$$

It is easy to verify that this equation admits the Lie groups of transformations with infinitesimal operators:

$$X_1 = y\partial_x \pm x\partial_y, \quad X_2 = y\partial_x \pm \frac{1}{2}\partial_y, \quad X_3 = x\partial_y \pm \frac{1}{2}\partial_x.$$

We find that X_1 has invariants $\widetilde{u} = u(z, t)$, $z = x^2 \pm y^2$ and $\widetilde{t} = t$. With respect to this Lie symmetry, Equation (29) is reduced to:

$$\widetilde{u}_t = \pm(-8z\widetilde{u}_z\widetilde{u}_{zz} + 4\widetilde{u}_z^2) \equiv \widetilde{M}_\pm[\widetilde{u}].$$

The operator $\widetilde{M}_\pm[\widetilde{u}]$ admits the invariant subspace $\widetilde{W}_3 = \mathcal{L}\{1, \sqrt{z}, z^2\}$ determined by the ODE:

$$\frac{d^3w}{dz^3} + \frac{1}{2z}\frac{d^2w}{dz^2} - \frac{1}{2z^2}\frac{dw}{dz} = 0,$$

and the invariant subspace $\widetilde{W}_3 = \mathcal{L}\{1, z, z^2\}$ determined by ODE $d^3w/dz^3 = 0$. Hence, the Monge–Ampère operator $M[u] = u_{xx}u_{yy} - u_{xy}^2$ admits the invariant subspaces:

$$W_3 = \mathcal{L}\{1, \sqrt{x^2 \pm y^2}, (x^2 \pm y^2)^2\}, \text{ and } W_3 = \mathcal{L}\{1, x^2 \pm y^2, (x^2 \pm y^2)^2\}$$

Similarly, under the Lie symmetries $X_{2,3}$, we obtain the following invariant subspaces preserved by the Monge–Ampère operator:

$$W_3 = \mathcal{L}\{1, (x \pm y^2)^{\frac{3}{2}}, (x \pm y^2)^3\}, \quad W_4 = \mathcal{L}\{1, (x \pm y^2), (x \pm y^2)^2, (x \pm y^2)^3\},$$

In general, assume that nonlinear evolution Equation (6) admits the Lie group of transformation with infinitesimal generator X, which has invariants:

$$z = p(x, y), \quad \widetilde{u} = \frac{u}{r(x, y)}, \quad \widetilde{t} = t,$$

and reduces it to the one-dimensional nonlinear evolution equation:

$$\widetilde{u}_t = \widetilde{F}(z, \widetilde{u}, \widetilde{u}_z, \widetilde{u}_{zz}, \cdots) \equiv \widetilde{F}[\widetilde{u}].$$

We then obtain the following proposition.

Proposition 5. *If the nonlinear differential operator $\widetilde{F}$ admits the invariant subspaces $\widetilde{W}_n = \mathcal{L}\{f_1(z), \cdots, f_n(z)\}$, then two-dimensional nonlinear differential operator F preserves the invariant subspaces $W_n = \mathcal{L}\{r(x, y)f_1(p(x, y)), \cdots, r(x, y)f_n(p(x, y))\}$.*

The proof is similar to that of Proposition 4. Clearly, in this approach, the estimate on the dimension of invariant subspace obeys Theorem 1.

5. Concluding Remarks

In this paper, several approaches are developed to obtain invariant subspaces of the two-dimensional nonlinear operators, including two direct extensions to the invariant subspace method in $\mathbb{R}$, the method of the general change of variables and the one-dimensional invariant subspace method combined with the Lie symmetry method. In particular, we find that the subspaces W_{nm}^{xy} and W_{n+m-1}^{xy} of the two-dimensional nonlinear differential operators are extensions of the invariant subspaces for one-dimensional nonlinear differential operators, which are determined by the spaces of solutions of ODEs completely. In $\mathbb{R}^2$, the invariant subspaces admitted by the quadratic operator $A[u]$ and their applications are considered. In general, the extensions of the concept of invariant subspaces in $\mathbb{R}^N$ could be introduced. Assume that $\{f_{j1}(x_j), \cdots, f_{jm_j}(x_j)\}$ is a finite set of linearly independent functions, and $W_{m_j}^{x_j}$ denotes their linear span $W_{m_j}^{x_j} = \mathcal{L}\{f_{j1}(x_j), \cdots, f_{jm_j}(x_j)\}$, where $j = 1, \cdots, N$. The $(m_1 \cdot \cdots \cdot m_N)$-dimensional subspace:

$$\widetilde{W} = \left\{ \sum_{i_1, \cdots, i_N} C_{i_1 \cdots i_N} f_{1i_1}(x_1) \cdot \cdots \cdot f_{Ni_N}(x_N), \forall C_{i_1 \cdots i_N} \in \mathbb{R}, i_j = 1, \cdots, m_j, j = 1, \cdots, N \right\}$$

can be introduced as an extension to the subspace W_{nm}^{xy} in $\mathbb{R}^N$. Consider the N-dimensional nonlinear operator:

$$F[u] \equiv F(u, Du, D^2u, \cdots, D^k u),$$

where $Du = (u_{x_1}, \cdots, u_{x_N})$, $D^2u = (u_{x_1x_1}, \cdots, u_{x_1x_N}, u_{x_2x_2}, \cdots, u_{x_2x_N}, \cdots, u_{x_Nx_N})$, etc. Assume that the subspace $W_{m_j}^{x_j}$ is the space of solutions of the ODE:

$$L_{x_j}[v_j] \equiv \frac{d^{m_j} v_j}{dx_j^{m_j}} + a_{jm_j-1}(x_j) \frac{d^{m_j-1} v_j}{dx_j^{m_j-1}} + \cdots + a_{j0}(x_j) v_j = 0, \quad j = 1, \cdots, N.$$

Then, the invariance condition of the subspace $\widetilde{W}$ preserved by the operator $F[u]$ (i.e., $F[\widetilde{W}] \subseteq \widetilde{W}$) is:

$$L_{x_j}[F[u]]|_{[\widetilde{H}]} \equiv 0, \quad j = 1, \cdots, N,$$

where $[\widetilde{H}]$ denotes $L_{x_j}[u] = 0$, and their differential consequences with respect to x_i, $i, j = 1, \cdots, N$. Similarly, we assume that $\{1, f_{j1}(x_j), \cdots, f_{jm_j}(x_j)\}$ is a set of basis of solutions of the ODE system:

$$\overline{L}_{x_j}[v_j] \equiv \frac{d^{m_j} v_j}{dx_j^{m_j}} + a_{jm_j-1}(x_j) \frac{d^{m_j-1} v_j}{dx_j^{m_j-1}} + \cdots + a_{j1}(x_j) \frac{dv_j}{dx_j} = 0, \quad j = 1, \cdots, N.$$

Let $\overline{W}_{m_j}^{x_j} = \mathcal{L}\{1, f_{j1}(x_j), \cdots, f_{jm_j}(x_j)\}$ denote the space of solutions of this ODE, where $j = 1, \cdots, N$. We can introduce the $(m_1 + \cdots + m_N - N + 1)$-dimensional subspace:

$$\overline{W} = \mathcal{L}\{1, f_{12}(x_1), \cdots, f_{1m_1}(x_1), \cdots, f_{N2}(x_N), \cdots, f_{Nm_N}(x_N)\}$$

as an extension of W_{n+m-1}^{xy} in $\mathbb{R}^N$. Then, the invariance condition of the subspace $\overline{W}$ preserved by the operator $F[u]$ (i.e., $F[\overline{W}] \subseteq \overline{W}$) is:

$$\overline{L}_{x_j}[F[u]]|_{[\bar{H}]} \equiv 0, \quad (F[u])_{x_ix_j}|_{[\bar{H}]} \equiv 0,$$

where $[\bar{H}]$ denotes $\overline{L}_{x_j}[u] = 0$, $u_{x_i x_j} = 0$, and their differential consequences with respect to x_i, $i, j = 1, \cdots, N, i \neq j$. The invariant subspaces obtained by this method can be regarded as original subspaces and used to obtain new ones by the general changes of variables in Section 3.

To obtain more invariant subspaces of nonlinear differential operators, we adopt the direct sum of invariant subspaces, which was used by Galaktionov and Svirshchevskii [1] to obtain the new invariant subspaces preserved by a given operator. For example, in Proposition 6.1 of [1], it was shown that the direct sum of the subspaces $W_n^q = \mathcal{L}\{1, x_i x_j, 1 \leqslant i \leqslant j \leqslant N\}$ and $W_N^{\text{lin}} = \mathcal{L}\{x_1, \cdots, x_N\}$ is preserved by the operator $K[u]$. It is expected that the formulation of the direct sum can be used to obtain the invariant subspaces W_{91} and W_{92} of $J[U]$ by them. Indeed, the following result is always true.

Proposition 6. *Given a nonlinear differential operator F. If the linear subspaces W_n and W_m are preserved by the operator F and $W_n \cap W_m = \{0\}$, then the direct sum of W_n and W_m, i.e., $W_n \oplus W_m$ is invariant or partially invariant under the operator F.*

Clearly, for the "nonlinear property" of the nonlinear operator, $F[W_n \oplus W_m] \subseteq W_n \oplus W_m$ is not always true. However, in the case of $F[W_n \oplus W_m] \nsubseteq W_n \oplus W_m$, it is said to be partially invariant under the operator F (see [1]). The linear space W_n is partially invariant under the operator F, i.e., $F[W_n] \nsubseteq W_n$, but for some part M of W_n, $F[M] \subseteq W_n$. If the subspace W_n is partially invariant under a given operator, then the corresponding evolution equation can be reduced to an over-determined system of ODEs. One can verify whether the direct sum of two invariant subspaces is invariant under the given operator by a direct computation.

The following result is a further extension to Proposition 6.

Proposition 7. *Let F be a given nonlinear differential operator. If the linear subspaces $W_{n_1}^1, \cdots, W_{n_m}^m$ are preserved by the operator F, then the subspace $W_{n_1}^1 \cup \cdots \cup W_{n_m}^m$ is invariant or partially invariant under the operator F.*

Let us return to the invariant subspaces W_{91} and W_{92}. We can express:

$$W_{91} = W_3^1 \cup W_3^2 \cup W_3^3 \cup W_3^4,$$

where:

$$\begin{aligned} &W_3^1 = \mathcal{L}\{1, x, y\}, \quad W_3^2 = \mathcal{L}\{x^2, xy, y^2\}, \\ &W_3^3 = \mathcal{L}\{x^2, x(x^2+y^2), (x^2+y^2)^2\}, \quad W_3^4 = \mathcal{L}\{y^2, y(x^2+y^2), (x^2+y^2)^2\}. \end{aligned}$$

and express:

$$W_{92} = W_5^1 \cup W_3^2 \cup W_3^3 \cup W_3^4 \cup W_3^5,$$

where:

$$\begin{aligned} &W_5^1 = \mathcal{L}\{1, \cos 2y, \sin 2y, \cosh 2x, \sinh 2x\}, \\ &W_3^2 = \mathcal{L}\{\cos^2 y, \cosh x \cos y, \cosh^2 x\}, \quad W_3^3 = \mathcal{L}\{\cos^2 y, \sinh x \cos y, \sinh^2 x\}, \\ &W_3^4 = \mathcal{L}\{\sin^2 y, \cosh x \sin y, \cosh^2 x\}, \quad W_3^5 = \mathcal{L}\{\sin^2 y, \sinh x \sin y, \sinh^2 x\}. \end{aligned}$$

Note that $2\cos^2 y - 1 = -2\sin^2 y + 1 = \cos 2y$, $\sinh^2 x = (\sinh 2x - 1)/2$, $\cosh^2 x = (\cosh 2x + 1)/2$, and every component of W_{91} and W_{92} can be obtained by the knowledge of algebra and ODEs (see Sections 2 and 4). The following invariant subspace $\widehat{\widehat{W}}{}_{3+3-1}^{xy}$ of $J[U]$ can be obtained from $\widehat{W}_{3+3-1}^{xy}$ by the discrete symmetry $x \to y$, $y \to x$. Indeed, we have:

(1) $\widehat{W}^{xy}_{3+3-1} = W_3^1 \cup W_3^2$, with:

$$W_3^1 = \mathcal{L}\{(x^2+y^2)^2, (x^2+y^2)^2\cos(b_1\frac{x}{x^2+y^2}), (x^2+y^2)^2\sin(b_1\frac{x}{x^2+y^2})\},$$
$$W_3^2 = \mathcal{L}\{(x^2+y^2)^2, (x^2+y^2)^2\exp(b_1\frac{y}{x^2+y^2}), (x^2+y^2)^2\exp(-b_1\frac{y}{x^2+y^2})\};$$

(2) $\widehat{\widehat{W}}^{xy}_{3+3-1} = W_3^3 \cup W_3^4$, with:

$$W_3^3 = \mathcal{L}\{(x^2+y^2)^2, (x^2+y^2)^2\cos(b_1\frac{y}{x^2+y^2}), (x^2+y^2)^2\sin(b_1\frac{y}{x^2+y^2})\},$$
$$W_3^4 = \mathcal{L}\{(x^2+y^2)^2, (x^2+y^2)^2\exp(b_1\frac{x}{x^2+y^2}), (x^2+y^2)^2\exp(-b_1\frac{x}{x^2+y^2})\}.$$

Here, $W_3^i (i = 1, \cdots, 4)$ can be obtained by the method in Section 4. Similarly, we can check that both the operator $(UU_x)_y$ and $J_p[U]$ admit the invariant subspace $\mathcal{L}\{1, x, y, x^2, xy, y^2\} = \mathcal{L}\{1, x, y\} \cup \mathcal{L}\{x^2, xy, y^2\}$. Hence, the porous medium Equation (26) has the exact solution of the more general form:

$$u = (c_1(t) + c_2(t)x + c_3(t)y + c_4(t)x^2 + c_5(t)xy + c_6(t)y^2)^{\frac{1}{p}}.$$

On the other hand, it was shown that the operator $J_{-\frac{4}{3}}[U]$ admits the following invariant subspaces (see Example 5):

$$W_3^1 = \mathcal{L}\{x^2, xy, y^2\}, \quad W_3^2 = \mathcal{L}\{x^2, y^2, x\sqrt{x^2+y^2}\}, \quad W_3^3 = \mathcal{L}\{x^2, y^2, y\sqrt{x^2+y^2}\}$$

By direct calculation, one can check that the operator $J_{-\frac{4}{3}}[U]$ admits another invariant subspace:

$$W_5 = \mathcal{L}\{x^2, xy, y^2, x\sqrt{x^2+y^2}, y\sqrt{x^2+y^2}\} = W_3^1 \cup W_3^2 \cup W_3^3.$$

Hence, for $p = -4/3$, the porous medium Equation (26) has another solution of the form:

$$u = (c_1(t)x^2 + c_2(t)xy + c_3(t)y^2 + c_4(t)x\sqrt{x^2+y^2} + c_5(t)y\sqrt{x^2+y^2})^{-\frac{3}{4}}.$$

Finally, we would like to address some open questions. Firstly, although we have several operable approaches to obtain the invariant subspaces of two-dimensional nonlinear operators, we do not have a systematic approach to obtain the invariant subspaces of $J[U]$ as W_{91} and W_{92} and those of the Monge–Ampère operator as:

$$W_{13} = \mathcal{L}\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^2y^2, y^4\}.$$

Secondly, as mentioned in the Introduction, what is the maximal dimension of the certain types of invariant subspaces of multi-dimensional k-th order nonlinear differential operators? All of these questions will be the content of our future research.

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