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Continuity of Fuzzified Functions Using the Generalized Extension Principle

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Abstract: To fuzzify the crisp functions, the extension principle has been widely used for performing this fuzzification. The purpose of this paper is to investigate the continuity of fuzzified function using the more generalized extension principle. The Hausdorff metric will be invoked to study the continuity of fuzzified function. We also apply the principle of continuity of fuzzified function to the fuzzy topological vector space.

Keywords: Hausdorff metric; normed space; normable topological vector space; generalized *t*-norm; generalized extension principle

1. Introduction

Let U be a universal set. A fuzzy subset $\tilde{A}$ of U is defined as a set of ordered pairs:

$$\tilde{A} = \{(x, \xi_{\tilde{A}}(x)) : x \in U\},$$

where $\xi_{\tilde{A}} : U \to [0,1]$ is called the *membership function* of $\tilde{A}$. The set of all fuzzy subsets of U is denoted by $\mathcal{F}(U)$.

We consider an onto function $f : U \to V$, where V is another universal set. This function is also called as a *crisp function*. The purpose is to fuzzify the crisp function f as a *fuzzy function* $\tilde{f} : \mathcal{F}(U) \to \mathcal{F}(V)$; that is, for any $\tilde{A} \in \mathcal{F}(U)$, it means $\tilde{f}(\tilde{A}) \in \mathcal{F}(V)$. We remark that this fuzzy function is completely different from the concept of fuzzy function studied in Hajek [1], Demirci [2] and Höhle et al. [3] in which the fuzzy function is treated as a fuzzy relation.

The principle for fuzzifying the crisp functions is called the *extension principle*, which was proposed by Zadeh [4–6]. In particular, if $V = \mathbb{R}$, then $f : U \to \mathbb{R}$ is called a real-valued function, and $\tilde{f} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ induced from f is called a *fuzzy-valued function*.

For any $\tilde{A} \in \mathcal{F}(U)$, the extension principle says that the membership function of $\tilde{B} \equiv \tilde{f}(\tilde{A})$ is defined by the following supremum:

$$\xi_{\tilde{B}}(y) = \xi_{\tilde{f}(\tilde{A})}(y) = \sup_{\{x:y=f(x)\}} \xi_{\tilde{A}}(x). \tag{1}$$

Suppose that U and V are taken as the topological spaces such that the original crisp function $f : (U, \tau_U) \to (V, \tau_V)$ is continuous. In this paper, we shall investigate the continuity of fuzzified function $\tilde{f} : \mathcal{F}(U) \to \mathcal{F}(V)$. Román-Flores et al. [7] has studied the continuity of this fuzzified function when U and V are taken as the Euclidean space $\mathbb{R}^n$. In this paper, we are going to extend these results to the case of normed spaces and normable topological vector spaces using the generalized extension principle discussed in Wu [8].

Let U_1 and U_2 be two universal sets. We consider the onto crisp function $f : U_1 \times U_2 \to V$. The purpose is to fuzzify the crisp function f as a fuzzy function $\tilde{f} : \mathcal{F}(U_1) \times \mathcal{F}(U_2) \to \mathcal{F}(V)$.

For any two fuzzy subsets $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ of U_1 and U_2, respectively, the membership function of $\tilde{B} \equiv \tilde{f}(\tilde{A}^{(1)}, \tilde{A}^{(2)}) \in \mathcal{F}(V)$ is defined by:

$$\xi_{\tilde{B}}(y) = \xi_{\tilde{f}(\tilde{A}^{(1)},\tilde{A}^{(2)})}(y) = \sup_{\{(x_1,x_2):y=f(x_1,x_2)\}} \min\left\{\xi_{\tilde{A}^{(1)}}(x_1), \xi_{\tilde{A}^{(2)}}(x_2)\right\}. \tag{2}$$

Nguyen [9] has obtained the following result.

For $\alpha \in (0,1]$, the α-level set of $\tilde{A}$ is defined and denoted by:

$$\tilde{A}_\alpha = \{x \in U : \xi_{\tilde{A}}(x) \geq \alpha\}.$$

If the universal set U is endowed with a topology, then the 0-level set of $\tilde{A}$ is defined by:

$$\tilde{A}_0 = \mathrm{cl}\left(\{x \in U : \xi_{\tilde{A}}(x) > 0\}\right),$$

which is the closure of the *support* $\{x \in U : \xi_{\tilde{A}}(x) > 0\}$ of $\tilde{A}$. However, if U is not assumed to be a topological space, then the 0-level set is usually taken to be the whole set U.

Theorem 1. *(Nguyen [9]) Let $f : U_1 \times U_2 \to V$ be an onto crisp function defined on $U_1 \times U_2$ and let $\tilde{f} : \mathcal{F}(U_1) \times \mathcal{F}(U_2) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (2). For $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$, $i = 1,2$, the following equality:*

$$\left(\tilde{f}(\tilde{A}^{(1)}, \tilde{A}^{(2)})\right)_\alpha = f\left(\tilde{A}^{(1)}_\alpha, \tilde{A}^{(2)}_\alpha\right) \tag{3}$$

holds true for each $\alpha \in (0,1]$ if and only if, for each $y \in V$, the following supremum:

$$\sup_{\{(x_1,x_2):y=f(x_1,x_2)\}} \min\left\{\xi_{\tilde{A}^{(1)}}(x_1), \xi_{\tilde{A}^{(2)}}(x_2)\right\} \tag{4}$$

is attained; that is, we have:

$$\sup_{\{(x_1,x_2):y=f(x_1,x_2)\}} \min\left\{\xi_{\tilde{A}^{(1)}}(x_1), \xi_{\tilde{A}^{(2)}}(x_2)\right\} = \max_{\{(x_1,x_2):y=f(x_1,x_2)\}} \min\left\{\xi_{\tilde{A}^{(1)}}(x_1), \xi_{\tilde{A}^{(2)}}(x_2)\right\}.$$

Fullér and Keresztfalvi [10] generalized Theorem 1 by considering the t-norm. In this case, the extension principle presented in Equation (2) can be generalized in the following form:

$$\xi_{\tilde{B}}(y) = \xi_{\tilde{f}(\tilde{A}^{(1)},\tilde{A}^{(2)})}(y) = \sup_{\{(x_1,x_2):y=f(x_1,x_2)\}} t\left(\xi_{\tilde{A}^{(1)}}(x_1), \xi_{\tilde{A}^{(2)}}(x_2)\right), \tag{5}$$

since $\min\{x,y\}$ is a t-norm. Therefore, Theorem 1 can be generalized as follows.

Theorem 2. *(Fullér and Keresztfalvi [10]) Let $f : U_1 \times U_2 \to V$ be an onto function defined on $U_1 \times \cdots \times U_n$ and let $\tilde{f} : \mathcal{F}(U_1) \times \mathcal{F}(U_2) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (5). For $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$, $i = 1,2$, the following equality:*

$$\left(\tilde{f}(\tilde{A}^{(1)}, \tilde{A}^{(2)})\right)_\alpha = \bigcup_{\{(\alpha_1,\alpha_2):t(\alpha_1,\alpha_2)\geq\alpha\}} f\left(\tilde{A}^{(1)}_{\alpha_1}, \tilde{A}^{(2)}_{\alpha_2}\right) \tag{6}$$

holds true for each $\alpha \in (0,1]$ if and only if, for each $y \in V$, the following supremum:

$$\sup_{\{(x_1,x_2):y=f(x_1,x_2)\}} t\left(\xi_{\tilde{A}^{(1)}}(x_1), \xi_{\tilde{A}^{(2)}}(x_2)\right) \tag{7}$$

is attained.

Fullér and Keresztfalvi [10] also obtained the following interesting result.

Theorem 3. *(Fullér and Keresztfalvi [10]) Let U_1, U_2, V be locally compact topological spaces. Let $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$ for $i = 1, 2$ such that the membership functions $\xi_{\tilde{A}^{(i)}}$ are upper semicontinuous and the 0-level sets $\tilde{A}^{(i)}_0$ are compact subsets of U_i for $i = 1, 2$. Let $f : U_1 \times U_2 \to V$ be a continuous and onto crisp function and let $\tilde{f} : \mathcal{F}(U_1) \times \mathcal{F}(U_2) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (5). If the t-norm t is upper semicontinuous, then the following equality:*

$$\left(\tilde{f}(\tilde{A}^{(1)}, \tilde{A}^{(2)})\right)_\alpha = \bigcup_{\{(\alpha_1,\alpha_2):t(\alpha_1,\alpha_2)\geq\alpha\}} f\left(\tilde{A}^{(1)}_{\alpha_1}, \tilde{A}^{(2)}_{\alpha_2}\right) \tag{8}$$

holds true for each $\alpha \in (0, 1]$.

Based on the Hausdorff space, Wu [8] generalizes Theorems 2 and 3 to the case of generalized t-norm $T_n : [0,1]^n \to [0,1]$ that is recursively defined by:

$$T_n(x_1, \cdots, x_n) = t(T_{n-1}(x_1, \cdots, x_{n-1}), x_n).$$

Let $U_1, \cdots, U_n, V$ be universal sets and let $f : U_1 \times \cdots \times U_n \to V$ be an onto crisp function defined on $U_1 \times \cdots \times U_n$. For $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$, $i = 1, \cdots, n$, the membership function of the fuzzy subset $\tilde{B} \equiv \tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})$ of V is defined by:

$$\xi_{\tilde{B}}(y) = \xi_{\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})}(y) = \sup_{\{(x_1,\cdots,x_n):y=f(x_1,\cdots,x_n)\}} T_n\left(\xi_{\tilde{A}^{(1)}}(x_1), \cdots, \xi_{\tilde{A}^{(n)}}(x_n)\right) \tag{9}$$

for each $y \in V$. This definition extends the definition given in Equation (7). In the sequel, we are going to consider the extension principle using an operator called W_n that is more general than T_n.

Let $W_n : [0,1]^n \to [0,1]$ be a function defined on $[0,1]^n$, which does not assume any extra conditions. Let $U_1, \cdots, U_n, V$ be universal sets and let $f : U_1 \times \cdots \times U_n \to V$ be an onto crisp function defined on $U_1 \times \cdots \times U_n$. For $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$, $i = 1, \cdots, n$, the membership function of the fuzzy subset $\tilde{B} \equiv \tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})$ of V is defined by:

$$\xi_{\tilde{B}}(y) = \xi_{\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})}(y) = \sup_{\{(x_1,\cdots,x_n):y=f(x_1,\cdots,x_n)\}} W_n\left(\xi_{\tilde{A}^{(1)}}(x_1), \cdots, \xi_{\tilde{A}^{(n)}}(x_n)\right) \tag{10}$$

for each $y \in V$. Of course, this definition extends the definition given in Equation (9). In this paper, we are going to investigate the continuity of this kind of fuzzy function. We also remark that the operator W_n is a kind of aggregation operator studied in Calvo et al. [11] and Grabisch et al. [12].

2. Generalized Extension Principle on Normed Spaces

Let $(X, \|\cdot\|)$ be a normed space. Then, we see that the norm $\|\cdot\|$ can induce a topology $\hat{\tau}$ such that $(X, \hat{\tau})$ becomes a Hausdorff topological vector space in which $\hat{\tau}$ is also called a **norm topology**.

Let $(U_i, \|\cdot\|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space. Since each normed space $(U_i, \|\cdot\|_{U_i})$ can induce a Hausdorff topological vector space $(U_i, \hat{\tau}_{U_i})$, we can form a product topological vector space $(\mathbf{U}, \tau_{U_1\times\cdots\times U_n})$ using $(U_i, \hat{\tau}_{U_i})$ for $i = 1, \cdots, n$, where $\tau_{U_1\times\cdots\times U_n}$ is the product topology. On the other hand, we can define a product norm on the product vector space $\mathbf{U}$ using the norms $\|\cdot\|_{U_i}$. For example, for $(u_1, \cdots, u_n) \in \mathbf{U}$, we can define the maximum norm:

$$\| (u_1, \cdots, u_n) \|^{(1)}_{U_1\times\cdots\times U_n} = \max\left\{\| u_1 \|_{U_1}, \cdots, \| u_n \|_{U_n}\right\}$$

which is shown in Kreyszig ([13], p. 71) or the p-norm:

$$\| (u_1, \cdots, u_n) \|^{(2)}_{U_1\times\cdots\times U_n} = (\| u_1 \|^p + \cdots + \| u_n \|^p)^{1/p},$$

which is shown in Conway ([14], p. 72), where $1 \leq p < \infty$ and $\| \cdot \|_{U_i}$ are taken to be the same norm $\| \cdot \|_{U_i} = \| \cdot \|$ for all $i = 1, \cdots, n$. We also see that the product norms $\| \cdot \|^{(1)}_{U_1 \times \cdots \times U_n}$ and $\| \cdot \|^{(2)}_{U_1 \times \cdots \times U_n}$ can induce the norm topologies $\hat{\tau}_1$ and $\hat{\tau}_2$, respectively. We can show that $\tau_{U_1 \times \cdots \times U_n} = \hat{\tau}_1 = \hat{\tau}_2$. However, in general, the norm topology generated by the product norm does not necessarily equal to the product topology $\tau_{U_1 \times \cdots \times U_n}$. Therefore, it is needed to investigate the generalized extension principle in the case of normed space separately.

We can simply regard $\mathbf{U}$ as a vector space over $\mathbb{R}$. Therefore, we can define a norm to make it as a normed space $(\mathbf{U}, \| \cdot \|_{\mathbf{U}})$. In this case, we can induce a norm topology $\hat{\tau}_{\mathbf{U}}$. Alternatively, we can consider the product norm for the product vector space $\mathbf{U}$ over $\mathbb{R}$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function defined on $\mathbb{R}^n$. In general, the product norm on $\mathbf{U}$ can be defined as:

$$\| (u_1, \cdots, u_n) \|_{U_1 \times \cdots \times U_n} = h(\| u_1 \|_{U_1}, \cdots, \| u_n \|_{U_n}). \tag{11}$$

Of course, the product norm is also a norm for $\mathbf{U}$. Based on this product norm, we can also induce a product norm topology $\hat{\tau}_{U_1 \times \cdots \times U_n}$. If we take $h(x_1, \cdots, x) = \max\{x_1, \cdots, x_n\}$, then we obtain the product norm $\| \cdot \|^{(1)}_{U_1 \times \cdots \times U_n}$, and if we take $h(x_1, \cdots, x_n) = (x_1^p + \cdots + x_n^p)^{1/p}$ for $1 \leq p < \infty$, then we can obtain the product norm $\| \cdot \|^{(2)}_{U_1 \times \cdots \times U_n}$.

For $u_i \in U_i$ and $\epsilon > 0$, the open ball in the normed space $(U_i, \| \cdot \|_{U_i})$ is defined by:

$$B_i(u_i; \epsilon) = \left\{ u \in U_i : \| u - u_i \|_{U_i} < \epsilon \right\}.$$

For $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbf{U}$, the open ball in the normed space $(\mathbf{U}, \| \cdot \|_{\mathbf{U}})$ is given by:

$$\mathbf{B}(\mathbf{u}; \epsilon) = \{ \mathbf{v} \in \mathbf{U} : \| \mathbf{v} - \mathbf{u} \|_{\mathbf{U}} < \epsilon \} = \{ \mathbf{v} \in \mathbf{U} : \| (v_1 - u_1, \cdots, v_n - u_n) \|_{\mathbf{U}} < \epsilon \},$$

and, by referring to Equation (11), the open ball in the product normed space $(\mathbf{U}, \| \cdot \|_{U_1 \times \cdots \times U_n})$ is given by:

$$\mathbf{B}(h, \mathbf{u}; \epsilon) = \left\{ \mathbf{v} \in \mathbf{U} : \| \mathbf{v} - \mathbf{u} \|_{U_1 \times \cdots \times U_n} < \epsilon \right\} = \left\{ \mathbf{v} \in \mathbf{U} : h(\| v_1 - u_1 \|_{U_1}, \cdots, \| v_n - u_n \|_{U_n}) < \epsilon \right\}.$$

We have the following interesting result.

Proposition 1. *Let $(U_i, \| \cdot \|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space which is endowed with a norm $\| \cdot \|_{\mathbf{U}}$. Given any $\epsilon > 0$ and $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbf{U}$, if there exist $\hat{\epsilon}_1, \hat{\epsilon}_2 > 0$ such that the following inclusions hold true:*

$$\mathbf{B}(\mathbf{u}; \hat{\epsilon}_1) \subseteq B_1(u_1; \epsilon) \times \cdots \times B_n(u_n; \epsilon) \tag{12}$$

and:

$$B_1(u_1; \hat{\epsilon}_2) \times \cdots \times B_n(u_n; \hat{\epsilon}_2) \subseteq \mathbf{B}(\mathbf{u}; \epsilon), \tag{13}$$

then $\hat{\tau}_{\mathbf{U}} = \tau_{U_1 \times \cdots \times U_n}$, where $\hat{\tau}_{\mathbf{U}}$ is the norm topology induced by the norm $\| \cdot \|_{\mathbf{U}}$ and $\tau_{U_1 \times \cdots \times U_n}$ is the product topology. If the product vector space $\mathbf{U}$ is endowed with the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have $\hat{\tau}_{U_1 \times \cdots \times U_n} = \tau_{U_1 \times \cdots \times U_n}$, where $\hat{\tau}_{U_1 \times \cdots \times U_n}$ is the product norm topology.

Proof. Since $\tau_{U_1 \times \cdots \times U_n}$ is the product topology for $U_1 \times \cdots \times U_n$, we recall that $\mathbf{O} \in \tau_{U_1 \times \cdots \times U_n}$ if and only if, for any $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbf{O}$, there exist open neighborhoods N_i of u_i for $i = 1, \cdots, n$ such that $N_1 \times \cdots \times N_n \subseteq \mathbf{O}$. Since N_i is an open neighborhood of u_i, there exists $\epsilon_i > 0$ such that

$B_i(u_i;\epsilon_i) \subseteq N_i$. Let $\epsilon = \min\{\epsilon_1, \cdots, \epsilon_n\}$. Then, we have $B_i(u_i;\epsilon) \subseteq B_i(u_i;\epsilon_i) \subseteq N_i$ for all $i = 1, \cdots, n$. By the assumption in Equation (12), it follows that

$$\mathbf{B}(\mathbf{u};\hat{\epsilon}_1) \subseteq B_1(u_1;\epsilon) \times \cdots \times B_n(u_n;\epsilon) \subseteq N_1 \times \cdots \times N_n \subseteq \mathbf{O}.$$

Therefore, we conclude that $\mathbf{O} \in \hat{\tau}_{\mathbf{U}}$. Conversely, for any open set $\mathbf{O} \in \hat{\tau}_{\mathbf{U}}$ and any element $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbf{O}$, there exists $\epsilon > 0$ such that $\mathbf{B}(\mathbf{u};\epsilon) \subseteq \mathbf{O}$. By the assumption in Equation (13), we see that $B_1(u_1;\hat{\epsilon}_2) \times \cdots \times B_n(u_n;\hat{\epsilon}_2) \subseteq \mathbf{B}(\mathbf{u};\epsilon) \subseteq \mathbf{O}$. Since $B_i(u_i;\hat{\epsilon}_2)$ is an open neighborhood of u_i for each $i = 1, \cdots, n$, we conclude that $\mathbf{O} \in \tau_{U_1 \times \cdots \times U_n}$. This completes the proof. □

Proposition 2. *Let $(U_i, \| \cdot \|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space. Then, the following statements hold true.*

(i) We consider the normed space $(\mathbf{U}, \| \cdot \|_{\mathbf{U}})$. Given any $\epsilon > 0$, if $\| (u_1, \cdots, u_n) \|_{\mathbf{U}} < \epsilon$ if and only if $\| u_i \|_{U_i} < \epsilon$ for all $i = 1, \cdots, n$, then $\hat{\tau}_{\mathbf{U}} = \tau_{U_1 \times \cdots \times U_n}$.

(ii) We consider the product normed space $(\mathbf{U}, \| \cdot \|_{U_1 \times \cdots \times U_n})$, where the product norm is defined by Equation (11). Given any $\epsilon > 0$, if $h(x_1, \cdots, x_n) < \epsilon$ if and only if $x_i < \epsilon$ for all $i = 1, \cdots, n$, then $\hat{\tau}_{U_1 \times \cdots \times U_n} = \tau_{U_1 \times \cdots \times U_n}$.

Proof. It suffices to prove the case of (ii). By definition, we have:

$$\begin{aligned} \mathbf{B}(\mathbf{u};\epsilon) &= \left\{ \mathbf{v} \in \mathbf{U} : \| \mathbf{v} - \mathbf{u} \|_{U_1 \times \cdots \times U_n} < \epsilon \right\} \\ &= \left\{ \mathbf{v} \in \mathbf{U} : h\left(\| v_1 - u_1 \|_{U_1}, \cdots, \| v_n - u_n \|_{U_n} \right) < \epsilon \right\} \\ &= \left\{ \mathbf{v} \in \mathbf{U} : \| v_i - u_i \|_{U_i} < \epsilon \text{ for } i = 1, \cdots, n \right\} \text{ (by the asumption of } h) \\ &= B_1(u_1;\epsilon) \times \cdots \times B_n(u_n;\epsilon). \end{aligned}$$

The results follow immediately from Proposition 1 by taking $\hat{\epsilon}_1 = \hat{\epsilon}_2 = \epsilon$. □

Remark 1. *Note that if the product norm is taken as the maximum norm $\| \cdot \|^{(1)}_{U_1 \times \cdots \times U_n}$ or the p-norm $\| \cdot \|^{(2)}_{U_1 \times \cdots \times U_n}$ defined above, then the assumption in part (ii) of Proposition 2 is satisfied automatically.*

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be two normed spaces. Recall that the function $f : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y)$ is continuous at x_0 if and only if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\| x - x_0 \|_X < \delta$ implies $\| f(x) - f(x_0) \|_Y < \epsilon$. The function f is continuous on X if and only if f is continuous at each point $x_0 \in X$. Then, we have the following easy observation.

Remark 2. *Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be two normed spaces such that $\hat{\tau}_X$ and $\hat{\tau}_Y$ are two norm topologies induced by the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. It is well-known that the function $f : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y)$ is continuous if and only if $f : (X, \hat{\tau}_X) \to (Y, \hat{\tau}_Y)$ is continuous. The continuity of $f : (X, \hat{\tau}_X) \to (Y, \hat{\tau}_Y)$ means that if $O \in \tau_Y$, then $f^{-1}(O) \in \tau_X$.*

Remark 3. *Let $(U_i, \| \cdot \|_{U_i})$ and $(V, \| \cdot \|_V)$ be the normed spaces for $i = 1, \cdots, n$. Then three kinds of continuity for the function $f : U_1 \times \cdots \times U_n \to V$ can be presented below.*

- *Suppose that the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ is endowed with the norm $\| \cdot \|_{\mathbf{U}}$. Then the function $f : (\mathbf{U}, \| \cdot \|_{\mathbf{U}}) \to (V, \| \cdot \|_V)$ is continuous at x_0 if and only if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\| \mathbf{u} - \mathbf{u}_0 \|_{\mathbf{U}} < \delta$ implies $\| f(\mathbf{u}) - f(\mathbf{u}_0) \|_V < \epsilon$, where $\mathbf{u} = (u_1, \cdots, u_n)$ and $\mathbf{u}_0 = (u_{10}, \cdots, u_{n0})$ are elements of $\mathbf{U}$. Since $\| \cdot \|_{\mathbf{U}}$ can induce a norm topology $\hat{\tau}_{\mathbf{U}}$, Remark 2 says that $f : (\mathbf{U}, \| \cdot \|_{\mathbf{U}}) \to (V, \| \cdot \|_V)$ is continuous if and only if $f : (\mathbf{U}, \hat{\tau}_{\mathbf{U}}) \to (V, \hat{\tau}_V)$ is continuous.*
- *Suppose that the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ is endowed with the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$. Then the continuity of the function $f : (\mathbf{U}, \| \cdot \|_{U_1 \times \cdots \times U_n}) \to (V, \| \cdot \|_V)$ can be*

similarly realized. We also see that $f : (\mathbf{U}, \|\cdot\|_{U_1\times\cdots\times U_n}) \rightarrow (V, \|\cdot\|_V)$ is continuous if and only if $f : (\mathbf{U}, \hat{\tau}_{U_1\times\cdots\times U_n}) \rightarrow (V, \hat{\tau}_V)$ is continuous.

- *Since we can form a product topological vector space $(\mathbf{U}, \tau_{U_1\times\cdots\times U_n})$ from the normed spaces $(U_i, \|\cdot\|_{U_i})$ for $i = 1, \cdots, n$, we say that the function $f : U_1 \times \cdots \times U_n \rightarrow V$ is continuous if and only if the function $f : (\mathbf{U}, \tau_{U_1\times\cdots\times U_n}) \rightarrow (V, \tau_V)$ is continuous in the topological sense. Propositions 1 and 2 say that this kind of continuity will be equivalent to one of the above two continuities under some suitable conditions.*

We say that W_n is *nondecreasing* if and only if $\alpha_i \geq \beta_i$ for all $i = 1, \cdots, n$ imply $W_n(\alpha_1, \cdots, \alpha_n) \geq W_n(\beta_1, \cdots, \beta_n)$. This definition does not necessarily say that $W_n(\alpha_1, \cdots, \alpha_n) \geq W_n(\beta_1, \cdots, \beta_n)$ implies $\alpha_i \geq \beta_i$ for all $i = 1, \cdots, n$. In the subsequent discussion, the function W_n will satisfy some of the following conditions:

(a) $W_n(\alpha_1, \cdots, \alpha_n) > 0$ if and only if $\alpha_i > 0$ for all $i = 1, \cdots, n$.
(b) For each $\alpha \in (0, 1]$, $W_n(\alpha_1, \cdots, \alpha_n) \geq \alpha$ if and only if $\alpha_i \geq \alpha$ for all $i = 1, \cdots, n$.
(c) W_n is upper semicontinuous and nondecreasing.
(d) if any one of $\{\alpha_1, \cdots, \alpha_n\}$ is zero, then $W_n(\alpha_1, \cdots, \alpha_n) = 0$.
(e) $W_n(1, \cdots, 1) = 1$.
(f) $W_n(\min\{a_1, b_1\}, \cdots, \min\{a_n, b_n\}) \geq \min\{W_n(a_1, \cdots, a_n), W_n(b_1, \cdots, b_n)\}$.

Next, we are going to present the generalized extension principle on normed spaces.

Theorem 4. *Let $f : U_1 \times \cdots \times U_n \rightarrow V$ be an onto crisp function defined on $U_1 \times \cdots \times U_n$ and let $\tilde{f} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \rightarrow \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Assume that $(U_i, \|\cdot\|_{U_i})$ and $(V, \|\cdot\|_V)$ are taken to be the normed spaces for $i = 1, \cdots, n$, and that the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ is endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. We also assume that the following supremum:*

$$\sup_{\{(x_1,\cdots,x_n):y=f(x_1,\cdots,x_n)\}} W_n\left(\xi_{\tilde{A}^{(1)}}(x_1), \cdots, \xi_{\tilde{A}^{(n)}}(x_n)\right) \tag{14}$$

is attained for each $y \in V$. Then, the following equality:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_\alpha = \left\{f(x_1, \cdots, x_n) : W_n\left(\xi_{\tilde{A}^{(1)}}(x_1), \cdots, \xi_{\tilde{A}^{(n)}}(x_n)\right) \geq \alpha\right\}$$

holds true for each $\alpha \in (0, 1]$. The results for the 0-level sets are given below.

- *We have:*

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_0 = \mathrm{cl}\left(\left\{f(x_1, \cdots, x_n) : W_n\left(\xi_{\tilde{A}^{(1)}}(x_1), \cdots, \xi_{\tilde{A}^{(n)}}(x_n)\right) > 0\right\}\right)$$

- *If we further assume that condition (a) for W_n is satisfied, then:*

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_0 = \mathrm{cl}\left(f\left(\tilde{A}^{(1)}_{0+}, \cdots, \tilde{A}^{(n)}_{0+}\right)\right) \subseteq \mathrm{cl}\left(f\left(\tilde{A}^{(1)}_0, \cdots, \tilde{A}^{(n)}_0\right)\right).$$

- *If we further assume that the function $f : (\mathbf{U}, \|\cdot\|_{\mathbf{U}}) \rightarrow (V, \|\cdot\|_V)$ is continuous and that condition (a) for W_n is satisfied, then:*

$$\begin{aligned}\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_0 &= \mathrm{cl}\left(\left\{f(x_1, \cdots, x_n) : W_n\left(\xi_{\tilde{A}^{(1)}}(x_1), \cdots, \xi_{\tilde{A}^{(n)}}(x_n)\right) > 0\right\}\right)\\ &= \mathrm{cl}\left(f\left(\tilde{A}^{(1)}_{0+}, \cdots, \tilde{A}^{(n)}_{0+}\right)\right) = f\left(\tilde{A}^{(1)}_0, \cdots, \tilde{A}^{(n)}_0\right).\end{aligned}$$

If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the *inclusions Equations (12) and (13) are not needed when we say that the function $f: U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$.*

Proof. Since a normed space can induce a Hausdorff topological space, the results follow immediately from Remarks 2 and 3, Proposition 1 and Wu ([8], Theorem 5.1). □

By referring to Remark 1, from part (ii) of Proposition 2, we see that if the product norm is taken as the maximum norm $\|\cdot\|^{(1)}_{U_1\times\cdots\times U_n}$ or the p-norm $\|\cdot\|^{(2)}_{U_1\times\cdots\times U_n}$, then Theorem 4 is applicable for these norms.

Theorem 5. *Let $f: U_1 \times \cdots \times U_n \to V$ be an onto crisp function defined on $U_1 \times \cdots \times U_n$ and let $\tilde{f}: \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Suppose that the following supremum:*

$$\sup_{\{(x_1,\cdots,x_n):y=f(x_1,\cdots,x_n)\}} W_n\left(\xi_{\tilde{A}^{(1)}}(x_1),\cdots,\xi_{\tilde{A}^{(n)}}(x_n)\right)$$

is attained for each $y \in V$, and that that condition (b) for W_n is satisfied. Then, for each $\alpha \in (0,1]$, we have the following equalities.

$$\left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_\alpha = f\left(\tilde{A}^{(1)}_\alpha,\cdots,\tilde{A}^{(n)}_\alpha\right) \text{ and } \left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_{0+} = f\left(\tilde{A}^{(1)}_{0+},\cdots,\tilde{A}^{(n)}_{0+}\right).$$

Let $(U_i,\|\cdot\|_{U_i})$ and $(V,\|\cdot\|_V)$ be now taken to be the normed spaces for $i = 1,\cdots,n$, and let the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. If we further assume that the function $f: (\mathbf{U},\|\cdot\|_{\mathbf{U}}) \to (V,\|\cdot\|_V)$ is continuous, then we also have the following equality:

$$\left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_0 = f\left(\tilde{A}^{(1)}_0,\cdots,\tilde{A}^{(n)}_0\right).$$

If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f: U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$.

Proof. Since a normed space can induce a Hausdorff topological space, the results follow immediately from Remarks 2 and 3, Proposition 1 and Wu ([8], Theorem 5.2). □

Theorem 6. *Let $(U_i,\|\cdot\|_{U_i})$ and $(V,\|\cdot\|_V)$ be normed spaces for $i = 1,\cdots,n$, and let the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. Let $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$ for all $i = 1,\cdots,n$ such that the membership functions $\xi_{\tilde{A}^{(i)}}$ are upper semicontinuous and the 0-level sets $\tilde{A}^{(i)}_0$ are compact subsets of U_i for all $i = 1,\cdots,n$. Let $f: (\mathbf{U},\|\cdot\|_{\mathbf{U}}) \to (V,\|\cdot\|_V)$ be a continuous and onto crisp function, and let $\tilde{f}: \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Suppose that conditions (c) and (d) for W_n are satisfied. Then, the following statements hold true.*

- *(i) The membership function $\xi_{\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})}$ is upper semicontinuous.*
- *(ii) For each $\alpha \in (0,1]$, we have the following equality:*

$$\left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_\alpha = \bigcup_{\{(\alpha_1,\cdots,\alpha_n):W_n(\alpha_1,\cdots,\alpha_n)\geq\alpha\}} f\left(\tilde{A}^{(1)}_{\alpha_1},\cdots,\tilde{A}^{(n)}_{\alpha_n}\right).$$

For the 0*-level sets, if we further assume that conditions (a) for* W_n *is satisfied, then:*

$$\left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_{0+}=f\left(\tilde{A}^{(1)}_{0+},\cdots,\tilde{A}^{(n)}_{0+}\right) \text{ and } \left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_{0}=f\left(\tilde{A}^{(1)}_{0},\cdots,\tilde{A}^{(n)}_{0}\right).$$

Under this further assumption, the α*-level sets* $(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)}))_\alpha$ *of* $\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})$ *are closed and compact subsets of* V *for all* $\alpha\in[0,1]$.

(iii) *If we further assume that conditions (b) for* W_n *is satisfied, then:*

$$\left(\tilde{f}(\tilde{A}^{(1)},\cdots,\tilde{A}^{(n)})\right)_{\alpha}=f\left(\tilde{A}^{(1)}_{\alpha},\cdots,\tilde{A}^{(n)}_{\alpha}\right)$$

for each $\alpha\in[0,1]$.

If the product vector space $\mathbf{U}$ *is endowed with the product norm* $\|\cdot\|_{U_1\times\cdots\times U_n}$ *such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function* $f:U_1\times\cdots\times U_n\to V$ *is continuous directly in topological sense without considering the norm* $\|\cdot\|_{\mathbf{U}}$ *and the product norm* $\|\cdot\|_{U_1\times\cdots\times U_n}$.

Proof. Since a normed space can induce a Hausdorff topological space, the results follow immediately from Remarks 2 and 3, Proposition 1 and Wu ([8], Theorem 5.3 and Corollary 5.1). □

Since we are going to consider the concept of convexity, we need to impose the vector addition and scalar multiplication upon the universal set U. Therefore, the universal set U is taken as a vector space. For any fuzzy subset $\tilde{A}$ of U, we say that $\tilde{A}$ is *convex* if and only if each α-level set $\tilde{A}_\alpha=\{x:\xi_{\tilde{A}}(x)\}$ is a convex subset of U for each $\alpha\in(0,1]$. It is well-known that:

$$\tilde{A} \text{ is convex if and only if } \xi_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)\geq\min\{\xi_{\tilde{A}}(x_1),\xi_{\tilde{A}}(x_2)\}; \tag{15}$$

that is, the membership function $\xi_{\tilde{A}}$ is quasi-concave.

Let U be a vector space over $\mathbb{R}$ which is endowed with a topology τ and let $\tilde{A}$ be a fuzzy subset of U. Since $\tilde{A}_0=\text{cl}(\bigcup_{\alpha>0}\tilde{A}_\alpha)$, we see that if $\tilde{A}$ is convex, then its 0-level set $\tilde{A}_0$ is also a convex subset of U.

Definition 1. *Let* U *be a vector space over* $\mathbb{R}$ *which is endowed with a topology* τ*. We denote by* $\mathcal{F}_{cc}(U)$ *the set of all fuzzy subsets of* U *such that each* $\tilde{a}\in\mathcal{F}_{cc}(U)$ *satisfies the following conditions:*

- $\tilde{a}$ *is normal, i.e.,* $\xi_{\tilde{a}}(x)=1$ *for some* $x\in U$;
- $\tilde{a}$ *is convex;*
- *the membership function* $\xi_{\tilde{a}}$ *is upper semicontinuous;*
- *the* 0*-level set* $\tilde{a}_0$ *is a compact subset of* U.

For $\tilde{a}\in\mathcal{F}_{cc}(U)$, we see that, for each $\alpha\in[0,1]$, the α-level set $\tilde{a}_\alpha$ is a compact and convex subset of U. Each element of $\mathcal{F}_{cc}(U)$ is called a *fuzzy element*. If $U=\mathbb{R}$, then each element of $\mathcal{F}_{cc}(\mathbb{R})$ is called a *fuzzy number*. In addition, if $U=\mathbb{R}^n$, then each element of $\mathcal{F}_{cc}(\mathbb{R}^n)$ is called a *fuzzy vector*. Moreover, for any fuzzy number $\tilde{a}\in\mathcal{F}_{cc}(\mathbb{R})$, we see that each of its α-level sets $\tilde{a}_\alpha$ is a closed, bounded and convex subset of $\mathbb{R}$, i.e., a closed interval in $\mathbb{R}$.

Theorem 7. *Let* $(U_i,\|\cdot\|_{U_i})$ *and* $(V,\|\cdot\|_V)$ *be normed spaces for* $i=1,\cdots,n$*, and let the product vector space* $\mathbf{U}=U_1\times\cdots\times U_n$ *be endowed with a norm* $\|\cdot\|_{\mathbf{U}}$ *such that the inclusions Equations (12) and (13) are satisfied. Let* $f:(\mathbf{U},\|\cdot\|_{\mathbf{U}})\to(V,\|\cdot\|_V)$ *be a continuous and onto crisp function. We also assume that* f *is linear in the sense of:*

$$\lambda f(x_1,\cdots,x_n)+(1-\lambda)f(y_1,\cdots,y_n)=f(\lambda x_1+(1-\lambda)y_1,\cdots,\lambda x_n+(1-\lambda)y_n).$$

Let $\tilde{f} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Suppose that conditions (a), (c), (d), (e) and (f) for W_n are satisfied. For $\tilde{A}^{(i)} \in \mathcal{F}_{cc}(U_i)$, $i = 1, \cdots, n$, the following statements hold true.

(i) We have $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_{cc}(V)$. The α-level sets $(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}))_\alpha$ are compact, convex and closed subsets of V for all $\alpha \in [0, 1]$.

(ii) For each $\alpha \in (0, 1]$, we have the following equality:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_\alpha = \bigcup_{\{(\alpha_1, \cdots, \alpha_n) : W_n(\alpha_1, \cdots, \alpha_n) \geq \alpha\}} f\left(\tilde{A}^{(1)}_{\alpha_1}, \cdots, \tilde{A}^{(n)}_{\alpha_n}\right).$$

For the 0-level sets, we also have:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_{0+} = f\left(\tilde{A}^{(1)}_{0+}, \cdots, \tilde{A}^{(n)}_{0+}\right) \text{ and } \left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_0 = f\left(\tilde{A}^{(1)}_0, \cdots, \tilde{A}^{(n)}_0\right).$$

(iii) If we further assume that conditions (b) for W_n is satisfied, then:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_\alpha = f\left(\tilde{A}^{(1)}_\alpha, \cdots, \tilde{A}^{(n)}_\alpha\right)$$

for each $\alpha \in [0, 1]$.

If the product vector space $\mathbf{U}$ is endowed with the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f : U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\| \cdot \|_{\mathbf{U}}$ and the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$.

Proof. Since a normed space can induce a Hausdorff topological space, the results follow immediately from Remarks 2 and 3, Proposition 1 and Wu ([8], Theorem 6.1 and Corollary 6.1). □

The linearity in Theorem 7 can be replaced by assuming the convexity.

Theorem 8. *Let $(U_i, \| \cdot \|_{U_i})$ and $(V, \| \cdot \|_V)$ be normed spaces for $i = 1, \cdots, n$, and let the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ be endowed with a norm $\| \cdot \|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. Let $f : (\mathbf{U}, \| \cdot \|_{\mathbf{U}}) \to (V, \| \cdot \|_V)$ be a continuous and onto crisp function and let $\tilde{f} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). We further assume that $f(A_1, \cdots, A_n)$ is a convex subset of V for any convex subsets A_i of U_i, $i = 1, \cdots, n$. Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. For $\tilde{A}^{(i)} \in \mathcal{F}_{cc}(U_i)$, $i = 1, \cdots, n$, the following statements hold true.*

(i) We have $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_{cc}(V)$. The α-level sets $(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}))_\alpha$ are compact, convex and closed subsets of V for all $\alpha \in [0, 1]$.

(ii) For each $\alpha \in (0, 1]$, we have the following equality:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_\alpha = f\left(\tilde{A}^{(1)}_\alpha, \cdots, \tilde{A}^{(n)}_\alpha\right).$$

For the 0-level sets, we also have:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_{0+} = f\left(\tilde{A}^{(1)}_{0+}, \cdots, \tilde{A}^{(n)}_{0+}\right) \text{ and } \left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_0 = f\left(\tilde{A}^{(1)}_0, \cdots, \tilde{A}^{(n)}_0\right).$$

If the product vector space $\mathbf{U}$ is endowed with the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f : U_1 \times \cdots \times U_n \to V$

is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ *and the product norm* $\|\cdot\|_{U_1\times\cdots\times U_n}$.

Proof. Since a normed space can induce a Hausdorff topological space, the results follow immediately from Remarks 2 and 3, Proposition 1 and Wu ([8], Theorem 6.2). □

Remark 4. *For* $n=2$, *the following functions:*

$$W_2(x,y)=\min\{x,y\} \text{ and } W_2(x,y)=\frac{xy}{\max\{x,y,\alpha\}} \text{ for some constant } \alpha\in[0,1]$$

satisfy the conditions of Theorem 8.

Let $(X,\|\cdot\|)$ be a normed space. We recall that the norm $\|\cdot\|$ can induce a norm topology $\hat{\tau}$ such that $(X,\hat{\tau})$ becomes a Hausdorff topological vector space. Conversely, if (X,τ) is a topological vector space, we say that X is *normable* if and only if there exists a norm $\|\cdot\|$ which can induce a norm topology $\hat{\tau}$ such that $\tau=\hat{\tau}$. A subset Y of a topological vector space (X,τ) is called *bounded* if and only if, for each neighborhood N of θ, where θ is the zero element of X, there is a real number r such that $Y\subseteq rN$. We have the following interesting results.

Proposition 3. *(Kelley and Namioka ([15], p. 44)) The following statements hold true.*

(i) *A topological Hausdorff vector space is normable if and only if there is a bounded convex neighborhood of* 0.
(ii) *A finite product of normable spaces is normable.*

From part (ii) of Proposition 3, we see that Theorems 4–8 are still valid if the normed spaces are replaced by the normable topological vector spaces. However, if we consider the continuity of the function $f:(\mathbf{U},\tau_{U_1\times\cdots\times U_n})\to(V,\tau_V)$ instead of the continuity of the function $f:(\mathbf{U},\|\cdot\|_{\mathbf{U}})\to(V,\|\cdot\|_V)$, then the assumption satisfying the inclusions Equations (12) and (13) is not needed, since, in this case, we can just apply Wu ([8], Theorems 5.1, 5.2, 5.3, 6.1 and 6.2) by considering the product topology $\tau_{U_1\times\cdots\times U_n}$ instead of the norm topology $\hat{\tau}_{\mathbf{U}}$ or the product norm topology $\hat{\tau}_{U_1\times\cdots\times U_n}$.

3. Continuity of Fuzzified Functions

Let $(X,\|\cdot\|)$ be a normed space. We denote by $\mathcal{K}(X)$ the family of all compact subsets of X in the sense of norm topology $\hat{\tau}_X$ induced by the norm $\|\cdot\|$. Let A and B be any two compact subsets of X. We can define the Hausdorff metric d_H between A and B as follows:

$$d_H=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

If X is a normable topological vector space, then we can also define the Hausdorff metric d_H. Now we have the following simple observation.

Lemma 1. *The function* $f:(X,\|\cdot\|)\to\mathbb{R}$ *defined by* $f(x)=\|x-x_0\|$ *is continuous for any given* $x_0\in X$.

Proof. We have:

$$|f(x)-f(y)|=\big|\|x-x_0\|-\|y-x_0\|\big|\leq\|(x-x_0)-(y-x_0)\|=\|x-y\|.$$

The continuity follows immediately from the above inequality. □

We need some useful properties from topological space.

Proposition 4. *The following statements hold true.*

(i) *(Royden ([16], p. 158)) If X is a Hausdorff space, then a compact subset of X is closed.*
(ii) *(Royden ([16], p. 158)) If f is a continuous function from the topological space X to another topological space Y, then the image $f(K)$ is a compact subset of Y when K is a compact subset of X.*
(iii) *(Royden ([16], p. 161)) If f is an upper semicontinuous real-valued function defined on X, then f assumes its maximum on a compact subset of X. If f is a lower semicontinuous real-valued function defined on X, then f assumes its minimum on a compact subset of X.*
(iv) *(Royden ([16], p. 158)) Let X be a topological space and let K be a compact subset of X. If A is a closed subset of X and is also a subset of K, then A is also a compact subset of X.*
(v) *(Royden ([16], p. 166)) (Tychonoff's Theorem) Let (X_i, τ_{X_i}) be n topological spaces and let K_i be compact subsets of X_i for $i = 1, \cdots, n$. Then the product $K_1 \times \cdots \times K_n$ is a compact subset of the product topological space $(X_1 \times \cdots \times X_n, \tau_{X_1 \times \cdots \times X_n})$, where $\tau_{X_1 \times \cdots \times X_n}$ is the product topology for $X_1 \times \cdots \times X_n$.*

Proposition 5. *Let the function $f : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y)$ be continuous. Let d_{H_X} and d_{H_Y} be the Hausdorff metrics defined by $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. Then the function $F : (\mathcal{K}(X), d_{H_X}) \to (\mathcal{K}(Y), d_{H_Y})$ defined by:*

$$F(A) = f(A) = \{f(a) : a \in A\}$$

is uniformly continuous.

Proof. For any $A, B \in \mathcal{K}(X)$, i.e., A and B are compact subsets of $(X, \| \cdot \|_X)$, since f is continuous, by part (ii) of Proposition 4, we see that $C = f(A) = F(A)$ and $D = f(B) = F(B)$ are also compact subsets of Y, i.e., $C, D \in \mathcal{K}(Y)$. We need to show that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $A, B \in \mathcal{K}(X)$, $d_{H_X}(A, B) < \delta$ implies $d_{H_Y}(C, D) < \epsilon$. By definition, for any $a_0 \in A$ and $b_0 \in B$, $d_{H_X}(A, B) < \delta$ implies:

$$\inf_{a \in A} \| a - b_0 \|_X < \delta \text{ and } \inf_{b \in B} \| a_0 - b \|_X < \delta.$$

By Lemma 1 and part (iii) of Proposition 4, we see that the above infimum are attained, i.e., $\| a^* - b_0 \|_X < \delta$ and $\| a_0 - b^* \| < \delta$ for some $a^* \in A$ and $b^* \in B$. Since f is continuous at a_0 and b_0, we have $\| f(a^*) - f(b_0) \|_Y < \epsilon/2$ and $\| f(a_0) - f(b^*) \|_Y < \epsilon/2$. Since a_0 and b_0 are any elements of A and B, respectively, we have:

$$\sup_{d \in D} \| f(a^*) - d \|_Y = \sup_{b_0 \in B} \| f(a^*) - f(b_0) \|_Y \leq \epsilon/2 < \epsilon \tag{16}$$

and:

$$\sup_{c \in C} \| c - f(b^*) \|_Y = \sup_{a_0 \in A} \| f(a_0) - f(b^*) \|_Y \leq \epsilon/2 < \epsilon. \tag{17}$$

Since C and D are compact subsets of Y, there exist $c^* \in C$ and $d^* \in D$ such that:

$$\| c^* - d \|_Y = \inf_{c \in C} \| c - d \|_Y \leq \| f(a^*) - d \|_Y \text{ and } \| c - d^* \|_Y = \inf_{d \in D} \| c - d \|_Y \leq \| c - f(b^*) \|_Y .$$

From Equations (16) and (17), we obtain:

$$\sup_{d \in D} \inf_{c \in C} \| c - d \|_Y \leq \sup_{d \in D} \| f(a^*) - d \|_Y < \epsilon \text{ and } \sup_{c \in C} \inf_{d \in D} \| c - d \|_Y \leq \sup_{c \in C} \| c - f(b^*) \|_Y < \epsilon,$$

which implies $d_{H_Y}(C, D) < \epsilon$. This completes the proof. □

Let (X, τ) be a normable topological vector space. Then, we can topologize $\mathcal{K}(X)$ by considering the Hausdorff metric d_H. This topological space is denoted by $(\mathcal{K}(X), \tau_H)$.

Proposition 6. *Let (X, τ_X) and (Y, τ_Y) be two normable topological vector spaces. Let τ_{H_X} and τ_{H_Y} be the topologies generated by the Hausdorff metrics d_{H_X} and d_{H_Y}, respectively. If the function $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then the function $F : (\mathcal{K}(X), \tau_{H_X}) \to (\mathcal{K}(Y), \tau_{H_Y})$ defined by $F(A) = f(A) = \{f(a) : a \in A\}$ is uniformly continuous; that is, the function $F : (\mathcal{K}(X), d_{H_X}) \to (\mathcal{K}(Y), d_{H_Y})$ is uniformly continuous.*

Proof. The result follows immediately from Proposition 5. □

To study the continuity of fuzzified function, we are going to consider the space $\mathcal{F}_c(U)$ instead of $\mathcal{F}(U)$, which is defined below. Let U be a topological space. We denote by $\mathcal{F}_c(U)$ the set of all fuzzy subsets of U such that, for each $\tilde{A} \in \mathcal{F}_c(U)$, its α-level sets $\tilde{A}_\alpha$ are compact subsets of U for all $\alpha \in [0, 1]$. Let $(U, \| \cdot \|_U)$ be a normed space. Then, we can define the Hausdorff metric d_{H_U} on $\mathcal{K}(U)$. For any $\tilde{A} \in \mathcal{F}_c(U)$, we see that $\tilde{A}_\alpha \in \mathcal{K}(U)$ for all $\alpha \in [0, 1]$. Therefore, for any $\tilde{A}, \tilde{B} \in \mathcal{F}_c(U)$, we can define a distance $d_{\mathcal{F}_c(U)}$ between $\tilde{A}$ and $\tilde{B}$ as:

$$d_{\mathcal{F}_c(U)}(\tilde{A}, \tilde{B}) = \sup_{\alpha \in [0,1]} d_{H_U}\left(\tilde{A}_\alpha, \tilde{B}_\alpha\right). \tag{18}$$

Then, we can show that $(\mathcal{F}_c(U), d_{\mathcal{F}_c(U)})$ forms a metric space. Based on this metric $d_{\mathcal{F}_c(U)}$, we can induce a topology $\tau_{\mathcal{F}_c(U)}$ that is called a *metric topology* for $\mathcal{F}_c(U)$.

Let $(U_i, \| \cdot \|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space that is endowed with the norm $\| \cdot \|_{\mathbf{U}}$ or the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$. In this case, we can define a Hausdorff metric $d_{H_{\mathbf{U}}}$ on $\mathcal{K}(\mathbf{U})$ based on the norm $\| \cdot \|_{\mathbf{U}}$ or the product norm $\| \cdot \|_{U_1 \times \cdots \times U_n}$. We write $\mathcal{F}_c(\mathbf{U}) = \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n)$. The element of $\mathcal{F}_c(\mathbf{U})$ can be realized as $\tilde{\mathbf{A}} = (\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})$, where $\tilde{A}^{(i)} \in \mathcal{F}_c(U_i)$ for $i = 1, \cdots, n$. We also write $\tilde{\mathbf{A}}_\alpha = (\tilde{A}^{(1)}_\alpha, \cdots, \tilde{A}^{(n)}_\alpha)$ for all $\alpha \in [0, 1]$. Using the Tychonoff's theorem in part (v) of Proposition 4, we see that $\tilde{\mathbf{A}}_\alpha \in \mathcal{K}(\mathbf{U})$ for all $\alpha \in [0, 1]$; that is, the product set $\tilde{\mathbf{A}}_\alpha$ is a compact subset of the product space $\mathbf{U}$. For any $\tilde{\mathbf{A}}, \tilde{\mathbf{B}} \in \mathcal{F}_c(\mathbf{U})$, we define $d_{\mathcal{F}_c(\mathbf{U})}$ between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ as follows:

$$d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) = \sup_{\alpha \in [0,1]} d_{H_{\mathbf{U}}}\left(\tilde{\mathbf{A}}_\alpha, \tilde{\mathbf{B}}_\alpha\right).$$

Then, we can also show that $(\mathcal{F}_c(\mathbf{U}), d_{\mathcal{F}_c(\mathbf{U})})$ forms a metric space. Based on this metric $d_{\mathcal{F}_c(\mathbf{U})}$, we can induce a metric topology $\tau_{\mathcal{F}_c(\mathbf{U})}$ for $\mathcal{F}_c(\mathbf{U})$. Now, if we assume that $U_1, \cdots, U_n$ are normable topological vector spaces and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space, then part (ii) of Proposition 3 says that $\mathbf{U}$ is also a normable topological vector space. Therefore, we still can define a Hausdorff metric $d_{H_{\mathbf{U}}}$ on $\mathcal{K}(\mathbf{U})$. In this case, we can also obtain the metric space $(\mathcal{F}_c(\mathbf{U}), d_{\mathcal{F}_c(\mathbf{U})})$.

Let $U_1, \cdots, U_n, V$ be topological spaces such that $f : U_1 \times \cdots \times U_n \to V$ is an onto crisp function. Then, we can induce a fuzzy function $\tilde{f} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$ from f via the extension principle defined in Equation (10). However, even f is continuous, we cannot always induce a fuzzy function $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}_c(V)$, where the spaces $\mathcal{F}_c(U_i)$ and $\mathcal{F}_c(V)$, instead of $\mathcal{F}(U_i)$ and $\mathcal{F}(V)$, are considered for $i = 1. \cdots, n$. In fact, we can just induce a fuzzy function $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}(V)$, where the range is the space $\mathcal{F}(V)$. In other words, given $\tilde{A}^{(i)} \in \mathcal{F}_c(U_i)$ for $i = 1, \cdots, n$, we can just have $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}(V)$, and we cannot always guarantee $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_c(V)$. The purpose is to present the sufficient conditions to guarantee this desired result. First of all, we need a useful lemma.

Lemma 2. *(Wu [8]) Let $U_1, \cdots, U_n$ be topological spaces and let V be a Hausdorff space. Let $f : (U_1 \times \cdots \times U_n, \tau_{U_1 \times \cdots \times U_n}) \to (V, \tau_V)$ be a continuous and onto crisp function defined on $U_1 \times \cdots \times U_n$ and let $\tilde{f} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Assume that $\tilde{A}^{(i)} \in \mathcal{F}(U_i)$ such that its membership function $\xi_{\tilde{A}^{(i)}}$ is upper semicontinuous and each 0-level set $\tilde{A}^{(i)}_0$ of $\tilde{A}^{(i)}$ is a compact subset of U_i for all $i = 1, \cdots, n$. Suppose that conditions (c) and (d) for W_n are satisfied. Then, the supremum given in Equation (14) is attained.*

Proposition 7. *Let $U_1, \cdots, U_n, V$ be Hausdorff spaces, and let $f : (U_1 \times \cdots \times U_n, \tau_{U_1\times\cdots\times U_n}) \to (V, \tau_V)$ be a continuous and onto crisp function. Suppose that conditions (b), (c) and (d) for W_n are satisfied. If $\tilde{A}^{(i)} \in \mathcal{F}_c(U_i)$ for all $i = 1, \cdots, n$, then $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_c(V)$.*

Proof. Since the α-level sets $\tilde{A}^{(i)}_\alpha$ of $\tilde{A}^{(i)}$ are compact subsets of U_i for all $\alpha \in [0, 1]$ and $i = 1, \cdots, n$, part (i) of Proposition 4 says that $\tilde{A}^{(i)}_\alpha$ are also closed subsets of U_i for all $\alpha \in [0, 1]$ and $i = 1, \cdots, n$. In other words, the membership functions $\xi_{\tilde{A}^{(i)}}$ are upper semicontinuous for all $i = 1, \cdots, n$. Therefore, applying Lemma 2, we see that the supremum given in Equation (14) is attained for each $y \in V$. According to Wu ([8], Theorem 5.2), since f is continuous, we have:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_\alpha = f\left(\tilde{A}^{(1)}_\alpha, \cdots, \tilde{A}^{(n)}_\alpha\right)$$

for each $\alpha \in [0, 1]$, which is a compact subset of V by part (ii) of Proposition 4 and the Tychonoff's theorem in part (v) of Proposition 4, i.e., $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_c(V)$. This completes the proof. □

Under the assumptions of Proposition 7, we can indeed induce a fuzzy function $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}_c(V)$ based on the spaces $\mathcal{F}_c(U_i)$ and $\mathcal{F}_c(V)$, instead of the spaces $\mathcal{F}(U_i)$ and $\mathcal{F}(V)$ for $i = 1, \cdots, n$. Now, if we take $(U_i, \|\cdot\|_{U_i})$ and $(V, \|\cdot\|_V)$ to be the normed spaces, $i = 1, \cdots, n$, then we need to apply Theorem 5 to induce the fuzzy function $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}_c(V)$. Now we have the following result.

Theorem 9. *Let $(U_i, \|\cdot\|_{U_i})$ and $(V, \|\cdot\|_V)$ be the normed spaces for $i = 1, \cdots, n$, and let the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. We assume that $f : (\mathbf{U}, \|\cdot\|_{\mathbf{U}}) \to (V, \|\cdot\|_V)$ is a continuous and onto crisp function. Suppose that conditions (b), (c) and (d) for W_n are satisfied. Let $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}_c(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Then, the fuzzy function $\tilde{f} : (\mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n), d_{\mathcal{F}_c(\mathbf{U})}) \to (\mathcal{F}_c(V), d_{\mathcal{F}_c(V)})$ is continuous; that is, $\tilde{f} : (\mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n), \tau_{\mathcal{F}_c(\mathbf{U})}) \to (\mathcal{F}_c(V), \tau_{\mathcal{F}_c(V)})$ is continuous. If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f : U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$.*

Proof. Since a normed space can induce a Hausdorff topological space, from Remarks 2 and 3, Proposition 1 and the arguments of Proposition 7, Lemma 2 says that the supremum given in Equation (14) is attained for each $y \in V$. Therefore, from Theorem 5, we have:

$$\left(\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)})\right)_\alpha = f\left(\tilde{A}^{(1)}_\alpha, \cdots, \tilde{A}^{(n)}_\alpha\right)$$

for each $\alpha \in [0, 1]$. From the arguments of Proposition 7, we can induce a fuzzy function $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}_c(V)$. Suppose that $\tilde{\mathbf{A}}_m, \tilde{\mathbf{A}}_0 \in \mathcal{F}_c(\mathbf{U})$ for $m = 1, 2, \cdots$ satisfy $d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}}_m, \tilde{\mathbf{A}}_0) \to 0$ as $m \to \infty$. We need to show:

$$d_{\mathcal{F}_c(V)}\left(\tilde{f}(\tilde{\mathbf{A}}_m), \tilde{f}(\tilde{\mathbf{A}}_0)\right) \to 0$$

as $m \to \infty$. For $\mathbf{A}, \mathbf{B} \in \mathcal{K}(\mathbf{U})$, Proposition 6 says that, for any $\epsilon > 0$, there exists $\delta > 0$ such that:

$$d_{H_{\mathbf{U}}}(\mathbf{A}, \mathbf{B}) < \delta \text{ implies } d_{H_V}(f(\mathbf{A}), f(\mathbf{B})) = d_{H_V}(F(\mathbf{A}), F(\mathbf{B})) < \epsilon/2. \quad (19)$$

Since $d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}}_m, \tilde{\mathbf{A}}_0) \to 0$ as $m \to 0$, there exists $m_0 \in \mathbb{N}$ such that:

$$d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}}_m, \tilde{\mathbf{A}}_0) = \sup_{\alpha \in [0,1]} d_{H_\mathbf{U}}\left(\tilde{\mathbf{A}}_{m\alpha}, \tilde{\mathbf{A}}_{0\alpha}\right) < \delta$$

for $m > m_0$, where $\tilde{\mathbf{A}}_{m\alpha}$ and $\tilde{\mathbf{A}}_{0\alpha}$ denote the α-level sets of $\tilde{\mathbf{A}}_m$ and $\tilde{\mathbf{A}}_0$, respectively, which also says that $d_{H_\mathbf{U}}(\tilde{\mathbf{A}}_{m\alpha}, \tilde{\mathbf{A}}_{0\alpha}) < \delta$ for all $\alpha \in [0,1]$. Therefore, according to Equation (19), we have $d_{H_V}(f(\tilde{\mathbf{A}}_{m\alpha}), f(\tilde{\mathbf{A}}_{0\alpha})) < \epsilon/2$ for all $\alpha \in [0,1]$. Applying Lemma 2, we see that the supremum given in Equation (14) is attained for each $y \in V$. From Theorem 5, we also see that, for each $\alpha \in [0,1]$, $(\tilde{f}(\tilde{\mathbf{A}}_m))_\alpha = f(\tilde{\mathbf{A}}_{m\alpha})$ and $(\tilde{f}(\tilde{\mathbf{A}}_0))_\alpha = f(\tilde{\mathbf{A}}_{0\alpha})$. Therefore, we obtain:

$$\begin{aligned} d_{\mathcal{F}_c(V)}\left(\tilde{f}(\tilde{\mathbf{A}}_m), \tilde{f}(\tilde{\mathbf{A}}_0)\right) &= \sup_{\alpha \in [0,1]} d_{H_V}\left((\tilde{f}(\tilde{\mathbf{A}}_m))_\alpha, (\tilde{f}(\tilde{\mathbf{A}}_0))_\alpha\right) \\ &= \sup_{\alpha \in [0,1]} d_{H_V}\left(f(\tilde{\mathbf{A}}_{m\alpha}), f(\tilde{\mathbf{A}}_{0\alpha})\right) \le \epsilon/2 < \epsilon. \end{aligned}$$

for $m > m_0$. This completes the proof. □

Theorem 10. *Let $U_1, \cdots, U_n, V$ be normable topological vector spaces. We also assume that V is a Hausdorff space. Let $f : (U_1 \times \cdots \times U_n, \tau_{U_1 \times \cdots \times U_n}) \to (V, \tau_V)$ be a continuous and onto crisp function. Suppose that conditions (b), (c) and (d) for W_n are satisfied. Let $\tilde{f} : \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n) \to \mathcal{F}_c(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Then, the fuzzy function $\tilde{f} : (\mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n), d_{\mathcal{F}_c(\mathbf{U})}) \to (\mathcal{F}_c(V), d_{\mathcal{F}_c(V)})$ is continuous; that is, $\tilde{f} : (\mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n), \tau_{\mathcal{F}_c(\mathbf{U})}) \to (\mathcal{F}_c(V), \tau_{\mathcal{F}_c(V)})$ is continuous.*

Proof. From Lemma 2 and Wu ([8], Theorem 5.2), we see that, for each $\alpha \in [0,1]$, $(\tilde{f}(\tilde{\mathbf{A}}_m))_\alpha = f(\tilde{\mathbf{A}}_{m\alpha})$ and $(\tilde{f}(\tilde{\mathbf{A}}_0))_\alpha = f(\tilde{\mathbf{A}}_{0\alpha})$. Since the normable topological vector spaces are also Hausdorff spaces, the result follows immediately from the same arguments of Theorem 9. □

Example 1. *To apply Theorem 10, we take $V = \mathbb{R}^m$ and $U_i = \mathbb{R}$ for $i = 1, \cdots, n$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous and onto function, where the continuity is based on the usual topologies $\tau_{\mathbb{R}^n}$ and $\tau_{\mathbb{R}^m}$. Suppose that conditions (b), (c) and (d) for W_n are satisfied. For convenient, we write:*

$$\mathcal{F}_c(\mathbb{R}) \times \cdots \times \mathcal{F}_c(\mathbb{R}) = \mathcal{F}_c^n(\mathbb{R}).$$

Let $\tilde{f} : \mathcal{F}_c^n(\mathbb{R}) \to \mathcal{F}_c(\mathbb{R}^m)$ be a fuzzy function induced from f via the extension principle defined in (10). Then the fuzzy function $\tilde{f} : (\mathcal{F}_c^n(\mathbb{R}), d_{\mathcal{F}_c(\mathbb{R}^n)}) \to (\mathcal{F}_c(\mathbb{R}^m), d_{\mathcal{F}_c(\mathbb{R}^m)})$ is continuous.

Example 2. *Continued from Example 1, we consider a continuous and onto function $f : [0, 2\pi] \times [0, 2\pi] \to [-2, 2]$ defined by $f(x, y) = \sin x + \cos y$. Now we take $W_2(a, b) = \min\{a, b\}$. Then, conditions (b), (c) and (d) are satisfied automatically. In this case, the fuzzy function $\tilde{f} : (\mathcal{F}_c^2(\mathbb{R}), d_{\mathcal{F}_c(\mathbb{R}^2)}) \to (\mathcal{F}_c(\mathbb{R}), d_{\mathcal{F}_c(\mathbb{R})})$ is continuous, where the membership function of $\tilde{C} = \tilde{f}(\tilde{A}, \tilde{B})$ for $\tilde{A}, \tilde{B} \in \mathcal{F}_c(\mathbb{R})$ is given by:*

$$\xi_{\tilde{C}}(z) = \sup_{\{(x,y):z=f(x,y)\}} \min\left\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(y)\right\} = \sup_{\{(x,y):z=\sin x + \cos y\}} \min\left\{\xi_{\tilde{A}}(x), \xi_{\tilde{B}}(y)\right\}.$$

Let U be a topological space. We denote by $\mathcal{F}_0(U)$ the set of all fuzzy subsets of U such that, for each $\tilde{A} \in \mathcal{F}_0(U)$, its membership function $\xi_{\tilde{A}}$ is upper semicontinuous and the 0-level set $\tilde{A}_0$ is a compact subset of U.

Remark 5. *We have the following observations.*

- *Let U be a topological space. For any $\tilde{A} \in \mathcal{F}_0(U)$, by the upper semicontinuity of $\xi_{\tilde{A}}$, we see that $\tilde{A}_\alpha$ are closed subsets of U for all $\alpha \in (0,1]$. Since $\tilde{A}_\alpha \subseteq \tilde{A}_0$ for all $\alpha \in (0,1]$ and $\tilde{A}_0$ is a compact subset of U, part (iv) of Proposition 4 says that $\tilde{A}_\alpha$ are also compact subsets of U for all $\alpha \in (0,1]$. Therefore we conclude that $\mathcal{F}_0(U) \subseteq \mathcal{F}_c(U)$.*
- *Let U be a Hausdorff space. For any $\tilde{A} \in \mathcal{F}_c(U)$, we see that $\tilde{A}_\alpha$ are compact subsets of U for all $\alpha \in [0,1]$. From Proposition 4 (i), the α-level sets $\tilde{A}_\alpha$ are also closed subsets of U for all $\alpha \in (0,1]$, i.e., the membership function $\xi_{\tilde{A}}$ is upper semicontinuous. Therefore we conclude that $\mathcal{F}_0(U) = \mathcal{F}_c(U)$.*
- *If U is taken as a normable topological vector space or $(U, \|\cdot\|)$ is taken as the normed space, then $\mathcal{F}_0(U) = \mathcal{F}_c(U)$, since the normable topological vector space and normed space are also the Hausdorff spaces.*

Let $U_1, \cdots, U_n$ be topological spaces and let V be a Hausdorff space. Let $f : (U_1 \times \cdots \times U_n, \tau_{U_1\times\cdots\times U_n}) \to (V, \tau_V)$ be a continuous and onto crisp function. In general, we cannot induce a fuzzy function $\tilde{f} : \mathcal{F}_0(U_1) \times \cdots \times \mathcal{F}_0(U_n) \to \mathcal{F}_0(V)$ from f via the extension principle defined in Equation (10). The following proposition presents the sufficient conditions to guarantee this property.

Proposition 8. *Let $U_1, \cdots, U_n$ be topological spaces and let V be a Hausdorff space. Let $f : (U_1 \times \cdots \times U_n, \tau_{U_1\times\cdots\times U_n}) \to (V, \tau_V)$ be a continuous and onto crisp function. Suppose that conditions (c) and (d) for W_n are satisfied. If $\tilde{A}^{(i)} \in \mathcal{F}_0(U_i)$ for all $i = 1, \cdots, n$, then $\tilde{f}(\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_0(V)$.*

Proof. The result follows immediately from Wu ([8], Theorem 5.3). □

Under the assumptions of Proposition 8, we can induce a fuzzy function $\tilde{f} : \mathcal{F}_0(U_1) \times \cdots \times \mathcal{F}_0(U_n) \to \mathcal{F}_0(V)$. We also have the following interesting observations.

Remark 6. *We remark that Theorems 9 and 10 are still valid if $\mathcal{F}_c(U_i)$ and $\mathcal{F}_c(V)$ are replaced by $\mathcal{F}_0(U_i)$ and $\mathcal{F}_0(V)$ for all $i = 1, \cdots, n$, since the third observation of Remark 5 says that $\mathcal{F}_c(U_i) = \mathcal{F}_0(U_i)$ and $\mathcal{F}_c(V) = \mathcal{F}_0(V)$ for all $i = 1, \cdots, n$.*

Theorems 9 and 10 present the continuity of fuzzified functions based on the spaces $\mathcal{F}_c(U_i)$ and $\mathcal{F}_c(V)$ for $i = 1, \cdots, n$. In the sequel, we are going to investigate the continuity of fuzzified functions based on the spaces $\mathcal{F}_{cc}(U_i)$ and $\mathcal{F}_{cc}(V)$ for $i = 1, \cdots, n$.

Theorem 11. *Let $(U_i, \|\cdot\|_{U_i})$ and $(V, \|\cdot\|_V)$ be normed spaces for $i = 1, \cdots, n$, and let the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. We assume that $f : (\mathbf{U}, \|\cdot\|_{\mathbf{U}}) \to (V, \|\cdot\|_V)$ is a linear, continuous and onto crisp function. Suppose that conditions (b), (c), (d), (e) and (f) for W_n are satisfied. Let $\tilde{f} : \mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n) \to \mathcal{F}_{cc}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Then the fuzzy function $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), d_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), d_{\mathcal{F}_{cc}(V)})$ is continuous; that is, $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), \tau_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), \tau_{\mathcal{F}_{cc}(V)})$ is continuous. If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f : U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$.*

Proof. From Remarks 2 and 3 and part (iii) of Theorem 7, we see that, for each $\alpha \in [0,1]$,

$$\left(\tilde{f}(\tilde{\mathbf{a}}_m)\right)_\alpha = f(\tilde{\mathbf{a}}_{m\alpha}) \text{ and } \left(\tilde{f}(\tilde{\mathbf{a}}_0)\right)_\alpha = f(\tilde{\mathbf{a}}_{0\alpha}). \quad (20)$$

The result follows immediately from the same arguments of Theorem 9. □

Theorem 12. *Let $U_1, \cdots, U_n, V$ be normable topological vector spaces. We also assume that V is a Hausdorff space. Let $f : (U_1 \times \cdots \times U_n, \tau_{U_1 \times \cdots \times U_n}) \to (V, \tau_V)$ be a linear, continuous and onto crisp function. Suppose that conditions (b), (c), (d), (e) and (f) for W_n are satisfied. Let $\tilde{f} : \mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n) \to \mathcal{F}_{cc}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Then the fuzzy function $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), d_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), d_{\mathcal{F}_{cc}(V)})$ is continuous; that is, $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), \tau_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), \tau_{\mathcal{F}_{cc}(V)})$ is continuous.*

Proof. From Wu ([8], Corollary 6.1), we see that the equalities in (20) are satisfied for each $\alpha \in [0,1]$. The result follows immediately from the same arguments of Theorem 9. □

The linearity in Theorems 11 and 12 can be replaced by assuming the convexity.

Theorem 13. *Let $(U_i, \|\cdot\|_{U_i})$ and $(V, \|\cdot\|_V)$ be normed spaces for $i = 1, \cdots, n$, and let the product vector space $\mathbf{U} = U_1 \times \cdots \times U_n$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions Equations (12) and (13) are satisfied. Let $f : (\mathbf{U}, \|\cdot\|_{\mathbf{U}}) \to (V, \|\cdot\|_V)$ be a continuous and onto crisp function. We further assume that $f(A_1, \cdots, A_n)$ is a convex subset of V for any convex subsets A_i of U_i, $i = 1, \cdots, n$. Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Let $\tilde{f} : \mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n) \to \mathcal{F}_{cc}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Then the fuzzy function $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), d_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), d_{\mathcal{F}_{cc}(V)})$ is continuous; that is, $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), \tau_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), \tau_{\mathcal{F}_{cc}(V)})$ is continuous. If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U_1 \times \cdots \times U_n}$ such that the inclusions Equations (12) and (13) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f : U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U_1 \times \cdots \times U_n}$.*

Proof. From Remarks 2 and 3 and part (ii) of Theorem 8, we see that the equalities in Equation (20) are satisfied for each $\alpha \in [0,1]$. The result follows immediately from the same arguments of Theorem 9. □

Theorem 14. *Let $U_1, \cdots, U_n, V$ be normable topological vector spaces. We also assume that V is a Hausdorff space. Let $f : (U_1 \times \cdots \times U_n, \tau_{U_1 \times \cdots \times U_n}) \to (V, \tau_V)$ be a continuous and onto crisp function. We further assume that $f(A_1, \cdots, A_n)$ is a convex subset of V for any convex subsets A_i of U_i, $i = 1, \cdots, n$. Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Let $\tilde{f} : \mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n) \to \mathcal{F}_{cc}(V)$ be a fuzzy function induced from f via the extension principle defined in Equation (10). Then, the fuzzy function $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), d_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), d_{\mathcal{F}_{cc}(V)})$ is continuous; that is, $\tilde{f} : (\mathcal{F}_{cc}(U_1) \times \cdots \times \mathcal{F}_{cc}(U_n), \tau_{\mathcal{F}_{cc}(\mathbf{U})}) \to (\mathcal{F}_{cc}(V), \tau_{\mathcal{F}_{cc}(V)})$ is continuous.*

Proof. From Wu ([8], Theorem 6.2), we see that the equalities in (20) are satisfied for each $\alpha \in [0,1]$. The result follows immediately from the same arguments of Theorem 9. □

4. Applications to Fuzzy Topological Vector Spaces

Before introducing the concept of fuzzy vector space, we need to consider the fuzzy addition and fuzzy scalar multiplication. Let U be a vector space over $\mathbb{R}$. For any $\tilde{a}, \tilde{b} \in \mathcal{F}(U)$, the membership function of $\tilde{a} \oplus \tilde{b}$ is defined by:

$$\xi_{\tilde{a} \oplus \tilde{b}}(z) = \sup_{\{(x,y):z=x+y\}} W_2\left(\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\right).$$

Let $\tilde{\lambda} \in \mathcal{F}(\mathbb{R})$. The membership function of $\tilde{\lambda} \otimes \tilde{a}$ is defined by:

$$\xi_{\tilde{\lambda} \otimes \tilde{a}}(z) = \sup_{\{(\lambda,x):z=\lambda x\}} W_2\left(\xi_{\tilde{\lambda}}(\lambda), \xi_{\tilde{a}}(x)\right).$$

If $\tilde{\lambda}$ is taken to be the crisp number $\tilde{1}_{\{\lambda\}}$ with value $\lambda \in \mathbb{R}$, i.e.,

$$\tilde{1}_{\{\lambda\}}(r) = \begin{cases} 1 & \text{if } r = \lambda \\ 0 & \text{if } r \neq \lambda \end{cases}$$

then, we simply write $\tilde{1}_{\{\lambda\}} \otimes \tilde{a}$ as $\lambda\tilde{a}$ with the membership function given by:

$$\xi_{\lambda\tilde{a}}(z) = \sup_{\{(k,x):z=kx\}} W_2\left(\xi_{\tilde{1}_{\{\lambda\}}}(k), \xi_{\tilde{a}}(x)\right).$$

We have the following interesting observation.

Remark 7. *Suppose that W_2 satisfies the following conditions:*

$$W_2(0, a) = 0 \text{ and } W_2(1, a) = a \tag{21}$$

for any $a \in [0, 1]$. Then, we have

$$\xi_{\lambda\tilde{a}}(z) = \sup_{\{(k,x):z=kx\}} W_2\left(\xi_{\tilde{1}_{\{\lambda\}}}(k), \xi_{\tilde{a}}(x)\right) = \xi_{\tilde{a}}(z/\lambda).$$

If we take $W_2(a, b) = \min\{a, b\}$, then conditions in Equation (21) are satisfied automatically.

The following result will be used in the further discussion.

Proposition 9. *(Wu [8]) Let U be a Hausdorff topological vector space over $\mathbb{R}$. Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Then, we have the following results.*

- *(i) If $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(U)$, then $\tilde{a} \oplus \tilde{b} \in \mathcal{F}_{cc}(U)$ and $(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha \oplus \tilde{b}_\alpha$ for any $\alpha \in [0, 1]$.*
- *(ii) If $\tilde{\lambda} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\tilde{a} \in \mathcal{F}_{cc}(U)$, then $(\tilde{\lambda} \otimes \tilde{a})_\alpha = \tilde{\lambda}_\alpha \tilde{a}_\alpha$ for any $\alpha \in [0, 1]$.*
- *(iii) If we take $\tilde{\lambda}$ to be a nonnegative or nonpositive fuzzy number, then $\tilde{\lambda} \otimes \tilde{a} \in \mathcal{F}_{cc}(U)$.*
- *(iv) For $\lambda \in \mathbb{R}$, we have $\lambda\tilde{a} = \tilde{1}_{\{\lambda\}} \otimes \tilde{a} \in \mathcal{F}_{cc}(U)$, where $\tilde{1}_{\{\lambda\}}$ is a crisp number with value λ.*
- *(v) If $\tilde{\lambda}, \tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, then $\tilde{\lambda} \otimes \tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$.*

Definition 2. *Let U be a vector space over $\mathbb{R}$ and let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$. We say that $\mathcal{F}$ is a* fuzzy vector space over $\mathbb{R}$ *if and only if $\tilde{a} \oplus \tilde{b} \in \mathcal{F}$ and $\lambda\tilde{a} \in \mathcal{F}$ for any $\tilde{a}, \tilde{b} \in \mathcal{F}$ and $\lambda \in \mathbb{R}$. In other words, $\mathcal{F}$ is closed under the fuzzy addition and scalar multiplication.*

Proposition 10. *Let U be a Hausdorff topological vector space over $\mathbb{R}$. Then, the following statements hold true.*

- *(i) Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Then $\mathcal{F}_{cc}(U)$ is a fuzzy vector space over $\mathbb{R}$.*
- *(ii) Suppose that conditions (c) and (d) for W_n are satisfied. Then $\mathcal{F}_0(U) = \mathcal{F}_c(U)$ is a fuzzy vector space over $\mathbb{R}$.*

Proof. Part (i) follows immediately from parts (i) and (iv) of Proposition 9. Part (ii) follows immediately from Proposition 8 and the arguments of Proposition 9. □

If we consider $\mathcal{F}(\mathbb{R})$ instead of $\mathbb{R}$, then we can introduce another concept of fuzzy vector space in which we consider the so-called fuzzy scalar multiplication instead of scalar multiplication.

Definition 3. *Let U be a vector space over $\mathbb{R}$. Let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$ and $\mathcal{F}_{\mathbb{R}}$ be a subset of $\mathcal{F}(\mathbb{R})$. We say that $\mathcal{F}$ is a* fuzzy vector space over $\mathcal{F}_{\mathbb{R}}$ *if and only if $\tilde{a} \oplus \tilde{b} \in \mathcal{F}$ and $\tilde{\lambda}\tilde{a} \in \mathcal{F}$ for $\tilde{a}, \tilde{b} \in \mathcal{F}$ and $\tilde{\lambda} \in \mathcal{F}_{\mathbb{R}}$.*

In other words, $\mathcal{F}$ is closed under the fuzzy addition and fuzzy scalar multiplication, where the fuzzy scalar multiplication should be defined in another way.

Proposition 11. *Let U be a Hausdorff topological vector space over $\mathbb{R}$. Suppose that conditions (c) and (d) for W_n are satisfied. If the fuzzy scalar multiplication is defined as $\tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$, then $\mathcal{F}_0(U) = \mathcal{F}_c(U)$ is a fuzzy vector space over $\mathcal{F}_0(\mathbb{R}) = \mathcal{F}_c(\mathbb{R})$.*

Proof. The result follows immediately from Remark 5, Proposition 8 and the arguments of Proposition 9. □

We say that the fuzzy number $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$ is *nonnegative* if and only if $\xi_{\tilde{a}}(r) = 0$ for all $r < 0$, and *nonpositive* if and only if $\xi_{\tilde{a}}(r) = 0$ for all $r > 0$. We denote by $\mathcal{F}_{cc}^{+}(\mathbb{R})$ the set of all nonnegative fuzzy numbers, and by $\mathcal{F}_{cc}^{-}(\mathbb{R})$ the set of all nonpositive fuzzy numbers. Let $\tilde{a}$ be a fuzzy number. We define the membership functions of $\tilde{a}^{+}$ and $\tilde{a}^{-}$ as follows:

$$\xi_{\tilde{a}^{+}}(r) = \begin{cases} \xi_{\tilde{a}}(r) & \text{if } r > 0 \\ 1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r') < 1 \text{ for all } r' > 0 \\ \xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists a } r' > 0 \text{ such that } \xi_{\tilde{a}}(r') = 1 \\ 0 & \text{otherwise} \end{cases}$$

and:

$$\xi_{\tilde{a}^{-}}(r) = \begin{cases} \xi_{\tilde{a}}(r) & \text{if } r < 0 \\ 1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r') < 1 \text{ for all } r' < 0 \\ \xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists a } r' < 0 \text{ such that } \xi_{\tilde{a}}(r') = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\tilde{a}^{+}$ is a nonnegative fuzzy number and $\tilde{a}^{-}$ is a nonpositive fuzzy number, since the α-level sets $\tilde{a}_{\alpha}^{+}$ and $\tilde{a}_{\alpha}^{-}$ are closed intervals for all $\alpha \in [0, 1]$; that is, their membership functions $\xi_{\tilde{a}^{+}}(r)$ and $\xi_{\tilde{a}^{-}}(r)$ are upper semicontinuous (the other conditions in Definition 1 are obviously true). Furthermore, we have:

$$\tilde{a}_{\alpha} = \tilde{a}_{\alpha}^{+} \oplus \tilde{a}_{\alpha}^{-} = (\tilde{a}^{+} \oplus \tilde{a}^{-})_{\alpha}$$

for all $\alpha \in [0, 1]$. Thus $\tilde{a} = \tilde{a}^{+} \oplus \tilde{a}^{-}$. We call $\tilde{a}^{+}$ and $\tilde{a}^{-}$ the *positive part* and the *negative part* of $\tilde{a}$, respectively.

Proposition 12. *Let U be a Hausdorff topological vector space over $\mathbb{R}$. Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Then, the following statements hold true.*

(i) Let $\mathcal{F}_{cc}^{\pm}(\mathbb{R}) = \mathcal{F}_{cc}^{+}(\mathbb{R}) \cup \mathcal{F}_{cc}^{-}(\mathbb{R})$. If the fuzzy scalar multiplication is defined as $\tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$, then $\mathcal{F}_{cc}(U)$ is a fuzzy vector space over $\mathcal{F}_{cc}^{\pm}(\mathbb{R})$.

(ii) If the fuzzy scalar multiplication is defined as:

$$\tilde{\lambda}\tilde{a} = \begin{cases} \tilde{\lambda} \otimes \tilde{a} & \text{if } \tilde{\lambda} \in \mathcal{F}_{cc}^{\pm}(\mathbb{R}) \\ (\tilde{\lambda}^{+} \otimes \tilde{a}) \oplus (\tilde{\lambda}^{-} \otimes \tilde{a}) & \text{if } \tilde{\lambda} \in \mathcal{F}_{cc}(\mathbb{R}) \setminus \mathcal{F}_{cc}^{\pm}(\mathbb{R}), \end{cases}$$

where $\tilde{\lambda} = \tilde{\lambda}^{+} \oplus \tilde{\lambda}^{-}$, then $\mathcal{F}_{cc}(U)$ is a fuzzy vector space over $\mathcal{F}_{cc}(\mathbb{R})$.

Proof. Part (i) follows immediately from part (iii) of Proposition 9. To prove part (ii), we just need to claim that $\tilde{\lambda}\tilde{a} \in \mathcal{F}_{cc}(U)$ for $\tilde{\lambda} \in \mathcal{F}_{cc}(\mathbb{R}) \setminus \mathcal{F}_{cc}^{\pm}(\mathbb{R})$. By definition, we see that $\tilde{\lambda}^{+} \otimes \tilde{a}, \tilde{\lambda}^{-} \otimes \tilde{a} \in \mathcal{F}_{cc}(U)$ by part (iii) of Proposition 9, since $\tilde{\lambda}^{+}, \tilde{\lambda}^{-} \in \mathcal{F}_{cc}(\mathbb{R})$. By part (i) of Proposition 9 again, we see that $\tilde{\lambda}\tilde{a} = (\tilde{\lambda}^{+} \otimes \tilde{a}) \oplus (\tilde{\lambda}^{-} \otimes \tilde{a}) \in \mathcal{F}_{cc}(U)$. This completes the proof. □

In the sequel, we are going to introduce the concept of the fuzzy topological vector space. Recall that, for each normed space $(U_i, \|\cdot\|_{U_i})$, we can induce a metric space $(\mathcal{F}_c(U_i), d_{\mathcal{F}_c(U_i)})$ for $i = 1, \cdots, n$, where the metric $d_{\mathcal{F}_c(U_i)}$ is defined in (18). Given any $\epsilon > 0$, the open ball with center $\tilde{A} \in \mathcal{F}_c(U_i)$ is given by:

$$B_i(\tilde{A};\epsilon) = \left\{\tilde{B} \in \mathcal{F}_c(U_i) : d_{\mathcal{F}_c(U_i)}(\tilde{A},\tilde{B}) < \epsilon\right\}.$$

Based on these open balls, we can induce a topological space $(\mathcal{F}_c(U_i), \tau_{\mathcal{F}_c(U_i)})$, where $\tau_{\mathcal{F}_c(U_i)}$ is the so-called metric topology. Based on the topological spaces $(\mathcal{F}_c(U_i), \tau_{\mathcal{F}_c(U_i)})$ for $i = 1, \cdots, n$, we can also form a topological space $(\mathcal{F}_c(\mathbf{U}), \tau_{\mathcal{F}_c(U_1)\times\cdots\times\mathcal{F}_c(U_n)})$, where $\mathcal{F}_c(\mathbf{U}) = \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n)$ and $\tau_{\mathcal{F}_c(U_1)\times\cdots\times\mathcal{F}_c(U_n)}$ is the product topology. On the other hand, we can induce another metric space $(\mathcal{F}_c(\mathbf{U}), d_{\mathcal{F}_c(\mathbf{U})})$ by considering the norm $\|\cdot\|_{\mathbf{U}}$ or the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$. The open ball with center $\tilde{\mathbf{A}} \in \mathcal{F}_c(\mathbf{U})$ is given by:

$$\mathbf{B}(\tilde{\mathbf{A}};\epsilon) = \left\{\tilde{\mathbf{B}} \in \mathcal{F}_c(\mathbf{U}) : d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}},\tilde{\mathbf{B}}) < \epsilon\right\}.$$

Based on these open balls, we can also induce the metric topology $\tau_{\mathcal{F}_c(\mathbf{U})}$ for $\mathcal{F}_c(\mathbf{U})$. To introduce the concept of the fuzzy topological vector space, we need to provide some suitable conditions to guarantee the equality $\tau_{\mathcal{F}_c(U_1)\times\cdots\times\mathcal{F}_c(U_n)} = \tau_{\mathcal{F}_c(\mathbf{U})}$; that is, the product topology is equal to the metric topology.

Proposition 13. *Let $(U_i, \|\cdot\|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space which is endowed with a norm $\|\cdot\|_{\mathbf{U}}$. Given any $\epsilon > 0$ and:*

$$\tilde{\mathbf{A}} = (\tilde{A}^{(1)}, \cdots, \tilde{A}^{(n)}) \in \mathcal{F}_c(\mathbf{U}) = \mathcal{F}_c(U_1) \times \cdots \times \mathcal{F}_c(U_n),$$

if there exist $\hat{\epsilon}_1, \hat{\epsilon}_2 > 0$ such that the following inclusions hold true:

$$\mathbf{B}(\tilde{\mathbf{A}};\hat{\epsilon}_1) \subseteq B_1(\tilde{A}^{(1)};\epsilon) \times \cdots \times B_n(\tilde{A}^{(n)};\epsilon) \tag{22}$$

and:

$$B_1(\tilde{A}^{(1)};\hat{\epsilon}_2) \times \cdots \times B_n(\tilde{A}^{(n)};\hat{\epsilon}_2) \subseteq \mathbf{B}(\tilde{\mathbf{A}};\epsilon), \tag{23}$$

then the product topology $\tau_{\mathcal{F}_c(U_1)\times\cdots\times\mathcal{F}_c(U_n)}$ is equal to the metric topology $\tau_{\mathcal{F}_c(\mathbf{U})}$ for $\mathcal{F}_c(\mathbf{U})$. If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U_1\times\cdots\times U_n}$ such that the inclusions Equations (22) and (23) are satisfied, then the product topology is also equal to the metric topology.

Proof. The results follow immediately from the same arguments of Proposition 1. □

Lemma 3. *Let $(U_i, \|\cdot\|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space. Then, the following statements hold true.*

(i) We consider the normed space $(\mathbf{U}, \|\cdot\|_{\mathbf{U}})$. Given any $\epsilon > 0$, assume that $\|(u_1, \cdots, u_n)\|_{\mathbf{U}} < \epsilon$ if and only if $\|u_i\|_{U_i} < \epsilon$ for all $i = 1, \cdots, n$. Then, we have the following inclusions:

$$\mathbf{B}(\tilde{\mathbf{A}};\epsilon/2) \subseteq B_1(\tilde{A}^{(1)};\epsilon) \times \cdots \times B_n(\tilde{A}^{(n)};\epsilon) \tag{24}$$

and

$$B_1(\tilde{A}^{(1)};\epsilon/2) \times \cdots \times B_n(\tilde{A}^{(n)};\epsilon/2) \subseteq \mathbf{B}(\tilde{\mathbf{A}};\epsilon). \tag{25}$$

(ii) We consider the product normed space $(\mathbf{U}, \|\cdot\|_{U_1\times\cdots\times U_n})$, where the product norm is defined by (11). Given any $\epsilon > 0$, assume that $h(x_1, \cdots, x_n) < \epsilon$ if and only if $x_i < \epsilon$ for all $i = 1, \cdots, n$. Then, we also have the inclusions in Equations (24) and (25).

Proof. To prove part (i), for $\tilde{\mathbf{B}} = (\tilde{B}^{(1)}, \cdots, \tilde{B}^{(n)}) \in \mathbf{B}(\tilde{\mathbf{A}}; \epsilon/2)$, i.e., $d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) < \epsilon/2$, we have $d_{H_{\mathbf{U}}}(\tilde{\mathbf{A}}_\alpha, \tilde{\mathbf{B}}_\alpha) < \epsilon/2$ for all $\alpha \in [0,1]$; that is,

$$\sup_{\mathbf{a} \in \tilde{\mathbf{A}}_\alpha} \inf_{\mathbf{b} \in \tilde{\mathbf{B}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} < \epsilon/2 \text{ and } \sup_{\mathbf{b} \in \tilde{\mathbf{B}}_\alpha} \inf_{\mathbf{a} \in \tilde{\mathbf{A}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} < \epsilon/2$$

for all $\alpha \in [0,1]$. It suffices to consider the case of:

$$\sup_{\mathbf{a} \in \tilde{\mathbf{A}}_\alpha} \inf_{\mathbf{b} \in \tilde{\mathbf{B}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} < \epsilon/2$$

for all $\alpha \in [0,1]$. In this case, we see that $\inf_{\mathbf{b} \in \tilde{\mathbf{B}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} < \epsilon/2$ for all $\mathbf{a} \in \tilde{\mathbf{A}}_\alpha$ and $\alpha \in [0,1]$. Since $\tilde{\mathbf{B}}_\alpha$ is a compact subset of $\mathbf{U}$, by Lemma 1 and part (iii) of Proposition 4, we see that the above infimum is attained, i.e., $\| \mathbf{a} - \mathbf{b}^* \|_{\mathbf{U}} < \epsilon/2$ for some $\mathbf{b}^* \in \tilde{\mathbf{B}}_\alpha$, where $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b}^* = (b_1^*, \cdots, b_n^*)$ are in $\mathbf{U}$. By the assumption of $\| \cdot \|_{\mathbf{U}}$, we have $\| a_i - b_i^* \|_{U_i} < \epsilon/2$ for all $i = 1, \cdots, n$. Therefore, we also have:

$$\inf_{b_i \in \tilde{B}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} \leq \| a_i - b_i^* \|_{U_i} < \epsilon/2$$

for all $a_i \in \tilde{A}_\alpha^{(i)}$ and $\alpha \in [0,1]$, which also says that:

$$\sup_{a_i \in \tilde{A}_\alpha^{(i)}} \inf_{b_i \in \tilde{B}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} \leq \epsilon/2$$

for all $\alpha \in [0,1]$. By considering another case, we can similarly show that:

$$\sup_{b_i \in \tilde{B}_\alpha^{(i)}} \inf_{a_i \in \tilde{A}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} \leq \epsilon/2$$

for all $\alpha \in [0,1]$. Then, we have:

$$d_{\mathcal{F}_c(U_i)}(\tilde{A}^{(i)}, \tilde{B}^{(i)}) = \sup_{\alpha \in [0,1]} \max \left\{ \sup_{a_i \in \tilde{A}_\alpha^{(i)}} \inf_{b_i \in \tilde{B}_\alpha^{(i)}} \| a_i - b_i \|_{U_i}, \sup_{b_i \in \tilde{B}_\alpha^{(i)}} \inf_{a_i \in \tilde{A}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} \right\} \leq \epsilon/2 < \epsilon,$$

which says that $\tilde{B}^{(i)} \in B_i(\tilde{A}^{(i)}; \epsilon)$ for all $i = 1, \cdots, n$. Therefore, we conclude the inclusion:

$$\mathbf{B}(\tilde{\mathbf{A}}; \epsilon/2) \subseteq B_1(\tilde{A}^{(1)}; \epsilon) \times \cdots \times B_n(\tilde{A}^{(n)}; \epsilon).$$

On the other hand, for $\tilde{B}^{(i)} \in B_i(\tilde{A}^{(i)}; \epsilon/2)$, $i = 1, \cdots, n$, we have:

$$\sup_{a_i \in \tilde{A}_\alpha^{(i)}} \inf_{b_i \in \tilde{B}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} < \epsilon/2 \text{ and } \sup_{b_i \in \tilde{B}_\alpha^{(i)}} \inf_{a_i \in \tilde{A}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} < \epsilon/2$$

for all $\alpha \in [0,1]$. We consider the case of:

$$\sup_{a_i \in \tilde{A}_\alpha^{(i)}} \inf_{b_i \in \tilde{B}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} < \epsilon/2$$

for all $\alpha \in [0,1]$ and $i = 1, \cdots, n$. This says that $\inf_{b_i \in \tilde{B}_\alpha^{(i)}} \| a_i - b_i \|_{U_i} < \epsilon/2$ for all $a_i \in \tilde{A}_\alpha^{(i)}$, $\alpha \in [0,1]$ and $i = 1, \cdots, n$. By Lemma 1 and part (iii) of Proposition 4, we see that $\| a_i - b_i^* \|_{U_i} < \epsilon/2$ for some

$b_i^* \in \tilde{B}_\alpha^{(i)}$. By the assumption of $\| \cdot \|_{\mathbf{U}}$, there exists $\mathbf{b}^* = (b_1^*, \cdots, b_n^*) \in \tilde{\mathbf{B}}_\alpha$ such that $\| \mathbf{a} - \mathbf{b}^* \|_{\mathbf{U}} < \epsilon/2$ for all $\mathbf{a} \in \tilde{\mathbf{A}}_\alpha$ and $\alpha \in [0,1]$. Therefore, we obtain:

$$\sup_{\mathbf{a}\in\tilde{\mathbf{A}}_\alpha} \inf_{\mathbf{b}\in\tilde{\mathbf{B}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} \leq \sup_{\mathbf{a}\in\tilde{\mathbf{A}}_\alpha} \| \mathbf{a} - \mathbf{b}^* \|_{\mathbf{U}} \leq \epsilon/2$$

for all $\alpha \in [0,1]$. By considering another case, we can similarly show that:

$$\sup_{\mathbf{b}\in\tilde{\mathbf{B}}_\alpha} \inf_{\mathbf{a}\in\tilde{\mathbf{A}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} \leq \epsilon/2$$

for all $\alpha \in [0,1]$. Then, we have:

$$d_{\mathcal{F}_c(\mathbf{U})}(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) = \sup_{\alpha\in[0,1]} \max \left\{ \sup_{\mathbf{a}\in\tilde{\mathbf{A}}_\alpha} \inf_{\mathbf{b}\in\tilde{\mathbf{B}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}}, \sup_{\mathbf{b}\in\tilde{\mathbf{B}}_\alpha} \inf_{\mathbf{a}\in\tilde{\mathbf{A}}_\alpha} \| \mathbf{a} - \mathbf{b} \|_{\mathbf{U}} \right\} \leq \epsilon/2 < \epsilon.$$

Therefore, we conclude the inclusion:

$$B_1(\tilde{A}^{(1)}; \epsilon/2) \times \cdots \times B_n(\tilde{A}^{(n)}; \epsilon/2) \subseteq \mathbf{B}(\tilde{\mathbf{A}}; \epsilon).$$

The above same arguments can similarly apply to part (ii). This completes the proof. □

Proposition 14. *Let $(U_i, \| \cdot \|_{U_i})$ be normed spaces for $i = 1, \cdots, n$ and let $\mathbf{U} = U_1 \times \cdots \times U_n$ be the product vector space. Then, the following statements hold true.*

- *(i) We consider the normed space $(\mathbf{U}, \| \cdot \|_{\mathbf{U}})$. Given any $\epsilon > 0$, assume that $\| (u_1, \cdots, u_n) \|_{\mathbf{U}} < \epsilon$ if and only if $\| u_i \|_{U_i} < \epsilon$ for all $i = 1, \cdots, n$. Then the product topology $\tau_{\mathcal{F}_c(U_1)\times\cdots\times\mathcal{F}_c(U_n)}$ is equal to the metric topology $\tau_{\mathcal{F}_c(\mathbf{U})}$ for $\mathcal{F}_c(\mathbf{U})$.*
- *(ii) We consider the product normed space $(\mathbf{U}, \| \cdot \|_{U_1\times\cdots\times U_n})$, where the product norm is defined by (11). Given any $\epsilon > 0$, assume that $h(x_1, \cdots, x_n) < \epsilon$ if and only if $x_i < \epsilon$ for all $i = 1, \cdots, n$. Then the product topology $\tau_{\mathcal{F}_c(U_1)\times\cdots\times\mathcal{F}_c(U_n)}$ is equal to the metric topology $\tau_{\mathcal{F}_c(\mathbf{U})}$ for $\mathcal{F}_c(\mathbf{U})$.*

Proof. The results follow immediately from Proposition 13 and Lemma 3. □

We recall that if (U, τ) is a topological vector space, then the mappings of vector addition $(x, y) \mapsto x + y$ and scalar multiplication $(\lambda, x) \mapsto \lambda x$ are continuous under this topology τ. Therefore, we can also introduce the concept of fuzzy topological vector space as follows.

Definition 4. *Let U be a vector space over $\mathbb{R}$ and let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$. We say that $(\mathcal{F}, \tau_{\mathcal{F}})$ is a* fuzzy topological vector space over $\mathbb{R}$ *if and only if the following conditions are satisfied:*

- *$\mathcal{F}$ is a fuzzy vector space over $\mathbb{R}$;*
- *the mapping of fuzzy addition $(\mathcal{F} \times \mathcal{F}, \tau_{\mathcal{F}\times\mathcal{F}}) \to (\mathcal{F}, \tau_{\mathcal{F}})$ defined by $(\tilde{a}, \tilde{b}) \mapsto \tilde{a} \oplus \tilde{b}$ is continuous, where $\tau_{\mathcal{F}\times\mathcal{F}}$ is the product topology for $\mathcal{F} \times \mathcal{F}$;*
- *the mapping of scalar multiplication $(\mathbb{R} \times \mathcal{F}, \tau_{\mathbb{R}\times\mathcal{F}}) \to (\mathcal{F}, \tau_{\mathcal{F}})$ defined by $(\lambda, \tilde{a}) \mapsto \lambda\tilde{a}$ is continuous, where $\tau_{\mathbb{R}\times\mathcal{F}}$ is the product topology for $\mathbb{R} \times \mathcal{F}$.*

Remark 8. *Let $(U, \| \cdot \|_U)$ be a normed space, which can also induce a norm topology $\hat{\tau}_U$. It is well-known that the mapping $f_1 : (U \times U, \tau_{U\times U}) \to (U, \hat{\tau}_U)$ defined by $(x, y) \mapsto x + y$ and the mapping $f_2 : (\mathbb{R} \times U, \tau_{\mathbb{R}\times U}) \to (U, \hat{\tau}_U)$ defined by $(\lambda, x) \mapsto \lambda x$ are continuous, where $\tau_{U\times U}$ is the product topology for $U \times U$ formed by the norm topology $\hat{\tau}_U$ and $\tau_{\mathbb{R}\times U}$ is the product topology for $\mathbb{R} \times U$ formed by the norm topology $\hat{\tau}_U$ and the usual topology for $\mathbb{R}$. In this case, the normed space $(U, \| \cdot \|_U)$ becomes a Hausdorff topological vector space $(U, \hat{\tau}_U)$. Let $\mathbf{U} = U \times U$ be the product vector space such that $\mathbf{U}$ is endowed with the norm $\| \cdot \|_{\mathbf{U}}$ or the product norm $\| \cdot \|_{U\times U}$, which can also induce the norm topology $\hat{\tau}_{\mathbf{U}}$ or the product norm* *topology $\hat{\tau}_{U\times U}$. Let $U_{\mathbb{R}} = \mathbb{R} \times U$ be the product vector space such that $U_{\mathbb{R}}$ is endowed with the norm $\|\cdot\|_{U_{\mathbb{R}}}$ or the product norm $\|\cdot\|_{\mathbb{R}\times U}$, which can also induce the norm topology $\hat{\tau}_{U_{\mathbb{R}}}$ or the product norm topology $\hat{\tau}_{\mathbb{R}\times U}$. In this case, the mapping $f_1 : (U \times U, \hat{\tau}_{\mathbf{U}}) \to (U, \hat{\tau}_U)$ or $f_1 : (U \times U, \hat{\tau}_{U\times U}) \to (U, \hat{\tau}_U)$ defined by $(x, y) \mapsto x + y$ and the ampping $f_2 : (\mathbb{R} \times U, \hat{\tau}_{U_{\mathbb{R}}}) \to (U, \hat{\tau}_U)$ or $f_2 : (\mathbb{R} \times U, \hat{\tau}_{\mathbb{R}\times U}) \to (U, \hat{\tau}_U)$ defined by $(\lambda, x) \mapsto \lambda x$ are not necessarily continuous unless the inclusions in Equations (12) and (13) are satisfied.*

Theorem 15. *Let $(U, \|\cdot\|_U)$ be a normed space and let the product vector space $\mathbf{U} = U \times U$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions in Equations (12), (13), (22) and (23) are satisfied. Then, the following statements hold true.*

(i) Suppose that conditions (b), (c) and (d) for W_n are satisfied. Then,

$$(\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) = (\mathcal{F}_0(U), \tau_{\mathcal{F}_0(U)})$$

is a fuzzy topological vector space over $\mathbb{R}$.

(ii) Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Then, $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ is a fuzzy topological vector space over $\mathbb{R}$.

If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U\times U}$ such that the inclusions in Equations (12), (13), (22) and (23) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when we say that the function $f : U_1 \times \cdots \times U_n \to V$ is continuous directly in topological sense without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U\times U}$.

Proof. To prove part (i), using part (ii) of Proposition 10, it follows that $\mathcal{F}_c(U) = \mathcal{F}_0(U)$ is a fuzzy vector space over $\mathbb{R}$. From Remark 8, Proposition 13 and Theorem 9, we see that the mappings of fuzzy addition:

$$(\mathcal{F}_c(U) \times \mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)\times\mathcal{F}_c(U)}) \to (\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) \text{ by } (\tilde{a}, \tilde{b}) \mapsto \tilde{a} \oplus \tilde{b},$$

and scalar multiplication:

$$(\mathbb{R} \times \mathcal{F}_c(U), \tau_{\mathbb{R}\times\mathcal{F}_c(U)}) \text{ by } (\lambda, \tilde{a}) \mapsto \lambda\tilde{a}$$

are continuous, where $\lambda \in \mathbb{R}$ is regarded as the crisp number $\tilde{1}_{\{\lambda\}}$.

To prove part (ii), we consider the mapping $f_1 : U \times U \to U$ defined by $(x_1, x_2) \mapsto x_1 + x_2$ and the mapping $f_2 : \mathbb{R} \times U \to U$ defined by $(\lambda, x) \mapsto \lambda x$. From the arguments of Proposition 9, we see that, for any convex subsets A_1, A_2 of U and any convex subset A_3 of $\mathbb{R}$, $f_1(A_1, A_2)$ and $f_2(A_3, A_1)$ are also convex subsets of U. Therefore, the result follows immediately from Remark 8, Proposition 13 and Theorem 13. This completes the proof. □

Corollary 1. *Let $(U, \|\cdot\|_U)$ be a normed space and let the product vector space $\mathbf{U} = U \times U$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that, given any $\epsilon > 0$, $\| (u_1, u_2) \|_{\mathbf{U}} < \epsilon$ if and only if $\| u_i \|_{U_i} < \epsilon$ for $i = 1, 2$. Then, the following statements hold true.*

(i) Suppose that conditions (b), (c) and (d) for W_n are satisfied. Then,

$$(\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) = (\mathcal{F}_0(U), \tau_{\mathcal{F}_0(U)})$$

is a fuzzy topological vector space over $\mathbb{R}$.

(ii) Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Then $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ is a fuzzy topological vector space over $\mathbb{R}$.

Proof. The results follow immediately from Remark 8, part (i) of Proposition 2 part (i) of Proposition 14 and Theorem 15. □

Corollary 2. *Let $(U, \|\cdot\|_U)$ be a normed space and let the product vector space $\mathbf{U} = U \times U$ be endowed with a product norm $\|\cdot\|_{U\times U}$ that is defined by (11) such that, given any $\epsilon > 0$, $h(x_1, x_2) < \epsilon$ if and only if $x_i < \epsilon$ for $i = 1, 2$. Then, the following statements hold true.*

(i) Suppose that conditions (b), (c) and (d) for W_n are satisfied. Then:

$$(\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) = (\mathcal{F}_0(U), \tau_{\mathcal{F}_0(U)})$$

is a fuzzy topological vector space over $\mathbb{R}$.

(ii) Suppose that conditions (b), (c), (d) and (e) for W_n are satisfied. Then, $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ is a fuzzy topological vector space over $\mathbb{R}$.

Proof. The results follow immediately from Remark 8, part (ii) of Proposition 2 (ii), part (ii) of Proposition 14 and Theorem 15. □

In the sequel, we consider the case of fuzzy scalar multiplication.

Definition 5. *Let U be a vector space over $\mathbb{R}$. Let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$ and let $\mathcal{F}_{\mathbb{R}}$ be a subset of $\mathcal{F}(\mathbb{R})$. We say that $(\mathcal{F}, \tau_{\mathcal{F}})$ is a* fuzzy topological vector space over $\mathcal{F}_{\mathbb{R}}$ *if and only if the following conditions are satisfied:*

- *$\mathcal{F}$ is a fuzzy vector space over $\mathcal{F}_{\mathbb{R}}$;*
- *the mapping of fuzzy addition $(\mathcal{F} \times \mathcal{F}, \tau_{\mathcal{F}\times\mathcal{F}}) \to (\mathcal{F}, \tau_{\mathcal{F}})$ defined by $(\tilde{a}, \tilde{b}) \mapsto \tilde{a} \oplus \tilde{b}$ is continuous, where $\tau_{\mathcal{F}\times\mathcal{F}}$ is the product topology for $\mathcal{F} \times \mathcal{F}$; and*
- *the mapping of fuzzy scalar multiplication $(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}, \tau_{\mathcal{F}_{\mathbb{R}}\times\mathcal{F}}) \to (\mathcal{F}, \tau_{\mathcal{F}})$ defined by $(\tilde{\lambda}, \tilde{a}) \mapsto \tilde{\lambda}\tilde{a}$ is continuous, where $\tau_{\mathcal{F}_{\mathbb{R}}\times\mathcal{F}}$ is the product topology for $\mathcal{F}_{\mathbb{R}} \times \mathcal{F}$.*

Theorem 16. *Let $(U, \|\cdot\|_U)$ be a normed space and let the product vector space $\mathbf{U} = U \times U$ be endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that the inclusions in Equations (12), (13), (22) and (23) are satisfied. Suppose that conditions (b), (c) and (d) for W_n are satisfied. If the fuzzy scalar multiplication is defined as $\tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$, then*

$$(\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) = (\mathcal{F}_0(U), \tau_{\mathcal{F}_0(U)})$$

is a fuzzy topological vector space over $\mathcal{F}_c(\mathbb{R}) = \mathcal{F}_0(\mathbb{R})$. If the product vector space $\mathbf{U}$ is endowed with the product norm $\|\cdot\|_{U\times U}$ such that the inclusions in Equations (12), (13), (22) and (23) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when the normed space $(U, \|\cdot\|_U)$ is directly regarded as a Hausdorff topological vector space over $\mathbb{R}$ without considering the norm $\|\cdot\|_{\mathbf{U}}$ and the product norm $\|\cdot\|_{U\times U}$.

Proof. Proposition 11 says that $\mathcal{F}_c(U) = \mathcal{F}_0(U)$ is a fuzzy vector space over $\mathcal{F}_c(\mathbb{R}) = \mathcal{F}_0(\mathbb{R})$. From Remark 8, Proposition 13 and Theorem 9, we see that the mappings of fuzzy addition:

$$(\mathcal{F}_c(U) \times \mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)\times\mathcal{F}_c(U)}) \to (\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) \text{ defined by } (\tilde{a}, \tilde{b}) \mapsto \tilde{a} \oplus \tilde{b},$$

and fuzzy scalar multiplication:

$$(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(U), \tau_{\mathcal{F}_c(\mathbb{R})\times\mathcal{F}_c(U)}) \text{ defined by } (\tilde{\lambda}, \tilde{a}) \mapsto \tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$$

are continuous. This completes the proof. □

Corollary 3. *Let $(U, \|\cdot\|_U)$ be a normed space and let $\mathbf{U} = U \times U$ be the product vector space such that one of the following conditions is satisfied:*

- *the product vector space $\mathbf{U}$ is endowed with a norm $\|\cdot\|_{\mathbf{U}}$ such that, given any $\epsilon > 0$, $\|(u_1, u_2)\|_{\mathbf{U}} < \epsilon$ if and only if $\| u_i \|_{U_i} < \epsilon$ for $i = 1, 2$; and*

- *the product vector space* $\mathbf{U}$ *is endowed with a product norm* $\| \cdot \|_{U \times U}$ *that is defined by Equation (11) such that, given any* $\epsilon > 0$, $h(x_1, x_2) < \epsilon$ *if and only if* $x_i < \epsilon$ *for* $i = 1, 2$.

  If the fuzzy scalar multiplication is defined as $\tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$, *then:*

$$(\mathcal{F}_c(U), \tau_{\mathcal{F}_c(U)}) = (\mathcal{F}_0(U), \tau_{\mathcal{F}_0(U)})$$

is a fuzzy topological vector space over $\mathcal{F}_c(\mathbb{R}) = \mathcal{F}_0(\mathbb{R})$.

Proof. The results follow immediately from Remark 8, Propositions 2, 14 and Theorem 16. □

Theorem 17. *Let* $(U, \| \cdot \|_U)$ *be a normed space and let the product vector space* $\mathbf{U} = U \times U$ *be endowed with a norm* $\| \cdot \|_{\mathbf{U}}$ *such that the inclusions in Equations (12), (13), (22) and (23) are satisfied. Suppose that conditions (b), (c), (d) and (e) for* W_n *are satisfied. Then, the following statements hold true.*

- (i) *If the fuzzy scalar multiplication is defined by* $\tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$, *then* $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ *is a fuzzy topological vector space over* $\mathcal{F}_{cc}^{\pm}(\mathbb{R})$.
- (ii) *If the fuzzy scalar multiplication is defined by:*

$$\tilde{\lambda}\tilde{a} = \begin{cases} \tilde{\lambda} \otimes \tilde{a} & \text{if } \tilde{\lambda} \in \mathcal{F}_{cc}^{\pm}(\mathbb{R}) \\ (\tilde{\lambda}^{+} \otimes \tilde{a}) \oplus (\tilde{\lambda}^{-} \otimes \tilde{a}) & \text{if } \tilde{\lambda} \in \mathcal{F}_{cc}(\mathbb{R}) \setminus \mathcal{F}_{cc}^{\pm}(\mathbb{R}), \end{cases}$$

  where $\tilde{\lambda} = \tilde{\lambda}^{+} \oplus \tilde{\lambda}^{-}$, *then* $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ *is a fuzzy topological vector space over* $\mathcal{F}_{cc}(\mathbb{R})$.

If the product vector space $\mathbf{U}$ *is endowed with the product norm* $\| \cdot \|_{U \times U}$ *such that the inclusions in Equations (12), (13), (22) and (23) are satisfied, then we also have the same results. The assumptions satisfying the inclusions Equations (12) and (13) are not needed when the normed space* $(U, \| \cdot \|_U)$ *is directly regarded as a Hausdorff topological vector space over* $\mathbb{R}$ *without considering the norm* $\| \cdot \|_{\mathbf{U}}$ *and the product norm* $\| \cdot \|_{U \times U}$.

Proof. Part (i) of Proposition 12 says that $\mathcal{F}_{cc}(U)$ is a fuzzy vector space over $\mathcal{F}_{cc}^{\pm}(\mathbb{R})$. We consider the mapping $f_1 : U \times U \to U$ defined by $(x_1, x_2) \mapsto x_1 + x_2$ and the mapping $f_2 : \mathbb{R} \times U \to U$ defined by $(\lambda, x) \mapsto \lambda x$. From the arguments of Proposition 9, we see that, for any convex subsets A_1, A_2 of U and any convex subset A_3 of $\mathbb{R}$, $f_1(A_1, A_2)$ and $f_2(A_3, A_1)$ are also convex subsets of U. Therefore, the result follows immediately from Remark 8, Proposition 13 and Theorem 13. This completes the proof. □

Corollary 4. *Let* $(U, \| \cdot \|_U)$ *be a normed space and let* $\mathbf{U} = U \times U$ *be the product vector space such that one of the following conditions is satisfied:*

- *the product vector space* $\mathbf{U}$ *is endowed with a norm* $\| \cdot \|_{\mathbf{U}}$ *such that, given any* $\epsilon > 0$, $\| (u_1, u_2) \|_{\mathbf{U}} < \epsilon$ *if and only if* $\| u_i \|_{U_i} < \epsilon$ *for* $i = 1, 2$; *and*
- *the product vector space* $\mathbf{U}$ *is endowed with a product norm* $\| \cdot \|_{U \times U}$ *that is defined by Equation (11) such that, given any* $\epsilon > 0$, $h(x_1, x_2) < \epsilon$ *if and only if* $x_i < \epsilon$ *for* $i = 1, 2$.

  Then, the following statements hold true.

- (i) *If the fuzzy scalar multiplication is defined by* $\tilde{\lambda}\tilde{a} = \tilde{\lambda} \otimes \tilde{a}$, *then* $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ *is a fuzzy topological vector space over* $\mathcal{F}_{cc}^{\pm}(\mathbb{R})$.
- (ii) *If the fuzzy scalar multiplication is defined by:*

$$\tilde{\lambda}\tilde{a} = \begin{cases} \tilde{\lambda} \otimes \tilde{a} & \text{if } \tilde{\lambda} \in \mathcal{F}_{cc}^{\pm}(\mathbb{R}) \\ (\tilde{\lambda}^{+} \otimes \tilde{a}) \oplus (\tilde{\lambda}^{-} \otimes \tilde{a}) & \text{if } \tilde{\lambda} \in \mathcal{F}_{cc}(\mathbb{R}) \setminus \mathcal{F}_{cc}^{\pm}(\mathbb{R}), \end{cases}$$

  where $\tilde{\lambda} = \tilde{\lambda}^{+} \oplus \tilde{\lambda}^{-}$, *then* $(\mathcal{F}_{cc}(U), \tau_{\mathcal{F}_{cc}(U)})$ *is a fuzzy topological vector space over* $\mathcal{F}_{cc}(\mathbb{R})$.

Proof. The results follow immediately from Remark 8, Propositions 2, 14 and Theorem 17. □

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References

1. Hajek, P. *Metamathematics of Fuzzy Logic*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1998.
2. Demirci, M. Fuzzy functions and their fundamental properties. *Fuzzy Sets Syst.* **1999**, *106*, 239–246.
3. Höhle, U.; Porst, H.-E.; Sostak, A. Fuzzy functions: A fuzzy extension of the category SET and some related categories. *Appl. Gen. Topol.* **2000**, *1*, 115–127.
4. Zadeh, L.A. The concept of linguistic variable and its application to approximate reasoning I. *Inf. Sci.* **1975**, *8*, 199–249.
5. Zadeh, L.A. The concept of linguistic variable and its application to approximate reasoning II. *Inf. Sci.* **1975**, *8*, 301–357.
6. Zadeh, L.A. The concept of linguistic variable and its application to approximate reasoning III. *Inf. Sci.* **1975**, *9*, 43–80.
7. Román-Flores, H.; Barros, L.C.; Bassanezi, R.C. A note on Zadeh's extensions. *Fuzzy Sets Syst.* **2001**, *117*, 327–331.
8. Wu, H.-C. Generaized extension principle. *Fuzzy Optim. Decis. Mak.* **2010**, *9*, 31–68.
9. Nguyen, H.T. A note on the extension principle for fuzzy sets. *J. Math. Anal. Appl.* **1978**, *64*, 369–380.
10. Fullér, R.; Keresztfalvi, T. On generalization of Nguyen's theorem. *Fuzzy Sets Syst.* **1990**, *41*, 371–374.
11. Calvo, T.; Mayor, G.; Mesiar, R. *Aggregation Operators: New Trends and Applications*; Physica-Verlag: Heidelberg, Germany; New York, NY, USA, 2002.
12. Grabisch, M.; Marichal, J.-L.; Mesiar, R.; Pap, E. *Aggregation Functions*; Cambridge University Press: Cambridge, UK, 2009.
13. Kreyszig, E. *Introductory Functional Analysis with Applications*; John Wiley and Sons: New York, NY, USA, 1978.
14. Conway, J.B. *A Course in Functional Analysis*; Springer: New York, NY, USA, 1990.
15. Kelley, J.L.; Namioka, I. *Linear Topological Spaces*; Springer: New York, NY, USA, 1961.
16. Royden, H.L. *Real Analysis*, 2nd ed.; Macmillan Publishing Company: London, UK, 1968.