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Reflection Negative Kernels and Fractional Brownian Motion

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Abstract: In this article we study the connection of fractional Brownian motion, representation theory and reflection positivity in quantum physics. We introduce and study reflection positivity for affine isometric actions of a Lie group on a Hilbert space $\mathcal{E}$ and show in particular that fractional Brownian motion for Hurst index $0 < H \leq 1/2$ is reflection positive and leads via reflection positivity to an infinite dimensional Hilbert space if $0 < H < 1/2$. We also study projective invariance of fractional Brownian motion and relate this to the complementary series representations of $\mathrm{GL}_2(\mathbb{R})$. We relate this to a measure preserving action on a Gaussian L^2-Hilbert space $L^2(\mathcal{E})$.

Keywords: fractional brownian motion; reflection positivity; reflection negative kernels; representations of $SL_2(\mathbb{R})$

1. Introduction

In this paper we continue our investigations of the representation theoretic aspects of *reflection positivity* and its relations to stochastic processes ([1,2]). This is a basic concept in constructive quantum field theory [3–6], where it arises as a requirement on the euclidean side to establish a duality between euclidean and relativistic quantum field theories [7]. It is closely related to "Wick rotations" or "analytic continuation" in the time variable from the real to the imaginary axis.

The underlying structure is that of a *reflection positive Hilbert space*, introduced in [8]. This is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, $\theta : \mathcal{E} \to \mathcal{E}$ is a unitary involution and $\mathcal{E}_+$ is a closed subspace of $\mathcal{E}$ which is θ-positive in the sense that the hermitian form $\langle u, \theta v \rangle$ is positive semidefinite on $\mathcal{E}_+$. We write $\widehat{\mathcal{E}}$ for the corresponding Hilbert space and $q \colon \mathcal{E}_+ \to \widehat{\mathcal{E}}, \xi \mapsto \widehat{\xi}$ for the canonical map.

To relate this to group representations, let us call a triple (G, S, τ) a *symmetric semigroup* if G is a Lie group, τ is an involutive automorphism of G and $S \subseteq G$ a subsemigroup invariant under the involution $s \mapsto s^\sharp := \tau(s)^{-1}$. The Lie algebra $\mathfrak{g}$ of G decomposes into τ-eigenspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and we obtain the *Cartan dual Lie algebra* $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. We write G^c for a Lie group with Lie algebra $\mathfrak{g}^c$. The prototypical pair (G, G^c) consists of the euclidean motion group $E(d) = \mathbb{R}^d \rtimes \mathrm{O}_d(\mathbb{R})$ and the orthochronous Poincaré group $P(d)^\uparrow = \mathbb{R}^d \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})^\uparrow$. If (G, H, τ) is a symmetric Lie group and $(\mathcal{E}, \mathcal{E}_+, \theta)$ a reflection positive Hilbert space, then we say that a unitary representation $U \colon G \to \mathrm{U}(\mathcal{E})$ is *reflection positive with respect to* (G, S, τ) if

$$U_{\tau(g)} = \theta U_g \theta \quad \text{for } g \in G \quad \text{and} \quad U_S \mathcal{E}_+ \subseteq \mathcal{E}_+. \tag{1}$$

If $(\pi, \mathcal{E})$ is a reflection positive representation of G on $(\mathcal{E}, \mathcal{E}_+, \theta)$, then $\widehat{U}_s q(v) := q(U_s v)$ defines a representation $(\widehat{U}, \widehat{\mathcal{E}})$ of the involutive semigroup $(S, \sharp)$ by contractions ([8] Lemma 1.4, [4] or [9], Prop. 3.3.3). However, if S has interior points, we would like to have a unitary representation U^c of a Lie group G^c with Lie algebra $\mathfrak{g}^c$ on $\widehat{\mathcal{E}}$ whose derived representation is compatible with the representation of S. If such a representation exists, then we call $(U, \mathcal{E})$ a *euclidean realization* of the representation $(U^c, \widehat{\mathcal{E}})$ of G^c. Sufficient conditions for the existence of U^c have been developed in [10].

Althought this is a rather general framework, the present paper is only concerned with very concrete aspect of reflection positivity. The main new aspect we introduce is a notion of reflection positivity for affine isometric actions of a symmetric semigroup (G, S, τ) on a real Hilbert space. Here $\mathcal{E}_+$ is naturally defined by the closed subspace generated by the S-orbit of the origin. On the level of positive definite functions, this leads to the notion of a reflection negative function. For $(G, S, \tau) = (\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$, reflection negative functions ψ are easily determined because reflection negativity is equivalent to $\psi|_{(0,\infty)}$ being a Bernstein function ([11]). An announcement of some of the results in the present paper appeared in [2].

For a group G, affine isometric actions $\alpha_g \xi = U_g \xi + \beta_g$ on a real Hilbert space $\mathcal{E}$ are encoded in real-valued negative definite functions $\psi(g) = \|\beta_g\|^2$ satisfying $\psi(e) = 0$ (cf. [12,13]). Especially for $G = \mathbb{R}$, these structures have manifold applications in various fields of mathematics (see for instance [14–16], and also [17] for the generalization to *spirals* which corresponds to actions of $\mathbb{R}$ by affine conformal maps). For the group $G = (\mathbb{R}, +)$, the homogeneous function $\psi(x) = |x|^{2H}$ is negative definite if and only if $0 \leq H \leq 1$, and this leads to the positive definite kernels

$$C^H(s,t) := \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}),$$

which for $0 < H < 1$ are the covariance kernels of fractional Brownian motion with Hurst index H ([18–22]).

One of the central results of this paper is an extension of the well-known projective invariance of Brownian motion in the sense of P. Lévy (cf. [23] §I.2, and [24]) to fractional Brownian motion. Here we use the identification of $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ with the real projective line, which leads to the action of $\mathrm{GL}_2(\mathbb{R})$ by Möbius transformations $g.x = \frac{ax+b}{cx+d}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Starting from a realization of fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$ with Hurst index $H \in (0,1)$ in a suitable Hilbert space $\mathcal{H}_H$ by the functions

$$b_t^H := \mathrm{sgn}(t)\chi_{[t\wedge 0, t\vee 0]} = \chi_{[0,\infty)} - \chi_{[t,\infty)}, \qquad t \in \mathbb{R}, \tag{2}$$

we associate to every pair of distinct points α, γ in $\mathbb{R}_\infty$ a normalized process whose covariance kernels $C^H_{\alpha,\gamma}$ transform naturally under Möbius transformations in the sense that

$$C^H_{g.\alpha, g.\beta}(g.s, g.t) = C^H_{\alpha,\beta}(s,t). \tag{3}$$

Here the normalized fractional Brownian motion $\widetilde{B}_t^H = |t|^{-H} B_t^H$ has the covariance kernel $C^H_{0,\infty}$ and the transformed process $\widetilde{B}^H_{g.t}$ is equivalent to the original one.

The structure of this paper is as follows. In Section 2 we briefly recall the general background of reflection positive Hilbert spaces and representations and in Section 3 we introduce reflection positive affine isometric actions $U\colon G \to \mathrm{Mot}(\mathcal{E})$ on real Hilbert spaces $\mathcal{E}$. Since the group $\mathrm{Mot}(\mathcal{E})$ has a natural unitary representation on the Fock space $\Gamma(\mathcal{E})$, the L^2-space of the canonical Gaussian measure of $\mathcal{E}$, affine isometric representations are closely linked with symmetries of Gaussian stochastic processes for which G acts on the corresponding index set. This is made precise in Appendix B.1, where we discuss the measure preserving G-action corresponding to a stochasic process with stationary increments. For square integrable processes, this connects with affine isometric actions on Hilbert spaces.

To pave the way for the analysis of the interaction of fractional Brownian motion with unitary representations, we introduce in Section 4 a family of unitary representations $(U^H, \mathcal{H}_H)_{0<H<1}$ of $\mathrm{GL}_2(\mathbb{R})$, respectively its projective quotient $\mathrm{PGL}_2(\mathbb{R})$, i.e., the group of Möbius transformations on the real projective line. For $H = \frac{1}{2}$ this is the natural representation on $L^2(\mathbb{R})$ (belonging to the principal series), whereas for $H \neq \frac{1}{2}$ it belongs to the complementary series ([4,25]). The Hilbert spaces $\mathcal{H}_H$ are obtained from positive definite distribution kernels by completion of $\mathcal{S}(\mathbb{R})$ with respect to the scalar product

$$\langle \xi, \eta \rangle_H = -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\xi'(x)} \eta'(y) |x-y|^{2H} \, dx \, dy.$$

In Section 5 we realize fractional Brownian motion in a very natural way in terms of the cocycle (2) defining an affine isometric action of the translation group $(\mathbb{R}, +)$ on $\mathcal{H}_H$. Acting with the group $\mathrm{GL}_2(\mathbb{R})$ on these functions leads naturally to the projective invariance of fractional Brownian motion, both on the level of the normalized kernels as in (3), and with respect to our concrete realization (Theorem 1).

Reflection positivity is then explored in Section 6. For $\alpha \neq \gamma$ in $\mathbb{R}_\infty \cong \mathbb{S}^1$ we consider a reflection θ with a fixed point and exchanging α and γ. Here our main result is Theorem 2, asserting that the normalized kernels $C^H_{\alpha,\gamma}$ on the complement of the two-element set $\{\alpha, \gamma\}$ in $\mathbb{R}_\infty \cong \mathbb{S}^1$ is reflection positive with respect to θ if and only if $H \leq \frac{1}{2}$. In particular, this implies reflection positivity for a Brownian bridge on a real interval $[\alpha, \gamma]$ with respect to the reflection in the midpoint. Reflection positivity for the complementary series representations of $\mathrm{SL}_2(\mathbb{R})$ has already been observed in [4], where the representation U^c is identified as a holomorphic discrete series representation.

Reflection positivity for the affine action of the translation group in $\mathcal{H}_H$ defined by the cocycle b^H_t realizing fractional Brownian motion is studied in Section 7. Although we always have involutions that lead to reflection positive Hilbert spaces in a natural way, only for $H \leq \frac{1}{2}$ we obtain reflection positive affine actions of $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id})$. We conclude Section 7 with a discussion of the increments of a 1-cocycle $(\beta_t)_{t\in\mathbb{R}}$ defining an affine isometric action. In particular, we characterize cocycles with orthogonal increments as those corresponding to multiples of Brownian motion. Note that the increments of fractional Brownian motion are positively correlated for $H \geq \frac{1}{2}$ and negatively correlated for $H \leq \frac{1}{2}$. We conclude this paper with a brief discussion of some related results concerning higher dimensional spaces in Section 8. We plan to return to the corresponding representation theoretic aspects in the near future.

In order not to distract the reader from the main line of the paper, we moved several auxiliary tools and some definitions and calculations into appendices: Appendix A deals with affine isometries and positive definite kernels and Appendix B reviews some properties of stochastic processes. In particular, we provide in Proposition A3 a representation theoretic proof for the Lévy–Khintchine formula for the real line, which represents a negative definite function in terms of its spectral measure ([19,20,23,26] Thm. 32). Appendix C briefly recalls the measure theoretic perspective on Second Quantization, Appendix D contains the verification that the representations U^H mentioned above are unitary, and Appendix E contains a calculation of the spectral measure for fractional Brownian motion.

A different kind of projective invariance, in the path parameter t, for one-dimensional Brownian motion has been observed by S. Takenaka in [27]: For a Brownian motion $(B_t)_{t\in\mathbb{R}}$, the process

$$B^g_t := (ct+d)B_{g.t} - ct \cdot B_{g.\infty} - d \cdot B_{g.0}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \qquad t \in \mathbb{R}_\infty, g.t \neq \infty,$$

also is a Brownian motion, and the relation $(B^g)^h = B^{gh}$ leads to a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on the realization space. From that he derives the projective invariance in the sense of Lévy, and he argues that his method does not extend to fractional Brownian motion. In [28], Takenaka shows that the representation of $\mathrm{SL}_2(\mathbb{R})$ he obtains belongs to the discrete series, so that it is differet from ours. He also hints at the possibility of extending Hida's method [24] to fractional Brownian motion, and in a certain sense this is carried out in the present paper.

2. Reflection Positive Functions and Representations

Since our discussion is based on positive definite kernels and the associated Hilbert spaces ([29,30] Ch. I, [9]), we first recall the pertinent definitions. As customary in physics, we follow the convention that the inner product of a complex Hilbert space is linear in the second argument.

Definition 1. *(a) Let X be a set. A kernel $Q\colon X \times X \to \mathbb{C}$ is called hermitian if $Q(x,y) = \overline{Q(y,x)}$. A hermitian kernel Q is called positive definite if for $x_1, \ldots, x_n \in X, c_1, \ldots, c_n \in \mathbb{C}$, we have $\sum_{j,k=1}^{n} c_j \overline{c_k} Q(x_j, x_k) \geq 0$. It is called negative definite if $\sum_{j,k=1}^{n} c_j \overline{c_k} Q(x_j, x_k) \leq 0$ holds for $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum c_j = 0$ ([31]).*

(b) If $(S, *)$ is an involutive semigroup, then $\varphi\colon S \to \mathbb{C}$ is called positive (negative) definite if the kernel $(\varphi(st^*))_{s,t \in S}$ is positive (negative) definite. If G is a group, then we consider it as an involutive semigroup with $g^* := g^{-1}$ and definite positive/negative definite functions accordingly.

We shall use the following lemma to translate between positive definite and negative definite kernels ([31], Lemma 3.2.1):

Lemma 1. *Let X be a set, $x_0 \in X$ and $Q\colon X \times X \to \mathbb{C}$ be a hermitian kernel. Then the kernel*

$$K(x,y) := Q(x,x_0) + Q(x_0,y) - Q(x,y) - Q(x_0,x_0)$$

is positive definite if and only if Q is negative definite.

Remark 1. *According to Schoenberg's Theorem ([31] Thm. 3.2.2), a kernel $Q\colon X \times X \to \mathbb{C}$ is negative definite if and only if, for every $h > 0$, the kernel e^{-hQ} is positive definite.*

Remark 2. *Let X be a set, $K\colon X \times X \to \mathbb{C}$ be a positive definite kernel and $\mathcal{H}_K \subseteq \mathbb{C}^X$ be the corresponding reproducing kernel Hilbert space. This is the unique Hilbert subspace of $\mathbb{C}^X$ on which all point evaluations $f \mapsto f(x)$ are continuous and given by*

$$f(x) = \langle K_x, f \rangle \quad \text{for} \quad K(x,y) = K_y(x) = \langle K_x, K_y \rangle.$$

Then the map $\gamma\colon X \to \mathcal{H}_K, \gamma(x) = K_x$ has total range and satisfies $K(x,y) = \langle \gamma(x), \gamma(y) \rangle$. The latter property determines the pair $(\gamma, \mathcal{H}_K)$ up to unitary equivalence ([30] Ch. I).

Definition 2. *A reflection positive Hilbert space is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, θ a unitary involution and $\mathcal{E}_+$ is a closed subspace which is θ-positive in the sense that the hermitian form $\langle \xi, \eta \rangle_\theta := \langle \xi, \theta\eta \rangle$ is positive semidefinite on $\mathcal{E}_+$.*

For a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$, let $\mathcal{N} := \{\xi \in \mathcal{E}_+ \colon \langle \xi, \theta\xi \rangle = 0\}$ and write $\widehat{\mathcal{E}}$ for the completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\theta$. We write $q\colon \mathcal{E}_+ \to \widehat{\mathcal{E}}, \xi \mapsto \widehat{\xi}$ for the canonical map.

Example 1. *Suppose that $K\colon X \times X \to \mathbb{C}$ is a positive definite kernel and $\tau\colon X \to X$ is an involution leaving K invariant and that $X_+ \subseteq X$ is a subset with the property that the kernel $K^\tau(x,y) := K(x, \tau y)$ is also positive definite on X_+. We call such kernels K reflection positive with respect to (X, X_+, τ). Then the closed subspace $\mathcal{E}_+ \subseteq \mathcal{E} := \mathcal{H}_K$ generated by $(K_x)_{x \in X_+}$ is θ-positive for $(\theta f)(x) := f(\tau x)$. We thus obtain a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$.*

In this context, the space $\widehat{\mathcal{E}}$ can be identified with the reproducing kernel space $\mathcal{H}^{K^\tau} \subseteq \mathbb{C}^{X_+}$, where q corresponds to the map

$$q\colon \mathcal{E}_+ \to \mathcal{H}_{K^\tau}, \quad q(f)(x) := f(\tau(x))$$

([9] Lemma 2.4.2).

For a symmetric semigroup (G, S, τ), we obtain natural classes of reflection positive kernels:

Definition 3. *A function $\varphi\colon G \to \mathbb{C}$ on a group G is called reflection positive ([11]) if the kernel $K(x,y) := \varphi(xy^{-1})$ is reflection positive with respect to (G,S,τ) in the sense of Example 1 with $X = G$ and $X_+ = S$. These are two simultaneous positivity conditions, namely that the kernel $\varphi(gh^{-1})_{g,h\in G}$ is positive definite on G and that the kernel $\varphi(st^\sharp)_{s,t\in S}$ is positive definite on S.*

The usal Gelfand–Naimark–Segal construction naturally extends to reflection positive functions and provides a correspondence with reflection positive representations (see [9] Thm. 3.4.5).

Definition 4. *For a symmetric semigroup (G,S,τ), a unitary representation U of G on a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$ is called reflection positive if $\theta U_g \theta = U_{\tau(g)}$ for $g \in G$ and $U_s\mathcal{E}_+ \subseteq \mathcal{E}_+$ for every $s \in S$.*

Remark 3. *(a) If $(U_g)_{g\in G}$ is a reflection positive representation of (G,S,τ) on $(\mathcal{E},\mathcal{E}_+,\theta)$, then we obtain contractions $(\widehat{U}_s)_{s\in S}$ on $\widehat{\mathcal{E}}$, determined by*

$$\widehat{U}_s \circ q = q \circ U_s|_{\mathcal{E}_+} \quad \textit{for} \quad s \in S,$$

and this leads to an involutive representation $(\widehat{U}, \widehat{\mathcal{E}})$ of S by contractions (cf. [32] Cor. 3.2, [8] or [9]). We then call $(U, \mathcal{E}, \mathcal{E}_+, \theta)$ a euclidean realization of $(\widehat{U}, \widehat{\mathcal{E}})$.

(b) For $(G,S,\tau) = (\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$, continuous reflection positive unitary one-parameter groups $(U_t)_{t\in\mathbb{R}}$ lead to a strongly continuous semigroup $(\widehat{U}, \widehat{\mathcal{E}})$ of hermitian contractions and every such semigroup $(C, \mathcal{H})$ has a natural euclidean realization obtained as the GNS representation associated to the positive definite operator-valued function $\varphi(t) := C_{|t|}$, $t \in \mathbb{R}$ ([33] [Prop. 6.1]).

Example 2. *On $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id})$, we have:*

- *(a) For $0 \le \alpha \le 2$, the function $|x|^\alpha$ on $(\mathbb{R}, +)$ is negative definite by [31] Cor. 3.2.10 because x^2 is obviously negative definite.*
- *(b) For $\alpha \ge 0$, the function $|x|^\alpha$ is reflection negative if and only if $0 \le \alpha \le 1$ ([11] Ex. 4.3(a)).*
- *(c) The function $-|x|^\alpha$ is reflection negative for $1 \le \alpha \le 2$ ([31] Ex. 6.5.15, [11] Ex. 4.4(a)).*

3. Reflection Positivity for Affine Actions

In this section we introduce reflection positive affine isometric actions $U\colon G \to \mathrm{Mot}(\mathcal{E})$ on real Hilbert spaces $\mathcal{E}$ and relate it to the corresponding measure preserving action on the Gaussian L^2-space $\Gamma(\mathcal{E})$.

Let (G,S,τ) be a symmetric semigroup and $\mathcal{E}$ be a real Hilbert space, endowed with an isometric involution θ. We consider an affine isometric action

$$\alpha_g v = U_g v + \beta_g \quad \text{for} \quad g \in G, v \in \mathcal{E}, \tag{4}$$

where $U\colon G \to \mathrm{O}(\mathcal{E})$ is an orthogonal representation and $\beta\colon G \to \mathcal{E}$ a 1-cocycle, i.e.,

$$\beta_{gh} = \beta_g + U_g\beta_h = \alpha_g \beta_h \quad \text{for} \quad g,h \in G. \tag{5}$$

Note that (5) in particular implies $\beta_e = 0$ and thus $\beta_{g^{-1}} = -U_g^{-1}\beta_g$. We further assume that $\theta\alpha_g\theta = \alpha_{\tau(g)}$, which is equivalent to

$$\theta U_g \theta = U_{\tau(g)} \quad \text{and} \quad \theta\beta_g = \beta_{\tau(g)} \quad \text{for} \quad g \in G. \tag{6}$$

If β_G is total in $\mathcal{E}$, then we can realize $\mathcal{E}$ as a reproducing kernel Hilbert space $\mathcal{H}_C \subseteq \mathbb{R}^G$ with kernel

$$C(s,t) := \langle \beta_s, \beta_t \rangle, \qquad s,t \in G.$$

For the function
$$\psi\colon G \to \mathbb{R}, \quad \psi(g) := \|\beta_g\|^2 = C(g,g),$$
we then obtain
$$\psi(s^{-1}t) = \|\beta_{s^{-1}t}\|^2 = \|\beta_{s^{-1}} + U_s^{-1}\beta_t\|^2 = \|U_s\beta_{s^{-1}} + \beta_t\|^2 = \|\beta_t - \beta_s\|^2 = \psi(s) + \psi(t) - 2C(s,t),$$
so that
$$C(s,t) = \frac{1}{2}\big(\psi(s) + \psi(t) - \psi(s^{-1}t)\big) \quad \text{and} \quad \psi(s^{-1}t) = C(s,s) + C(t,t) - 2C(t,s). \tag{7}$$

In view of (7), ψ is negative definite by Lemma 1. Equation (7) implies that, if β_G is total, then the affine action α can be recovered completely from the function ψ and every real-valued negative definite function $\psi\colon G \to \mathbb{R}$ with $\psi(e) = 0$ is of this form (cf. [12,13]). We also note that $\theta\beta_g = \beta_{\tau(g)}$ implies that $\psi \circ \tau = \psi$.

Definition 5. *(Reflection positive affine actions) The closed subspace $\mathcal{E}_+$ generated by $(\beta_s)_{s\in S}$ is invariant under the affine action of S on $\mathcal{E}$ because $\alpha_s\beta_t = \beta_{st}$ for $s,t \in S$. We call the affine action $(\alpha,\mathcal{E})$ reflection positive with respect to (G,S,τ) if $\mathcal{E}_+$ is θ-positive.*

Example 3. *(A universal example) Let $(\mathcal{E},\mathcal{E}_+,\theta)$ be a reflection positive real Hilbert space, $\mathcal{E}_- := \theta(\mathcal{E}_+)$ and write $\mathrm{Mot}(\mathcal{E}) \cong \mathcal{E} \rtimes \mathrm{O}(\mathcal{E})$ for its motion group. We define an involution on $\mathrm{Mot}(\mathcal{E})$ by $\tau(b,g) := (\theta b, \theta g\theta)$. For $\gamma \in \mathrm{Mot}(\mathcal{E})$ we put $\gamma^\sharp := \tau(\gamma)^{-1}$. Then*
$$S := \{\gamma \in \mathrm{Mot}(\mathcal{E})\colon \gamma\mathcal{E}_+ \subseteq \mathcal{E}_+, \gamma^\sharp\mathcal{E}_+ \subseteq \mathcal{E}_+\} = \{\gamma \in \mathrm{Mot}(\mathcal{E})\colon \gamma\mathcal{E}_+ \subseteq \mathcal{E}_+, \gamma\mathcal{E}_- \supseteq \mathcal{E}_-\}$$
is a $\sharp$-invariant subsemigroup of $\mathrm{Mot}(\mathcal{E})$ with
$$S \cap S^{-1} = \{\gamma \in \mathrm{Mot}(\mathcal{E})\colon \gamma(\mathcal{E}_+) = \mathcal{E}_+, \gamma(\mathcal{E}_-) = \mathcal{E}_-\}.$$

By construction, the affine action of $\mathrm{Mot}(\mathcal{E})$ on $\mathcal{E}$ is reflection positive in the sense of Definition 5.

For $\gamma = (b,g)$, the relation $\gamma(\mathcal{E}_+) = \mathcal{E}_+$ is equivalent to $b \in \mathcal{E}_+$ and $g\mathcal{E}_+ = \mathcal{E}_+$. This shows that $(b,g) \in S \cap S^{-1}$ is equivalent to $b \in \mathcal{E}_+ \cap \theta(\mathcal{E}_+) = (\mathcal{E}_+)^\theta$ (because of θ-positivity) and to the condition that the restrictionss of g to $\mathcal{E}_\pm$ are unitary.

The positive definite kernel $Q(x,y) := e^{-\|x-y\|^2/2}$ (Appendix C) is reflection positive with respect to (G,S,τ) because the kernel $Q^\theta(x,y) = Q(x,\theta y) = e^{-\|x-\theta y\|^2/2}$ is positive definite on $\mathcal{E}_+$ (cf. Example 1). From the $\mathrm{Mot}(\mathcal{E})$-invariance of Q, we thus obtain a reflection positive representation of $(\mathrm{Mot}(\mathcal{E}),S,\tau)$ on the corresponding reflection positive Hilbert space $(\Gamma(\mathcal{E}),\Gamma(\mathcal{E}_+),\Gamma(\theta))$.

It is instructive to make the corresponding space $\widehat{\Gamma}(\mathcal{E})$ more explicit and to see how it identifies with $\Gamma(\widehat{\mathcal{E}})$. From
$$\langle \widehat{e^{i\varphi(v)}}, \widehat{e^{i\varphi(w)}}\rangle = \langle e^{i\varphi(v)}, \Gamma(\theta)e^{i\varphi(w)}\rangle = \langle e^{i\varphi(v)}, e^{i\varphi(\theta w)}\rangle = e^{-\frac{1}{2}\|v-\theta w\|^2} = e^{-\frac{1}{2}(\|v\|^2+\|w\|^2)+\langle v,\theta w\rangle}$$
and
$$\langle e^{i\varphi(\widehat{v})}, e^{i\varphi(\widehat{w})}\rangle = e^{-\frac{1}{2}\|\widehat{v}-\widehat{w}\|^2} = e^{-\frac{1}{2}(\langle v,\theta v\rangle + \langle w,\theta w\rangle)+\langle v,\theta w\rangle} \quad \text{in} \quad \Gamma(\widehat{\mathcal{E}}),$$
we derive that
$$e^{i\varphi(\widehat{v})} = e^{\frac{1}{2}(\|v\|^2-\langle v,\theta v\rangle)}\widehat{e^{i\varphi(v)}} \quad \text{and} \quad \widehat{e^{i\varphi(v)}} = e^{\frac{1}{2}(\langle v,\theta v\rangle-\|v\|^2)}e^{i\varphi(\widehat{v})}. \tag{8}$$

For $\gamma \in S$, this leads to
$$\widehat{\gamma}e^{i\varphi(\widehat{v})} = e^{\frac{1}{2}(\|v\|^2-\langle v,\theta v\rangle)}\widehat{e^{i\varphi(\gamma v)}} = e^{\frac{1}{2}(\|v\|^2-\langle v,\theta v\rangle+\langle\gamma v,\theta\gamma v\rangle-\|\gamma v\|^2)}e^{i\varphi(\widehat{\gamma v})}.$$

In particular, the cyclic subrepresentation generated by the constant function $1 = e^{i\varphi(0)}$ is determined for $\gamma = (b, g)$ by the positive definite function

$$\varphi(b,g) = \langle e^{i\varphi(0)}, \widehat{\gamma} e^{i\varphi(0)} \rangle = e^{\frac{1}{2}(\langle b,\theta b\rangle - \|b\|^2)} \langle e^{i\varphi(0)}, e^{i\varphi(\widehat{b})} \rangle = e^{\frac{1}{2}(\langle b,\theta b\rangle - \|b\|^2)} e^{-\frac{1}{2}\|\widehat{b}\|^2}$$
$$= e^{\frac{1}{2}(\langle b,\theta b\rangle - \|b\|^2)} e^{-\frac{1}{2}\langle b,\theta b\rangle} = e^{-\frac{1}{2}\|b\|^2}.$$

It follows that the function $\varphi(b, g) = e^{-\frac{1}{2}\|b\|^2}$ on $\mathrm{Mot}(\mathcal{E})$ is reflection positive for $(\mathrm{Mot}(\mathcal{E}), S, \tau)$.

The following lemma provides a characterization of reflection positive affine actions in terms of kernels.

Lemma 2. *Let (G, S, τ) be a symmetric semigroup and $(\alpha, \mathcal{E})$ be an affine isometric action of (G, τ) on the real Hilbert space $\mathcal{E}$. We write $\mathcal{E}_+ := \overline{\mathrm{span}\, \beta_S}$ for the closed subspace generated by $\alpha_S(0) = \beta_S$. Then the following are equivalent:*

(a) *The kernel $Q^\theta(x, y) = Q(x, \theta y) = e^{-\|x-\theta y\|^2/2}$ is positive definite on $\mathcal{E}_+$.*
(b) *$(\alpha, \mathcal{E})$ is reflection positive with respect to (G, S, τ), i.e., $\mathcal{E}_+$ is θ-positive.*
(c) *The kernel $C^\tau(s, t) := C(s, \tau(t)) = \frac{1}{2}\big(\psi(s) + \psi(t) - \psi(s^\sharp t)\big)$ is positive definite on $(S, \sharp)$.*
(d) *The function $\psi|_S \colon S \to \mathbb{R}$ is negative definite on $(S, \sharp)$.*

Proof. (a) ⇔ (b): In view of $Q(x, \theta y) = e^{-\frac{\|x\|^2}{2}} e^{-\frac{\|\theta y\|^2}{2}} e^{\langle x,\theta y\rangle} = e^{-\frac{\|x\|^2}{2}} e^{-\frac{\|y\|^2}{2}} e^{\langle x,\theta y\rangle}$, the kernel Q^θ is positive definite on $\mathcal{E}_+$ if and only if the kernel $e^{\langle x,\theta y\rangle}$ is positive definite on $\mathcal{E}_+$, but this is equivalent to $\mathcal{E}_+$ being θ-positive ([33] Rem. 2.8).

(b) ⇔ (c): Since $\mathcal{E}_+$ is generated by $(\beta_s)_{s\in S}$, this follows from (7) and the definition of C.

(c) ⇔ (d): By Lemma 1, the kernel C^τ is positive definite if and only if the kernel $(\psi(s^\sharp t))_{s,t\in S}$ is negative definite, which is (d). □

This leads us to the following concept:

Definition 6. *We call a continuous function $\psi \colon G \to \mathbb{R}$ reflection negative with respect to (G, S, τ) if ψ is a negative definite function on G and $\psi|_S$ is a negative definite function on the involutive semigroup $(S, \sharp)$ (Definition 1).*

From Schoenberg's Theorem for kernels (Remark 1) we immediately obtain from Lemma 2:

Corollary 1. *Let $(\alpha, \mathcal{E})$ be a reflection positive affine action of (G, S, τ). Then, for every $h > 0$, the function $\varphi_h(g) := e^{-h\|\beta_g\|^2}$ is reflection positive, i.e., the function $\|\beta_g\|^2$ is reflection negative.*

Remark 4. *(a) Let $\mathcal{H}$ be a real Hilbert space. For $h > 0$, the function $\varphi_h(b, g) := e^{-h\|b\|^2}$ on $\mathrm{Mot}(\mathcal{H})$ is positive definite. A corresponding cyclic representation can be realized as follows. We consider the unitary representation of $\mathrm{Mot}(\mathcal{H})$ on $L^2(\mathcal{H}^*, \gamma_h)$ given by*

$$\rho(b,g)F = e^{i\varphi(b)} g_* F, \quad \text{i.e.,} \quad (\rho(b,g)F)(\alpha) = e^{i\alpha(b)} F(\alpha \circ g), \quad \alpha \in \mathcal{H}^*,$$

where γ_h is the Gaussian measure on $\mathcal{H}^*$ with Fourier transform $\widehat{\gamma}_h(v) = e^{-h\|v\|^2}$ and $\varphi(v)(\alpha) = \alpha(v)$ as in Definition A6 (see also Remark A3). Then the constant function 1 is a cyclic vector, and the corresponding positive definite function is

$$\langle 1, \rho(b,g)1 \rangle = \mathbb{E}(e^{i\varphi(b)}) = e^{-h\|b\|^2} = \varphi_h(b,g). \tag{9}$$

(b) We conclude that, for every reflection positive affine action $(\alpha, \mathcal{E})$, for (G, S, τ), a cyclic reflection positive representation of (G, S, τ) corresponding to $\varphi_h(g) = e^{-h\|\beta_g\|^2}$ is obtained on the cyclic subspace of $L^2(\mathcal{E}^*, \gamma_h)$ generated by the constant function 1*.*

4. Some Unitary Representations of $\mathrm{GL}_2(\mathbb{R})$

In this section we introduce a family of unitary representations $(U^H, \mathcal{H}_H)_{0<H<1}$ of $\mathrm{GL}_2(\mathbb{R})$, respectively of the projective group $\mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times$.

We identify the real projective line $\mathbb{P}_1(\mathbb{R}) \cong \mathbb{S}^1$ of one-dimensional linear subspaces of $\mathbb{R}^2$ with $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. On this space the group $G := \mathrm{GL}_2(\mathbb{R})$ acts naturally by fractional linear maps

$$g.x = g(x) = \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that

$$g'(x) = \frac{ad-bc}{(cx+d)^2}, \tag{10}$$

which shows that g acts on the circle $\mathbb{R}_\infty$ in an orientation preserving fashion if and only if $\det g > 0$.

Definition 7. *For two different elements $\alpha \neq \gamma \in \mathbb{R}_\infty$, we write (α, γ) for the open interval between α and γ with respect to the cyclic order. For $\gamma < \alpha$ in $\mathbb{R}$ this means that*

$$(\alpha, \gamma) = (\alpha, \infty) \cup \{\infty\} \cup (-\infty, \gamma).$$

Definition 8. *For the action of G on $\mathbb{R}_\infty$, Lebesgue measure λ on $\mathbb{R}$, resp., the corresponding measure on $\mathbb{S}^1 \cong \mathbb{R}_\infty$ with $\lambda(\{\infty\}) = 0$ is quasi-invariant with $\frac{d(g_*^{-1}\lambda)}{d\lambda}(x) = |g'(x)|$. A unitary representation of* $\mathrm{GL}_2(\mathbb{R})$ *(resp., of* $\mathrm{PGL}_2(\mathbb{R})$*) on $L^2(\mathbb{R}) = L^2(\mathbb{R}, \lambda)$ is given by*

$$(U_g\xi)(x) = \mathrm{sgn}(\det g)\frac{|ad-bc|^{1/2}}{|cx+d|}\xi\Big(\frac{ax+b}{cx+d}\Big) \quad \textit{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{11}$$

We could as well work without the $\mathrm{sgn}(\det g)$*-factor, but we shall see below that it is more natural this way when it comes to the relation with fractional Brownian motion.*

We now explain how this representation can be embedded into a family of unitary representations $(U^H)_{0<H<1}$. For $H \neq \frac{1}{2}$, these representations belong to the so-called *complementary series* (cf. [4,8,25]). For $H > \frac{1}{2}$, the corresponding Hilbert space $\mathcal{H}_H$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R})$ with respect to the inner product

$$\langle \xi, \eta \rangle_H := H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(x)}\eta(y)\frac{dx\,dy}{|x-y|^{2-2H}}. \tag{12}$$

Note that $2-2H \in (0,1)$, so that the kernel $|x-y|^{2H-2}$ is locally integrable and defines a positive definite distribution kernel on $\mathbb{R}$. This implies in particular that (12) makes sense for any pair of compactly supported bounded measurable functions on $\mathbb{R}$ and that any such function defines an element of $\mathcal{H}_H$. In Appendix D we show that

$$\langle \xi, \eta \rangle_H = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi'(x)}\eta'(y)|x-y|^{2H}\,dx\,dy. \tag{13}$$

Definition 9. *As we have seen in Example 2(a), the continuous function $D^H(x) = |x|^{2H}$ on $\mathbb{R}$ is negative definite for $0 < H \leq 1$. Therefore (13) defines for $0 < H < 1$ a positive semidefinite form on $\mathcal{S}(\mathbb{R})$. We write*

$\mathcal{H}_H$ for the corresponding Hilbert space. Here we use that the total integrals of ξ' and η' vanish (cf. Remark 12). Note that this definition also makes sense for $H = 0$ and $H = 1$, but $\mathcal{H}_0 = \{0\}$ and $\mathcal{H}_1$ is one-dimensional.

Definition 10. *We obtain unitary representations of $\mathrm{GL}_2(\mathbb{R})$ (resp., the quotient $\mathrm{PGL}_2(\mathbb{R})$) on $\mathcal{H}_H$, $0 < H < 1$ by*

$$(U^H_g \xi)(x) = \mathrm{sgn}(\det g) \frac{|ad - bc|^H}{|cx + d|^{2H}} \xi\Big(\frac{ax + b}{cx + d}\Big) \quad \textit{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14}$$

For the verification of unitarity we refer to Appendix D. For $H = \frac{1}{2}$, we obtain the representation on $L^2(\mathbb{R}) \cong \mathcal{H}_{1/2}$ from (11).

Remark 5. *(a) Considering the singularities of the factors in the formula for U^H_g, we see that the operators U^H_g preserve the class of locally bounded measurable functions for which*

$$\sup_{x \in \mathbb{R}} |x|^{2H} |\xi(x)| < \infty.$$

For $H > \frac{1}{2}$, all these functions are contained in $\mathcal{H}_H$, so that we obtain a dense subspace of $\mathcal{H}_H$ invariant under the operators U^H_g.

(b) We note that the representation $(U^H, \mathcal{H}_H)$ is equivalent to $(U^{1-H}, \mathcal{H}_{1-H})$, as can be seen by realizing these representations on $\mathbb{S}^1$ (see [9] Ch. 7). We will not use this duality here.

Remark 6. *The unitary representations $(U^H)_{0<H<1}$ of $\mathrm{GL}_2(\mathbb{R})$ yield in particular three important one-parameter groups:*

- *Translations: $(S^H_t \xi)(x) = \xi(x - t)$ for $t \in \mathbb{R}$.*
- *Dilations: $(\tau^H_a \xi)(x) = \mathrm{sgn}(a)|a|^H \xi(ax)$ for $a \in \mathbb{R}^\times$.*
- *Inverted translations: $(\kappa^H_t \xi)(x) = \frac{1}{|1-tx|^{2H}} \xi(\frac{x}{1-tx})$ for $t \in \mathbb{R}$.*

Note that

$$\tau^H_{r^{-1}} S^H_t \tau^H_r = S^H_{rt} \quad \textit{for} \quad t \in \mathbb{R}, r \in \mathbb{R}^\times. \tag{15}$$

For $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with $\sigma.x = \frac{1}{x}$, we have

$$U^H_\sigma(\xi)(x) = -|x|^{-2H} \xi(x^{-1}) \quad \textit{and} \quad U^H_\sigma S^H_t U^H_\sigma = \kappa^H_t. \tag{16}$$

5. Fractional Brownian Motion

In this section we introduce fractional Brownian motion in terms of its covariance kernel. We then show that the unitary representations $(U^H)_{0<H<1}$ of $\mathrm{GL}_2(\mathbb{R})$ and a realization of fractional Brownian motion in the Hilbert space $\mathcal{H}_H$, resp., on its Fock space, can be used to obtain in a very direct and simple fashion the projective invariance of fractional Brownian motion.

5.1. A Realization of Fractional Brownian Motion

Definition 11. *Fractional Brownian motion with Hurst index $H \in (0, 1)$ is a real-valued Gaussian process $(B^H_t)_{t \in \mathbb{R}}$ with zero means and covariance kernel*

$$C^H(s, t) = \mathbb{E}(B^H_s B^H_t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s - t|^{2H}) \quad \textit{for} \quad s, t \in \mathbb{R}$$

(cf. [34] Satz 7 for the determination of those parameters for which this kernel is positive definite). A curve $\gamma \colon \mathbb{R} \to \mathcal{H}$ with values in a Hilbert space $\mathcal{H}$ satisfying $\langle \gamma(s), \gamma(t) \rangle = C^H(s, t)$ is called a fractional Wiener spiral.

Brownian motion arises for $H = 1/2$*, and in this case*

$$C^{1/2}(s,t) = \frac{1}{2}(|s| + |t| - |s-t|) = \begin{cases} |t| \wedge |s| & \text{for } st \geq 0. \\ 0 & \text{for } st < 0. \end{cases}$$

We refer to the monograph [22] for a stochastic calculus for fractional Brownian motion.

Example 4. *(Bifractional Brownian motion) For* $0 < H \leq 1$ *and* $0 < K \leq 1$*, the kernel*

$$C(s,t) = (|t|^{2H} + |s|^{2H})^K - |t-s|^{2HK}$$

on $\mathbb{R}$ *is positive definite (Lemma 1). The corresponding centered Gaussian process* $B^{H,K}$ *is called bifractional Brownian motion ([35]). For* $K = 1$ *we obtain fractional Brownian motion which has stationary increments, but for* $K < 1$ *the process* $B^{H,K}$ *does not have this property since the kernel*

$$D(t,s) = C(t,t) + C(s,s) - 2C(t,s) = 2^K(|t|^{2HK} + |s|^{2HK}) - (|t|^{2H} + |s|^{2H})^K + |t-s|^{2HK}$$

on $\mathbb{R}$ *is not translation invariant.*

For a concept of trifractional Brownian motion and decompositions of fractional Brownian motion into independent bifractional and trifractional components we refer to [36].

Remark 7. *For* $0 < H < 1$*, the kernel* C^H *satisfies*

$$C^H(\lambda s, \lambda t) = |\lambda|^{2H} C(s,t) \quad \text{and} \quad C^H(s^{-1}, t^{-1}) = |st|^{-2H} C^H(s,t) \quad \text{for} \quad s,t \in \mathbb{R}^\times, \lambda \in \mathbb{R}. \tag{17}$$

These transformation rules show that:

(a) *For a fractional Brownian motion* $(B_t^H)_{t\in\mathbb{R}}$ *with Hurst index* H*, the centered Gaussian process* $(X_t)_{t\in\mathbb{R}}$ *defined by*

$$X_0 := 0 \quad \text{and} \quad X_t := |t|^{2H} B_{1/t} \quad \text{for} \quad t \neq 0$$

also is a fractional Brownian motion with Hurst index H*.*

(b) *For* $c \in \mathbb{R}^\times$ *and* $X_t := |c|^{-H} B_{ct}^H$*, the process* $(X_t)_{t\in\mathbb{R}}$ *is a fractional Brownian motion with Hurst index* H*.*

(c) *For* $h \in \mathbb{R}$*, the process* $X_t := B_{t+h}^H - B_h^H$ *also is a fractional Brownian motion with Hurst index* H*.*

Lemma 3. *For* $t \in \mathbb{R}$ *and* $0 < H < 1$*, consider the random variables*

$$B_t^H = \varphi(b_t^H) \quad \text{for} \quad b_t^H := \text{sgn}(t)\chi_{[t\wedge 0, t\vee 0]} = \chi_{[0,\infty)} - \chi_{[t,\infty)} \in \mathcal{H}_H.$$

Then $\langle b_s^H, b_t^H \rangle_H = C^H(s,t)$*, i.e.,* $(B_t^H)_{t\in\mathbb{R}}$ *is a realization of fractional Brownian motion with Hurst index* H*.*

Proof. **Case** $H \geq \frac{1}{2}$**:** As $C^H(s,t) = C^H(-s,-t) = C^H(t,s)$, we only have to show that

$$C^H(s,t) = H(2H-1) \int_0^t \int_0^s \frac{dx\,dy}{|x-y|^{2-2H}} \quad \text{for} \quad 0 < s \leq t$$

and

$$C^H(s,t) = -H(2H-1) \int_t^0 \int_0^s \frac{dx\,dy}{|x-y|^{2-2H}} \quad \text{for} \quad t < 0 < s.$$

This is an elementary calculation.

Case $H < \frac{1}{2}$**:** In this case we can calculate the scalar product (A12) by using the formula $(b_t^H)' = \delta_0 - \delta_t$ (a difference of two point evaluations). This leads to

$$\begin{aligned}\langle b_s^H, b_t^H\rangle_H &= -\frac{1}{2}\int_{\mathbb{R}} (b_s^H)'(y)(|y|^{2H} - |t-y|^{2H})\,dy \\ &= -\frac{1}{2}\big(-|t|^{2H} - (|s|^{2H} - |t-s|^{2H})\big) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big).\end{aligned}$$

For $H > \frac{1}{2}$, the preceding lemma follows from [21] (p. 168) and for $H = \frac{1}{2}$ it is already contained in [34] (p. 117). Other realizations of fractional Brownian motion are discussed in [18]. □

Remark 8. *For Brownian motion* ($H = \frac{1}{2}$)*, an alternative realization is obtained by* $b_t' := \chi_{[t\wedge 0, t\vee 0]}$ *for* $t \in \mathbb{R}$ *(see [37] p. 130). For* $H \neq 1/2$ *this sign change does no longer work because* $C^H(s,t) \neq 0$ *for* $st < 0$*.*

5.2. Projective Invariance of the Covariance Kernels

Recall the *cross ratio*

$$\mathrm{CR}(z, z_1, z_2, z_3) := \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \text{of four different elements} \quad z, z_1, z_2, z_3 \in \mathbb{R}_\infty$$

and that it is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$. As $\mathrm{CR}(z, 0, 1, \infty) = z$, we obtain for $g(\alpha, \beta, \gamma) = (0, 1, \infty)$ the relation

$$g(z) = \mathrm{CR}(g(z), 0, 1, \infty) = \mathrm{CR}(z, \alpha, \beta, \gamma) = \frac{z-\alpha}{z-\gamma}\cdot\frac{\beta-\gamma}{\beta-\alpha}, \tag{18}$$

expressing g as a cross ratio. Accordingly, we obtain for each triple (α, β, γ) of mutually different elements of $\mathbb{R}_\infty$ the following kernel

$$\begin{aligned}C^H_{\alpha,\beta,\gamma}(s,t) &:= C^H(g(s), g(t)) = C^H\Big(\frac{(s-\alpha)(\beta-\gamma)}{(s-\gamma)(\beta-\alpha)}, \frac{(t-\alpha)(\beta-\gamma)}{(t-\gamma)(\beta-\alpha)}\Big) \\ &= \Big|\frac{\beta-\gamma}{\beta-\alpha}\Big|^{2H} C^H\Big(\frac{s-\alpha}{s-\gamma}, \frac{t-\alpha}{t-\gamma}\Big),\end{aligned} \tag{19}$$

where the last expression only makes sense for $\alpha, \beta, \gamma \in \mathbb{R}$. By construction we then have

$$C^H_{h.\alpha,h.\beta,h.\gamma}(h(s), h(t)) = C^H_{\alpha,\beta,\gamma}(s,t) \quad \text{for} \quad h \in \mathrm{GL}_2(\mathbb{R}), s, t \neq \gamma. \tag{20}$$

Note that $C^H_{0,1,\infty} = C^H$ and that, for $\beta \in \mathbb{R}^\times$, $\alpha = 0$ and $\gamma = \infty$, we obtain in particular for the dilation $g.x = \beta^{-1}x$:

$$C^H_{0,\beta,\infty}(s,t) = C^H(\beta^{-1}s, \beta^{-1}t) = |\beta|^{-2H} C^H(s,t),$$

which is a multiple of C^H. In particular, normalization of C^H and $C^H_{0,\beta,\infty}$ leads on $\mathbb{R}^\times$ to the same kernels. We also observe that

$$C^H_{\infty,\beta,0}(s,t) = C^H\Big(\frac{\beta}{s}, \frac{\beta}{t}\Big) = \frac{|\beta|^{2H}}{|st|^{2H}} C^H(s,t)$$

implies the equality of the normalized kernels $\widetilde{C}^H_{\infty,\beta,0}(s,t) = \widetilde{C}^H_{0,\beta,\infty}(s,t)$.

From (19) and the preceding discussion we obtain immediately:

Lemma 4. *For* $\alpha \neq \gamma$ *in* $\mathbb{R}_\infty$*, the normalized kernel* $C^H_{\alpha,\gamma} := \widetilde{C}^H_{\alpha,\beta,\gamma}$ *on* $\mathbb{R}_\infty \setminus \{\alpha, \gamma\}$ *does not depend on* β *and satisfies the symmetry condition* $C^H_{\alpha,\gamma} = C^H_{\gamma,\alpha}$*.*

Proposition 1. *For $g \in \mathrm{GL}_2(\mathbb{R})$ we have*

$$C^H_{g.\alpha,g.\gamma}(g(s),g(t)) = C^H_{\alpha,\gamma}(s,t) \quad \textit{for} \quad s,t \notin \{\alpha,\gamma\}. \tag{21}$$

In particular, if $g \in \mathrm{GL}_2(\mathbb{R})$ preserves the 2-element set $\{\alpha,\gamma\}$, then the kernel $C^H_{\alpha,\gamma}$ on $\mathbb{R}_\infty \setminus \{\alpha,\gamma\}$ is g-invariant.

Proof. Equation (21) follows directly from (20) and the remainder is a consequence of Lemma 4.

The preceding proposition expresses the projective invariance of fractional Brownian motion in the sense of P. Lévy. For $H = 1/2$, this is classical ([23] §I.2, [24,37] Thm. 5.2). □

Remark 9. *The identity component $G_0^{\alpha,\gamma}$ of the stabilizer $G^{\alpha,\gamma}$ of $\{\alpha,\gamma\}$ in $G = \mathrm{PGL}_2(\mathbb{R})$ is a (hyperbolic) one-parameter group of $\mathrm{SL}_2(\mathbb{R})$ whose fixed points are α and β (these are the orientation preserving transformations mapping the interval (α,γ) onto itself). The full stabilizer of the pair (α,γ) is isomorphic to $\mathbb{R}^\times$. It also contains an involution in $\mathrm{PSL}_2(\mathbb{R})$ exchanging the two connected components of $\mathbb{R}_\infty \setminus \{\alpha,\gamma\}$.*

Moreover, there exists for each $\beta \neq \alpha,\gamma$ a unique involution $\theta^{\alpha,\beta,\gamma} \in \mathrm{GL}_2(\mathbb{R})$ exchanging α and γ and fixing β. It satisfies

$$\theta^{\alpha,\beta,\gamma} g \theta^{\alpha,\beta,\gamma} = g^{-1} \quad \textit{and} \quad g\theta^{\alpha,\beta,\gamma} g^{-1} = \theta^{\alpha,g.\beta,\gamma} \quad \textit{for} \quad g \in G_0^{\alpha,\gamma}.$$

The subgroup $G^{\alpha,\gamma} \subseteq \mathrm{PGL}_2(\mathbb{R})$ has four connected components.

5.3. Projective Invariance of the Realization

We now link the projective invariance of fractional Brownian motion to the specific realization in the Hilbert space $\mathcal{H}_H$. Formula (b) in the theorem below connects the normalized projective transforms of the kernel C^H to the unitary representation U^H of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{H}_H$.

Theorem 1. *For a triple (α,t,γ) of mutually different points in $\mathbb{R}_\infty$, there exists a uniquely determined Möbius transformation $g_t \in \mathrm{PSL}_2(\mathbb{R})$ with $(g_t(\alpha),g_t(t),g_t(\gamma)) = (0,1,\infty)$. We thus obtain functions of the form*

$$f_t^{\alpha,\gamma}(x) := (U^H_{g_t^{-1}}\chi_{[0,1]})(x) = \mathrm{sgn}(\det g_t)\frac{|\det g_t|^H}{|cx+d|^{2H}}\chi_{g_t^{-1}([0,1])}(x) \quad \textit{for} \quad t \in \mathbb{R}_\infty \setminus \{\alpha,\gamma\}. \tag{22}$$

Then the following assertions hold:

(a) *All functions $f_t^{\alpha,\gamma}$ are unit vectors in $\mathcal{H}_H$.*
(b) $\langle f_s^{\alpha,\gamma}, f_t^{\alpha,\gamma}\rangle_{\mathcal{H}_H} = C^H_{\alpha,\gamma}(s,t)$ *for $s,t \notin \{\alpha,\gamma\}$.*

Proof. (a) As $\|\chi_{[0,1]}\|_{\mathcal{H}_H} = 1$ and the representation U^H is unitary, the functions $f_t^{\alpha,\gamma}$ are unit vectors in $\mathcal{H}_H$.

(b) For $g = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ with $g.x = tx$,
we have

$$U_g\chi_{[0,1]} = |t|^{-H} b_t^H, \quad \text{resp.} \quad b_t^H = |t|^H \cdot U_g\chi_{[0,1]} \quad \text{for} \quad t \in \mathbb{R}^\times. \tag{23}$$

This relation is the reason for the $\mathrm{sgn}(\det g)$-factor in the definition of U^H. □

For $s, t \notin \{\alpha, \gamma\}$, the element $g_s g_t^{-1}$ fixes 0 and ∞, hence is linear and given by multiplication with $(g_s g_t^{-1})(1) = g_s(t)$. We thus obtain with Remark 7, (18) and (23)

$$\begin{aligned}
\langle f_s^{\alpha,\gamma}, f_t^{\alpha,\gamma}\rangle_{\mathcal{H}_H} &= \langle U^H_{g_s^{-1}}\chi_{[0,1]}, U^H_{g_t^{-1}}\chi_{[0,1]}\rangle_{\mathcal{H}_H} = \langle \chi_{[0,1]}, U^H_{g_s g_t^{-1}}\chi_{[0,1]}\rangle_{\mathcal{H}_H} \\
&= \langle b_1^H, |g_s(t)|^{-H} b^H_{g_s(t)}\rangle_{\mathcal{H}_H} = |g_s(t)|^{-H} C^H(1, g_s(t)) \\
&= \frac{|t-\gamma|^H|s-\alpha|^H}{|t-\alpha|^H|s-\gamma|^H} C^H\Big(1, \frac{(t-\alpha)(s-\gamma)}{(t-\gamma)(s-\alpha)}\Big) \\
&= \frac{|t-\gamma|^H|s-\gamma|^H}{|t-\alpha|^H|s-\alpha|^H} C^H\Big(\frac{s-\alpha}{s-\gamma}, \frac{t-\alpha}{t-\gamma}\Big) \\
&= \frac{|t-\gamma|^H|s-\gamma|^H}{|t-\alpha|^H|s-\alpha|^H} \frac{|\beta-\alpha|^{2H}}{|\beta-\gamma|^{2H}} C^H_{\alpha,\beta,\gamma}(s,t).
\end{aligned}$$

Since the kernel on the left hand side is normalized, (b) follows.

From Proposition 1 and Theorem 1, we obtain:

Corollary 2. *The normalized stochastic process defined by* $(f_t^{\alpha,\gamma})_{t\neq\alpha,\gamma}$ *is stationary with respect to the stabilizer of the two-point set* $\{\alpha,\gamma\}$ *in* $\mathrm{GL}_2(\mathbb{R})$.

Remark 10. *Since the representations* $(U^H, \mathcal{H}_H)$ *of* $\mathrm{GL}_2(\mathbb{R})$ *are irreducible, [1] (Prop. 5.20) implies that the space* $\mathcal{H}_H^\infty$ *of smooth vectors is nuclear. Therefore [1] (Cor. 5.19) shows that the Gaussian measure* $\gamma_{\mathcal{H}_H}$ *can be realized on the space* $\mathcal{H}_H^{-\infty}$ *of distribution vectors for this representation (cf. Appendix C). Therefore our construction leads to a realization of fractional Brownian motion on the topological dual space* $\mathcal{H}_H^{-\infty}$ *of the* $\mathrm{GL}_2(\mathbb{R})$*-invariant subspace* $\mathcal{H}_H^\infty$ *of smooth vectors.*

From the proof of [1] (Prop. 5.20(b)), we further derive that an element $\xi \in \mathcal{H}_H$ *is a smooth vector if and only if it is a smooth vector for the compact subgroup* $K = \mathrm{O}_2(\mathbb{R})$. *Considering* $\mathcal{H}_H$ *as a space of distributions on the circle* $\mathbb{S}^1$, *it is not hard to see that* $\mathcal{H}_H^\infty = C^\infty(\mathbb{S}^1)$ *and hence that* $\mathcal{H}_H^{-\infty} = C^{-\infty}(\mathbb{S}^1)$ *is the space of distributions on the circle.*

6. Fractional Brownian Motion and Reflection Positivity

We now turn to reflection positivity in connection with fractional Brownian motion. Our main result is Theorem 2 on the reflection positivity of the normalized kernels $C^H_{\alpha,\gamma}$ for $H \leq \frac{1}{2}$. We start with the normalization of the kernel C^H, which corresponds to the pair $(\alpha,\gamma) = (0,\infty)$.

Proposition 2. *The kernel*

$$C^H_{0,\infty}(s,t) = \widetilde{C}^H(s,t) = \frac{C^H(s,t)}{|s|^H|t|^H} \quad \text{on} \quad X := \mathbb{R}^\times$$

is invariant under the involution $\theta(x) = x^{-1}$. *It is reflection positive on* $X_+ := (-1,1) \cap \mathbb{R}^\times$ *if and only of* $0 < H \leq \frac{1}{2}$. *If this is the case, then* $\widehat{\mathcal{E}} \cong L^2((0,\infty),\mu)$, *with the measure*

$$\mu = \delta_{2H} + \sum_{k=1}^\infty \binom{2H}{k}(-1)^{k-1}\delta_k.$$

For $H = \frac{1}{2}$, *we have* $\mu = 2\delta_1$, *and* $\widehat{\mathcal{E}}$ *is one-dimensional.*

Proof. Reflection positivity with respect to (X, X_+, θ) is equivalent to the positive definiteness of the kernel

$$C^H_{0,\infty}(s,t^{-1}) = |t|^H|s|^{-H}C^H(s,t^{-1}) = |t|^{-H}|s|^{-H}C^H(st,1) = \frac{1}{2|t|^H|s|^H}(1+|st|^{2H}-(1-st)^{2H})$$

for $|t|, |s| < 1$ (cf. Example 1). This kernel is positive definite on $(-1,1)$ if and only if the function

$$\varphi(t) := 1 + t^{2H} - (1-t)^{2H} = t^{2H} - \sum_{k=1}^{\infty} \binom{2H}{k} (-1)^k t^k$$

on the multiplicative semigroup $S = ((0,1), \mathrm{id})$ is positive definite. For $k \geq 1$ we have

$$(-1)^{k-1} \binom{2H}{k} = \frac{2H(1-2H)(2-2H)\cdots(k-1-2H)}{k!}.$$

For $\frac{1}{2} < H < 1$, we have $1 - 2H < 0$ and all other factors are positive. As φ is increasing, it is positive definite if and only if there exists a positive Radon measure μ on $[0,\infty)$ with

$$\varphi(t) = \int_0^\infty t^\lambda \, d\mu(\lambda)$$

(use [31] (Prop. 4.4.2) or apply [30] (Cor. VI.2.11) to $S \cong ((0,\infty), +)$). Therefore φ is positive definite if and only if $H \leq \frac{1}{2}$. In this case the description of $\widehat{\mathcal{E}}$ follows from the proof of [30] (Thm. VI.2.10). □

Remark 11. *(Reflection positivity of fractional Brownian motion) For* $\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *with* $\theta.x = \theta^{0,1,\infty}(x) = x^{-1}$, *we have* $(U_\theta^{1/2}\xi)(x) = -|x|^{-2H}\xi(x^{-1})$. *In particular,*

$$(U_\theta^H \chi_{[0,t]})(x) = -|x|^{-2H} \chi_{[t^{-1},\infty)}.$$

Therefore U_θ^H *is not the unique unitary involution* $\widehat{\theta}$ *of* $\mathcal{H}_H$ *transforming* b_t^H *into* $|t|^{2H} b_{1/t}$ *(cf. Remark 7).*

With the kernels $C_{\alpha,\gamma}^H$ (Lemma 4), we obtain a family of normalized Gaussian processes, covariant with respect to the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}_\infty$. The following proposition shows that, for $H \leq \frac{1}{2}$, these kernels are reflection positive with respect to involutions exchanging α and γ.

Theorem 2. *Let* α, β, γ *in* $\mathbb{R}_\infty$ *be mutually different and let* $\theta := \theta^{\alpha,\beta,\gamma}$ *be the projective involution exchanging* α *and* γ *and fixing* β. *Let* $X := \mathbb{R}_\infty \setminus \{\alpha, \gamma\}$ *and* $X_\pm \subseteq X$ *be the intersection of* X *with the two connected components of the complement of the fixed point set of* θ *(which consists of two points). Then the kernel* $C_{\alpha,\gamma}^H$ *is reflection positive with respect to* (X, X_+, θ) *if and only if* $H \leq \frac{1}{2}$.

Proof. Since the family of kernels $C_{\alpha,\gamma}^H$ is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$ and

$$g\theta^{\alpha,\beta,\gamma}g^{-1} = \theta^{g.\alpha, g.\beta, g.\gamma},$$

it suffices to verify the assertion for $(\alpha, \beta, \gamma) = (0, 1, \infty)$. Then $\theta(x) = x^{-1}$ and we may put $X_+ = (-1,1) \setminus \{0\}$. Hence the assertion follows from Proposition 2. □

Example 5. *For* $(\alpha, \beta, \gamma) = (-1, 0, 1)$, *the involution* $\theta := \theta^{\alpha,\beta,\gamma}$ *is given by* $\theta(x) = -x$. *It has the two fixed points* 0 *and* ∞. *From* (19) *we obtain*

$$C(s,t) := C_{-1,0,1}^H(s,t) = C^H\Big(\frac{s+1}{s-1}, \frac{t+1}{t-1}\Big) = C^H\Big(\frac{1+s}{1-s}, \frac{1+t}{1-t}\Big)$$

and

$$\widetilde{C}(s,t) = \Big(\frac{1-s}{1+s}\Big)^H \Big(\frac{1-t}{1+t}\Big)^H C^H\Big(\frac{1+s}{1-s}, \frac{1+t}{1-t}\Big).$$

Theorem 2 now implies that the kernel $\widetilde{C}$ *is reflection positive with respect to* $(\mathbb{R}^\times, \mathbb{R}_+^\times, \theta)$.

Brownian Bridges

As we shall see below, for $H = \frac{1}{2}$, the covariance kernels $C^{1/2}_{\alpha,\gamma}$ turn out to correspond to Brownian bridges.

Definition 12. *([37] Def. 2.8, p. 109) (a) A Gaussian process $(X_t)_{\alpha \le t \le \gamma}$ is called a Brownian bridge if $m(t) := \mathbb{E}(X_t)$ is an affine function and*

$$C(t,s) := \mathbb{E}((X_t - m(t))(X_s - m(s))) = \frac{(t \wedge s - \alpha)(\gamma - t \vee s)}{\gamma - \alpha}.$$

If $m(t) = 0$ for every t, then $(X_t)_{\alpha \le t \le \beta}$ is called a pinned Brownian motion.

(b) A normalized Brownian bridge is a Brownian bridge whose variance is normalized to 1, so that its covariance kernels is

$$\widetilde{C}(t,s) = \frac{(s \wedge t - \alpha)(\gamma - s \vee t)}{\sqrt{(s-\alpha)(\gamma - s)(t-\alpha)(\gamma - t)}} = \sqrt{\frac{(s \wedge t - \alpha)(\gamma - s \vee t)}{(s \vee t - \alpha)(\gamma - s \wedge t)}}. \tag{24}$$

Proposition 3. (Reflection positivity of the Brownian bridge) *For $\alpha < \gamma$ in $\mathbb{R}$ and $H = \frac{1}{2}$, the kernel $C^{1/2}_{\alpha,\gamma}$ is the covariance of a normalized Brownian bridge on the interval $[\alpha,\gamma]$. This kernel is reflection positive for (X, X_+, θ), where $X = [\alpha,\gamma]$, $X_+ = [\alpha,\beta]$ and $\theta(t) = \alpha + \gamma - t$ is the reflection in the midpoint. The corresponding Hilbert space $\widehat{\mathcal{E}}$ is one-dimensional.*

Proof. First we observe that

$$C^{1/2}\Big(\frac{s-\alpha}{s-\gamma}, \frac{t-\alpha}{t-\gamma}\Big) = C^{1/2}\Big(\frac{s-\alpha}{\gamma-s}, \frac{t-\alpha}{\gamma-t}\Big) = \frac{s-\alpha}{\gamma-s} \wedge \frac{t-\alpha}{\gamma-t} = \frac{s \wedge t - \alpha}{\gamma - s \wedge t},$$

so that we obtain for the associated normalized kernel

$$C(s,t) := C^{1/2}_{\alpha,\gamma}(s,t) = \frac{\frac{s\wedge t-\alpha}{\gamma - s\wedge t}}{\big(\frac{s-\alpha}{\gamma-s}\big)^{1/2}\big(\frac{t-\alpha}{\gamma-t}\big)^{1/2}} = \sqrt{\frac{s\wedge t-\alpha}{s \vee t - \alpha}}\sqrt{\frac{\gamma - s \vee t}{\gamma - s \wedge t}}.$$

This is the kernel (24) of a normalized Brownian bridge on $[\alpha,\gamma]$.

For $\beta := \frac{\alpha+\gamma}{2}$, the reflection $\theta^{\alpha,\beta,\gamma}$ is given by $\theta(t) := \alpha + \gamma - t$, which leaves the kernel $C^{1/2}_{\alpha,\gamma}$ invariant by Proposition 1. For $\alpha \le t, s \le \beta$, we have

$$C^\theta(s,t) = C(s,\theta(t)) = \sqrt{\frac{(s \wedge \theta(t) - \alpha)(\gamma - s \vee \theta(t))}{(s \vee \theta(t) - \alpha)(\gamma - s \wedge \theta(t))}} = \sqrt{\frac{(s-\alpha)(\gamma - \theta(t))}{(\gamma - s)(\theta(t) - \alpha)}} = \sqrt{\frac{s-\alpha}{\gamma-s}}\sqrt{\frac{t-\alpha}{\gamma-t}}.$$

This is a positive definite kernel defining a one-dimensional Hilbert space. We conclude that the kernel C is reflection positive for (X, X_+, θ), where $X = [\alpha,\gamma]$ and $X_+ = [\alpha,\beta]$. □

7. Affine Actions and Fractional Brownian Motion

In this section we discuss reflection positivity for the affine isometric action of $\mathbb{R}$ corresponding to fractional Brownian motion $(B^H_t)_{t\in\mathbb{R}}$. In Subsection 7.2 we shall encounter the curious phenomenon that, for every H there exists a natural unitary involution θ that leads to a reflection positive Hilbert space, but only for $H \le \frac{1}{2}$ it can be implemented in such a way that $\theta b^H_t = b^H_{-t}$, so that we obtain a reflection positive affine action of $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id})$. In a third subsection we discuss increments of a 1-cocycle $(\beta_t)_{t\in\mathbb{R}}$ defining an affine isometric action and characterize cocycles with orthogonal increments as those corresponding to multiples of Brownian motion.

7.1. Generalities

If $(\alpha, \mathcal{E})$ with $\alpha_t\xi = U_t\xi + \beta_t$ is an affine isometric action of $\mathbb{R}$ on the complex Hilbert space $\mathcal{E}$, then Proposition A2 implies that, up to unitary equivalence, $\mathcal{E} \cong L^2(\mathbb{R}, \sigma)$ for a Borel measure σ on $\mathbb{R}$. We may assume that

$$(U_t f)(x) = e^{itx} f(x) \quad \text{and} \quad \beta_t(x) = e_t(x) := \begin{cases} \frac{e^{itx}-1}{ix} & \text{for } x \neq 0 \\ t & \text{for } x = 0. \end{cases}$$

Then second quantization leads to the centered Gaussian process $X_t := \varphi(\beta_t)$ whose covariance kernel is given by

$$C_\sigma(s,t) = \langle e_s, e_t \rangle_{\mathcal{E}} = r(t) + \overline{r(s)} - r(t-s) \quad \text{for} \quad r(t) = \int_{\mathbb{R}} \Big(1 - e^{itu} + \frac{itu}{1+u^2}\Big) \frac{d\sigma(u)}{u^2} \tag{25}$$

(Proposition A3). Below we only consider real Hilbert spaces. This corresponds to the situation where the measure σ is symmetric. Then the function is also real and given by

$$r(t) = \int_{\mathbb{R}} (1 - \cos(tu)) \frac{d\sigma(u)}{u^2} = \frac{1}{2} C_\sigma(t,t).$$

Lemma 5. *Let $C\colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a continuous positive definite kernel with $C_0 = C(\cdot, 0) = 0$. For $\xi \in C_c^\infty(\mathbb{R})$, we put*

$$C_\xi(x) := \int_{\mathbb{R}} \xi(t) C_t(x)\, dt = \int_{\mathbb{R}} C(x,t)\xi(t)\, dt.$$

Then the subspace

$$\{C_\xi \colon \xi \in C_c^\infty(\mathbb{R})_0\} \quad \textit{with} \quad C_c^\infty(\mathbb{R})_0 := \Big\{\xi \in C_c^\infty(\mathbb{R}) \colon \int_{\mathbb{R}} \xi(t)\, dt = 0\Big\}$$

is dense in the corresponding reproducing kernel Hilbert space $\mathcal{H}_C$.

Proof. From the existence of the $\mathcal{H}_C$-valued integral defining C_ξ, it follows that these are elements of $\mathcal{H}_C$. Let $\mathcal{H}_1$ denote the closed subspace generated by the elements C_ξ, $\xi \in C_c^\infty(\mathbb{R})_0$.

If $\delta_n \in C_c^\infty(\mathbb{R})$ is a δ-sequence, we obtain $C_{\delta_n} \to 0$. For $\xi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \xi(t)\, dt = 1$, we have $\xi - \delta_n \in C_c^\infty(\mathbb{R})_0$, so that $C_\xi - C_{\delta_n} \in \mathcal{H}_1$ and $C_{\delta_n} \to 0$ imply that $C_\xi \in \mathcal{H}_1$. Using a sequences of the form $\xi_n := \delta_n(\cdot - t) \in C_c^\infty(\mathbb{R})$, which converges to δ_t, we see that $C_{\xi_n} \to C_t$ for $t \in \mathbb{R}$, hence that $\mathcal{H}_1 = \mathcal{H}_C$. □

In the following we write $\mathcal{S}(\mathbb{R})_0 := \{\xi \in \mathcal{S}(\mathbb{R}) \colon \int_{\mathbb{R}} \xi = 0\}$.

Corollary 3. *Let $\alpha_t\xi = U_t\xi + \beta_t$ define a continuous affine isometric action of $\mathbb{R}$ on the real Hilbert space $\mathcal{H}$. For $\xi \in \mathcal{S}(\mathbb{R})$ we put $\beta_\xi := \int_{\mathbb{R}} \xi(t)\beta_t\, dt$. Then $\{\beta_\xi \colon \xi \in \mathcal{S}(\mathbb{R})_0\}$ generates the same closed subspace as $(\beta_t)_{t\in\mathbb{R}}$.*

Remark 12. *(a) A function $D\colon \mathbb{R} \to \mathbb{R}$ with $D(0) = 0$ is negative definite if and only if*

$$C(s,t) := \frac{1}{2}(D(s) + D(t) - D(s-t)) \tag{26}$$

is a positive definite kernel (Lemma 1). Then $D(s) = C(s,s)$ yields

$$D(s-t) = C(s,s) + C(t,t) - 2C(s,t)$$

On $\mathcal{S}(\mathbb{R})_0$ we therefore obtain with (26)

$$-\frac{1}{2}\int_{\mathbb{R}} \overline{\xi(x)}\eta(y)D(x-y)\,dx\,dy = \int_{\mathbb{R}} \overline{\xi(x)}\eta(y)C(x,y)\,dx\,dy \quad \text{for} \quad \xi,\eta \in \mathcal{S}(\mathbb{R})_0 \tag{27}$$

(cf. [38] (p. 222) for a corresponding statement on more general homogeneous spaces).

(b) Write $C(s,t) = \langle \beta_s, \beta_t \rangle$ and $D(t) = C(t,t)$ for a cocycle $(\beta_t)_{t\in\mathbb{R}}$ of an orthogonal representation $(U_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on the real Hilbert space $\mathcal{E}$ (see Section 3 and [12,13]). We also assume that the family $(\beta_t)_{t\in\mathbb{R}}$ is total in $\mathcal{E}$. Then

$$\langle \beta_\xi, \beta_\eta \rangle_{\mathcal{E}} = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(s)}\eta(t)C(s,t)\,dt\,ds = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(s)}\eta(t)D(s-t)\,dt\,ds \quad \text{for} \quad \xi,\eta \in \mathcal{S}(\mathbb{R})_0.$$

As $\mathcal{E}$ is generated by the $(\beta_t)_{t\in\mathbb{R}}$, the Hilbert space $\mathcal{E}$ can be identified with the reproducing kernel Hilbert space $\mathcal{H}_C \subseteq \mathcal{S}'(\mathbb{R})$ corresponding to the positive definite distribution C, but the preceding argument adds another picture. It can also be identified with the Hilbert space $\mathcal{H}_D$ obtained by completing $\mathcal{S}(\mathbb{R})_0$ with respect to the scalar product (27). Taking into account that $\mathcal{S}(\mathbb{R})_0 = \{\xi' \colon \xi \in \mathcal{S}(\mathbb{R})\}$, this is how we introduced the Hilbert space $\mathcal{H}_H$ in (A12).

7.2. Fractional Brownian Motion

For fractional Brownian motion with Hurst index $H \in (0,1)$, we have

$$C^H(s,t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}) \quad \text{and} \quad D^H(t) = C^H(t,t) = |t|^{2H}.$$

As in (25), the spectral measure σ (a Borel measure on $\mathbb{R}$) of fractional Brownian motion is determined by

$$C^H(s,t) = \int_{\mathbb{R}} \overline{e_s(\lambda)}e_t(\lambda)\,d\sigma(\lambda) \quad \text{for} \quad e_t(\lambda) = \begin{cases} \frac{e^{i\lambda t}-1}{i\lambda} & \text{for } \lambda \neq 0 \\ t & \text{for } \lambda = 0. \end{cases}$$

The corresponding realization is obtained by $\beta_t := e_t \in L^2(\mathbb{R},\sigma) =: \mathcal{E}$ (Proposition A2). According to [18] p. 40, the measure σ is given by

$$d\sigma(\lambda) = \frac{\sin(\pi H)\Gamma(1+2H)}{2\pi} \cdot |\lambda|^{1-2H}\,d\lambda$$

(see (A22) in Appendix E for a derivation of this formula).

Example 6. *For $H = \frac{1}{2}$, the spectral measure σ is a multiple of Lebesgue measure:*

$$d\sigma(\lambda) = \frac{\sin(\pi/2)\Gamma(2)}{2\pi}\,d\lambda = \frac{d\lambda}{2\pi}.$$

This leads to a natural realization of Brownian motion by a cocycle of the multiplication representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ and, by Fourier transform, to the realization of Brownian motion as a cocycle for the translation representation of $\mathbb{R}$ on $L^2(\mathbb{R})$.

Remark 13. *Combining Remark 12 with (A12) in Section 4, we see that the Hilbert space $\mathcal{H}_H$ can alternatively be constructed from the scalar product*

$$\langle \xi, \eta \rangle_{C^H} = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(x)}\eta(y)|x-y|^{2H}\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(x)}\eta(y)C^H(x,y)\,dx\,dy \quad \text{for} \quad \xi,\eta \in \mathcal{S}(\mathbb{R})_0$$

as the completion of $\mathcal{S}(\mathbb{R})_0$*. This implies that the map*

$$D\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})_0, \quad \xi \mapsto \xi' \quad \text{satisfies} \quad \|D\xi\|_{C^H} = \|\xi\|_H,$$

so that D *extends to a unitary operator* $D\colon \mathcal{H}_H \to \mathcal{H}_{C^H}$*. Here we write* $\mathcal{H}_{C^H} \subseteq \mathcal{S}'(\mathbb{R})$ *for the Hilbert space of distributions defined by* C^H*, obtained by completing* $\mathcal{S}(\mathbb{R})$ *with respect to the scalar product* $\langle \cdot, \cdot \rangle_{C^H}$*, defined by the positive definite distribution kernel* C^H *and associating to* $\xi \in \mathcal{S}(\mathbb{R})_0$ *the distribution* $C^H_\xi := \langle \cdot, \xi \rangle_{C^H}$*. Note that*

$$D(b^H_t) = D(\chi_{[0,\infty)} - \chi_{[t,\infty)}) = \delta_0 - \delta_t,$$

where we consider the δ*-functionals as elements of* $\mathcal{H}_{C^H}$ *with*

$$\langle \delta_s, \xi \rangle_{C^H} = \int_{\mathbb{R}} \int_{\mathbb{R}} \delta_s(x)\xi(y)C^H(x,y)\, dx\, dy = \int_{\mathbb{R}} C^H(s,y)\xi(y)\, dy = C^H_\xi(s).$$

As a distribution, δ_s *corresponds to the function* C^H_s *which corresponds to the evaluation in* s *in the reproducing kernel Hilbert space* $\mathcal{H}_{C^H} \subseteq \mathcal{S}'(\mathbb{R})$*. Compare also with the corresponding discussion in [8,9] Ch. 7.*

The inverse of the unitary operator $D\colon \mathcal{H}_H \to \mathcal{H}_{C^H}$ *is given by*

$$I\colon \mathcal{H}_{C^H} \to \mathcal{H}_H, \quad I(\xi)(x) = \int_{-\infty}^{x} \xi(y)\, dy = -\int_{x}^{\infty} \xi(y)\, dy \quad \text{for} \quad \xi \in \mathcal{S}(\mathbb{R})_0.$$

Proposition 4. *Consider the realization* $(b^H_t)_{t\in\mathbb{R}}$ *of fractional Brownian motion in the Hilbert space* $\mathcal{E} := \mathcal{H}_H$*, the affine isometric* $\mathbb{R}$*-action defined by*

$$\alpha^H_t \xi = S^H_t \xi + b^H_t,$$

where S^H_t *denotes the translation by* t *on* $\mathcal{H}_H$*, and the closed subspace* $\mathcal{E}_+$ *generated by* $(b^H_t)_{t\geq 0}$*. Then*

$$(\theta\xi)(x) := -\xi(-x)$$

defines a unitary involution with $\theta b^H_t = b^H_{-t}$ *for* $t \in \mathbb{R}$*. Now* $(\mathcal{E}, \mathcal{E}_+, \theta)$ *is reflection positive if and only if* $H \leq \frac{1}{2}$*, so that we obtain in this case a reflection positive affine action. For* $H \geq \frac{1}{2}$*, the triple* $(\mathcal{E}, \mathcal{E}_+, -\theta)$ *is also reflection positive, but it does not lead to a reflection positive affine action because* $-\theta b^H_t \neq -b^H_t \neq b^H_t$ *for* $t > 0$*.*

Proof. By Example 2(a), the function $D^H(t) = |t|^{2H}$ is negative definite on the additive group $(\mathbb{R}, +)$ and it is reflection positive for $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id})$ if and only if $H \leq 1/2$ by Example 2(b). Accordingly, the reflection $\sigma(t) = -t$ on $\mathbb{R}$ leads to the twisted kernel

$$C^H(s,-t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t+s|^{2H})$$

on $\mathbb{R}_+$ which is positive definite if and only if D^H is negative definite on $(\mathbb{R}_+, \mathrm{id})$ (Lemma 1), which in turn is equivalent to $H \leq \frac{1}{2}$.

For $H \geq \frac{1}{2}$, the unitary involution $-\theta$ satisfies $-\theta b^H_t = -b^H_{-t}$, which leads to the twisted kernel

$$(s,t) \mapsto \langle b^H_s, -\theta b^H_t \rangle = \langle b^H_s, -b^H_{-t} \rangle = -C^H(s,-t) = -\frac{1}{2}(|s|^{2H} + |t|^{2H} - |t+s|^{2H}).$$

As $-D^H(t) = -|t|^{2H}$ is negative definite on the semigroup $(\mathbb{R}_+, \mathrm{id})$ if $\frac{1}{2} \leq H \leq 1$ (Example 2(c)), the assertion follows. □

We conclude that the affine actions of $\mathbb{R}$ corresponding to fractional Brownian motion with Hurst parameter $H \leq \frac{1}{2}$ leads to a reflection positive affine action, and from the calculation in Example 3 we derive that the reflection positive function on $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id})$ corresponding to the constant function

$1 = e^{i\varphi(0)} \in \Gamma(\mathcal{E})$ is given by $\varphi(t) = e^{-\|b_t^H\|^2/2} = e^{-t^{2H}/2}$. The preceding proposition also explains why we obtain trivial reflection positivity for $H = \frac{1}{2}$ since in this case $(\mathcal{E}, \mathcal{E}_+, \pm\theta)$ are both reflection positive.

Remark 14. *For $h > 0$ and $H \leq \frac{1}{2}$ and the kernel $Q^h(x,y) := e^{-h\|x-y\|^2}$ on $\mathcal{E}$, we obtain with the same arguments the positive definite functions $\varphi(t) = e^{-ht^{2H}}$ on $\mathbb{R}_+$.*

7.3. Cocycles with Orthogonal Increments

In this subsection we discuss the question when a cocycle $(\beta_t)_{t\in\mathbb{R}}$ for an orthogonal representation $(U, \mathcal{H})$ of $\mathbb{R}$ has *orthogonal increments* in the sense that, for $t_1 \leq t_2 \leq t_3 \leq t_4$ we have

$$\langle \beta_{t_2} - \beta_{t_1}, \beta_{t_4} - \beta_{t_3} \rangle = 0$$

(cf. [34]).

Proposition 5. *The following are equivalent:*

(i) *β has orthogonal increments.*
(ii) *For $s, t > 0$, we have $\langle \beta_t, \beta_{-s} \rangle = 0$.*
(iii) *There exists a $c \geq 0$ such that $C(t,s) := \langle \beta_t, \beta_s \rangle = \frac{c}{2}(|s| + |t| - |s-t|)$ for all $t, s \in \mathbb{R}$. If $c > 0$, then $(c^{-1/2}\varphi(\beta_t))_{t\in\mathbb{R}}$ realizes a two-sided Brownian motion in $\Gamma(\mathcal{H})$.*

Proof. (cf. [34] Satz 8 for a variant of this observation)

(i) ⇒ (ii) follows with $t_1 = -s, t_2 = t_3 = 0$ and $t_4 = t$.

(ii) ⇒ (i): Let $\mathcal{H}_\pm \subseteq \mathcal{H}$ be the closed subspaces generated by the β_t for $\pm t \geq 0$. Then (ii) means that $\mathcal{H}_+ \perp \mathcal{H}_-$. For $t_1 \leq t_2 \leq t_3 \leq t_4$ we now observe that

$$\beta_{t_1} - \beta_{t_2} = \beta_{t_2+(t_1-t_2)} - \beta_{t_2} = U_{t_2}\beta_{t_1-t_2}$$

and similary $\beta_{t_4} - \beta_{t_3} = U_{t_3}\beta_{t_4-t_3}$. Therefore

$$\begin{aligned} \langle \beta_{t_1} - \beta_{t_2}, \beta_{t_4} - \beta_{t_3} \rangle &= \langle U_{t_2}\beta_{t_1-t_2}, U_{t_3}\beta_{t_4-t_3} \rangle = \langle \beta_{t_1-t_2}, U_{t_3-t_2}\beta_{t_4-t_3} \rangle \\ &= \langle \beta_{t_1-t_2}, \beta_{t_3-t_2+t_4-t_3} - \beta_{t_3-t_2} \rangle = 0. \end{aligned}$$

(ii) ⇒ (iii): Put $\psi(t) := C(t,t) = \|\beta_t\|^2$ and note that this function is increasing for $t \geq 0$. For $0 \leq s \leq t$ we have

$$C(s,t) = \langle \beta_s, \beta_t \rangle = \langle \beta_s, (\beta_t - \beta_s) + \beta_s \rangle = \langle \beta_s, \beta_s \rangle = \psi(s),$$

and therefore

$$C(s,t) = \psi(s \wedge t) \quad \text{for} \quad t, s \geq 0. \tag{28}$$

Further,

$$\|\beta_t - \beta_s\|^2 = C(t,t) + C(s,s) - 2C(t,s) = \psi(t) + \psi(s) - 2\psi(s \wedge t) = |\psi(t) - \psi(s)|,$$

so that translation invariance of this kernel leads to

$$\|\beta_t - \beta_s\|^2 = \psi(|t-s|).$$

From the orthogonality of the increments, we further derive for $0 \leq s \leq t$ the relation

$$\psi(t) - \psi(s) = \psi(t-s),$$

so that

$$\psi(a+b) = \psi(a) + \psi(b) \quad \text{for} \quad a, b \geq 0.$$

Since ψ is continuous, there exists a $c \geq 0$ with $\psi(t) = ct$ for $t \geq 0$, and therefore (28) yields $C(s,t) = c \cdot s \wedge t$ for $t, s \geq 0$.

We likewise find some $c' \geq 0$ with

$$C(s,t) = c' \cdot |s| \wedge |t| \quad \text{for} \quad s, t \leq 0.$$

Now $\psi(-t) = \psi(t)$ implies that $c' = c$, and this completes the proof.

(iii) $\Rightarrow$ (ii) follows from the fact that $C(s,t) = 0$ for $ts < 0$ holds for the covariance kernel of Brownian motion. □

If $(\mathcal{E}, \mathcal{E}_+, \theta)$ is reflection positive for an affine $\mathbb{R}$-action, $\mathcal{E}_+$ is generated by $(\beta_t)_{t \geq 0}$ and $\theta\beta_t = \beta_{-t}$, then the space $\widehat{\mathcal{E}}$ is trivial if and only if

$$\langle \beta_s, \beta_t \rangle = 0 \quad \text{for} \quad ts < 0,$$

which in turn means that β has orthogonal increments by Proposition 5. In view pf Proposition 5(iii), Brownian motion can, up to positive multiples, be characterized as a process with stationary orthogonal increments.

Remark 15. *Consider the stochastic process $(\varphi(\beta_t))_{t\in\mathbb{R}}$ associated to the cocycle $(\beta_t)_{t\in\mathbb{R}}$ in $\mathcal{E}$. We say that the increments of this process are positively (negatively) correlated if, for $t_1 \leq t_2 \leq t_3 \leq t_4$, we have*

$$\pm\langle \beta_{t_2} - \beta_{t_1}, \beta_{t_4} - \beta_{t_3} \rangle \geq 0.$$

As

$$\langle \beta_{t_2} - \beta_{t_1}, \beta_{t_4} - \beta_{t_3} \rangle = \langle U_{t_1}\beta_{t_2-t_1}, U_{t_3}\beta_{t_4-t_3} \rangle = \langle U_{t_1-t_3}\beta_{t_2-t_1}, \beta_{t_4-t_3} \rangle = \langle \beta_{t_2-t_3} - \beta_{t_1-t_3}, \beta_{t_4-t_3} \rangle,$$

we may w.l.o.g. assume that $t_3 = 0$, i.e., $t_1 \leq t_2 \leq 0 \leq t_4$. Therefore the process has positively (negatively) correlated increments if and only if, for every $t \geq 0$, the functions

$$C_t(s) := C(s,t) := \langle \beta_s, \beta_t \rangle$$

are increasing (decreasing) on $(-\infty, 0]$. Note that this implies in particular that $C(s,t) \geq 0$, resp., ≤ 0 for $s \leq 0 \leq t$.

For fractional Brownian motion, we have for $s \leq 0 \leq t$

$$C^H(s,t) = \frac{1}{2}((-s)^{2H} + t^{2H} - (t-s)^{2H}).$$

As

$$\frac{\partial}{\partial s} C^H(s,t) = H((t-s)^{2H-1} - (-s)^{2H-1})$$

is non-negative for $H \geq \frac{1}{2}$ and non-positive for $H \leq \frac{1}{2}$, it follows that fractional Brownian motion has positively correlated increments for $H \geq \frac{1}{2}$ and negatively correlated increments for $H \leq \frac{1}{2}$.

8. Perspectives

In this final section we briefly discuss some results that are possibly related to far reaching generalizations of what we discuss in the present paper on the real line, resp., on its conformal compactification $\mathbb{S}^1$.

8.1. Helices and Hilbert Distances

Let G be a Lie group, $K \subseteq G$ be a closed subgroup. We write $X = G/K$ for the corresponding homogeneous space and $x_0 := eK$ for the base point in X.

In [38] Def. 2.3, a kernel $C\colon X \times X \to \mathbb{R}$ on $X = G/K$ is called a *Lévy–Schoenberg kernel* if

(LS1) C is positive definite and $C_{x_0} = 0$.
(LS2) The kernel $r(s,t) := C(s,s) + C(t,t) - 2C(s,t)$ is G-invariant.

Then $\psi(g) := r(x_0, x) = C(x,x)$, where $x = g.x_0$, defines a function on G with

$$C(gK, hK) = \frac{1}{2}(\psi(g) + \psi(h) - \psi(h^{-1}g)), \tag{29}$$

so that ψ is a negative definite K-biinvariant function with $\psi(e) = 0$ on G (Lemma 1). Conversely, every such function defines by (29) a Lévy–Schoenberg kernel on G/K.

For a Lévy–Schoenberg kernel C, there exists a map $\xi\colon X \to \mathcal{E}$ into a real Hilbert space $\mathcal{E}$ with $\xi(x_0) = 0$, unique up to orthogonal equivalence (Lemma A1), such that

$$r(x,y) = \|\xi(x) - \xi(y)\|^2 \quad \text{for} \quad x,y \in X.$$

Then ξ is called a *helix* and $\sqrt{r}$ is called an *invariant Hilbert distance* on X. The uniqueness of ξ further implies the existence of an affine isometric action $\alpha\colon G \to \mathrm{Mot}(\mathcal{E})$ for which ξ is equivariant. Writing $\alpha_g\xi = U_g\xi + \beta_g$, we then have $\xi(gK) = \beta_g$ for $g \in G$. In particular, any helix specifies an orthogonal representation $(U, \mathcal{E})$ of G.

Classification results for Lévy–Schoenberg kernels, resp., invariant Hilbert distances, resp., affine isometric actions of G with a K-fixed points, are mostly stated in terms of integral formulas (Lévy–Khintchine formulas). Results are nown in various contexts:

- for G locally compact and K compact ([39]); see [40,41] for locally compact abelian groups.
- for the euclidean motion group $G = E(d) \supseteq \mathrm{O}_d(\mathbb{R}) = K$ ([38,42] p. 135)
- for G compact ([38] Thm. 3.15); see [43] for $G = \mathrm{SO}_{d+1}(\mathbb{R})$ and $X = \mathbb{S}^d$.
- for G/K Riemannian symmetric ([38] (Thm. 3.31) and [38] (Thm. 4.1) for $G = \mathrm{SL}_2(\mathbb{R})$)
- for G the additive group of a Hilbert space and K a closed subspace ([44]).
- for $G = \mathrm{O}_{1,\infty}(\mathbb{R})$ and $K = \mathrm{O}_\infty(\mathbb{R})$ and X the infinite dimensional hyberbolic space ([39] Thm. 8.1). It is shown in particular that the kernel $Q(x,y) = \log\cosh(d(x,y))$ is negative definite, so that all kernels $e^{-sQ(x,y)} = \cosh(d(x,y))^{-s}$ are positive definite; they correspond to the spherical functions of X (cf. [13] Thm. 21, p. 79).

Example 7. *(a) If $\mathcal{E}$ is a real Hilbert space and $G = \mathrm{Mot}(\mathcal{E}) \cong \mathcal{E} \rtimes \mathrm{O}(\mathcal{E})$ its isometry group, then $\psi(b,g) := \|b\|^{2H}$ defines for $0 < H \leq 1$ a negative definite $\mathrm{O}(\mathcal{E})$-biinvariant function on G with $\psi(e) = 0$ ([31] Ex. 3.2.13(b)). The corresponding Lévy–Schoenberg kernel on $\mathcal{E}$ is*

$$C(s,t) := \frac{1}{2}(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}) \quad \textit{with} \quad r(s,t) = \|s-t\|^{2H}$$

(cf. [38] p. 135).

(b) The kernel

$$C(s,t) := \frac{1}{2}(d(s,e_0) + d(t,e_0) - d(s,t)) \quad \textit{with} \quad r(s,t) = d(s,t) \tag{30}$$

on the sphere $\mathbb{S}^d$, where d denotes the Riemannian metric on $\mathbb{S}^d$ and $e_0 \in \mathbb{S}^d$ is fixed. Here $G = \mathrm{O}_{d+1}(\mathbb{R})$ and $K = \mathrm{O}_d(\mathbb{R})$ is the stabilizer of e_0 ([38] p. 174, [23]). This means that the Riemannian metric on $\mathbb{S}^d$ is a negative definite kernel.

(c) From (a) it follows that for any real-valued negative definite function ψ satisfying $\psi(e) = 0$ on the group G, the functions ψ^H, $0 \leq H \leq 1$, are negative definite as well ([38] p. 189).

8.2. Brownian Motion on Metric Spaces

Definition 13. *Let (M,d) be a metric space. In [23] a real-valued Gaussian process $(B_m)_{m\in M}$ is called a Brownian motion with parameter space (M,d) if there exists a point $m_0 \in M$ with*

$$\mathbb{E}(B_n B_m) = \frac{1}{2}\big(d(m,m_0) + d(n,m_0) - d(m,n)\big) \quad \textit{for} \quad m,n \in M.$$

Remark 16. *For a metric space (M,d) a Brownian motion with parameter space (M,d) exists if and only if the metric $d\colon M\times M \to \mathbb{R}$ is a negative definite kernel, which is equivalent to the kernels*

$$C(n,m) = \frac{1}{2}\big(d(m,m_0) + d(n,m_0) - d(m,n)\big)$$

being positive definite for every $m_0 \in M$ (Lemma 1; see also [45] and [23] (Cor. 58)). This is verified for $M = \mathbb{R}^d$ in [45] (Thm. 7) and for $M = \mathbb{S}^d$ in [45] (Thm. 5).

Definition 14. *If d is negative definite, then there exists an isometric embedding $\eta\colon M \to \mathcal{E}$ into a real Hilbert space with $\eta(m_0) = 0$ and then*

$$C(n,m) = \frac{1}{2}\big(\|\eta(m)\| + \|\eta(n)\| - \|\eta(m) - \eta(n)\|\big).$$

Example 7(a) then implies that the kernels

$$C^H(n,m) = \frac{1}{2}\big(\|\eta(m)\|^{2H} + \|\eta(n)\|^{2H} - \|\eta(m) - \eta(n)\|^{2H}\big), \qquad 0 \le H \le 1,$$

are positive definite as well. This suggests to call a Gaussian process $(B^H_m)_{m\in M}$ a fractional Brownian motion with parameter space (M,d) and Hurst index $H \in (0,1)$ if there exists an $m_0 \in M$ with $\mathbb{E}(B^H_n B^H_m) = C^H(n,m)$ for $n,m \in M$. Note that $(\varphi(\eta(m)))_{m\in M}$ yields a realization of fractional Brownian motion with parameter space (M,d) in the Fock space $\Gamma(\mathcal{E})$. If $M = G/K$ is a homogeneous space, then the map η is called a fractional Brownian helix (cf. [14] for the terminology).

For various aspects of fractional Brownian motion on $\mathbb{R}^d$, we refer to [19] and [46].

Problem 1. *The natural analog of the function $\chi_{\mathbb{R}_+}$ which generates the realization of the fractional Brownian motion in $\mathcal{H}_H$ has a natural higher dimensional analog in $\chi_{\mathbb{R}^d_+}$, the characteristic function of a half space. Does this correspond to some "fractional Brownian motion" on $\mathbb{R}^d$?*

8.3. Complementary Series of the Conformal Group

The function $\|x\|^{-\alpha}$ on $\mathbb{R}^d$ is locally integrable if and only if $\alpha < d$, and it defines a positive definite distribution if and only if $\alpha \ge 0$ ([8] Lemma 2.13). We thus obtain a family of Hilbert subspaces $\mathcal{H}_\alpha \subseteq C^{-\infty}(\mathbb{R}^d)$ for $0 \le \alpha < d$. For $\alpha = 0$ this space is one-dimensional, consisting of constant functions.

From [8] (Prop. 6.1) we also know that, for $0 \le \alpha < d$, the distribution $\|x\|^{-\alpha}$ is reflection positive with respect to $\theta(x) = (-x_0, \mathbf{x})$ if and only if $\alpha = 0$ or $d - 2 \le \alpha < d$.

Let $G := \mathrm{Conf}(\mathbb{R}^d) \subseteq \mathrm{Diff}(\mathbb{S}^d)$ be the conformal group of $\mathbb{R}^d$, considered as a group of diffeomorphisms of the conformal compactification $\mathbb{S}^d$ (implemented by a stereographic projection). We consider the kernels

$$Q(x,y) := \|x-y\| \quad \text{and} \quad Q_\alpha(x,y) := \|x-y\|^{-\alpha}.$$

We then have

$$Q(g(x), g(y)) = \|\mathrm{d}g(x)\|^{1/2} Q(x,y) \|\mathrm{d}g(y)\|^{1/2} \quad \text{if} \quad g(x), g(y) \in \mathbb{R}^d. \tag{31}$$

In fact, this relation is obvious for affine maps $g(x) = Ax + b$, $A \in \mathbb{R}^\times \mathrm{O}_d(\mathbb{R})$. As the conformal group is generated by the affine conformal group $\mathbb{R}^d \rtimes (\mathbb{R}^\times \mathrm{O}_d(\mathbb{R}))$ and the inversion $\sigma(x) := \frac{x}{\|x\|^2}$ in the unit sphere, it now suffices to verify the relation also for σ. It is a consequence of

$$\begin{aligned}\|\sigma(x) - \sigma(y)\|^2 &= \Big\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \Big\|^2 = \frac{1}{\|x\|^2 \|y\|^2} \Big\| \frac{\|y\| x}{\|x\|} - \frac{\|x\| y}{\|y\|} \Big\|^2 \\ &= \frac{1}{\|x\|^2 \|y\|^2} (\|y\|^2 - 2\langle x, y\rangle + \|x\|^2) = \frac{\|y - x\|^2}{\|x\|^2 \|y\|^2},\end{aligned}$$

combined with

$$\mathrm{d}\sigma(x) y = \frac{y}{\|x\|^2} - 2\frac{\langle x, y\rangle}{\|x\|^4} x = \frac{r_x(y)}{\|x\|^2},$$

where r_x is the reflection in $x^\perp$, so that $\|d\sigma(x)\| = \|x\|^{-2}$.

As a consequence, we obtain

$$Q_\alpha(g.x, g.y) = \|\mathrm{d}g(x)\|^{-\alpha/2} Q_\alpha(x,y) \|\mathrm{d}g(y)\|^{-\alpha/2} \tag{32}$$

(see [8] (Lemma 5.8) for the corresponding relation on the sphere $\mathbb{S}^d$).

The transformation Formula (31) implies in particular that the *conformal cross ratio*

$$\mathrm{CR}(x, y, z, u) := \frac{Q(x,y) Q(z,u)}{Q(x,u) Q(z,y)}$$

is invariant under the conformal group.

In view of (32), we obtain with $J_g(x) := \|\mathrm{d}g(x)\|$ a representation

$$(U^\alpha_g \xi)(x) := J_{g^{-1}}(x)^{d - \frac{\alpha}{2}} \xi(g^{-1}.x)$$

on test functions. The same calculation as in [8] (Lemma 5.8) now implies that U^α defines a unitary representation of G on the space $\mathcal{H}_\alpha$, specified by the scalar product

$$\langle \xi, \eta\rangle_\alpha := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \overline{\xi(x)} \eta(y) Q_\alpha(x,y)\, dx\, dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \overline{\xi(x)} \eta(y) \|x - y\|^{-\alpha}\, dx\, dy \quad \text{for} \quad \xi, \eta \in \mathcal{S}(\mathbb{R}^d).$$

For $g^{-1}(x) = Ax + b$, we have in particular

$$(U^\alpha_{g^{-1}} \xi)(x) = \|A\|^{d - \frac{\alpha}{2}} \xi(Ax + b),$$

and for the involution $\sigma(x) = \|x\|^{-2} x$ we have

$$(U^\alpha_\sigma \xi)(x) = \|x\|^{\alpha - 2d} \xi(\sigma(x)).$$

Remark 17. *Up to the factor* $\mathrm{sgn}(\det g)$, *this specializes for* $d = 1$ *and* $\alpha = 2(1 - H)$ *to the representation* U^H *for* $\frac{1}{2} < H < 1$. *We refer to Appendix D for more detailed discussion of this case.*

As the kernel $D^H(x,y) := \|x - y\|^{2H}$, $0 < H \leq 1$, on $\mathbb{R}^d$ is negative definite, the corresponding kernel

$$C^H(x,y) := \frac{1}{2}\big(\|x\|^{2H} + \|y\|^{2H} - \|x - y\|^{2H}\big)$$

is positive definite. We thus obtain on $\mathcal{S}(\mathbb{R}^d)_0 = \{\xi \in \mathcal{S}(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} \xi(x)\,dx = 0\}$ a positive semidefinite hermitian form by

$$\begin{aligned}\langle \xi, \eta \rangle_{C^H} &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \overline{\xi(x)}\eta(y) C^H(x,y)\,dx\,dy \\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \overline{\xi(x)}\eta(y)\|x-y\|^{2H}\,dx\,dy, \qquad \xi,\eta \in \mathcal{S}(\mathbb{R})_0. \end{aligned} \tag{33}$$

Example 8. *For $H = 1$, we have*

$$C^1(x,y) = \frac{1}{2}\big(\|x\|^2 + \|y\|^2 - \|x-y\|^2\big) = \langle x, y\rangle,$$

so that the corresponding reproducing kernel Hilber space is $\mathcal{H}_{C^1} \cong \mathbb{C}^d$. For $\xi, \eta \in \mathcal{S}(\mathbb{R}^d)$, we then have

$$\langle \xi, \eta\rangle_{C^1} = \langle [\xi], [\eta]\rangle, \quad \textit{where} \quad [\xi] := \int_{\mathbb{R}^d} \xi(x) x\,dx \in \mathbb{R}^d$$

is the center of mass of the measure $\xi\,dx$.

Remark 18. *In [47] (Thm. 7) Takenaga derives some "conformal invariance" of Brownian motion in $\mathbb{R}^d$ but it seems that his method only works on the parabolic subgroups of the conformal group stabilizing either 0 or ∞. So it would be interesting to use the complementary series representations of the conformal group to derive a more complete conformal invariance in the spirit of the present paper for $d > 1$.*

Remark 19. *Similar arguments as in Remark 10 apply in the higher dimensional context: Since the complementary series representations $(U^\alpha, \mathcal{H}_\alpha)$ of $\mathrm{O}_{1,d}(\mathbb{R})^\uparrow$ are irreducible, [1] (Prop. 5.20) implies that the space $\mathcal{H}_\alpha^\infty$ of smooth vectors is nuclear. From the proof of [1] (Prop. 5.20(b)), we further derive that an element $\xi \in \mathcal{H}_\alpha$ is a smooth vector if and only if it is a smooth vector for the maximal compact subgroup $K \cong \mathrm{O}_d(\mathbb{R})$. Considering $\mathcal{H}_\alpha$ as a space of distributions on the sphere $\mathbb{S}^d$, it is not hard to see that $\mathcal{H}_\alpha^\infty = C^\infty(\mathbb{S}^d)$ and hence that $\mathcal{H}_\alpha^{-\infty} = C^{-\infty}(\mathbb{S}^d)$ is the space of distributions on the sphere.*

8.4. The Ornstein—Uhlenbeck Process

In this section we describe shortly the connection to the *Ornstein–Uhlenbeck process*. For that let $H = \frac{1}{2}$. Then $Y_t := \varphi(\tau_{e^t}^{1/2}\chi_{[0,1]})$, $t \in \mathbb{R}$, is a stationary Gaussian process realized in $\mathcal{H}_{1/2} \cong L^2(\mathbb{R})$. It is the *Ornstein–Uhlenbeck process*. The corresponding covariance kernel is

$$C(t,s) := \mathbb{E}(Y_t Y_s) = \int_0^{e^{s-t}} e^{(t-s)/2}\,du = e^{(s-t)/2} = e^{-|s-t|/2} \quad \text{for} \quad s \le t$$

which is reflection positive with respect to $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$ because the kernel

$$C(-t,s) = e^{-(s+t)/2}$$

on $\mathbb{R}_+$ is positive definite leading to a one-dimensional Hilbert space via the Osterwalder-Schrader construction (cf. Example 1).

For $0 < H < 1$, we also obtain by $Y_t^H := \varphi(\tau_{e^t}^H \chi_{[0,1]}) = \varphi(e^{tH}\chi_{[0,e^{-t}]})$, $t \in \mathbb{R}$, in $\mathcal{H}_H$ a stationary Gaussian process. The corresponding covariance kernel is

$$\begin{aligned} C(t,s) &= \mathbb{E}(Y_s Y_t) = e^{(t+s)H} C^H(e^{-s}, e^{-t}) = \frac{e^{(t+s)H}}{2}\big(e^{-2sH} + e^{-2tH} - |e^{-t} - e^{-s}|^{2H}\big) \\ &= \frac{1}{2}\big(e^{(t-s)H} + e^{(s-t)H} - |e^{(s-t)/2} - e^{(t-s)/2}|^{2H}\big) \\ &= \cosh((t-s)H) - 2^{2H-1}|\sinh((s-t)/2)|^{2H} =: \varphi(s-t). \end{aligned}$$

We then have

$$C(s,-t)=\varphi(s+t)=\cosh((t+s)H)-2^{2H-1}|\sinh((s+t)/2)|^{2H}$$

and

$$\begin{aligned}2\varphi(x)&=e^{Hx}+e^{-Hx}-(e^{x/2}-e^{-x/2})^{2H}=e^{Hx}+e^{-Hx}-e^{Hx}(1-e^{-x})^{2H}\\&=e^{-Hx}+e^{Hx}\sum_{k=1}^{\infty}\binom{2H}{k}(-1)^{k-1}e^{-kx}.\end{aligned}$$

In Proposition 2 we have seen that this function is positive definite if and only if $0<H\leq 1/2$. Hence C is reflection positive for $0<H\leq 1/2$.

Now let $H=1/2$ so that $C(t,s)$ corresponds to the Ornstein–Uhlenbeck process. In this case $C(t,s)$ is invariant under the reflection $\theta(t)=-t$. As $\chi_{[0,1]}$ is cyclic in $L^2(\mathbb{R}_+)$ for the dilation group, there exists a unique unitary isometry V on $L^2(\mathbb{R}_+)$ with $V(\tau_{e^t}\chi_{[0,1]})=\tau_{e^{-t}}\chi_{[0,1]}$ for $t\in\mathbb{R}$. The latter relation is is equivalent to $e^{t/2}V(\chi_{[0,e^{-t}]})=e^{-t/2}\chi_{[0,e^t]}$, resp.,

$$V(\chi_{[0,t]})=t\chi_{[0,t^{-1}]}\quad\text{for}\quad t>0. \tag{34}$$

Therefore V coincides with the unitary involution $\widehat{\theta}$ corresponding to the symmetry of Brownian motion under inverstion of t (see Remark 7(a), and also Lemma 6 below).

Note that $(\theta\xi)(x)=\frac{1}{x}\xi\left(\frac{1}{x}\right)$ also defines an isometric involution on $L^2(\mathbb{R}_+^\times)$ having the same intertwining properties with the dilation group as $\widehat{\theta}$, but this involution does not fix $\chi_{[0,1]}$.

To derive a formula for the involution $\widehat{\theta}$, we recall the Sobolev space $H^1_*(\mathbb{R})$.

Definition 15. *Let $H^1_*(\mathbb{R})$ denote the Sobolev space of all absolutely continuous functions $F\colon\mathbb{R}\to\mathbb{R}$ satisfying $F(0)=0$ and $F'\in L^2(\mathbb{R})$. Then*

$$I\colon L^2(\mathbb{R}_+)\to H^1_*(\mathbb{R}),\quad I(f)(t):=\int_0^t f(s)\,ds=\langle b_t^{1/2},f\rangle$$

is a bijection. We define a real Hilbert space structure on $H^1_*(\mathbb{R})$ in such a way that I is isometric. The inverse isometry is then given by $F\mapsto F'$.

Remark 20. *(a) From the relation $I(f)(t)=\langle f,b_t^{1/2}\rangle$, it follows that $H^1_*(\mathbb{R})$ is the real reproducing kernel Hilbert space with kernel $C=C^{1/2}$, i.e., the covariance kernel of Brownian motion $(B_t)_{t\in\mathbb{R}}$.*

(b) We observe that $|F(t)|\leq\|F\|\sqrt{|t|}$ for $F\in H^1_*(\mathbb{R})$ and $t\in\mathbb{R}$ follows immediately from the Cauchy–Schwarz inequality and $\|b_t^{1/2}\|_2^2=C^{1/2}(t,t)=|t|$.

Lemma 6. *There exists a uniquely determined isometric involution $\widehat{\theta}$ on $L^2(\mathbb{R}_+)$ satisfying*

$$\widehat{\theta}(\chi_{[0,t]})=t\chi_{[0,t^{-1}]}\quad\textit{for}\quad t>0.$$

It is given by

$$(\widehat{\theta}\xi)(t)=\int_0^{1/t}\xi(s)\,ds-\frac{1}{t}\xi\left(\frac{1}{t}\right)\quad\textit{for}\quad t>0. \tag{35}$$

Proof. First we observe that the family $b_t=\chi_{[0,t]}$, $t>0$, is total in $L^2(\mathbb{R}_+)$. Since the family $\widetilde{b}_t:=tb_{1/t}$ satisfies $\langle\widetilde{b}_t,\widetilde{b}_s\rangle=\langle b_t,b_s\rangle=t\wedge s$ (Remark 7), there exists a uniquely determined isometry $\widehat{\theta}$ with $\widehat{\theta}(b_t)=\widetilde{b}_t$ for $t>0$. As $(\widetilde{b}_t)_{t>0}$ is also total, $\widehat{\theta}$ is surjective. Now $\widehat{\theta}(\widetilde{b}_t)=t\frac{1}{t}b_t=b_t$ for $t>0$ implies $\widehat{\theta}^2=\mathbf{1}$.

Let $\widetilde{\theta}$ denote the involutive isometry of $H^1_*(\mathbb{R})$ specified by $\widetilde{\theta} \circ I = I \circ \widehat{\theta}$, resp., $\widehat{\theta}(F') = \widetilde{\theta}(F)'$. Then we obtain

$$(\widetilde{\theta} I(f))(t) = I(\widehat{\theta} f)(t) = \langle \widehat{\theta} f, \chi_{[0,t]} \rangle = \langle f, \widehat{\theta} \chi_{[0,t]} \rangle = t \langle f, \chi_{[0,t^{-1}]} \rangle = t I(f)(t^{-1})$$

and thus

$$(\widetilde{\theta} F)(t) = t F(t^{-1}).$$

For $f = F'$, this leads to

$$(\widehat{\theta} f)(t) = (\widehat{\theta} F')(t) = (\widetilde{\theta} F)'(t) = F(t^{-1}) - \frac{1}{t} F'(t^{-1}) = \int_0^{t^{-1}} f(s)\, ds - \frac{1}{t} f(t^{-1}).$$

This proves (35). □

Remark 21. *(a) Note that* (35) *has a striking similarity with the formula for T_J one finds in [28] p. 273. This suggests these operators correspond to a discrete series representation of* $\mathrm{SL}_2(\mathbb{R})$*, hence cannot be implemented in the complementary series representation that we consider.*

(b) From the explicit formula for $\widehat{\theta}$, we can also make the natural map q from $\mathcal{E}_+ := L^2([0,1]) \subseteq \mathcal{E} = L^2(\mathbb{R}_+)$ to the space $\widehat{\mathcal{E}} \cong \mathbb{C}$ more explicit. It is given by

$$q(f) = \int_0^1 f(x)\, dx.$$

This follows from

$$\langle f, \widehat{\theta} f \rangle = \int_0^1 \int_0^{1/x} f(u)\, du\, \overline{f(x)}\, dx - \int_0^1 \overline{f(x)} \frac{1}{x} f(x^{-1})\, dx = \int_0^1 \int_0^1 f(u) \overline{f(x)}\, du = \Big| \int_0^1 f(x)\, dx \Big|^2.$$

Here $\chi_{[0,1]} \in \mathcal{E}_+$ spans the one-dimensional subspace of $\widehat{\theta}$-fixed points, so that $q \colon \mathcal{E}_+ \to \widehat{\mathcal{E}}$ can be identified with the projection onto $\mathbb{C}\chi_{[0,1]}$.

Takenaga's formula ([27])

$$B^g_t := (ct+d) B_{g.t} - ct \cdot B_{g.\infty} - d \cdot B_{g.0}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \qquad t \in \mathbb{R}_\infty, g.t \neq \infty,$$

defines for each $g \in \mathrm{SL}_2(\mathbb{R})$ on $H^1_*(\mathbb{R}) \cong \mathcal{H}_{C^{1/2}}$ a unitary operator which acts on the point evaluations $(b_t)_{t \in \mathbb{R}}$ by

$$U_g^{-1} b_t := b^g_t := (ct+d) b_{g.t} - ct \cdot b_{g.\infty} - d \cdot b_{g.0},$$

where we put $cb_{g.\infty} = 0$ for $c = 0$ ($g.\infty = \infty$), $d \cdot b_{g.0} = 0$ for $d = 0$ ($g.0 = \infty$) and $(ct+d) b_{g.t} = 0$ for $ct + d = 0$ ($g.t = \infty$). On general functions $F \in H^1_*(\mathbb{R})$, the operator U_g acts by

$$(U_g F)(t) = \langle b_t, U_g F \rangle = \langle U_g^{-1} b_t, F \rangle = (ct+d) F(g^{-1}.t) - ct F(a/c) - d F(b/d), \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that all summands are well defined for $c = 0$, $d = 0$, resp., $ct + d = 0$ because $|F(t)| \leq \|F\| \cdot |t|^{1/2}$ implies

$$\lim_{t \to 0} t F(s/t) = 0 \quad \text{for} \quad s \in \mathbb{R}.$$

The relation $(b^g)^h_t = b^{gh}_t$ for $g, h \in \mathrm{SL}_2(\mathbb{R})$ now leads to

$$U_{gh}^{-1} b_t = b^{gh}_t = U_h^{-1} b^g_t = U_h^{-1} U_g^{-1} b_t \quad \text{for} \quad g, h \in \mathrm{SL}_2(\mathbb{R}), t \in \mathbb{R},$$

and hence to $U_g U_h = U_{gh}$. We thus obtain on $H^1_*(\mathbb{R})$ a continuous unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Note that $U_{-\mathbf{1}} = -\mathbf{1}$, so that this representation does NOT factor through a representation of $\mathrm{PSL}_2(\mathbb{R})$. The most economical way to verify the assertion that the operators U_g are unitary is to do that for $g.t = at + b$ and $\sigma.t = -t^{-1}$ and then to verify that $(b^g)^h_t = b^{gh}_t$ holds for all $g \in \mathrm{SL}_2(\mathbb{R})$ and $h.t = at + b$ or $h.t = -t^{-1}$. As $\mathrm{SL}_2(\mathbb{R})$ is generated by elements of this form, it follows that U defines a unitary representation on $H^1_*(\mathbb{R}) \cong \mathcal{H}_{C^{1/2}}$.

With the aforementioned conventions concerning expressions of the form $tF(s/t) = 0$ for $t = 0$, the representation is given by

$$\begin{aligned}(U_g F)(t) &= (ct + d)F(g^{-1}.t) - ctF(a/c) - dF(b/d) \\ &= (ct + d)F(g^{-1}.t) - ct \cdot F(g^{-1}.\infty) - d \cdot F(g^{-1}.0).\end{aligned} \tag{36}$$

If $g^{-1}.t = \alpha t + \beta$ with $\alpha > 0$ is affine, then $g^{-1} = \begin{pmatrix} \sqrt{\alpha} & \sqrt{\alpha}\beta \\ 0 & \sqrt{\alpha}^{-1} \end{pmatrix}$ and we get

$$(U_g F)(t) = \alpha^{-1/2}(F(\alpha t + \beta) - F(\beta)).$$

For $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $J^{-1} = -J$, so that

$$(U_J F)(t) = -t(F(-t^{-1}) - F(0)) = -tF(-t^{-1}).$$

We also note that (36) leads to

$$\begin{aligned}(U_g F)'(t) &= c(g_* F)(t) + (ct + d)(g_* F')(t)\frac{1}{(ct + d)^2} - cF(a/c) \\ &= c\big(F(g^{-1}.t) - F(g^{-1}.\infty)\big) + (ct + d)^{-1}(g_* F')(t).\end{aligned}$$

If $g^{-1}.t = \alpha t + \beta$ is affine, then

$$(U_g F')(t) = \alpha^{1/2} F'(\alpha t + \beta)$$

yields the usual action of $\mathrm{Aff}(\mathbb{R})$ on $L^2(\mathbb{R})$.

Remark 22. *Since we want to express this in terms of the derivatives, we observe that, formally, we expect something like*

$$\begin{aligned}F(g^{-1}.t) - F(g^{-1}.\infty) &= \int_{g^{-1}.\infty}^{g^{-1}.t} F'(x)\,dx = \int_{\infty}^{t} F'(g^{-1}.x)\,(g^{-1})'(x)\,dx \\ &= -\int_t^{\infty} \frac{F'(g^{-1}.x)}{(cx + d)^2}\,dx = \int_{-\infty}^{t} \frac{F'(g^{-1}.x)}{(cx + d)^2}\,dx.\end{aligned}$$

In particular, we have

$$(U_J F)'(t) = -F(-t^{-1}) - t \cdot t^{-2} F'(-t^{-1}) = -F(-t^{-1}) - t^{-1}F'(-t^{-1}) = -\int_0^{-t^{-1}} F'(x)\,dx - t^{-1}F'(-t^{-1}).$$

For $\xi = F'$, this reads

$$(U_J \xi)(t) = -t^{-1}\xi(-t^{-1}) - \int_0^{-t^{-1}} \xi(x)\,dx \quad \text{for} \quad t \neq 0.$$

This formula describes the unique unitary involution on $L^2(\mathbb{R})$ mapping b_t to $b_{-t^{-1}}$ (cf. Lemma 6).

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Appendix A. Existence of Affine Isometries

For a map $\gamma\colon X \to \mathcal{H}$ into a Hilbert space, the closed subspace $\mathcal{H}_\gamma$ generated by all differences $\gamma(x) - \gamma(y)$, $x, y \in X$, is called the *chordal space of* γ (cf. [17]). The following lemma is an abstraction of [48] Satz 1,3.

Lemma A1. *Let X be a non-empty set, $\mathcal{H}$ be a real or complex Hilbert space and $\gamma\colon X \to \mathcal{H}$ and $\gamma'\colon X \to \mathcal{H}'$ be maps with $\mathcal{H}_\gamma = \mathcal{H}$ and $\mathcal{H}_{\gamma'} = \mathcal{H}'$. For $x_0 \in X$, consider the kernel*

$$K_\gamma^{x_0}(x,y) := \langle \gamma(x) - \gamma(x_0), \gamma(y) - \gamma(x_0) \rangle \quad on \quad X \times X.$$

Then the following are equivalent:

(i) *There exists an affine isometry $V\colon \mathcal{H} \to \mathcal{H}'$ with $V \circ \gamma = \gamma'$.*
(ii) $K_\gamma^{x_0} = K_{\gamma'}^{x_0}$ *for every $x_0 \in X$.*
(iii) $K_\gamma^{x_0} = K_{\gamma'}^{x_0}$ *for some $x_0 \in X$.*

If $\mathcal{H}$ and $\mathcal{H}'$ are real, then these conditions are equivalent to

(iv) $\|\gamma(x) - \gamma(y)\|^2 = \|\gamma'(x) - \gamma'(y)\|^2$ *for $x, y \in X$.*

If (i)–(iii) are satisfied, then the affine isometry V in (i) is uniquely determined by the relation $V \circ \gamma = \gamma'$.

Proof. (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): From [30] (Ch. I) it follows that there exists a unique unitary operator $U\colon \mathcal{H} \to \mathcal{H}$ with

$$U(\gamma(x) - \gamma(x_0)) = \gamma'(x) - \gamma'(x_0) \quad \text{for all} \quad x \in X.$$

Then we put $V\xi := U\xi - U(\gamma(x_0)) + \gamma'(x_0)$.

(i) $\Rightarrow$ (ii): If $V\xi = U\xi + b$ is an affine isometry with $V \circ \gamma = \gamma'$, then

$$\begin{aligned} K_{\gamma'}^{x_0}(x,y) &= \langle \gamma'(x) - \gamma'(x_0), \gamma'(y) - \gamma'(x_0) \rangle = \langle U\gamma(x) - U\gamma(x_0), U\gamma(y) - U\gamma(x_0) \rangle \\ &= \langle \gamma(x) - \gamma(x_0), \gamma(y) - \gamma(x_0) \rangle = K_\gamma^{x_0}(x,y). \end{aligned}$$

(iv) $\Leftrightarrow$ (iii): The kernel $D_\gamma(x,y) := \|\gamma(x) - \gamma(y)\|^2$ satisfies

$$\begin{aligned} D_\gamma(x,y) &= \|\gamma(x) - \gamma(x_0) + \gamma(x_0) - \gamma(y)\|^2 \\ &= \|\gamma(x) - \gamma(x_0)\|^2 + \|\gamma(x_0) - \gamma(y)\|^2 + 2\,\mathrm{Re}\, K_\gamma^{x_0}(x,y) \\ &= K_\gamma^{x_0}(x,x) + K_\gamma^{x_0}(y,y) + 2\,\mathrm{Re}\, K_\gamma^{x_0}(x,y) \end{aligned}$$

and, conversely,

$$\mathrm{Re}\, K_\gamma^{x_0}(x,y) = \frac{1}{2}\big(D_\gamma(x,y) - D_\gamma(x,x_0) - D_\gamma(y,x_0)\big).$$

Therefore (iv) is equivalent to $\mathrm{Re}\, K_\gamma^{x_0} = \mathrm{Re}\, K_{\gamma'}^{x_0}$. If $\mathcal{H}$ is real, this is equivalent to (iii). □

Appendix B. Stochastic Processes

Definition A1. *Let (Q, Σ, μ) be a probability space and $(B, \mathfrak{B})$ be a measurable space. A stochastic process with state space $(B, \mathfrak{B})$ is a family $(X_t)_{t \in T}$ of measurable functions $X_t\colon Q \to B$, where T is a set.*

(a) We call the stochastic process $(X_t)_{t\in T}$ full if, up to sets of measure 0, Σ is the smallest σ-algebra for which all functions X_t are measurable.

(b) For $B = \mathbb{R}$ or $\mathbb{C}$, we say that $(X_t)_{t\in T}$ is square integrable if every X_t is square integrable. Then the covariance kernel

$$C(s,t) := \mathbb{E}(\overline{X_s}X_t)$$

on T is positive definite. If $C(t,t) = \mathbb{E}(|X_t|^2) > 0$ for every $t \in T$, then $\widetilde{X}_t := X_t/\sqrt{\mathbb{E}(|X_t|^2)}$ is called the associated normalized process. Its covariance kernel is

$$\widetilde{C}(s,t) = \frac{C(s,t)}{\sqrt{C(s,s)C(t,t)}} \quad \text{for} \quad s,t \in T.$$

(c) On the product space B^T of all maps $T \to B$, there exists a unique probability measure ν with the property that, for $t_1, \ldots, t_n \in T$, the image of ν under the evaluation map $\mathrm{ev}_{t_1,\ldots,t_n} : B^T \to B^n$ is the image of μ under the map $(X_{t_1}, \ldots, X_{t_n})$. We call ν the distribution of the process $(X_t)_{t\in T}$ ([37], Thm. 1.5).

Definition A2. *Let $(X_t)_{t\in T}$ be a centered $\mathbb{K}$-valued stochastic process and $\sigma\colon G \times T \to T$ be a group acting on T.*

(a) The process $(X_t)_{t\in T}$ is called stationary if, for every $g \in G$, the process $(X_{g.t})_{t\in T}$ has the same distribution. Then we obtain a measure preserving G-action on the underlying path space $\mathbb{K}^T$ by $(g.\omega)(t) := \omega(g^{-1}.t)$, resp., $g.X_t = X_{g.t}$.

(b) The process $(X_t)_{t\in T}$ is said to have stationary increments if, for $t_0, t_1, \ldots, t_n, t \in T$, the random vectors

$$(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_0}) \quad \text{and} \quad (X_{g.t_1} - X_{g.t_0}, \ldots, X_{g.t_n} - X_{g.t_0})$$

have the same distribution (cf. [34]).

Definition A3. *([49] Def. 2.8.1) A square integrable process $(Z_t)_{t\geq 0}$ is said to be wide sense stationary if the function $t \mapsto \mathbb{E}(Z_t)$ is constant and there exists a function $C\colon \mathbb{R} \to \mathbb{C}$ such that $C(s,t) = \mathbb{E}(\overline{Z_s}Z_t) = C(s-t)$.*

Appendix B.1. Processes with stationary increments

Proposition A1. (The flow of a process with stationary increments) *Let $(X_t)_{t\in T}$ be a $\mathbb{K}$-valued stochastic process and $\sigma\colon G \times T \to T, (g,t) \mapsto g.t$ be a G-action on T. Then the following are equivalent:*

(i) *$(X_t)_{t\in T}$ has stationary increments.*
(ii) *For every $t_0 \in T$,*

$$(g.\omega)(t) := \omega(g^{-1}.t) + \omega(t_0) - \omega(g^{-1}.t_0) \tag{A1}$$

defines a measure preserving flow on the path space $\mathbb{K}^T$ satisfying

$$g.X_t = X_{g.t} + X_{t_0} - X_{g.t_0}. \tag{A2}$$

Proof. (i) $\Rightarrow$ (ii): For each $g \in G$, we consider the map

$$\overline{\sigma}_g\colon \mathbb{K}^T \to \mathbb{K}^T, \quad \sigma_g(\omega)(t) := \omega(g^{-1}.t) + \omega(t_0) - \omega(g^{-1}.t_0). \tag{A3}$$

Then

$$((\overline{\sigma}_g)_* X_t)(\omega) = X_t(\overline{\sigma}_{g^{-1}}\omega) = \omega(g.t) + \omega(t_0) - \omega(g.t_0), \tag{A4}$$

i.e.,

$$(\sigma_g)_* X_t = X_{g.t} + X_{t_0} - X_{g.t_0}, \quad \text{resp.,} \quad X_{g.t} = (\sigma_g)_* X_t + X_{g.t_0} - X_{t_0}.$$

Since, for every finite subset $F \subseteq T$, the random vector $(X_{g.t} - X_{g.t_0})_{t\in F}$ has the same distribution as $(X_t - X_{t_0})_{t\in F}$, the flow on $\mathbb{K}^T$ defined by σ is measure preserving.

(ii) $\Rightarrow$ (i): If there exists a measure preserving G-action on $\mathbb{K}^T$ satisfying (A2), then the distribution of $(X_{g.t} + X_{t_0} - X_{g.t_0})_{t\in T}$ is the same as the distribution of $(X_t)_{t\in T}$. Subtracting X_{t_0}, it follows that the distribution of $(X_{g_t} - X_{g.t_0})_{t\in T}$ is the same as the distribution of $(X_t - X_{t_0})_{t\in T}$, i.e., that $(X_t)_{t\in T}$ has stationary increments. □

Remark A1. *(a) If the $\mathbb{K}$-valued process $(X_t)_{t\in T}$ on (Q, Σ, μ) is square integrable, then $(X_t)_{t\in T}$ generates a closed linear subspace $\mathcal{H}_1 \subseteq L^2(Q, \mu)$. The existence of a unitary representation $(U_g)_{g\in G}$ on $\mathcal{H}_1$ with $U_g X_t = X_{g.t}$ for $g \in G, t \in T$, is equivalent to the invariance of the covariance kernel*

$$C(s,t) := \mathbb{E}(\overline{X_s} X_t) = \langle X_s, X_t \rangle$$

(cf. [30] Ch. I). This condition is in particular satisfied if the process is stationary.

(b) For a square integrable process, it likewise follows that the existence of an action of G by affine isometries $(\alpha_g)_{g\in G}$ on the closed affine subspace $\mathcal{A} \subseteq L^2(Q, \mu)$ generated by $(X_t)_{t\in T}$ satisfying

$$X_{g.t} = \alpha_g X_t \quad \textit{for} \quad g \in G, t \in T$$

is equivalent to the independence from $g \in G$ of the kernel

$$Q^g(t,s) := \mathbb{E}((X_{g.t} - X_{g.t_0})(\overline{X_{g.s} - X_{g.t_0}})) \quad \textit{for} \quad s,t \in T,$$

for some $t_0 \in T$ (and hence for all $t_0 \in T$)) (Lemma A1). For a real-valued process ($\mathbb{K} = \mathbb{R}$), this condition is equivalent to the G-invariance of the kernel

$$D(t,s) := \mathbb{E}((X_t - X_s)^2) \quad \textit{for} \quad t,s \in T$$

on $T \times T$ (Lemma A1).

Lemma A2. *Let $(\alpha_t)_{t\in\mathbb{R}}$ be a continuous isometric affine $\mathbb{R}$-action of the form*

$$\alpha_t \xi = U_t \xi + \beta_t \quad \textit{for} \quad t \in \mathbb{R}, \xi \in \mathcal{H}$$

on the real or complex Hilbert space $\mathcal{H}$. If $\beta_{\mathbb{R}}$ is total in $\mathcal{H}$, then the unitary representation $(U_t)_{t\in\mathbb{R}}$ is cyclic.

Proof. **First proof:** We write $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0 = \mathcal{H}^U$ is the closed subspace of U-fixed vectors and $\mathcal{H}_1 := \mathcal{H}_0^\perp$. Accordingly, we write $\beta = \beta_0 + \beta_1$. Then $\beta_0 \colon \mathbb{R} \to \mathcal{H}_0$ is a continuous homomorphism, hence of the form $\beta_0(t) = tv_0$ for some $v_0 \in \mathcal{H}_0$. We conclude that $\dim \mathcal{H}_0 \leq 1$, so that it suffices to show that the representation on $\mathcal{H}_1$ is cyclic. We may therefore assume from now on that $\mathcal{H}^U = \{0\}$.

Step 1: First we assume that $\mathrm{Spec}(U)$ is compact and does not contain 0. Then there exists an $\varepsilon > 0$ such that the operators $U_t - \mathbf{1}$ are invertible for $|t| < \varepsilon$. For $|t|, |s| < \varepsilon$, we then have

$$(U_t - \mathbf{1})\beta_s = \beta_{t+s} - \beta_s - \beta_t = (U_s - \mathbf{1})\beta_t,$$

so that

$$v := (U_t - \mathbf{1})^{-1}\beta_t \quad \text{for} \quad |t| < \varepsilon$$

is independent of t. Now the relation $\beta_t = U_t v - v$ holds for $|t| < \varepsilon$, but since β is a continuous cocycle, it follows for all $t \in \mathbb{R}$. Clearly, $v \in \mathcal{H}$ is a U-cyclic vector.

Step 2: Now we consider the general case where $\mathcal{H}$ is complex. We write $\mathbb{R}^\times = \bigcup_{n\in\mathbb{N}} C_n$, where C_n is relatively compact with $0 \notin \overline{C_n}$. If P is the spectral measure of U, we accordingly obtain a U-invariant decomposition $\mathcal{H} = \widehat{\bigoplus}_{n\in\mathbb{N}} P(C_n)\mathcal{H}$ into subspace on which U has compact spectrum not containing 0.

Now our assumption implies that every $\mathcal{H}_n$ is generated by the values of the $\mathcal{H}_n$-component of β. Step 1 now implies that each $\mathcal{H}_n$ is cyclic, and since representations on the subspaces $\mathcal{H}_n$ are mutually disjoint, the representation on $\mathcal{H}$ is cyclic.

Step 3: Finally, we consider the general case where $\mathcal{H}$ is real. Then we may choose the sets $C_n \subseteq \mathbb{R}$ such that they are symmetric, i.e., $C_n = -C_n$. Then the corresponding spectral subspaces of $\mathcal{H}_{\mathbb{C}}$ are invariant under complex conjugation and we can proceed as in Step 2.

Alternative proof: A more direct argument can be derived from the work of P. Masani ([16]; see also [17]). For the element

$$\xi := \int_0^\infty e^{-t}\beta_t\, dt$$

one shows that the *shift operators*

$$T(a,b) := U_b - U_a - \int_a^b U_t\, dt = -T(b,a)$$

satisfy $\beta_t = T(t,0)\xi$. Here the main point is to verify first the *switching property* ([16] Lemma 2.18)

$$T(a,b)(\beta_c - \beta_d) = T(c,d)(\beta_a - \beta_b) \quad \text{for} \quad a,b,c,d \in \mathbb{R},$$

and that $\int_0^\infty e^{-t} T_U(t,0)\, ds = \mathbf{1}$ ([16] Thm. A.2). Then the assertion follows from

$$T(t,0)\xi = \int_0^\infty e^{-s}T(t,0)\beta_s\, ds = \int_0^\infty e^{-s}T(s,0)\beta_t\, ds = \beta_t$$

([16] Thm. 2.19). □

Proposition A2. (Normal form of cocycles) *Let $(U,\mathcal{H})$ be a continuous unitary one-parameter group and $\beta\colon \mathbb{R} \to \mathcal{H}$ be a continuous cocycle. Then there exists a Borel measure σ on $\mathbb{R}$ such that the triple $(U,\beta,\mathcal{H})$ is unitarily equivalent to the triple $(\widetilde{U},\widetilde{\beta},L^2(\mathbb{R},\sigma))$ with*

$$(\widetilde{U}_t f)(x) = e^{itx}f(x) \quad \textit{and} \quad \beta_t(x) = \begin{cases} \frac{e^{itx}-1}{ix} & \textit{for } x \neq 0 \\ t & \textit{for } x = 0. \end{cases}$$

Proof. In view of Lemma A2, we may assume that the representation $(U,\mathcal{H})$ is cyclic.

Step 1: First we assume that $\mathcal{H}^U = \{0\}$. According to Bochner's Theorem, any cyclic unitary one-parameter group $(U,\mathcal{H})$ with $\mathcal{H}^U = \{0\}$ is equivalent to the multiplication representation on some space $L^2(\mathbb{R}^\times,\mu)$ by $(U_t f)(x) = e^{itx}f(x)$. For this representation it is easy to determine the cocycles. They are of the form

$$\beta_t(x) = (e^{itx} - 1)u(x),$$

where $u\colon \mathbb{R} \to \mathbb{C}$ is a measurable function with the property that, for every $t \in \mathbb{R}$, the function $(e^{itx} - 1)u$ is square integrable. Replacing μ by the measure

$$d\sigma(x) = x^2|u(x)|^2\, d\mu(x),$$

we may assume that $u(x) = \frac{1}{ix}$, which leads to $\beta_t(x) = \frac{e^{itx}-1}{ix}$.

Step 2: If $\mathcal{H}^U = \mathcal{H}$, then $\beta\colon \mathbb{R} \to \mathcal{H}$ is a continuous homomorphism, hence of the form $\beta_t = tv$ for some $v \in \mathcal{H}$. The cyclicity assumption implies that $\mathcal{H} = \mathbb{C}v \cong L^2(\mathbb{R},\sigma)$ for the measure $\sigma = \|v\|^2\delta_0$. Here the vector v corresponds to the constant function 1, so that $\beta_t = tv = t$.

The assertion now follows by applying Steps 1 and 2 to the summands of the decomposition $\mathcal{H} = \mathcal{H}^U \oplus (\mathcal{H}^U)^\perp$. □

The following theorem is basically the Lévy–Khintchine Theorem for the group $G = \mathbb{R}$ (cf. [50] (Thm 5.5.1), [23] (Thm. 32), and [48] for a different form).

Proposition A3. *Let $(X_t)_{t\in\mathbb{R}}$ be a complex-valued zero mean Gaussian process on (Q, Σ, μ) with $X_0 = 0$ and stationary quadratic increments. Then there exists a uniquely determined Borel measure σ on $\mathbb{R}$ such that*

$$C_\sigma(s,t) = \mathbb{E}(X_s^* X_t) = \int_{\mathbb{R}} \overline{e_s(u)} e_t(u)\, d\sigma(u) \quad \textit{for} \quad e_t(u) = \frac{e^{itu} - 1}{iu} = \int_0^t e^{i\tau u}\, d\tau. \tag{A5}$$

A measure σ on $\mathbb{R}$ arises for such a process if and only if

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{1+u^2} < \infty. \tag{A6}$$

The function

$$r(t) := \int_{\mathbb{R}} \Big(1 - e^{itu} + \frac{itu}{1+u^2}\Big) \frac{d\sigma(u)}{u^2}$$

is negative definite and satisfies

$$r(t) + \overline{r(s)} - r(t-s) = C_\sigma(s,t). \tag{A7}$$

All other negative definite continuous functions satisfying (A7) are of the form $\widetilde{r}(t) = r(t) + it\mu$ for some $\mu \in \mathbb{R}$.

The measure σ is called the *spectral measure* of the process $(X_t)_{t\in\mathbb{R}}$.

Proof. Let $\mathcal{A} \subseteq L^2(Q, \Sigma, \mu)$ be the closed affine subspace generated by $(X_t)_{t\in\mathbb{R}}$. As $X_0 = 0$, this is actually a linear subspace. Now Lemma A1 implies the existence of an affine isometric action $(\alpha_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{A}$ satisfying $\alpha_t X_s = X_{s+t}$. In particular, $\beta_t := X_t$ is a corresponding cocycle. Now the existence of σ follows from Proposition A2.

Now we show that (A6) is equivalent to the square integrability of all $(e_t)_{t\neq 0}$ (Definition A1) and the continuity of the function $t \mapsto C_\sigma(t,t) = \|e_t\|_2^2$.

From

$$|e_t(u)|^2 = \Big|\frac{\cos(tu) - 1}{u}\Big|^2 + \Big|\frac{\sin(tu)}{u}\Big|^2 = \frac{1 + \cos^2(tu) + \sin^2(tu) - 2\cos(tu)}{u^2} = 2\frac{1-\cos(tu)}{u^2}$$

it follows that the square integrability of all e_t with respect to σ is equivalent to

$$f(t) := \int_{\mathbb{R}} \frac{1-\cos(tu)}{u^2}\, d\sigma(u) < \infty \quad \text{for all} \quad t \in \mathbb{R}.$$

If $r > 0$ and t is sufficiently small, then the integrand has a positive infimum on the interval $[-r, r]$. Therefore the finiteness of all $f(t)$ implies that all compact subsets of $\mathbb{R}$ have finite σ-measure. Since the function $f(t) = \frac{1}{2}C_\sigma(t,t)$ is continuous, for every $\varepsilon > 0$, we have

$$\infty > \int_0^\varepsilon f(t)\, dt = \int_{\mathbb{R}} \int_0^\varepsilon (1 - \cos(tu))\, dt\, \frac{d\sigma(u)}{u^2} = \int_{\mathbb{R}} \Big(\varepsilon - \frac{\sin \varepsilon u}{u}\Big) \frac{d\sigma(u)}{u^2}.$$

As the function $u \mapsto 1 - \frac{\sin(\varepsilon u)}{\varepsilon u}$ has a positive infimum on $[1, \infty)$, it follows that $\int_{|u|\geq 1} \frac{d\sigma(u)}{u^2} < \infty$. This implies that $\int_{\mathbb{R}} \frac{d\sigma(u)}{1+u^2} < \infty$.

Suppose, conversely, $\int_{\mathbb{R}} \frac{d\sigma(u)}{1+u^2} < \infty$ (cf. [50] Lemma 5.5.1). We claim that we obtain a continuous negative definite function

$$r(t) := \int_{\mathbb{R}} \Big(1 - e^{itu} + \frac{itu}{1+u^2}\Big) \frac{d\sigma(u)}{u^2} = \int_{\mathbb{R}} \Big(\frac{1 - e^{itu}}{u} + \frac{it}{1+u^2}\Big) \frac{d\sigma(u)}{u}. \tag{A8}$$

We first show that the integrals exist. To this end, we observe that

$$\Big(1 - e^{itu} + \frac{itu}{1+u^2}\Big)\frac{1+u^2}{u^2} = 1 - e^{itu} + \frac{1+itu-e^{itu}}{u^2}.$$

Since all three summands are bounded, the existence of the integral (A8) defining $r(t)$ follows. The first two summands are bounded independently of t, and the third summand can also be written as

$$\frac{1+itu-e^{itu}}{u^2} = h(tu)t^2,$$

where the function $h\colon \mathbb{R} \to \mathbb{C}$ is bounded. We conclude that all summands are locally uniformly bounded in t. Therefore the continuity of the function r follows from Lebesgue's Dominated Convergence Theorem. Moreover, r is negative definite because the functions $t \mapsto 1 - e^{itu}$ and $t \mapsto it$ are.

We further have the relation

$$r(t) + \overline{r(s)} - r(t-s) = C_\sigma(s,t) = \mathbb{E}(X_s^* X_t) = \int_{\mathbb{R}} \overline{e_s(u)} e_t(u)\, d\sigma(u), \tag{A9}$$

showing that C_σ is the positive definite kernel associated to the continuous negative definite function r, hence in particular continuous.

Appendix C. Second Quantization and Gaussian Processes

Definition A4. *([37] Def. 1.6) Let T be a set and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-valued stochastic process $(X_t)_{t\in T}$ is said to be Gaussian if, for all finite subsets $F \subseteq T$, the corresponding distribution of the random vector $X_F = (X_t)_{t\in F}$ with values in $\mathbb{K}^F$ is Gaussian.*

Definition A5. *Let $\mathcal{H}$ be a $\mathbb{K}$-Hilbert space. A Gaussian random process indexed by $\mathcal{H}$ is a random process $(\varphi(v))_{v\in\mathcal{H}}$ on a probability space (Q, Σ, P) indexed by $\mathcal{H}$ such that*

(GP1) *$(\varphi(v))_{v\in\mathcal{H}}$ is full, i.e., the random variables $\varphi(v)$ generate the σ-algebra Σ modulo zero sets.*
(GP2) *Each $\varphi(v)$ is a Gaussian random variable of mean zero.*
(GP3) *$\mathbb{E}(\varphi(v)\overline{\varphi(w)}) = \langle v, w\rangle_{\mathcal{H}}$ for $v, w \in \mathcal{H}$.*

Remark A2. *If T is a set, $\gamma\colon T \to \mathcal{H}$ a map and $(\varphi(v))_{v\in\mathcal{H}}$ is a Gaussian process indexed by $\mathcal{H}$, then $(\varphi(\gamma(t)))_{t\in T}$ is a Gaussian process indexed by T with zero means and covariance kernel*

$$C(s,t) = \mathbb{E}\big(\varphi(\gamma(t))\overline{\varphi(\gamma(s))}\big) = \langle \gamma(t), \gamma(s)\rangle.$$

For any function $m\colon T \to \mathbb{R}, t \mapsto m_t$, we obtain a Gaussian process $(X_t)_{t\in T}$ with mean vector $(m_t)_{t\in T}$ by

$$X_t := \varphi(\gamma(t)) + m_t.$$

If $\gamma(T)$ is total in $\mathcal{H}$, then the corresponding Gaussian process is full.

Conversely, every Gaussian process $(X_t)_{t\in T}$ with mean vector $(m_t)_{t\in T}$ is of this form. Here we may choose $\mathcal{H}$ as the subspace of $L^2(Q, \Sigma, \mu)$ generated by the $X_t - m_t$ ([37] Thm. 1.10).

Definition A6. *(Second quantization; [51]) For a real Hilbert space $\mathcal{H}$, we write $\mathcal{H}^*$ for its algebraic dual, i.e., the set of all linear functionals $\mathcal{H} \to \mathbb{R}$, continuous or not. Let $\Gamma(\mathcal{H}) := L^2(\mathcal{H}^*, \gamma, \mathbb{C})$ denote the canonical Gaussian measure space on $\mathcal{H}^*$. This measure is defined on the smallest σ-algebra $\Sigma = \Sigma_{\mathcal{H}^*}$ for which all evaluations $\varphi(v)(\alpha) := \alpha(v)$, $v \in \mathcal{H}$, are measurable. It is determined uniquely by*

$$\mathbb{E}(e^{i\varphi(v)}) = e^{-\|v\|^2/2} \quad \text{for} \quad v \in \mathcal{H}.$$

Considering the $\varphi(v)$ as random variables, we thus obtain the canonical centered Gaussian process $(\varphi(v))_{v\in\mathcal{H}}$ over $\mathcal{H}$. It satisfies

$$\mathbb{E}(\varphi(v)) = 0 \quad \text{and} \quad \mathbb{E}(\varphi(v)\varphi(w)) = \langle v, w\rangle \quad \text{for} \quad v, w \in \mathcal{H}.$$

Remark A3. *(The unitary representation of* $\mathrm{Mot}(\mathcal{H})$ *on* $\Gamma(\mathcal{H})$*) The group* $\mathrm{Mot}(\mathcal{H}) \cong \mathcal{H} \rtimes \mathrm{O}(\mathcal{H})$ *of bijective isometries of* $\mathcal{H}$ *has a natural unitary representation on* $\Gamma(\mathcal{H})$ *given by*

$$U_{(b,g)}F = e^{i\varphi(b)} g_* F, \quad \text{where} \quad (g_*F)(\alpha) = F(g^{-1}.\alpha) = F(\alpha \circ g).$$

In particular, the map

$$\mathcal{H} \to \Gamma(\mathcal{H}), \quad x \mapsto U_{(x,\mathbf{1})}1 = e^{i\varphi(x)}$$

is $\mathrm{Mot}(\mathcal{H})$*-equivariant with total range. The canonical Gaussian process over the real Hilbert space* $\mathcal{H}$ *satisfies*

$$Q(v, w) := \mathbb{E}(\overline{e^{i\varphi(v)}} e^{i\varphi(w)}) = \mathbb{E}(e^{i\varphi(w-v)}) = e^{-\frac{\|v-w\|^2}{2}}. \tag{A10}$$

Remark A4. *The canonical Gaussian measure γ on the algebraic dual $\mathcal{H}^*_{1/2} = L^2(\mathbb{R})^*$ is called white noise measure. The space of smooth vectors of the unitary representation $(U^{1/2}, L^2(\mathbb{R}))$ of* $\mathrm{GL}_2(\mathbb{R})$ *(see [9] (Ch. 7) for this concept) can be naturally identified with the space $C^\infty(\mathbb{S}^1)$, considered as a subspace of $\mathcal{H}_{1/2}$. It coincides with the space D_0 in Hida's book [37] (p. 304).*

Appendix D. The Hilbert Spaces $\mathcal{H}_H$, $0 < H < 1$

Appendix D.1. The Scalar Product on $\mathcal{H}_H$

In this section we give a short discussion about the complementary series representation in the one dimensional case. For detailed discussion see [32] (p. 28) and [4] (Sect. 9).

For $\frac{1}{2} < H < 1$ and $\xi, \eta \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
&(2H-1)\int_{\mathbb{R}} \overline{\xi(x)}|x-y|^{2H-2}\,dx \\
&= (2H-1)\Big(\int_{-\infty}^{y} \overline{\xi(x)}(y-x)^{2H-2}\,dx + \int_{y}^{\infty} \overline{\xi(x)}(x-y)^{2H-2}\,dx\Big) \\
&= \int_{-\infty}^{y} \overline{\xi'(x)}(y-x)^{2H-1}\,dx - \Big[\overline{\xi(x)}(y-x)^{2H-1}\Big|_{-\infty}^{y} \\
&\quad - \int_{y}^{\infty} \overline{\xi'(x)}(x-y)^{2H-1}\,dx + \Big[\overline{\xi(x)}(x-y)^{2H-1}\Big|_{y}^{\infty} \\
&= \int_{\mathbb{R}} \overline{\xi'(x)}\frac{\mathrm{sgn}(y-x)}{|x-y|^{1-2H}}\,dx\,dy
\end{aligned}$$

We accordingly obtain

$$\langle \xi, \eta\rangle_H = H\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi'(x)}\eta(y)\frac{\mathrm{sgn}(y-x)}{|x-y|^{1-2H}}\,dx\,dy \tag{A11}$$

and thus

$$\begin{aligned}
\lim_{H\to\frac{1}{2}} \langle \xi, \eta\rangle_H &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi'(x)}\eta(y)\,\mathrm{sgn}(y-x)\,dx\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}} \eta(y)\Big[\int_{-\infty}^{y}\overline{\xi'(x)}\,dx - \int_{y}^{\infty}\overline{\xi'(x)}\,dx\Big] = \int_{\mathbb{R}} \eta(y)\overline{\xi(y)}\,dy = \langle \xi, \eta\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

It therefore makes sense to put $\mathcal{H}_{1/2} := L^2(\mathbb{R})$, so that we have Hilbert spaces $\mathcal{H}_H$ for $\frac{1}{2} \le H < 1$.

In the form (A11), the scalar product $\langle \cdot, \cdot \rangle_H$ is defined by a distribution kernel which is locally integrable for any $H > 0$. We shall use this observation to define Hilbert spaces $\mathcal{H}_H$ for $0 < H < 1$. To find a more symmetric form of the scalar product, we calculate

$$\begin{aligned} \int_{\mathbb{R}} \eta(y)\,\mathrm{sgn}(y-x)|x-y|^{2H-1}\,dy &= -\int_{-\infty}^{x} \eta(y)(x-y)^{2H-1}\,dy + \int_{x}^{\infty} \eta(y)(y-x)^{2H-1}\,dy \\ &= \frac{1}{2H}\Big[\eta(y)(x-y)^{2H}\Big|_{-\infty}^{x} - \frac{1}{2H}\int_{-\infty}^{x} \eta'(y)(x-y)^{2H}\,dy \\ &\quad + \frac{1}{2H}\Big[\eta(y)(y-x)^{2H}\Big|_{x}^{\infty} - \frac{1}{2H}\int_{x}^{\infty} \eta'(y)(y-x)^{2H}\,dy \\ &= -\frac{1}{2H}\int_{\mathbb{R}} \eta'(y)|x-y|^{2H}\,dy. \end{aligned}$$

We thus obtain from (A11) the simple form

$$\langle \xi, \eta \rangle_H = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi'(x)}\eta'(y)|x-y|^{2H}\,dx\,dy. \tag{A12}$$

Appendix D.2. Unitarity of the Representations U^H*,* $0 < H < 1$

To verify the unitarity of the representations U^H, for $H > \frac{1}{2}$, we calculate for $\xi, \eta \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} \frac{\langle U_g^H\xi, U_g^H\eta\rangle}{H(2H-1)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(g^{-1}.x)}\eta(g^{-1}.y)\frac{|ad-bc|^{2H}}{|cx+d|^{2H}|cy+d|^{2H}}\frac{dx\,dy}{|x-y|^{2-2H}} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi(x)}\eta(y)\frac{|ad-bc|^{2H-2}}{|c(g.x)+d|^{2H-2}|c(g.y)+d|^{2H-2}}\frac{dx\,dy}{|g.x-g.y|^{2-2H}}, \end{aligned}$$

so that unitarity follows from

$$c\cdot(g.x)+d = \frac{ad-bc}{a-cx}, \quad \text{which implies} \quad \frac{|g.x-g.y|}{|x-y|} = \frac{|ad-bc|}{|a-cx||a-cy|} = \frac{|c(g.x)+d|\cdot|c(g.y)+d|}{|ad-bc|}.$$

For $H < \frac{1}{2}$ we use the fact that $\mathrm{PGL}_2(\mathbb{R})$ is generated by the affine group $\mathrm{Aff}(\mathbb{R})$ and the map $\sigma(x) = x^{-1}$. As the kernel function $|x-y|^{2H}$ is translation invariant, the translations define unitary operators $U_g^H(\xi)(x) := \xi(x+b)$, $b \in \mathbb{R}$. For dilations $g^{-1}(x) = ax$, we have

$$(U_g^H\xi)(x) = \mathrm{sgn}(a)|a|^H\xi(ax) \quad \text{and} \quad (U_g^H\xi)'(x) = |a|^{H+1}\xi'(ax).$$

This leads to

$$\langle U_g^H\xi, U_g^H\eta\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} |a|^{2H+2}\overline{\xi'(ax)}\eta'(ay)|x-y|^{2H}\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\xi'(x)}\eta'(y)|x-y|^{2H}\,dx\,dy = \langle \xi, \eta\rangle.$$

The most tricky part is to verify that the operator $(U\xi)(x) := -|x|^{-2H}\xi(x^{-1})$ is unitary on $\mathcal{H}_H$. Since U is an involution, it suffices to show that

$$\langle U\xi, \eta\rangle = \langle \xi, U\eta\rangle \quad \text{for} \quad \xi, \eta \in \mathcal{S}(\mathbb{R}). \tag{A13}$$

To verify this symmetry relation, we may assume that ξ and η are real-valued. We first observe that

$$(U\xi)'(x) = 2H\,\mathrm{sgn}(x)|x|^{-2H-1}\xi(x^{-1}) + |x|^{-2H-2}\xi'(x^{-1}).$$

This leads to

$$\langle U\xi, \eta\rangle = -\int_{\mathbb{R}}\int_{\mathbb{R}} H\,\mathrm{sgn}(x)|x|^{-2H-1}\xi(x^{-1})\eta'(y)|x-y|^{2H}\,dx\,dy \\ -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |x|^{-2H-2}\xi'(x^{-1})\eta'(y)|x-y|^{2H}\,dx\,dy. \tag{A14}$$

The second summand in (A14) equals

$$-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |x|^{2H+2}\xi'(x)\eta'(y)|x^{-1}-y|^{2H}\,\frac{dx}{x^2}\,dy = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \xi'(x)\eta'(y)|1-xy|^{2H}|\,dx\,dy,$$

which is symmetric in ξ and η. The first summand in (A14) equals

$$-\int_{\mathbb{R}}\int_{\mathbb{R}} H\,\mathrm{sgn}(x)|x|^{2H+1}\xi(x)\eta'(y)|x^{-1}-y|^{2H}\,\frac{dx}{x^2}\,dy = -\int_{\mathbb{R}}\int_{\mathbb{R}} H\xi(x)\eta'(y)|1-xy|^{2H}\,dx\,\frac{dy}{x}.$$

With

$$\int_{\mathbb{R}} \eta'(y)|1-xy|^{2H}\,\frac{dy}{x} = -2H\int_{\mathbb{R}} \eta(y)|1-xy|^{2H-1}\,\mathrm{sgn}(1-xy)(-x)\,\frac{dy}{x} \\ = 2H\int_{\mathbb{R}} \eta(y)|1-xy|^{2H-1}\,\mathrm{sgn}(1-xy)\,dy,$$

we obtain for the first summand in (A14)

$$-2H^2\int_{\mathbb{R}}\int_{\mathbb{R}} \xi(x)\eta(y)|1-xy|^{2H-1}\,\mathrm{sgn}(1-xy)\,dx\,dy.$$

Again, the symmetry of the integral kernel now implies that this expression is symmetric in ξ and η. We conclude that U defines a unitary operator on $\mathcal{H}_H$. This implies unitarity of the operators U_g^H for $0 < H < \frac{1}{2}$.

Appendix E. The Spectral Measure of Fractional Brownian Motion

For $\mathrm{Re}\, z > 0$, we have the integral representation of the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-\lambda}\lambda^{z-1}\,d\lambda. \tag{A15}$$

For $0 < \alpha < 1$, this leads to

$$\Gamma(1-\alpha) = \int_0^\infty e^{-\lambda}\frac{d\lambda}{\lambda^\alpha}. \tag{A16}$$

By partial integration, we further obtain

$$\int_0^\infty (1-e^{-\lambda})\lambda^{-1-\alpha}\,d\lambda = \frac{1}{\alpha}\int_0^\infty e^{-\lambda}\frac{d\lambda}{\lambda^\alpha} = \frac{\Gamma(1-\alpha)}{\alpha}. \tag{A17}$$

This in turn leads to

$$z^\alpha = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-e^{-z\lambda})\lambda^{-1-\alpha}\,d\lambda \quad \text{for} \quad 0<\alpha<1, z\in\mathbb{C}\setminus(-\infty,0] \tag{A18}$$

(cf. [31] p. 78). In fact, for $z > 0$ real, this follows from (A17) by dilation, so that the claim follows by analytic continuation from the holomorphy of both sides. For $z = i$, we obtain from (A18)

$$\cos\left(\frac{\alpha\pi}{2}\right) = \mathrm{Re}\, e^{i\alpha\pi/2} = \mathrm{Re}(i^\alpha) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-\cos\lambda)\lambda^{-1-\alpha}\,d\lambda \quad \text{for} \quad 0<\alpha<1. \tag{A19}$$

With

$$\Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin(\pi\alpha)} = \frac{\pi}{2\sin(\pi\alpha/2)\cos(\pi\alpha/2)} \tag{A20}$$

we obtain

$$2\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)}. \tag{A21}$$

Thus

$$\int_{\mathbb{R}} (1-\cos\lambda)|\lambda|^{-1-\alpha}\,d\lambda = \frac{\pi}{\alpha\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)} = \frac{\pi}{\Gamma(1+\alpha)\sin\left(\frac{\alpha\pi}{2}\right)} = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma(1+\alpha)}.$$

We further obtain for $t \in \mathbb{R}$ by dilation

$$\int_{\mathbb{R}} (1-\cos t\lambda)|\lambda|^{-1-\alpha}\,d\lambda = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma(1+\alpha)}|t|^{\alpha}.$$

For $H = 2\alpha$ this implies

$$\begin{aligned}
|t|^{2H} &= \frac{\Gamma(2H+1)}{\Gamma(H)\Gamma(1-H)} \int_{\mathbb{R}} \frac{1-\cos\lambda t}{|\lambda|^{2H}}\frac{d\lambda}{|\lambda|} \\
&= \frac{1}{2}\frac{\Gamma(2H+1)}{\Gamma(H)\Gamma(1-H)} \int_{\mathbb{R}} \frac{(1-\cos\lambda t)^2+\sin^2\lambda t}{\lambda^2}|\lambda|^{1-2H}\,d\lambda \\
&= \frac{1}{2}\frac{\Gamma(2H+1)}{\Gamma(H)\Gamma(1-H)} \int_{\mathbb{R}} |e_t(\lambda)|^2|\lambda|^{1-2H}\,d\lambda,
\end{aligned}$$

and therefore the spectral measure of fractional Brownian motion is

$$d\sigma(\lambda) = \frac{1}{2}\frac{\Gamma(2H+1)}{\Gamma(H)\Gamma(1-H)}\cdot|\lambda|^{1-2H}\,d\lambda = \frac{\sin(\pi H)\Gamma(2H+1)}{2\pi}\cdot|\lambda|^{1-2H}\,d\lambda. \tag{A22}$$

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