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Extension of Eigenvalue Problems on Gauss Map of Ruled Surfaces

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Abstract: A finite-type immersion or smooth map is a nice tool to classify submanifolds of Euclidean space, which comes from the eigenvalue problem of immersion. The notion of generalized 1-type is a natural generalization of 1-type in the usual sense and pointwise 1-type. We classify ruled surfaces with a generalized 1-type Gauss map as part of a plane, a circular cylinder, a cylinder over a base curve of an infinite type, a helicoid, a right cone and a conical surface of G-type.

Keywords: ruled surface; pointwise 1-type Gauss map; generalized 1-type Gauss map; conical surface of G-type

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1. Introduction

Nash's embedding theorem enables us to study Riemannian manifolds extensively by regarding a Riemannian manifold as a submanifold of Euclidean space with sufficiently high codimension. By means of such a setting, we can have rich geometric information from the intrinsic and extrinsic properties of submanifolds of Euclidean space. Inspired by the degree of algebraic varieties, B.-Y. Chen introduced the notion of order and type of submanifolds of Euclidean space. Furthermore, he developed the theory of finite-type submanifolds and estimated the total mean curvature of compact submanifolds of Euclidean space in the late 1970s ([1]).

In particular, the notion of finite-type immersion is a direct generalization of the eigenvalue problem relative to the immersion of a Riemannian manifold into a Euclidean space: Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a submanifold M into the Euclidean m-space $\mathbb{E}^m$ and Δ the Laplace operator of M in $\mathbb{E}^m$. The submanifold M is said to be of finite-type if x has a spectral decomposition by $x = x_0 + x_1 + ... + x_k$, where x_0 is a constant vector and x_i are the vector fields satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$ $(i = 1, 2, ..., k)$. If the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ are different, it is called k-type. Since this notion was introduced, many works have been made in this area (see [1,2]). This notion of finite-type immersion was naturally extended to that of pseudo-Riemannian manifolds in pseudo-Euclidean space and it was also applied to smooth maps, particularly the Gauss map defined on submanifolds of Euclidean space or pseudo-Euclidean space ([1,3–6]).

Regarding the Gauss map of finite-type, B.-Y. Chen and P. Piccini ([7]) studied compact surfaces with 1-type Gauss map, that is, $\Delta G = \lambda(G + \mathbb{C})$, where $\mathbb{C}$ is a constant vector and $\lambda \in \mathbb{R}$. Since then, many works regarding finite-type Gauss map have been established ([1,3,4,8–15]).

However, some surfaces have an interesting property concerning the Gauss map: The helicoid in $\mathbb{E}^3$ parameterized by

$$x(u, v) = (u \cos v, u \sin v, av), \quad a \neq 0$$

has the Gauss map and its Laplacian respectively given by

$$G = \frac{1}{\sqrt{a^2+u^2}}(a\sin v, -a\cos v, u)$$

and

$$\Delta G = \frac{2a^2}{(a^2+u^2)^2}G.$$

The right (or circular) cone C_a with parametrization

$$x(u,v) = (u\cos v, u\sin v, au), \quad a \geq 0$$

has the Gauss map

$$G = \frac{1}{\sqrt{1+a^2}}(a\cos v, a\sin v, -1)$$

which satisfies

$$\Delta G = \frac{1}{u^2}\left(G + \left(0, 0, \frac{1}{\sqrt{1+a^2}}\right)\right)$$

(Reference [8,10]). The Gauss maps above are similar to be of 1-type, but not to be of the 1-type Gauss map in the usual sense. Based upon such cases, B.-Y. Chen and the present authors defined the notion of pointwise 1-type Gauss map ([8]).

Definition 1. *A submanifold M in $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if the Gauss map G of M satisfies*

$$\Delta G = f(G + \mathbb{C})$$

for some non-zero smooth function f and a constant vector $\mathbb{C}$. In particular, if $\mathbb{C}$ is zero, then the Gauss map is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of pointwise 1-type of the second kind.

Let p be a point of $\mathbb{E}^3$ and $\beta = \beta(s)$ a unit speed curve such that p does not lie on β. A surface parameterized by

$$x(s,t) = p + t\beta(s)$$

is called a conical surface. A typical conical surface is a right cone and a plane.

Let us consider a following example of a conical surface.

Example 1 ([15]). *Let M be a surface in $\mathbb{E}^3$ parameterized by*

$$x(s,t) = (s\cos^2 t, s\sin t\cos t, s\sin t).$$

Then, the Gauss map G can be obtained by

$$G = \frac{1}{\sqrt{1+\cos^2 t}}(-\sin^3 t, (2-\cos^2 t)\cos t, -\cos^2 t).$$

After a considerably long computation, its Laplacian turns out to be

$$\Delta G = fG + g\mathbb{C}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$. The surface M is a kind of conical surfaces generated by a spherical curve $\beta(t) = (\cos^2 t, \sin t\cos t, \sin t)$ on the unit sphere $\mathbb{S}^2(1)$ centered at the origin.

Inspired by such an example, we would like to generalize the notion of pointwise 1-type Gauss map as follows:

Definition 2 ([15])**.** *The Gauss map G of a submanifold M in $\mathbb{E}^m$ is of generalized 1-type if the Gauss map G of M satisfies*

$$\Delta G = fG + g\mathbb{C} \tag{1}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$.

Especially, we define a conical surface of G-type.

Definition 3. *A conical surface with generalized 1-type Gauss map is called a conical surface of G-type.*

Remark 1 ([15])**.** *A conical surface of G-type is constructed by the functions f, g and the constant vector $\mathbb{C}$ by solving the differential equations generated by Equation (1).*

In [15], the authors classified flat surfaces with a generalized 1-type Gauss map in $\mathbb{E}^3$. In fact, flat surfaces are ruled surfaces which are locally cones, cylinders or tangent developable surfaces. In the present paper, without such an assumption of flatness, we prove that non-cylindrical ruled surfaces with a generalized 1-type Gauss map are flat and thus we completely classify ruled surfaces with generalized 1-type Gauss map in $\mathbb{E}^3$.

2. Preliminaries

Let M be a surface of $\mathbb{E}^3$. The map $G : M \to \mathbb{S}^2(1) \subset \mathbb{E}^3$ which maps each point p of M to a point G_p of $\mathbb{S}^2(1)$ by identifying the unit normal vector N_p to M at the point with G_p is called the Gauss map of the surface M, where $\mathbb{S}^2(1)$ is the unit sphere in $\mathbb{E}^3$ centered at the origin.

For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on M, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. Then the Laplacian Δ on M is in turn given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{\mathcal{G}}\, \tilde{g}^{ij} \frac{\partial}{\partial x^j}\Big). \tag{2}$$

Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ defined on an open interval I and $\beta = \beta(s)$ a transversal vector field to $\alpha'(s)$ along α. Then a ruled surface M can be parameterized by

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in \mathbb{R}$$

satisfying $\langle \alpha', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = 1$, where $'$ denotes d/ds. The curve α is called the base curve and β the director vector field or ruling. It is said to be cylindrical if β is constant, or, non-cylindrical otherwise.

Throughout this paper, we assume that all the functions and vector fields are smooth and surfaces under consideration are connected unless otherwise stated.

3. Cylindrical Ruled Surfaces in $\mathbb{E}^3$ with Generalized 1-Type Gauss Map

In this section, we study the cylindrical ruled surfaces with the generalized 1-type Gauss map in $\mathbb{E}^3$.

Let M be a cylindrical ruled surface in $\mathbb{E}^3$. We can parameterize M with a plane curve $\alpha = \alpha(s)$ and a constant vector β as

$$x(s,t) = \alpha(s) + t\beta.$$

Here the plane curve α is assumed to be defined by $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ with the arc length s and β a constant unit vector, namely $\beta = (0,0,1)$. In the sequel, the Gauss map G of M is given by

$$G = \alpha' \times \beta = (\alpha_2', -\alpha_1', 0) \tag{3}$$

and the Laplacian ΔG of the Gauss map G using Equation (2) is obtained by

$$\Delta G = (-\alpha_2''', \alpha_1''', 0), \tag{4}$$

where $'$ stands for d/ds.

From now on, $'$ denotes the differentiation with respect to the parameter s relative to the base curve.

Suppose that the Gauss map G of M is of generalized 1-type, i.e., G satisfies Equation (1). We now consider two cases either $f = g$ or $f \neq g$.

Case 1. $f = g$.

In this case, the Gauss map G is of pointwise 1-type described in Definition 1. According to Classification Theorem in [10,11], we have that the ruled surface M is part of a plane, a circular cylinder or a cylinder over a base curve of an infinite-type satisfying

$$\sin^{-1}\left(\frac{c^2 f^{-\frac{1}{3}} - 1}{\sqrt{c_1^2 + c_2^2}}\right) - \sqrt{c_1^2 + c_2^2 - \left(c^2 f^{-\frac{1}{3}} - 1\right)^2} = \pm c^3 (s + k), \tag{5}$$

where $\mathbb{C} = (c_1, c_2, 0)$, and $c\ (\neq 0)$ and k are constants.

Case 2. $f \neq g$.

By a direct computation using Equations (3) and (4), we see that the third component c_3 of the constant vector $\mathbb{C}$ is zero. We put $\mathbb{C} = (c_1, c_2, 0)$. Then, we have the following system of ordinary differential equations

$$\begin{aligned} -\alpha_2''' &= f\alpha_2' + gc_1, \\ \alpha_1''' &= -f\alpha_1' + gc_2. \end{aligned} \tag{6}$$

Since α is of unit speed, that is, $(\alpha_1')^2 + (\alpha_2')^2 = 1$, we may put

$$\alpha_1'(s) = \cos\theta(s) \quad \text{and} \quad \alpha_2'(s) = \sin\theta(s)$$

for a smooth function $\theta = \theta(s)$ of s. One can write Equation (6) as

$$\begin{aligned} (\theta')^2 \sin\theta - \theta'' \cos\theta &= f \sin\theta + gc_1, \\ (\theta')^2 \cos\theta + \theta'' \sin\theta &= f \cos\theta - gc_2, \end{aligned}$$

which give

$$(\theta')^2 = f + g(c_1 \sin\theta - c_2 \cos\theta), \tag{7}$$

$$-\theta'' = g(c_1 \cos\theta + c_2 \sin\theta). \tag{8}$$

Taking the derivative of Equation (7), we have

$$2\theta'\theta'' = f' + g'(c_1 \sin\theta - c_2 \cos\theta) + g(c_1 \cos\theta + c_2 \sin\theta)\theta'.$$

With the help of Equations (7) and (8) it implies that

$$\frac{3}{2}(\theta'^2)' = f' + \frac{g'}{g}((\theta')^2 - f).$$

Solving the above differential equation, we get

$$\theta'(s)^2 = kg^{\frac{2}{3}}(s) + \frac{2}{3} g^{\frac{2}{3}}(s) \int g^{-\frac{2}{3}}(s) f(s) \left(\frac{f'}{f} - \frac{g'}{g}\right) ds, \quad k(\neq 0) \in \mathbb{R}.$$

If we put

$$\theta'(s) = \pm\sqrt{p(s)}\,, \tag{9}$$

where $p(s) = |kg^{\frac{2}{3}}(s) + \frac{2}{3}g^{\frac{2}{3}}(s)\int g^{-\frac{2}{3}}(s)f(s)(\frac{f'}{f} - \frac{g'}{g})ds|$ for some non-zero constant k, we get a base curve α of M as follows:

$$\alpha(s) = \left(\int \cos\theta(s)ds, \int \sin\theta(s)ds, 0\right), \tag{10}$$

where $\theta(s) = \pm\int\sqrt{p(s)}\,ds$. In fact, θ' is the signed curvature of the curve α which is precisely determined by the given functions f, g and the constant vector $\mathbb{C}$.

Note that if f and g are constant, the Gauss map G is of 1-type in the usual sense. In this case, the signed curvature of α is non-zero constant and thus M is part of a circular cylinder.

Suppose that one of the functions f and g is not constant. Since a plane curve in $\mathbb{E}^3$ is of finite-type if and only if it is part of a straight line or a circle, the base curve α defined by Equation (10) is of an infinite-type ([2]). Thus, by putting together Cases 1 and 2, we have a classification theorem as follows:

Theorem 1. *Let M be a cylindrical ruled surface in $\mathbb{E}^3$ with the generalized 1-type Gauss map. Then it is an open part of a plane, a circular cylinder or a cylinder over a base curve of an infinite-type satisfying Equations* (5), (9) *and* (10).

4. Classification Theorem

In this section, we examine non-cylindrical ruled surfaces with generalized 1-type Gauss map in $\mathbb{E}^3$ and obtain a classification theorem.

Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ parameterized by a base curve α and a director vector field β. Up to a rigid motion, its parametrization is given by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\alpha', \beta\rangle = 0$, $\langle\beta,\beta\rangle = 1$ and $\langle\beta',\beta'\rangle = 1$. Then, we have an orthonormal frame $\{\beta, \beta', \beta\times\beta'\}$ along α. With the frame $\{x_s, x_t\}$, we define the smooth functions q, u, Q and R as follows:

$$q = \langle x_s, x_s\rangle, \quad u = \langle\alpha', \beta'\rangle, \quad Q = \langle\alpha', \beta\times\beta'\rangle, \quad R = \langle\beta'', \beta\times\beta'\rangle.$$

With such functions above, we can express the vector fields $\alpha', \beta'', \alpha'\times\beta$, $\beta\times\beta''$ in the following:

$$\begin{aligned}
\alpha' &= u\beta' + Q\beta\times\beta', \\
\beta'' &= -\beta + R\beta\times\beta', \\
\alpha'\times\beta &= Q\beta' - u\beta\times\beta', \\
\beta\times\beta'' &= -R\beta',
\end{aligned} \tag{11}$$

from which, the smooth function q and the Gauss map G are represented respectively as

$$q = t^2 + 2ut + u^2 + Q^2$$

and

$$G = \frac{x_s\times x_t}{||x_s\times x_t||} = q^{-1/2}\left(Q\beta' - (u+t)\beta\times\beta'\right). \tag{12}$$

Then, by straightforward computation, the mean curvature H and the Gaussian curvature K of M are respectively represented as:

$$\begin{aligned} H &= \frac{1}{2}q^{-3/2}(-Rt^2 - (2uR + Q')t + u'Q - Q^2R - u^2R - uQ'), \\ K &= -\frac{Q^2}{q^2}. \end{aligned} \tag{13}$$

Remark 2. *If $R = 0$, then the director vector field β is a plane curve.*

By the Gauss and Weingarten formulas, the following equation is easily obtained:

$$\Delta G = 2\nabla H + (\mathrm{tr} A^2)G,$$

where ∇H is the gradient of H and A denotes the shape operator of M. From Equation (13), we get

$$\begin{aligned} 2\nabla H &= 2e_1(H)e_1 + 2e_2(H)e_2 \\ &= q^{-3}B_1e_1 + q^{-5/2}A_1e_2 \\ &= q^{-7/2}\left(qA_1\beta + (u+t)B_1\beta' + QB_1\beta \times \beta'\right), \end{aligned}$$

where $e_1 = \frac{x_s}{||x_s||}$, $e_2 = \frac{x_t}{||x_t||}$,

$$\begin{aligned} A_1 =& Rt^3 + (3uR + 2Q')t^2 + (Q^2R - 3u'Q + 3u^2R + 4uQ')t \\ &+ (uQ^2R - 3uu'Q + u^3R + 2u^2Q' - Q^2Q'), \\ B_1 =& 3(u't + uu' + QQ')\{Rt^2 + (2uR + Q')t - u'Q + Q^2R + u^2R + uQ'\} \\ &+ (t^2 + 2ut + u^2 + Q^2)\{-R't^2 - (2u'R + 2uR' + Q'')t \\ &+ u''Q - 2QQ'R - Q^2R' - 2uu'R - u^2R' - uQ''\}. \end{aligned}$$

We also have

$$\mathrm{tr} A^2 = q^{-3}D_1,$$

where

$$D_1 = \{-Rt^2 - (2uR + Q')t - u(uR + Q') + Q(u' - QR)\}^2 + 2Q^2(t^2 + 2ut + u^2 + Q^2).$$

Thus we obtain the Laplacian ΔG of the Gauss map G of M given by

$$\Delta G = q^{-7/2}[qA_1\beta + ((u+t)B_1 + D_1Q)\,\beta' + (QB_1 - D_1(u+t))\,\beta \times \beta']. \tag{14}$$

Suppose that M has generalized 1-type Gauss map G. Then, with the help of Equations (1), (12) and (14), we obtain

$$\begin{aligned} &q^{-7/2}[qA_1\beta + \{(u+t)B_1 + D_1Q\}\beta' + \{QB_1 - D_1(u+t)\}\beta \times \beta'] \\ &= fq^{-1/2}\{Q\beta' - (u+t)\beta \times \beta'\} + g\mathbb{C}. \end{aligned} \tag{15}$$

By taking the inner product to Equation (15) with β, β' and $\beta \times \beta'$ respectively, we get the following:

$$q^{-5/2}A_1 = g\,\langle \mathbb{C}, \beta \rangle, \tag{16}$$

$$q^{-7/2}\{(u+t)B_1 + D_1Q\} = fq^{-1/2}Q + g\,\langle \mathbb{C}, \beta' \rangle, \tag{17}$$

$$q^{-7/2}\{QB_1 - (u+t)D_1\} = -fq^{-1/2}(u+t) + g\,\langle \mathbb{C}, \beta \times \beta' \rangle. \tag{18}$$

Combining Equations (16), (17) and (18), we have

$$qA_1\omega_2 - \{(u+t)B_1 + D_1Q\}\omega_1 + fq^3Q\omega_1 = 0, \tag{19}$$

$$qA_1\omega_3 - \{QB_1 - (u+t)D_1\}\omega_1 - fq^3(u+t)\omega_1 = 0, \tag{20}$$

$$\{(u+t)B_1 + D_1Q\}\omega_3 - \{QB_1 - (u+t)D_1\}\omega_2 - fq^3\{Q\omega_3 + (u+t)\omega_2\} = 0, \tag{21}$$

where we have put $\omega_1 = \langle \mathbb{C}, \beta\rangle$, $\omega_2 = \langle \mathbb{C}, \beta'\rangle$ and $\omega_3 = \langle \mathbb{C}, \beta \times \beta'\rangle$.

On the other hand, differentiating a constant vector $\mathbb{C} = \omega_1\beta + \omega_2\beta' + \omega_3\beta \times \beta'$ with respect to the parameter s and using Equation (11), we get

$$\begin{aligned} &\omega_1' - \omega_2 = 0,\\ &\omega_3' + \omega_2 R = 0,\\ &\omega_1 + \omega_2' - \omega_3 R = 0. \end{aligned} \tag{22}$$

Combining Equations (19) and (20), we obtain

$$A_1\{\omega_2(u+t) + \omega_3 Q\} - B_1\omega_1 = 0. \tag{23}$$

First of all, we consider the case of $R = 0$.

Theorem 2. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ with generalized 1-type Gauss map. If $R = 0$, then M is part of a plane or a helicoid.*

Proof. If the constant vector $\mathbb{C}$ is zero in the definition given by Equation (1), then the Gauss map G is of nothing but pointwise 1-type Gauss map of the first kind. By Characterization Theorem, M is part of a helicoid ([10]).

We now assume that the constant vector $\mathbb{C}$ is non-zero. In this case, we will show $Q = 0$ on M and thus M is part of a plane due to Equation (13).

Suppose that the open subset $U = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$ of $\mathbb{R}$ is not empty. Then, on a component U_C of U, we have from Equation (22) that ω_3 is a constant and $\omega_1'' = -\omega_1$. Since the left hand side of Equation (23) is a polynomial in t with functions of s as the coefficients, the leading coefficient consisting of functions of s must vanish and $\omega_1^2 Q'$ is a constant on U_C with the help of Equation (22).

Next, from the coefficient of t^2 in Equation (23), we obtain

$$3\omega_2 u'Q - 2\omega_3 QQ' + 3\omega_1 u'Q' + \omega_1 u''Q = 0. \tag{24}$$

Similar to the above, from the coefficient of the linear term in t of Equation (23) with the help of Equation (24), we get

$$\omega_2 QQ' + \omega_3 u'Q - \omega_1(u')^2 + \omega_1(Q')^2 = 0. \tag{25}$$

In addition, the constant term in Equation (23) relative to t is automatically zero. If we make use of Equation (24), we obtain

$$\begin{aligned} Q[\omega_1\{3u(u')^2 + 3u'QQ' - 3u(Q')^2 - u''Q^2\} - 3\omega_2 uQQ'&\\ - \omega_3(3uu'Q + Q^2Q')] &= 0. \end{aligned}$$

Hence, on U_C, we have

$$\begin{aligned} \omega_1\{3u(u')^2 + 3u'QQ' - 3u(Q')^2 - u''Q^2\} - 3\omega_2 uQQ'&\\ - \omega_3(3uu'Q + Q^2Q') &= 0. \end{aligned} \tag{26}$$

Using Equations (24) and (25), Equation (26) can be reduced to

$$2\omega_1 u' Q' + \omega_2 u' Q - \omega_3 Q Q' = 0. \tag{27}$$

Suppose that there is a point $s_0 \in U_C$ such that $u'(s_0) \neq 0$. Then, $u'(s) \neq 0$ everywhere on an open interval I containing s_0. So, Equation (25) yields

$$\omega_3 Q = \frac{1}{u'}\{\omega_1 (u')^2 - \omega_1 (Q')^2 - \omega_2 Q Q'\}. \tag{28}$$

Putting Equation (28) into (27), $(u'^2 + Q'^2)(\omega_2 Q + \omega_1 Q') = 0$, which implies $\omega_2 Q + \omega_1 Q' = 0$. Since $\omega_2 = \omega_1'$, we see that $\omega_1 Q$ is constant on I.

If $\omega_1 = 0$ on some subinterval J in I, $\omega_2 = 0$ on J. Equation (25) gives $\omega_3 = 0$ on J. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero vector, which is a contradiction. Thus, without loss of generality we may assume that $\omega_1 \neq 0$ everywhere on I and it is of the form $\omega_1 = k_1 \cos(s + s_1)$ for some non-zero constant k_1 and $s_1 \in \mathbb{R}$. Since $\omega_1^2 Q'$ is constant and $\omega_1 Q$ is constant on I, ω_1 must be zero on I, which contradicts $\omega_1 = k_1 \cos(s + s_1)$ for some non-zero constant k_1. Therefore, the open interval I is empty and thus $u' = 0$ on U_C. If we take into account Equations (25) and (27), we get $Q'(\omega_2 Q + \omega_1 Q') = 0$ and $\omega_3 Q' = 0$, respectively.

Suppose that $Q'(s_2) \neq 0$ at some point $s_2 \in U_C$. Then $\omega_3 = 0$ and $\omega_1 Q$ is a constant on an open interval J_1 containing s_2. Similar to the above argument, since $\omega_1^2 Q'$ and $\omega_1 Q$ are constant on J_1, it follows that $\omega_1 = 0$. By Equation (22), ω_2 is zero. Hence the constant vector $\mathbb{C}$ is zero, a contradiction. Thus J_1 is empty. Therefore, Q is constant on U_C. By continuity, Q is either a non-zero constant or zero on M. Because of Equation (13), M is minimal and it is an open part of a helicoid, which means that the Gauss map is of pointwise 1-type of the first kind. Therefore, the open subset U is empty. Consequently, Q is zero on M. Hence, M is an open part of a plane. □

Now, we assume that the function R is not vanishing everywhere.

If $f = g$, the Gauss map G of M is of pointwise 1-type. Thus, M is characterized as an open part of a right cone including the case that M is a plane or a helicoid depending upon whether the constant vector $\mathbb{C}$ is non-zero or zero ([9]).

From now on, we may assume the constant vector $\mathbb{C}$ is non-zero and $f \neq g$ unless otherwise stated. Similarly as before, Equation (23) yields

$$\omega_2 R + \omega_1 R' = 0. \tag{29}$$

Since $\omega_1' = \omega_2$ in Equation (22), we see that $\omega_1 R$ is constant. In addition, the coefficient of the term involving t^3 in Equation (23) must be zero.

With the help of Equation (29), we get

$$2\omega_2 Q' + \omega_3 Q R - \omega_1 u' R + \omega_1 Q'' = 0. \tag{30}$$

If we examine the coefficient of the term involving t^2 in Equation (23), using Equations (29) and (30) we obtain

$$\omega_1 Q^2 R' - 3\omega_2 u' Q + 2\omega_3 Q Q' - \omega_1 Q Q' R - 3\omega_1 u' Q' - \omega_1 u'' Q = 0. \tag{31}$$

Furthermore, from the coefficient of the linear term in t in Equation (23) with the help of Equations (29)–(31), we also get

$$Q\{\omega_2 Q Q' + \omega_3 u' Q - \omega_1 (u')^2 + \omega_1 (Q')^2\} = 0. \tag{32}$$

Suppose that the function Q is not zero, i.e., the open subset $V = \{s \in \text{dom}(\alpha) | Q(s) \neq 0\}$ of $\text{dom}(\alpha)$ is not empty. Equation (32) gives that

$$\omega_2 QQ' + \omega_3 u'Q - \omega_1 (u')^2 + \omega_1 (Q')^2 = 0. \tag{33}$$

Moreover, considering the constant term relative to t in Equation (23) and using Equations (29)–(31), we obtain

$$\begin{aligned} Q[\omega_1\{3u(u')^2 + 3u'QQ' - Q^2Q'R - 3u(Q')^2 - u''Q^2 + Q^3R'\} \\ - 3\omega_2 uQQ' - \omega_3(3uu'Q + Q^2Q')] = 0. \end{aligned}$$

Hence, on the open subset V in $\mathbb{R}$,

$$\begin{aligned} \omega_1\{3u(u')^2 + 3u'QQ' - Q^2Q'R - 3u(Q')^2 - u''Q^2 + Q^3R'\} \\ - 3\omega_2 uQQ' - \omega_3(3uu'Q + Q^2Q') = 0. \end{aligned} \tag{34}$$

Applying Equations (31) and (33) to Equation (34), we have

$$2\omega_1 u'Q' + \omega_2 u'Q - \omega_3 QQ' = 0. \tag{35}$$

On the other hand, since $\omega_3 R = \omega_1 + \omega_2{}'$ in Equation (22), Equation (30) becomes

$$(\omega_1 Q)'' + \omega_1 Q - \omega_1 u'R = 0. \tag{36}$$

Suppose that the function u is not constant, i.e., the open subset $V_1 = \{s \in V | u'(s) \neq 0\}$ is not empty. Then Equation (33) yields

$$\omega_3 Q = \frac{1}{u'}\{\omega_1(u')^2 - \omega_1(Q')^2 - \omega_2 QQ'\}. \tag{37}$$

Putting Equation (37) into (35), $(u'^2 + Q'^2)(\omega_2 Q + \omega_1 Q') = 0$ and thus $\omega_2 Q + \omega_1 Q' = 0$. Therefore, $\omega_1 Q$ is constant on a component $\mathcal{C}$ of V_1. From Equation (36), we get $\omega_1 Q = \omega_1 u'R$.

If $\omega_1 = 0$ on an open interval $\tilde{I} \subset \mathcal{C}$, the constant vector $\mathbb{C}$ is zero on M, a contradiction. Thus, $\omega_1 \neq 0$ and so $Q = u'R$ on $\mathcal{C}$. The fact that $\omega_1 Q$ and $\omega_1 R$ are constant on $\mathcal{C}$ implies that u' is a non-zero constant on $\mathcal{C}$. Then, Equations (31) and (35) are simplified as follows:

$$\omega_1 Q^2 R' + 2\omega_3 QQ' - \omega_1 QQ'R = 0, \tag{38}$$

$$\omega_1 u'Q' - \omega_3 QQ' = 0. \tag{39}$$

Putting $Q = u'R$ into Equation (38), $\omega_3 Q' = 0$ is derived. Thus, Equation (39) implies that $\omega_1 Q' = 0$ and so $Q' = 0$ on $\mathcal{C}$. Hence, Q and R are both non-zero constants on $\mathcal{C}$.

On the other hand, without difficulty, we can show that the torsion of the director vector field $\beta = \beta(s)$ viewed as a curve is zero and so β is part of a plane curve which is a small circle on the unit sphere centered at the origin with the normal curvature –1 and the geodesic curvature R on $\mathcal{C}$. Up to a rigid motion, we may put

$$\beta(s) = \frac{1}{p}(\cos ps, \sin ps, R)$$

on $\mathcal{C}$, where we have put $p = \sqrt{1 + R^2}$. Then, $u = \langle \alpha', \beta' \rangle = -\alpha_1' \sin ps + \alpha_2' \cos ps$, where $\alpha'(s) = (\alpha_1'(s), \alpha_2'(s), \alpha_3'(s))$. Therefore, on $\mathcal{C}$, we get

$$u' = -(\alpha_1'' + \alpha_2' p) \sin ps + (\alpha_2'' - \alpha_1' p) \cos ps,$$

from which, we see that $u' = 0$ on $\mathcal{C} \subset V_1$, a contradiction. Hence, V_1 is empty and so $u' = 0$ on V. Then, Equations (30), (33) and (35) can be respectively reduced to

$$2\omega_2 Q' + \omega_3 QR + \omega_1 Q'' = 0, \tag{40}$$

$$\omega_2 QQ' + \omega_1 (Q')^2 = 0, \tag{41}$$

$$\omega_3 QQ' = 0. \tag{42}$$

Suppose that $Q'(\tilde{s}_0) \neq 0$ at a point $\tilde{s}_0$ in V. From Equations (41) and (42), $\omega_3 = 0$ and $\omega_1 Q$ is a constant on an open interval $\tilde{J} \subset V$ containing $\tilde{s}_0$. Hence, ${\omega_2}' Q = 0$ is derived from Equation (40). Therefore, ${\omega_2}' = 0$ on $\tilde{J}$. The third equation of (22) yields $\omega_1 = 0$. It follows that $\omega_2 = 0$. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero on M, a contradiction. So, $Q' = 0$ on V. Thus, Q is non-zero constant on each component of V. If we consider Equations (30) and (31), we have

$$\omega_3 R = 0 \quad \text{and} \quad \omega_1 R' = 0.$$

Since $R \neq 0$, $\omega_3 = 0$ on each component of V. By Equation (29), $\omega_2 R = 0$, which yields that $\mathbb{C}$ is zero on M. It is a contradiction. Hence, the open subset V of $\mathbb{R}$ is empty and the function Q is vanishing on M. Thus, M is flat due to Equation (13). Since the ruled surface M is non-cylindrical, M is one of an open part of a tangent developable surface or a conical surface. One of the authors proved that tangential developable surfaces do not have a generalized 1-type Gauss map and a conical surface of G-type can be constructed by the given functions f, g and the constant vector $\mathbb{C}$ ([15]).

Consequently, we have

Theorem 3. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ with generalized 1-type Gauss map. Then, M is an open part of a plane, a helicoid, a right cone or a conical surface of G-type.*

Summing up our results, we obtain the following classification theorem.

Theorem 4. *(Classification) Let M be a ruled surface in $\mathbb{E}^3$ with a generalized 1-type Gauss map. Then, M is an open part of a plane, a circular cylinder, a cylinder over a base curve of an infinite-type satisfying Equations (5), (9) and (10), a helicoid, a right cone or a conical surface of G-type.*

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