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Hybrid Multivalued Type Contraction Mappings in α_K-Complete Partial b-Metric Spaces and Applications

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Abstract: In this paper, we initiate the notion of generalized multivalued $(\alpha_K^*, \Upsilon, \Lambda)$-contractions and provide some new common fixed point results in the class of α_K-complete partial b-metric spaces. The obtained results are an improvement of several comparable results in the existing literature. We set up an example to elucidate our main result. Moreover, we present applications dealing with the existence of a solution for systems either of functional equations or of nonlinear matrix equations.

Keywords: fixed point; triangular α_K^*-orbital admissible mappings; partial b-metric space; multivalued contraction mapping; functional equation; nonlinear matrix equation

1. Introduction and Preliminaries

Fixed point theory plays an essential role in functional and nonlinear analysis. Banach [1] proved a significant result for contraction mappings. Since then, many works dealing with fixed point results have been provided by various authors (see, for example, [2–42]).

On the one hand, Bakhtin [43] and Czerwik [34,35] gave generalizations of the known Banach fixed point theorem in the class of b-metric spaces. In 1994, Matthews [23,24] introduced the notion of a partial metric space, which is a generalization of metric spaces. Very recently, Shukla [41] introduced the notion of partial b-metric spaces by combining partial metric spaces and b-metric spaces.

On the other hand, Popescu [22] introduced triangular α-orbital admissible maps. Karapinar [42] gave some fixed point results for a generalized α-ψ-Geraghty contraction type mappings using triangular α-admissibility. Recently, Ameer et al. [32] initiated the concept of generalized α_*-ψ-Geraghty type multivalued contraction mappings and developed new common fixed point results in the class of α-complete b-metric spaces.

In this paper, we initiate the notion of generalized multivalued $(\alpha_K^*, \Upsilon, \Lambda)$-contraction pair of mappings. Some new common fixed point results are established for these mappings in the setting of α_K-complete partial b-metric spaces. Examples are also given to support the obtained results. Finally, we apply the obtained results to ensure the existence of a solution of either a pair of functional equations or nonlinear matrix equations.

Definition 1. [35] *Let ω be a non-empty. Take the real number $K \geq 1$. The function $d_b : \omega \times \omega \to [0, \infty)$ is a b-metric if for all $\zeta, \eta, v \in \omega$,*

(i) $d_b(\zeta, \eta) = 0$ *if and only if* $\zeta = \eta$.
(ii) $d_b(\zeta, \eta) = d_b(\eta, \zeta)$.
(ii) $d_b(\zeta, \eta) \leq K\,[d_b(\zeta, v) + d_b(v, \eta)]$.

Definition 2. [23] *Let ω be a nonempty set. The function $P : \omega \times \omega \to [0, \infty)$ is said to be a partial metric if for all $\zeta, \eta, z \in \omega$,*

(P_1) $P(\zeta, \zeta) = P(\zeta, \eta) = P(\eta, \eta)$ *if and only if* $\zeta = \eta$.
(P_2) $P(\zeta, \zeta) \leq P(\zeta, \eta)$.
(P_3) $P(\zeta, \eta) = P(\eta, \zeta)$.
(P_4) $P(\zeta, \eta) \leq P(\zeta, z) + P(z, \eta) - P(z, z)$.

Definition 3. [41] *Let $K \geq 1$ be a real number and $\omega \neq \emptyset$. The function $P_b : \omega \times \omega \to [0, \infty)$ satisfying the following for all $\zeta, \eta, z \in \omega$ is said to be a partial b-metric:*

(P_b1) $P_b(\zeta, \zeta) = P_b(\zeta, \eta) = P_b(\eta, \eta)$ *if and only if* $\zeta = \eta$.
(P_b2) $P_b(\zeta, \zeta) \leq P_b(\zeta, \eta)$.
(P_b3) $P_b(\zeta, \eta) = P_b(\eta, \zeta)$.
(P_b4) $P_b(\zeta, \eta) \leq K[P_b(\zeta, z) + P_b(z, \eta)] - P_b(z, z)$.

K is the coefficient of the partial b-metric space (ω, P_b).

Remark 1. *Obviously, a partial metric space is also a partial b-metric space with coefficient $K = 1$. A b-metric space is also a partial b-metric space with zero self-distance. However, the converse of these facts need not hold.*

Example 1. *Let $\omega = R^+$ and $k > 1$, the mapping $P_b : \omega \times \omega \to R^+$ defined by*

$$P_b(\zeta, \eta) = (\zeta \vee \eta)^k + |\zeta - \eta|^k, \text{ for all } \zeta, \eta \in \omega$$

is a partial b-metric on ω. Here, $K = 2^k$. For $\zeta = \eta$, $P_b(\zeta, \zeta) = \zeta^k \neq 0$, thus P_b is not a b-metric on ω.

Let $\zeta, \eta, z \in \omega$ be such that $\zeta > z > \eta$. The following inequality always holds

$$(\zeta - \eta)^k > (\zeta - z)^k + (z - \eta)^k.$$

Since $P_b(\zeta, \eta) = \zeta^k + (\zeta - \eta)^k$ and $P_b(\zeta, z) + P_b(z, \eta) - P_b(z, z) = \zeta^k + (\zeta - z)^k + (z - \eta)^k$, we have

$$P_b(\zeta, \eta) > P_b(\zeta, z) + P_b(z, \eta) - P_b(z, z).$$

This shows that P_b is not a partial metric on ω.

Definition 4. *Let (ω, P_b, K) be a partial b-metric space. The mapping $d_{P_b} : \omega \times \omega \to [0, \infty)$ defined by*

$$d_{P_b}(\zeta, \eta) = 2P_b(\zeta, \eta) - P_b(\zeta, \zeta) - P_b(\eta, \eta),$$

for all $\zeta, \eta \in \omega$, defines a metric on ω, called an induced metric.

Definition 5. [41] *Let (ω, P_b, K) be a partial b-metric space with a coefficient $K \geq 1$. Let $\{\zeta_n\}$ be a sequence in ω and $\zeta \in \omega$. Then,*

(i) $\{\zeta_n\}$ *is said to be convergent to* ζ *if* $\lim_{n\to\infty} P_b(\zeta_n, \zeta) = P_b(\zeta, \zeta)$.
(ii) $\{\zeta_n\}$ *is Cauchy if* $\lim_{n,m\to\infty} P_b(\zeta_n, \zeta_m)$ *exists and is finite.*

(iii) (ω, P_b) *is complete if every Cauchy sequence is convergent in* ω.

Lemma 1. *[41] Let* (ω, P_b, K) *be a partial b-metric space.*

(1) Every Cauchy sequence in (ω, d_{P_b}) is also Cauchy in (ω, P_b, K) and vice versa.
(2) (ω, P_b, K) is complete if and only if (ω, d_{P_b}) is a complete metric space.
(3) The sequence $\{\zeta_n\}$ is convergent to some $v \in \omega$ if and only if

$$\lim_{n\to\infty} P_b(\zeta_n, v) = P_b(v, v) = \lim_{n,m\to\infty} P_b(\zeta_n, \zeta_m).$$

Denote a metric space by MS.

Definition 6. *[21] Let* (ω, d) *be a MS.* $T : \omega \to \omega$ *is called an F-contraction self-mapping, if there exist* $\tau > 0$ *and* $F \in \digamma$ *such that*

$$\forall \zeta, \eta \in \omega,\ d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leq F(d(\zeta, \eta)),$$

where $\digamma$ *is the family of functions* $F : (0, \infty) \to (-\infty, \infty)$ *such that*
(F1) F is strictly increasing.
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \infty)$,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \Longleftrightarrow \lim_{n\to\infty} \alpha_n = 0.$$

(F3) There exists $\gamma \in (0, 1)$ *such that* $\lim_{t\longrightarrow 0^+} t^{\gamma} F(t) = 0$.

Theorem 1. *[21] Let* (ω, d) *be a complete MS and* $T : \omega \to \omega$ *be an F- contraction mapping. Then, T possesses a unique fixed point* $\zeta^* \in \omega$.

Piri and Kumam [17] modified the set of functions $F \in \digamma$.

Definition 7. *[17] Let* (ω, d) *be a MS.* $T : \omega \to \omega$ *is said to be a F-contraction self-mapping if there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\forall \zeta, \eta \in \omega, d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leq F(d(\zeta, \eta)),$$

where $\mathcal{F}$ *is the set of functions* $F : (0, \infty) \to (-\infty, \infty)$ *satisfying the following conditions:*
(F1) F is strictly increasing, i.e., for all $\zeta, \eta \in \mathbb{R}^+$ *with* $\zeta < \eta$, $F(\zeta) < F(\eta)$.
(F2) For each positive real sequence $\{\alpha_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0.$$

(F3) F is continuous.

On the other hand, recently Jleli and Samet [9,10] initiated the concept of θ-contractions.

Definition 8. *Let* (ω, d) *be a MS. A mapping* $T : \omega \to \omega$ *is said to be a* θ*-contraction, if there exist* $\theta \in \Theta$ *and a real constant* $k \in (0, 1)$ *such that*

$$\zeta, \eta \in \omega, d(T(\zeta), T(\eta)) \neq 0 \Longrightarrow \theta(d(T(\zeta), T(\eta))) \leq [\theta(d(\zeta, \eta)]^k,$$

where Θ *is the set of functions* $\theta : (0, \infty) \longrightarrow (1, \infty)$ *such that:*
($\Theta 1$*)* θ *is non-decreasing.*
($\Theta 2$*) for each positive sequence* $\{t_n\}$*,*

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+.$$

($\Theta 3$*) there exist* $r \in (0,1)$ *and* $\ell \in (0, \infty]$ *such that* $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$*.*
($\Theta 4$*)* θ *is continuous.*

The main result of Jleli and Samet [9] is the following.

Theorem 2. *[9] Let* (ω, d) *be a complete MS. Let* $T : \omega \to \omega$ *be a* θ*-contraction mapping. Then, there exists a unique fixed point of* T*.*

As in [13], the family of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ verifying:
$(\Theta 1)'$ θ is non-decreasing.
$(\Theta 2)'$ for each positive sequence $\{t_n\}$, $\inf_{t_n \in (0,\infty)} \theta(t_n) = 1$.
$(\Theta 3)'$ θ is continuous, is denoted by Ξ.

Theorem 3. *[13] Let* $T : \omega \to \omega$ *be a self-mapping on the complete MS* (ω, d)*. The following statements are equivalent:*
(i) T is a θ*-contraction mapping with* $\theta \in \Xi$*.*
(ii) T is a F-contraction mapping with $F \in \mathcal{F}$*.*

Liu et al. [13] initiated the concept of (Υ, Λ)-Suzuki contractions.

Definition 9. *Let* (ω, d) *be a MS. A mapping* $T : \omega \to \omega$ *is said to be a* (Υ, Λ)*-Suzuki contraction, if there exist a comparison function* Υ *and* $\Lambda \in \Phi$ *such that, for all* $\zeta, \eta \in \omega$ *with* $T(\zeta) \neq T(\eta)$

$$\frac{1}{2} d(\zeta, T(\zeta)) < d(\zeta, \eta) \Longrightarrow \Lambda(d(T(\zeta), T(\eta))) \leq \Upsilon[\Lambda(U(\zeta, \eta))],$$

where

$$U(\zeta, \eta) = \max\left\{ d(\zeta, \eta), d(\zeta, T(\zeta)), d(\eta, T(\eta)), \frac{d(\zeta, T(\eta)) + d(\eta, T(\zeta))}{2} \right\}.$$

Denote by Φ *the set of functions* $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ *verifying:*
($\Phi 1$*)* Λ *is non-decreasing.*
($\Phi 2$*) for each positive sequence* $\{t_n\}$*,*

$$\lim_{n\to\infty} \Lambda(t_n) = 0 \text{ if and only if } \lim_{n\to\infty} t_n = 0;$$

($\Phi 3$*)* Λ *is continuous.*

As in [2], a function $\Upsilon : (0, \infty) \longrightarrow (0, \infty)$ satisfying:
(i) Υ is monotone increasing, that is, $t_1 < t_2 \Longrightarrow \Upsilon(t_1) \leq \Upsilon(t_2)$.
(ii) $\lim_{n\to\infty} \Upsilon^n(t) = 0$ for all t$>$ 0, where Υ^n stands for the nth iterate of Υ,
is called a comparison function. Clearly, if Υ is a comparison function, then $\Upsilon(t) <$t for each $t > 0$.

Lemma 2. *[13] Let* $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ *be a continuous non-decreasing function such that* $\inf_{T \in (0,\infty)} \phi(T) = 0$*. Let* $\{t_k\}_k$ *be a positive sequence. Thus,*

$$\lim_{k\to\infty} \Lambda(t_k) = 0 \text{ if and only if } \lim_{k\to\infty} t_k = 0.$$

Example 2. [2] *The following functions* $\Upsilon : (0,\infty) \longrightarrow (0,\infty)$ *are comparison functions* :
(i) $\Upsilon(t) = at$ *with* $0 < a < 1$, *for each* $t > 0$.
(ii) $\Upsilon(t) = \frac{t}{t+1}$, *for each* $t > 0$.

For examples of functions in Φ, see [13]. For a MS (ω, d), $CB(\omega)$ stands for the collection of all closed and bounded subsets in ω.

Theorem 4. *Let* $S : \omega \longrightarrow CB(\omega)$ *be a multivalued mapping on the complete MS* (ω, d). *The two statements are equivalent:*
(i) S *is a multivalued* θ*-contraction mapping with* $\theta \in \Xi$.
(ii) S *is a multivalued* F*-contraction mapping with* $F \in \mathcal{F}$.

Proof. The proof of this theorem follows immediately from the proof of Theorem 3. □

Let (ω, P_b, K) be a partial b-metric space and $CB_{P_b}(\omega)$ be the family of all closed and bounded subsets of ω. For $\zeta \in \omega$ and $A, B \in CB_{P_b}(\omega)$, we define

$$D_{P_b}(\zeta, A) = \inf_{a \in A} P_b(\zeta, a), \quad D_{P_b}(A, B) = \sup_{a \in A} P_b(a, B).$$

Following [25,26], Felhi [44] Defined $H_{P_b} : CB_{P_b}(\omega) \times CB_{P_b}(\omega) \to [0, \infty)$ as

$$H_{P_b}(A, B) = \max\left\{D_{P_b}(A, B), D_{P_b}(B, A)\right\},$$

for every $A, B \in CB_{P_b}(\omega)$. It is clear that for $A, B \in CB_{P_b}(\omega)$ and $a \in A$, one has

$$D_{P_b}(a, B) = \inf_{b \in B} P_b(a, b) \le D_{P_b}(A, B) \le H_{P_b}(A, B).$$

Lemma 3. [44] *Let* $A, B \in CB_{P_b}(\omega)$, *where* (ω, P_b, K) *is a partial b-metric space. Set* $q > 1$. *Hence, for each* $u \in A$, *there exists* $v \in B$ *so that* $P_b(u, v) \le q H_{P_b}(A, B)$.

Lemma 4. [44] *Let* (ω, P_b, K) *be a partial b-metric space with coefficient* $K \ge 1$. *For* $A \in CB_{P_b}(\omega)$ *and* $\zeta \in \omega$, *then* $D_{P_b}(\zeta, A) = P_b(\zeta, \zeta)$ *if and only if* $\zeta \in \overline{A}$, *where* $\overline{A}$ *is the closure of* A.

Lemma 5. [44] *Let* (ω, P_b, K) *be a partial b-metric space. For all* $A, B, C \in CB_{P_b}(\omega)$, *the following inequalities hold:*

(H_1) $H_{P_b}(A, A) \le H_{P_b}(A, B)$.
(H_2) $H_{P_b}(A, B) = H_{P_b}(B, A)$.
(H_3) $H_{P_b}(A, B) \le K[H_{P_b}(A, C) + H_{P_b}(C, B)] - \inf_{c \in C} P_b(c, c)$.

Lemma 6. [44] *Let* (ω, P_b, K) *be a partial b-metric space with coefficient* $K \ge 1$ *and* $B \in CB_{P_b}(\omega)$. *Let* $\zeta \in \omega$ *such that* $D_{P_b}(\zeta, B) < c$ *with* $c > 0$, *then there exists* $\eta \in B$ *so that* $P_b(\zeta, \eta) < c$.

Definition 10. [28] *Given* $T : \omega \to CB(\omega)$ *and* $\alpha : \omega \times \omega \longrightarrow [0, +\infty)$ *be a given function. Such* T *is said* α_**-admissible if for* $\zeta, \eta \in \omega$ *with* $\alpha(\zeta, \eta) \ge 1$, *we have* $\alpha_*(T\zeta, T\eta) \ge 1$, *where* $\alpha_*(A, B) = \inf\{\alpha(\zeta, \eta) : \zeta \in A,\ \eta \in B\}$.

Definition 11. [32] *Given* $S, T : \omega \to CB(\omega)$ *and* $\alpha : \omega \times \omega \to [0, +\infty)$. *The pair* (S, T) *is triangular* α_**-admissible if:*

(i) the pair (S, T) is α_*-admissible, i.e., for $\zeta, \eta \in \omega$ with $\alpha(\zeta, \eta) \geq 1$, we have $\alpha_*(S\zeta, T\eta) \geq 1$ and $\alpha_*(T\zeta, S\eta) \geq 1$.
(ii) $\alpha(\zeta, x) \geq 1$ and $\alpha(x, \eta) \geq 1$ imply $\alpha(\zeta, \eta) \geq 1$.

Definition 12. *[32] Given $S, T : \omega \to CB(\omega)$ and $\alpha : \omega \times \omega \to [0, +\infty)$. The pair (S, T) is α_*-orbital admissible if:*
$\alpha_*(\zeta, S\zeta) \geq 1$ and $\alpha_*(\zeta, T\zeta) \geq 1$ imply $\alpha_*(S\zeta, T^2\zeta) \geq 1$ and $\alpha_*(T\zeta, S^2\zeta) \geq 1$.

Definition 13. *[32] Given $S, T : \omega \to CB(\omega)$ and $\alpha : \omega \times \omega \to [0, +\infty)$. The pair (S, T) is triangular α_*-orbital admissible, if:*
(i) (S, T) is α_*-orbital admissible.
(ii) $\alpha(\zeta, \eta) \geq 1$, $\alpha_*(\eta, S\eta) \geq 1$ and $\alpha_*(\eta, T\eta) \geq 1$ imply $\alpha_*(\zeta, S\eta) \geq 1$ and $\alpha_*(\zeta, T\eta) \geq 1$.

2. Main Results

We start with the following definitions.

Definition 14. *Given $K \geq 1$, $S, T : \omega \to CB_{P_b}(\omega)$ and $\alpha_K : \omega \times \omega \to [0, +\infty)$. The pair (S, T) is said to be triangular α_K^*-admissible if:*
(i) (S, T) is α_K^*-admissible, i.e., $\alpha_K(\zeta, \eta) \geq K^2$ implies $\alpha_K^*(S\zeta, T\eta) \geq K^2$ and $\alpha_K^*(T\zeta, S\eta) \geq K^2$, where

$$\alpha_K^*(A, B) = \inf\{\alpha(\zeta, \eta) : \zeta \in A,\ \eta \in B\}.$$

(ii) $\alpha_K(\zeta, u) \geq K^2$ and $\alpha_K(u, \eta) \geq K^2$ imply $\alpha_K(\zeta, \eta) \geq K^2$.

Definition 15. *Given $S, T : \omega \to CB_{P_b}(\omega)$ and $\alpha_K : \omega \times \omega \to [0, +\infty)$. The pair $(\check{S}, T)$ is said α_K^*-orbital admissible if:*
$\alpha_K^*(\zeta, S\zeta) \geq K^2$ and $\alpha_K^*(\zeta, T\zeta) \geq K^2$ imply $\alpha_K^*(S\zeta, TS\zeta) \geq K^2$ and $\alpha_K^*(T\zeta, ST\zeta) \geq K^2$.

Definition 16. *Given $S, T : \omega \to CB_{P_b}(\omega)$ and $\alpha_K : \omega \times \omega \to [0, +\infty)$. Then, the pair (S, T) is said to be triangular α_K^*-orbital admissible, if:*
(i) (S, T) is α_K^*-orbital admissible.
(ii) $\alpha_K(\zeta, \eta) \geq K^2$, $\alpha_K^*(\eta, S\eta) \geq K^2$ and $\alpha_K^*(\eta, T\eta) \geq K^2$ imply $\alpha_K^*(\zeta, S\eta) \geq K^2$ and $\alpha_K^*(\zeta, T\eta) \geq K^2$.

Lemma 7. *Given $S, T : \omega \to CB_{Pb}(\omega)$. Suppose that (S, T) is triangular α_K^*-orbital admissible and there exists $\zeta_0 \in \omega$ such that $\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2$. Define a sequence $\{\zeta_n\}$ in ω by $\zeta_{2i+1} \in S\zeta_{2i}$ and $\zeta_{2i+2} \in T\zeta_{2i+1}$, where $i = 0, 1, 2, ...$ Then, $\alpha_K(\zeta_n, \zeta_m) \geq K^2$ for all nonnegative integers n, m such that $m > n$.*

Proof. Since $\alpha_K^*(\zeta_0, S\zeta_0) = \inf\{\alpha(\zeta_0, \zeta_1) : \zeta_1 \in S\zeta_0\} \leq \alpha_K(\zeta_0, \zeta_1) \geq K^2$, using the triangular α_K^*-orbital admissibility of (S, T), we have

$$\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2 \text{ implies } \alpha_K^*(S\zeta_0, TS\zeta_0) \leq \alpha_K^*(\zeta_1, T\zeta_1) \leq \alpha_K(\zeta_1, \zeta_2) \geq K^2$$

and

$$\alpha_K^*(\zeta_1, T\zeta_1) \geq K^2 \text{ implies } \alpha_K^*(T\zeta_1, ST\zeta_1) \leq \alpha_K^*(\zeta_2, S\zeta_2) \leq \alpha_K(\zeta_2, \zeta_3) \geq K^2.$$

Thus, $\alpha_K(\zeta_n, \zeta_m) \geq K^2$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m = n + 1$. Using again the triangular α_K^*-orbital admissibility of (S, T), we get $\alpha_K(\zeta_n, \zeta_m) \geq K^2$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$. □

Definition 17. *Let (ω, P_b, K) be a partial b-metric space. Given $S : \omega \to CB_{P_b}(\omega)$ and $\alpha_K : \omega \times \omega \to [0, +\infty)$. Such S is α_K^*-P_b-continuous on $(CB_{P_b}(\omega), H_{P_b})$, if $\{\zeta_n\}$ is a sequence in ω such that $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ for each integer n and $\zeta \in \omega$ with $\lim_{n \longrightarrow \infty} P_b(\zeta_n, \zeta) = 0$, then $\lim_{n \longrightarrow \infty} H_{P_b}(S\zeta_n, S\zeta) = 0$.*

Now, we initiate the concept of generalized $(\alpha_K^*, \Upsilon, \Lambda)$-contraction multivalued pair of mappings as follows:

Definition 18. *Let (ω, P_b, K) be a partial b-metric space and $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ be a function. Given $S, T : \omega \longrightarrow CB_{P_b}(\omega)$. The pair (S, T) is called a generalized $(\alpha_K^*, \Upsilon, \Lambda)$-contraction multivalued pair of mappings if there exist a comparison function Υ and a function $\Lambda \in \Phi$ such that for $\zeta, \eta \in \omega$, $\alpha_K(\zeta, \eta) \geq K^2$,*

$$H_{P_b}(S(\zeta), T(\eta)) > 0 \Longrightarrow \Lambda\left(\alpha_K(\zeta,\eta) H_{P_b}(S(\zeta), T(\eta))\right) \leq \Upsilon\left(\Lambda\left(U_{P_b}(\zeta,\eta)\right)\right), \tag{1}$$

where

$$U_{P_b}(\zeta,\eta) = \max\left\{P_b(\zeta,\eta), D_{P_b}(\zeta, S(\zeta)), D_{P_b}(\eta, T(\eta)), \frac{D_{P_b}(\zeta, T(\eta)) + D_{P_b}(\eta, S(\zeta))}{2K}\right\}. \tag{2}$$

Our first main result is the following.

Theorem 5. *Let (ω, P_b, K) be a partial b-metric space. Given $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ and $S, T : \omega \longrightarrow CB_{P_b}(\omega)$. Suppose that*
(i) (ω, d, K) is an α_K-complete partial b-metric space.
(ii) (S, T) is a generalized $(\alpha_K^*, \Upsilon, \Lambda)$-contraction multivalued pair of mapping.
(iii) (S, T) is triangular α_K^*-orbital admissible.
(iv) There exists $\zeta_0 \in \omega$ such that $\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2$.
(v)
- *(a) S and T are α_K^*-P_b-continuous multivalued mappings.*
- *(b) If $\{\zeta_n\}$ is a sequence in ω such that $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ for each $n \in \mathbb{N}$ and $\zeta_n \longrightarrow \zeta^* \in \omega$ as $n \longrightarrow \infty$, then there exists a subsequence $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ for each $k \in \mathbb{N}$.*

If Υ is continuous, then there exists a common fixed point of S and T, e.g. $\zeta^* \in \omega$.

Proof. (a) Let $\zeta_0 \in \omega$ be such that $\alpha_K^*(\zeta_0, S(\zeta_0)) \geq K^2$. Choose $\zeta_1 \in S(\zeta_0)$ such that $\alpha_K(\zeta_0, \zeta_1) \geq K^2$ and $\zeta_1 \neq \zeta_0$. By Equation (1), it is easy to see that

$$0 < D_{P_b}(\zeta_1, T(\zeta_1)) \leq H_{P_b}(S(\zeta_0), T(\zeta_1)) \leq \alpha_K(\zeta_0,\zeta_1) H_{P_b}(S(\zeta_0), T(\zeta_1)).$$

Hence, there exists $\zeta_2 \in T(\zeta_1)$,

$$0 < P_b(\zeta_1, \zeta_2) \leq H_{P_b}(S(\zeta_0), T(\zeta_1)) \leq \alpha_K(\zeta_0,\zeta_1) H_{p_b}(S(\zeta_0), T(\zeta_1)). \tag{3}$$

Since Λ is nondecreasing, we have

$$\Lambda(P_b(\zeta_1,\zeta_2)) \leq \Lambda\left(H_{P_b}(S(\zeta_0), T(\zeta_1))\right) \leq \Lambda\left(\alpha_K(\zeta_0,\zeta_1) H_{P_b}(S(\zeta_0), T(\zeta_1))\right). \tag{4}$$

Hence, from Equation (3),

$$\begin{aligned} 0 &\leq \Lambda(P_b(\zeta_1,\zeta_2)) \leq \Lambda\left(\alpha_K(\zeta_0,\zeta_1) H_{P_b}\left(\check{S}(\zeta_0), T(\zeta_1)\right)\right) \\ &\leq \Upsilon\left(\Lambda\left(U_{P_b}(\zeta_0,\zeta_1)\right)\right), \end{aligned} \tag{5}$$

where

$$
\begin{aligned}
U_{P_b}(\zeta_0,\zeta_1) &= \max\left\{\begin{array}{c} P_b(\zeta_0,\zeta_1), D_{P_b}(\zeta_0,\check{S}(\zeta_0)), D_{P_b}(\zeta_1,T(\zeta_1)), \\ \frac{D_{P_b}(\zeta_0,T(\zeta_1))+D_{P_b}(\zeta_1,S(\zeta_0))}{2K} \end{array}\right\} \\
&\le \max\left\{P_b(\zeta_0,\zeta_1), D_{P_b}(\zeta_1,T(\zeta_1)), \frac{D_{P_b}(\zeta_0,T(\zeta_1))+P_b(\zeta_1,\zeta_1)}{2K}\right\} \\
&\le \max\left\{P_b(\zeta_0,\zeta_1), D_{P_b}(\zeta_1,T(\zeta_1))\right\}.
\end{aligned}
$$

If $\max\left\{P_b(\zeta_0,\zeta_1), D_{P_b}(\zeta_1,T(\zeta_1))\right\} = D_{P_b}(\zeta_1,T(\zeta_1))$, then from (5), we have

$$
\Lambda(P_b(\zeta_1,\zeta_2)) \le \Upsilon(\Lambda(P_b(\zeta_1,\zeta_2))) < \Lambda(P_b(\zeta_1,\zeta_2)),
$$

which is a contradiction. Thus, $\max\left\{P_b(\zeta_0,\zeta_1), D_{P_b}(\zeta_1,T(\zeta_1))\right\} = P_b(\zeta_0,\zeta_1)$. By Equation (5), we get that

$$
\Lambda(P_b(\zeta_1,\zeta_2)) \le \Upsilon(\Lambda(P_b(\zeta_0,\zeta_1))).
$$

Similarly, for $\zeta_2 \in T(\zeta_1)$ and $\zeta_3 \in S(\zeta_2)$. We have

$$
\begin{aligned}
\Lambda(P_b(\zeta_2,\zeta_3)) &= \Lambda\left(D_{P_b}(\zeta_2,S(\zeta_2))\right) \\
&\le \Lambda\left(H_{P_b}(T(\zeta_1),S(\zeta_2))\right) \\
&\le \Lambda\left(\alpha_K(\zeta_1,\zeta_2)H_{P_b}(T(\zeta_1),S(\zeta_2))\right) \\
&\le \Upsilon\left(\Lambda\left(U_{P_b}(\zeta_1,\zeta_2)\right)\right) \\
&\le \Upsilon(\Lambda(P_b(\zeta_1,\zeta_2))).
\end{aligned}
$$

This implies that

$$
\Lambda(P_b(\zeta_2,\zeta_3)) \le \Upsilon(\Lambda(P_b(\zeta_1,\zeta_2))). \tag{6}
$$

By continuing in this manner, we build a sequence $\{\zeta_n\}$ in ω in order that $\zeta_{2i+1} \in S(\zeta_{2i})$ and $\zeta_{2i+2} \in T(\zeta_{2i+1})$, $i = 0,1,2,\ldots$. $\alpha_K^*(\zeta_0,S(\zeta_0)) \ge K^2$ and (S,T) is triangular α_K^*-orbital admissible. By Lemma 7, we have

$$
\alpha_S(\zeta_n,\zeta_{n+1})) \ge S^2, \text{ for all } n \in \mathbb{N}\cup\{0\}.
$$

For $i \in \mathbb{N}$, we have,

$$
\begin{aligned}
0 &< \Lambda(P_b(\zeta_{2i+1},\zeta_{2i+2})) \le \Lambda\left(\alpha_K(\zeta_{2i},\zeta_{2i+1}) H_{P_b}(S(\zeta_{2i}),T(\zeta_{2i+1}))\right) \\
&\le \Upsilon\left(\Lambda\left(U_{P_b}(\zeta_{2i},\zeta_{2i+1})\right)\right)
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
U_{P_b}(\zeta_{2i},\zeta_{2i+1}) &= \max\left\{\begin{array}{c} P_b(\zeta_{2i},\zeta_{2i+1}), D_{P_b}(\zeta_{2i},S(\zeta_{2i})), D_{P_b}(\zeta_{2i+1},T(\zeta_{2i+1})), \\ \frac{D_{Pb}(\zeta_{2i},T(\zeta_{2i+1}))+D_{Pb}(\zeta_{2i+1},S(\zeta_{2i}))}{2K} \end{array}\right\} \\
&\le \max\left\{\begin{array}{c} P_b(\zeta_{2i},\zeta_{2i+1}), P_b(\zeta_{2i+1},\zeta_{2i+2}), \\ \frac{P_b(\zeta_{2i},\zeta_{2i+2})+D_{P_b}(\zeta_{2i+1},\zeta_{2i+1})}{2K} \end{array}\right\} \\
&\le \max\left\{P_b(\zeta_{2i},\zeta_{2i+1}), P_b(\zeta_{2i+1},\zeta_{2i+2})\right\}.
\end{aligned}
$$

If $\max\left\{P_b(\zeta_{2i},\zeta_{2i+1}), P_b(\zeta_{2i+1},\zeta_{2i+2})\right\} = P_b(\zeta_{2i+1},\zeta_{2i+2})$, then from (7) we have

$$
\begin{aligned}
\Lambda(P_b(\zeta_{2i+1},\zeta_{2i+2})) &\le \Upsilon(\Lambda(P_b(\zeta_{2i+1},\zeta_{2i+2}))) \\
&< \Lambda(P_b(\zeta_{2i+1},\zeta_{2i+2})),
\end{aligned}
$$

which is a contradiction. Thus,

$$\max \{P_b(\zeta_{2i}, \zeta_{2i+1}), P_b(\zeta_{2i+1}, \zeta_{2i+2})\} = P_b(\zeta_{2i}, \zeta_{2i+1}).$$

By Equation (7), we get that

$$\Lambda(P_b(\zeta_{2i}, \zeta_{2i+1})) < \Upsilon(\Lambda(P_b(\zeta_{2i}, \zeta_{2i+1}))).$$

This implies that

$$\Lambda(P_b(\zeta_{2n+1}, \zeta_{2n+2})) < \Upsilon(\Lambda(P_b(\zeta_{2n}, \zeta_{2n+1}))), \text{ for all } n \in \mathbb{N} \cup \{0\},$$

which implies

$$\begin{aligned} \Lambda(P_b(\zeta_{2n+1}, \zeta_{2n+2})) &\leq \Upsilon(\Lambda(P_b(\zeta_{2n+1}, \zeta_{2n+2}))) \leq \Upsilon^2(\Lambda(P_b(\zeta_{2n-1}, \zeta_{2n}))) \\ &\leq \ldots \leq \Upsilon^n(\Lambda(P_b(\zeta_0, \zeta_1))). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, we get

$$0 \leq \lim_{n \longrightarrow \infty} \Lambda(P_b(\zeta_{2n+1}, \zeta_{2n+2})) \leq \lim_{n \longrightarrow \infty} \Upsilon^n(\Lambda(P_b(\zeta_0, \zeta_1))) = 0,$$

implies

$$\lim_{n \longrightarrow \infty} \Lambda(P_b(\zeta_{2n+1}, \zeta_{2n+2})) = 0.$$

From (Φ2) and Lemma 2, we get

$$\lim_{n \longrightarrow \infty} P_b(\zeta_{2n+1}, \zeta_{2n+2}) = 0. \tag{8}$$

We claim that that $\{\zeta_n\}$ is Cauchy. We argue by contradiction. Suppose that there exist $\varepsilon > 0$ and a sequence $\left\{\hat{h}_n\right\}_{n=1}^{\infty}$ and $\{\hat{\jmath}_n\}_{n=1}^{\infty}$ such for each $n \in \mathbb{N}$, $\hat{h}_n > \hat{\jmath}_n > n$ with $P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{\jmath}(n)}\right) \geq \varepsilon$, $P_b\left(\zeta_{\hat{h}(n)-1}, \zeta_{\hat{\jmath}(n)}\right) < \varepsilon$. Therefore,

$$\begin{aligned} \varepsilon &\leq P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{\jmath}(n)}\right) \leq K\left[P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)-1}\right) + P_b\left(\zeta_{\hat{h}(n)-1}, \zeta_{\hat{\jmath}(n)}\right)\right] - P_b(\zeta_{\hat{h}(n)-1}, \zeta_{\hat{h}(n)-1}) \\ &\leq K\left[d_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)-1}\right) + P_b\left(\zeta_{\hat{h}(n)-1}, \zeta_{\hat{\jmath}(n)}\right)\right] \\ &< K\varepsilon + KP_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)-1}\right). \end{aligned} \tag{9}$$

Taking $n \to \infty$ in Equation (9), we get

$$\varepsilon < \lim_{n \to \infty} P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{\jmath}(n)}\right) < K\varepsilon. \tag{10}$$

From triangular inequality, we have

$$\begin{aligned} P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{\jmath}(n)}\right) &\leq K[P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{\jmath}(n)}\right)] - P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{h}(n)+1}\right) \\ &\leq K[P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{\jmath}(n)}\right)], \end{aligned} \tag{11}$$

and

$$\begin{aligned} P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{\jmath}(n)}\right) &\leq K[P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{\jmath}(n)}\right)] - P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)}\right) \\ &\leq K[P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{\jmath}(n)}\right)]. \end{aligned} \tag{12}$$

Applying the upper limit when $n \to \infty$ in (2.11) and applying Equation (8) together with Equation (10),

$$\varepsilon \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \leq K\left(\lim_{n\to\infty} \sup K_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)\right).$$

Again, the upper limit in Equation (12) yields that

$$\varepsilon < \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) \leq K\left(\lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right)\right) \leq K.K\varepsilon = K^2\varepsilon.$$

Thus ,

$$\frac{\varepsilon}{K} \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) \leq K^2\varepsilon. \quad (13)$$

Similarly,

$$\frac{\varepsilon}{K} \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)+1}\right) \leq K^2\varepsilon. \quad (14)$$

By triangular inequality, we have

$$\begin{aligned} P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) &\leq K[P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) + P_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)}\right)] - P_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \quad (15) \\ &\leq K[P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) + P_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)}\right)]. \end{aligned}$$

On letting $n \to \infty$ in Equation (15) and using the inequalities in Equations (8) and (13), we get

$$\frac{\varepsilon}{K^2} \leq \lim_{k\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right). \quad (16)$$

Similarly,

$$\lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \leq K^3\varepsilon. \quad (17)$$

From Equations (16) and (17), we get

$$\frac{\varepsilon}{K^2} \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \leq K^3\varepsilon. \quad (18)$$

From Equations (8) and (10), we can choose a positive integer $n_0 \geq 1$ such that for all $n \geq n_0$, from Equation (1), we get,

$$\begin{aligned} 0 &< \Lambda\left(\alpha_K(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)})P_b\left(\zeta_{\hat{h}(n)+2}, \zeta_{\hat{j}(n)+1}\right)\right) \leq \Lambda\left(\alpha_K(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)})H_{P_b}\left(S\left(\zeta_{\hat{h}(n)+1}\right), T\left(\zeta_{\hat{j}(n)}\right)\right)\right) \\ &\leq \psi\left(\phi\left(U_{P_b}\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)\right)\right), \text{ for all } n \geq n_0, \end{aligned}$$

where

$$\begin{aligned} U_{P_b}\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) &= \max\left\{\begin{array}{c} P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right), D_{P_b}\left(\zeta_{\hat{h}(n)+1}, \check{S}\left(\zeta_{\hat{h}(n)+1}\right)\right), D_{P_b}\left(\zeta_{\hat{j}(n)}, T\left(\zeta_{\hat{j}(n)}\right)\right), \\ \frac{D_{P_b}\left(\zeta_{\hat{h}(n)+1}, T\left(\zeta_{\hat{j}(n)}\right)\right) + D_{P_b}\left(\zeta_{\hat{j}(n)}, S\left(\zeta_{\hat{h}(n)+1}\right)\right)}{2K} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right), P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{h}(n)+2}\right), P_b\left(\zeta_{\hat{j}(n)}, \zeta_{\hat{j}(n)+1}\right), \\ \frac{P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) + P_b\left(\zeta_{\hat{j}(n)}, \zeta_{\hat{h}(n)+2}\right)}{2K} \end{array}\right\}. \end{aligned}$$

Taking the limit as $n \to \infty$ and using Equations (8), (10), (13) and (14), we get

$$\frac{\varepsilon}{K} = \max\left\{\frac{\varepsilon}{K}, \frac{K\varepsilon}{4}\right\} \leq \lim_{n\to\infty} \sup U_{P_b}\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) \leq \max\left\{K^2\varepsilon, \frac{K^2\varepsilon}{4}\right\} = K^2\varepsilon.$$

From Equation (16), (Φ2), and by Lemma 7 since $\alpha_K(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}) \geq K^2$, we get

$$\begin{aligned}\Lambda(K^2\varepsilon) &\leq \Lambda\left(\alpha_K(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}) \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+2}, \zeta_{\hat{j}(n)+1}\right)\right) \leq \lim_{n\to\infty} \Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)\right)\right) \\ &= \Upsilon\left(\Lambda(K^2\varepsilon)\right) < \Lambda(K^2\varepsilon).\end{aligned}$$

This is a contradiction. Therefore, $\{\zeta_n\}$ is Cauchy. The α_K-completeness of the partial b-metric space (ω, P_b, K) implies the α_K-completeness of the b-metric space (ω, d_{P_b}). Thus, there exists $\zeta^* \in \omega$ so that

$$\lim_{n\to\infty} d_{P_b}(\zeta_n, \zeta^*) = 0. \tag{19}$$

By Lemma 1,

$$\lim_{n\to\infty} P_b(\zeta_n, \zeta^*) = \lim_{n\to\infty} P_b(\zeta^*, \zeta^*) = \lim_{n\to\infty} P_b(\zeta_n, \zeta_m). \tag{20}$$

Since

$$d_b(\zeta, \eta) = 2P_b(\zeta, \eta) - P_b(\zeta, \zeta) - P_b(\eta, \eta). \tag{21}$$

Thus, from Equation (8) and axiom (P_b2) with Equation (19), we have

$$\lim_{n\to\infty} P_b(\zeta_n, \zeta_m) = 0. \tag{22}$$

Combining Equations (20) and (22)), we get

$$\lim_{n\to\infty} P_b(\zeta_n, \zeta^*) = \lim_{n\to\infty} P_b(\zeta^*, \zeta^*) = \lim_{n\to\infty} P_b(\zeta_n, \zeta_m) = 0.$$

Hence,

$$\lim_{n\to\infty} P_b(\zeta_n, \zeta^*) = 0,$$

which implies,

$$\lim_{n\to\infty} P_b(\zeta_{2i+1}, \zeta^*) = \lim_{n\to\infty} P_b(\zeta_{2i+2}, \zeta^*) = 0.$$

Since S is an α_K^*-P_b-continuous multivalued mapping, $\lim_{n\to\infty} H_{P_b}(S(\zeta_{2i+1}), S(\zeta^*)) = 0$. Thus,

$$D_{P_b}(\zeta^*, S(\zeta^*)) = \lim_{i\to\infty} D_{P_b}\left(\zeta_{2i+2}, \check{S}(\zeta^*)\right) \leq \lim_{i\to\infty} H_{P_b}(S(\zeta_{2i+1}), S(\zeta^*)) = 0,$$

and so, $\zeta^* \in S(\zeta^*)$ and, similarly, $\zeta^* \in T(\zeta^*)$. Therefore, S and T have a common fixed point $\zeta^* \in \omega$.
(b) From Case (a), we construct a sequence $\{\zeta_n\}$ in ω defined by $\zeta_{2i+1} \in S(\zeta_{2i})$ and $\zeta_{2i+2} \in T(\zeta_{2i+1})$ with $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$, for each $n \in \mathbb{N} \cup \{0\}$. In addition, $\{\zeta_n\}$ converges to $\zeta^* \in \omega$, and there exists a subsequence $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ for each k. Thus,

$$\begin{aligned}\Lambda\left(D_{P_b}\left(\zeta_{2n(k)+1}, T(\zeta^*)\right)\right) &\leq \Lambda\left(H_{Pb}\left(S\left(\zeta_{2n(k)}\right), T(\zeta^*)\right)\right) \\ &\leq \Lambda\left(\alpha_K(\zeta_{2n(k)}, \zeta^*) H_{Pb}\left(S\left(\zeta_{2n(k)}\right), T(\zeta^*)\right)\right) \\ &\leq \Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta_{2n(k)}, \zeta^*\right)\right)\right),\end{aligned} \tag{23}$$

where

$$\begin{aligned}U_{P_b}\left(\zeta_{2n(k)}, \zeta^*\right) &= \max\left\{\begin{array}{c} P_b\left(\zeta_{2n(k)}, \zeta^*\right), D_{P_b}\left(\zeta_{2n(k)}, S\left(\zeta_{2n(k)}\right)\right), D_{P_b}\left(\zeta^*, \check{T}(\zeta^*)\right), \\ \frac{D_{P_b}\left(\zeta_{2n(k)}, T(\zeta^*)\right) + D_{P_b}\left(\zeta^*, S\left(\zeta_{2n(k)}\right)\right)}{2K} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} P_b\left(\zeta_{2n(k)}, \zeta^*\right), P_b\left(\zeta_{2n(k)}, \zeta_{2n(k)+1}\right), D_{P_b}\left(\zeta^*, T(\zeta^*)\right), \\ \frac{D_{P_b}\left(\zeta_{2n(k)}, T(\zeta^*)\right) + D_{P_b}\left(\zeta^*, S\left(\zeta_{2n(k)}\right)\right)}{2K} \end{array}\right\}.\end{aligned}$$

Since

$$\lim_{k\longrightarrow\infty}\sup\frac{D_{P_b}\left(\zeta_{2n(k)},T\left(\zeta^*\right)\right)+D_{P_b}\left(\zeta^*,S\left(\zeta_{2n(k)}\right)\right)}{2K}\leq\frac{D_{P_b}\left(\zeta^*,T\left(\zeta^*\right)\right)+P_b\left(\zeta^*,\zeta^*\right)}{2K},$$

by letting $k\longrightarrow\infty$, we have $\lim_{k\longrightarrow\infty}U_{P_b}\left(\zeta_{2n(k)},\zeta^*\right)=D_{P_b}\left(\zeta^*,T\left(\zeta^*\right)\right)$. Suppose that $D_{P_b}\left(\zeta^*,T\left(\zeta^*\right)\right)>0$. From Equation (23),

$$\Lambda\left(P_b\left(\zeta_{2n(k)+1},T\left(\zeta^*\right)\right)\right)\leq\Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta_{2n(k)},\zeta^*\right)\right)\right).$$

Letting $k\longrightarrow\infty$ in the above inequality and by continuity of Λ and Υ, we obtain that

$$\Lambda\left(P_b\left(\zeta^*,T\left(\zeta^*\right)\right)\right)\leq\Upsilon\left(\Lambda\left(P_b\left(\zeta^*,T\left(\zeta^*\right)\right)\right)\right)<\Lambda\left(P_b\left(\zeta^*,T\left(\zeta^*\right)\right)\right),$$

a contradiction. Hence, $P_b\left(\zeta^*,T\left(\zeta^*\right)\right)=0$, and, due to (P_b1) and (P_b2), we obtain, $\zeta^*\in T\left(\zeta^*\right)$. Similarly, we can show that $\zeta^*\in S\left(\zeta^*\right)$. Thus, S and T have a common fixed point $\zeta^*\in\omega$. □

Corollary 1. *Let (ω,P_b,K) be a partial b-metric space. Given $\alpha_K:\omega\times\omega\longrightarrow[0,\infty)$ and $S:\omega\longrightarrow CB_{P_b}(\omega)$. Suppose that:*
(i) (ω,P_b,K) is an α_K-complete partial b-metric space.
(ii) S is a generalized $(\alpha_K^*,\Upsilon,\Lambda)$-contraction multivalued mapping, that is, if there exist a comparison function Υ and and a function $\Lambda\in\Phi$ such that, for $\zeta,\eta\in\omega$, $\alpha_K(\zeta,\eta)\geq K^2$,

$$H_{P_b}\left(S\left(\zeta\right),S\left(\eta\right)\right)>0\Longrightarrow\Lambda\left(\alpha_K(\zeta,\eta)H_{P_b}\left(S\left(\zeta\right),S\left(\eta\right)\right)\right)\leq\Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta,\eta\right)\right)\right),$$

where

$$U_{P_b}\left(\zeta,\eta\right)=\max\left\{P_b(\zeta,\eta),D_{P_b}\left(\zeta,S\left(\zeta\right)\right),D_{P_b}\left(\eta,S\left(\eta\right)\right),\frac{D_{P_b}\left(\zeta,S\left(\eta\right)\right)+D_{P_b}\left(\eta,S\left(\zeta\right)\right)}{2K}\right\}.$$

(iii) S is triangular α_K^*-orbital admissible.
(iv) There exists $\zeta_0\in\omega$ so that $\alpha_K^*\left(\zeta_0,S\zeta_0\right)\geq K^2$.
(v)
- *(a) S is an α_K^*-P_b-continuous multivalued mapping.*
- *(b) If $\{\zeta_n\}$ is a sequence in ω such that $\alpha_K\left(\zeta_n,\zeta_{n+1}\right)\geq K^2$ for all $n\in\mathbb{N}$ and $\zeta_n\longrightarrow\zeta^*\in\omega$ as $n\longrightarrow\infty$, then there exists $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\alpha_K\left(\zeta_{n(k)},\zeta^*\right)\geq K^2$ for all $k\in\mathbb{N}$.*

If Υ is continuous, then S has a fixed point $\zeta^*\in\omega$.

Proof. Set $S=T$ in Theorem 5. □

Example 3. *Let $\omega=[0,1]$. Take $P_b:\omega\times\omega\to[0,+\infty)$ by $P_b(\zeta,\eta)=|\zeta-\eta|^2+\left(\max\{\zeta,\eta\}\right)^2$, for all $\zeta,\eta\in\omega$. Clearly, (ω,P_b,K) is a complete partial b-metric spaces with $K=4$. Define $\Lambda:(0,\infty)\longrightarrow(0,\infty)$ by $\Lambda\left(t\right)=te^t$, for all $t>0$. Then, $\Lambda\in\Phi$. In addition, define $\Upsilon:(0,\infty)\longrightarrow(0,\infty)$ by $\Upsilon\left(t\right)=\frac{190t}{200}$, for each $t>0$. Then, Υ is a continuous comparison function. Define the mappings $S,T:\omega\longrightarrow CB_{P_b}\left(\omega\right)$ by*

$$S\left(\zeta\right)=\begin{cases}\left\{\frac{8\zeta}{1000}\right\}, & \text{if } 0\leq\zeta\leq\frac{1}{2}\\ \{1\}, & \text{if } \frac{1}{2}<\zeta\leq1.\end{cases}\quad\text{and } T\left(\zeta\right)=\{0\}\text{, for all } \zeta\in\omega.$$

In addition, we define the function $\alpha_K : \omega \times \omega \longrightarrow [0,\infty)$ *by*

$$\alpha_K(\zeta,\eta) = \begin{cases} K^2, & \text{if } 0 \leq \zeta,\eta \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

If the sequence $\{\zeta_n\}$ *is Cauchy with* $\alpha_K(\zeta_n,\zeta_{n+1}) \geq K^2$ *for each integer n, then* $\{\zeta_n\} \subseteq \left[0,\frac{1}{2}\right]$. *Since* $\left(\left[0,\frac{1}{2}\right], P_b, K\right)$ *is a complete partial b-metric space,* $\{\zeta_n\}$ *converges in* $\left[0,\frac{1}{2}\right] \subseteq \omega$. *Thus* (ω, P_b, K) *is an* α_K*-complete partial b-metric space. Let* $\alpha_K^*(\zeta, S\zeta) \geq K^2$ *and* $\alpha_K^*(\zeta, T\zeta) \geq K^2$, *thus* $\zeta \in \left[0,\frac{1}{2}\right]$ *and* $S\zeta$, $T\zeta \in \left[0,\frac{1}{2}\right]$ *and so* $S^2\zeta = S(S\zeta), T^2\zeta = T(T\zeta) \in \left[0,\frac{1}{2}\right]$, *then* $\alpha_K^*(S\zeta, T^2\zeta) \geq K^2$ *and* $\alpha_K^*(T\zeta, S^2\zeta) \geq K^2$. *Thus,* (S,T) *is* α_K^**-orbital admissible. Let* $\zeta,\eta \in \omega$ *be such that* $\alpha_K(\zeta,\eta) \geq K^2$, $\alpha_K^*(\eta, S\eta) \geq K^2$ *and* $\alpha_K^*(\eta, T\eta) \geq K^2$. *Clearly,* $\alpha_K^*(\zeta, S\eta) \geq K^2$ *and* $\alpha_K^*(\zeta, T\eta) \geq K^2$. *Therefore,* (S,T) *is triangular* α_K^**-orbital admissible. Let* $\{\zeta_n\}$ *be a Cauchy sequence so that* $\lim_{n\longrightarrow\infty} P_b(\zeta_n,\zeta) = 0$ *and* $\alpha_K(\zeta_n,\zeta_{n+1}) \geq K^2$ *for each* $n \in \mathbb{N}$. *Then,* $\{\zeta_n\} \subseteq \left[0,\frac{1}{2}\right]$ *for each* $n \in \mathbb{N}$. *Hence,* $\lim_{n\longrightarrow\infty} H_{P_b}(T\zeta_n, T\zeta) = \lim_{n\longrightarrow\infty} H_{P_b}\left(\left\{\frac{8\zeta_n}{1000}\right\}, T\zeta\right) = H_{P_b}\left(\left\{\frac{8\zeta}{1000}\right\}, T\zeta\right) = 0$. *Hence, T is an* α_K^**-*P_b*−continuous multivalued mapping. Similarly, we can show that S is an* α_K^**-*P_b*−continuous multivalued mapping. Let* $\zeta_0 = \frac{1}{4}$. *Then,*

$$\alpha_K^*\left(\frac{1}{4}, S\left(\frac{1}{4}\right)\right) = \alpha_K\left(\frac{1}{4}, 0\right) \geq K^2.$$

Let $\zeta,\eta \in \omega$ *be such that* $\alpha_K(\zeta,\eta) \geq K^2$. *Then,* $\zeta,\eta \in \left[0,\frac{1}{2}\right]$. *Suppose, without any loss of generality, that all* ζ,η *are nonzero and* $\zeta < \eta$. *Then,*

$$\begin{aligned}
\Lambda\left(\alpha_K(\zeta,\eta) H_{P_b}\left(\check{S}(\zeta), T(\eta)\right)\right) &= \Lambda\left(K^2 H_{P_b}\left(\left\{\frac{8\zeta}{1000}\right\}, \{0\}\right)\right) \\
&= \Lambda\left(16\left[\left|\frac{8\zeta}{1000}\right|^2 + \left(\frac{8\zeta}{1000}\right)^2\right]\right) \\
&= \Lambda\left(16\left[\left|\frac{8\zeta}{1000}\right|^2 + \left(\frac{8\zeta}{1000}\right)^2\right]\right) \\
&= \Lambda\left(\left(\frac{32}{1000}\right)^2\left[|\zeta|^2 + (\zeta)^2\right]\right) \\
&= \left(\frac{32}{1000}\right)^2\left[|\zeta|^2 + (\zeta)^2\right] e^{\left(\left(\frac{32}{1000}\right)^2\left[|\zeta|^2 + (\zeta)^2\right]\right)} \\
&\leq \frac{190}{200}\left[|\zeta-\eta|^2 + (\max\{\zeta,\eta\})^2\right] e^{\left(\frac{190}{300}\left[|\zeta-\eta|^2 + (\max\{\zeta,\eta\})^2\right]\right)} \\
&= \frac{190}{200}\left[P_b(\zeta,\eta)\right] e^{\left(\frac{190}{300}\left[P_b(\zeta,\eta)\right]\right)} \\
&\leq \frac{190}{200} U_{P_b}(\zeta,\eta) e^{U_b(\zeta,\eta)} \\
&= \frac{190}{200}\Lambda\left(U_{P_b}(\zeta,\eta)\right) = \Upsilon\left(\Lambda\left(U_b(\zeta,\eta)\right)\right).
\end{aligned}$$

Hence, all the hypotheses of Theorem 5 hold, and so S and T have a common fixed point.

Definition 19. *Let* (ω, P_b, K) *be a partial b-metric space. Given* $\alpha_K : \omega \times \omega \longrightarrow [0,\infty)$ *and* $S, T : \omega \longrightarrow CB_{P_b}(\omega)$. (S,T) *is called an* $(\alpha_K^*, \Upsilon, \Lambda)$*-contraction multivalued pair of mappings if there exist a comparison function* Υ *and a function* $\Lambda \in \Phi$ *such that for* $\zeta,\eta \in \omega$, $\alpha_K(\zeta,\eta) \geq K^2$,

$$H_{P_b}(S(\zeta), T(\eta)) > 0 \Longrightarrow \Lambda\left(\alpha_K(\zeta,\eta) H_{P_b}(S(\zeta), T(\eta))\right) \leq \Upsilon\left(\Lambda\left(P_b(\zeta,\eta)\right)\right).$$

Theorem 6. *Let (ω, P_b, K) be a partial b-metric space. Given $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ and $S, T : \omega \longrightarrow CB_{P_b}(\omega)$. Suppose that:*
(i) (ω, P_b, K) is an α_K-complete partial b-metric space.
(ii) (S, T) is an $(\alpha_K^*, \Upsilon, \Lambda)$-contraction multivalued pair of mappings.
(iii) (S, T) is triangular α_K^*-orbital admissible.
(iv) There exists $\zeta_0 \in \zeta$ such that $\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2$.
(v)

(a) S and T are α_K^*-P_b-continuous.

(b) If $\{\zeta_n\}$ is a sequence such that $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ and $\zeta_n \longrightarrow \zeta^* \in \omega$ as $n \longrightarrow \infty$, then there exists $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ for each $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \zeta$.

Corollary 2. *Let $\left(\acute{Y}, d_{P_b}, K\right)$ be a b-metric space. Given $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ and $S, T : \acute{Y} \longrightarrow CB_b\left(\acute{Y}\right)$. Suppose that:*
(i) $\left(\acute{Y}, d_{P_b}, K\right)$ is an α_S-complete b-metric space.
(ii) (S, T) is an $(\alpha_K^*, \Upsilon, \Lambda)$-contraction multivalued pair of mappings with respect to $\acute{Y}$.
(iii) (S, T) is triangular α_K^*-orbital admissible.
(iv) There exists $\acute{y}_0 \in \acute{Y}$ such that $\alpha_K^*(\acute{y}_0, S\acute{y}_0) \geq K^2$.
(v)

(a) S and T are α_K^*-d_{P_b}-continuous.

(b) If $\{\acute{y}_n\}$ is a sequence in $\acute{Y}$ such that $\alpha_K(\acute{y}_n, \acute{y}_{n+1}) \geq K^2$ for all $n \in \mathbb{N}$ and $\acute{y}_n \longrightarrow \acute{y}^* \in \acute{Y}$ as $n \longrightarrow \infty$, then there exists $\left\{\acute{y}_{n(k)}\right\}$ of $\{\acute{y}_n\}$ such that $\alpha_K\left(\acute{y}_{n(k)}, \acute{y}^*\right) \geq K^2$ for all $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\acute{y}^* \in \acute{Y}$.

Proof. Set $P_b(\zeta, \eta) = 0$, for each $\zeta \in \omega$ in Theorem 5. □

Theorem 7. *Let (ω, P_b, K) be a partial b-metric space. Given $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ and $S, T : \omega \longrightarrow CB_{P_b}(\omega)$. Suppose that:*
(i) (ω, P_b, K) is an α_K-complete partial b-metric space.
(ii) If there exists $\theta \in \tilde{\Theta}$ and $k \in (0, 1)$ such that, for all $\zeta, \eta \in \omega$, $\alpha_K(\zeta, \eta) \geq K^2$,

$$H_{P_b}(S(\zeta), T(\eta)) > 0 \Longrightarrow \theta\left(\alpha_K(\zeta, \eta) H_{P_b}(S(\zeta), T(\eta))\right) \leq \left[\theta\left(U_{P_b}(\zeta, \eta)\right)\right]^k,$$

and $U_{P_b}(\zeta, \eta)$ is defined as in Equation (2);
(iii) (S, T) is triangular α_K^*-orbital admissible.
(iv) There exists $\zeta_0 \in \omega$ such that $\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2$.
(v)

(a) S and T are α_K^*-P_b-continuous.

(b) If $\{\zeta_n\}$ is a sequence in ω such that $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ and $\zeta_n \longrightarrow \zeta^* \in \omega$ as $n \longrightarrow \infty$, then there exists $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ for each $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \omega$.

Proof. It suffices to take in Theorem 5, $\Upsilon(t) := (\ln k) t$ and $\Lambda(t) = \ln \theta : (0, \infty) \longrightarrow (0, \infty)$. □

Theorem 8. *Let (ω, P_b, K) be a partial b-metric space. Given $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ and $S, T : \omega \longrightarrow CB_{P_b}(\omega)$. Assume that:*

(i) (ω, P_b, K) *is an* α_K*-complete partial b-metric space.*
(ii) There exist $F \in \mathcal{F}$ *and* $\tau > 0$ *such that, for all* $\zeta, \eta \in \omega$, $\alpha_K(\zeta, \eta) \geq K^2$,

$$H_{P_b}(S(\zeta), T(\eta)) > 0 \Longrightarrow \tau + F\left(\alpha_K(\zeta, \eta) H_{P_b}(S(\zeta), T(\eta))\right) \leq F\left(U_{P_b}(\zeta, \eta)\right),$$

and $U_{P_b}(\zeta, \eta)$ *is defined as in Equation (2).*
(iii) (S, T) *is triangular* α_K^**-orbital admissible.*
(iv) There exists $\zeta_0 \in \omega$ *such that* $\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2$.
(v)

- (a) S *and* T *are* α_K^**-*P_b*-continuous.*
- (b) *If* $\{\zeta_n\}$ *is a sequence in* ω *such that* $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ *and* $\zeta_n \longrightarrow \zeta^* \in \omega$ *as* $n \longrightarrow \infty$*, then there exists* $\left\{\zeta_{n(k)}\right\}$ *of* $\{\zeta_n\}$ *such that* $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ *for each* $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \omega$.

Proof. The result follows from Theorem 5 by taking $\Upsilon(t) = e^{-\tau}t$ and $\Lambda(t) = e^F : (0, \infty) \longrightarrow (0, \infty)$. □

Theorem 9. *Let* (ω, P_b, K) *be a partial b-metric space. Given* $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ *and* $S, T : \omega \longrightarrow CB_{P_b}(\omega)$*. Assume that:*
(i) (ω, P_b, K) *is an* α_K*-complete partial b-metric space.*
(ii) If for all $\zeta, \eta \in \omega$, $\alpha_K(\zeta, \eta) \geq K^2$,

$$\alpha_K(\zeta, \eta) H_{P_b}(S(\zeta), T(\eta)) \leq \beta\left(U_{P_b}(\zeta, \eta)\right) U_{P_b}(\zeta, \eta),$$

$U_{P_b}(\zeta, \eta)$ *is defined as in (2) and* $\beta : [0, \infty) \to [0, \infty)$ *is such that* $\lim_{r \longrightarrow t^+} \beta(r) < 1$ *for each* $t \in (0, \infty)$.
(iii) (S, T) *is triangular* α_K^**-orbital admissible.*
(iv) There exists $\zeta_0 \in \omega$ *such that* $\alpha_K^*(\zeta_0, S\zeta_0) \geq K^2$.
(v)

- (a) S *and* T *are* α_K^**-*P_b*-continuous.*
- (b) *If* $\{\zeta_n\}$ *is a sequence in* ω *such that* $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ *and* $\zeta_n \longrightarrow \zeta^* \in \omega$ *as* $n \longrightarrow \infty$*, then there exists* $\left\{\zeta_{n(k)}\right\}$ *of* $\{\zeta_n\}$ *such that* $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ *for each* $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \omega$.

Proof. It follows from Theorem 5 by taking $\psi(t) := \beta(t)\, t$ and $\phi(t) = t : (0, \infty) \longrightarrow (0, \infty)$. □

3. Some Consequences

In this section, we obtain some fixed point results for singlevalued mappings when applying the corresponding results of Section 2.

Definition 20. *Let* (ω, P_b, K) *be a partial b-metric space. Given* $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ *and* $S, T : \omega \longrightarrow \omega$ *are two self-mappings.* (S, T) *is called a generalized* $(\alpha_K, \Upsilon, \Lambda)$*-contraction pair of mappings if there exist a comparison function* Υ *and a function* $\Lambda \in \Phi$ *such that for* $\zeta, \eta \in \omega$, $\alpha_K(\zeta, \eta) \geq K^2$,

$$\Lambda\left(\alpha_K(\zeta, \eta) P_b(S(\zeta), T(\eta))\right) \leq \Upsilon\left(\Lambda\left(U_{P_b}(\zeta, \eta)\right)\right),$$

where

$$U_{P_b}(\zeta, \eta) = \max\left\{P_b(\zeta, \eta), P_b(\zeta, S(\zeta)), P_b(\eta, T(\eta)), \frac{P_b(\zeta, T(\eta)) + P_b(\eta, S(\zeta))}{2K}\right\}.$$

Theorem 10. *Let (ω, P_b, K) be a partial b-metric space. Given $\alpha_K : \omega \times \omega \longrightarrow [0, \infty)$ and $S, T : \omega \longrightarrow \omega$. Assume that:*
(i) (ω, P_b, K) is an α_K-complete partial b-metric space.
(ii) (S, T) is an $(\alpha_K^*, \Upsilon, \Lambda)$-contraction pair of mappings.
(iii) (S, T) is triangular α_K-orbital admissible.
(iv) There exists $\zeta_0 \in \omega$ such that $\alpha_K(\zeta_0, S\zeta_0) \geq K^2$.
(v)

(a) S and T are α_K-P_b-continuous.
(b) If $\{\zeta_n\}$ is a sequence in ω such that $\alpha_K(\zeta_n, \zeta_{n+1}) \geq K^2$ and $\zeta_n \longrightarrow \zeta^* \in \omega$ as $n \longrightarrow \infty$, then there exists $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\alpha_K\left(\zeta_{n(k)}, \zeta^*\right) \geq K^2$ for each $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \omega$.

Corollary 3. *Let $(\omega, \preceq, P_b, K)$ be an ordered complete partial b-metric space. Assume that $S, T : \omega \longrightarrow \omega$ are weakly increasing mappings [that is, $S(\zeta) \preceq TS(\zeta)$ and $T(\eta) \preceq ST(\eta)$ hold for all $\zeta, \eta \in \omega$] and satisfy the following conditions:*
(i) If there exist a comparison function Υ and $\Lambda \in \Phi$ such that for all comparable $\zeta, \eta \in \omega$, (i.e., $\zeta \preceq \eta$ or $\eta \preceq \zeta$),

$$\Lambda\left(P_b\left(S\left(\zeta\right), T\left(\eta\right)\right)\right) \leq \Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta, \eta\right)\right)\right),$$

where

$$U_{P_b}\left(\zeta, \eta\right) = \max\left\{P_b(\zeta, \eta), P_b\left(\zeta, S\left(\zeta\right)\right), P_b\left(\eta, T\left(\eta\right)\right), \frac{P_b\left(\zeta, T\left(\eta\right)\right) + P_b\left(\eta, S\left(\zeta\right)\right)}{2K}\right\}.$$

(ii) There exists $\zeta_0 \in \omega$ such that $\zeta_0 \preceq S\zeta_0$.
(iii)

(a) Either S or T is continuous.
(b) If $\{\zeta_n\}$ is a nondecreasing sequence in ω such that $\zeta_n \longrightarrow \zeta^* \in \omega$ as $n \longrightarrow \infty$, then there exists $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\zeta_{n(k)} \preceq \zeta^*$ for each $k \in \mathbb{N}$.

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \omega$.

Proof. Define the relation $\preceq$ on ω by

$$\alpha_K(\zeta, \eta) = \begin{cases} K^2, & \zeta \preceq \eta \text{ or } \eta \preceq \zeta, \\ 0, & \text{otherwise.} \end{cases}$$

The proof follows from the proof of Theorem 5. □

Jachymski [45] initiated the graph structure on metric spaces.

Definition 21. *[45] $S : \omega \rightarrow \omega$ is a Banach G-contraction or simply a G-contraction if S preserves edges of G, i.e.,*

$$\zeta, \eta \in \omega,\ (\zeta, \eta) \in E(G) \text{ implies } \left(S\left(\zeta\right), S\left(\eta\right)\right) \in E(G)$$

and there exists $k \in (0, 1)$ such that

$$\zeta, \eta \in \omega, (\zeta, \eta) \in E(G) \text{ implies } d(S\left(\zeta\right), S\left(\eta\right))) \leq kd(\zeta, \eta)).$$

Definition 22. *[45] A mapping $S : \omega \rightarrow \omega$ is called G-continuous, if given $\zeta \in \omega$ and sequence $\{\zeta_n\}$ such that $\zeta_n \rightarrow \zeta$, as $n \rightarrow \infty$ and $(\zeta_n, \zeta_{n+1}) \in E(G)$ for each integer, implies $S\left(\zeta_n\right) \rightarrow S\left(\zeta\right)$.*

Corollary 4. *Let (ω, G, P_b, K) be a complete partial b-metric space endowed with a graph G. Assume $S, T : \omega \longrightarrow \omega$ satisfy the following conditions:*
(i) If there exist a comparison function Υ and $\Lambda \in \Phi$ such that, for all $\zeta, \eta \in \omega$, with $(\zeta, \eta) \in E(G)$,

$$\Lambda\left(P_b\left(S\left(\zeta\right), T\left(\eta\right)\right)\right) \leq \Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta, \eta\right)\right)\right),$$

where

$$U_{P_b}\left(\zeta, \eta\right) = \max\left\{P_b(\zeta, \eta), P_b\left(\zeta, S\left(\zeta\right)\right), P_b\left(\eta, T\left(\eta\right)\right), \frac{P_b\left(\zeta, T\left(\eta\right)\right) + P_b\left(\eta, S\left(\zeta\right)\right)}{2K}\right\}.$$

(ii) For $\zeta, \eta \in \omega$, $(\zeta, \eta) \in E(G)$ implies $(S(\zeta), TS(\zeta)) \in E(G))$ and $(T(\eta), ST(\eta)) \in E(G))$.
(iii) There exists $\zeta_0 \in \omega$ such that $(\zeta_0, S(\zeta_0)) \in E(G)$.
(iv)

- *(a) Either S or T is G-continuous.*
- *(b) If $\{\zeta_n\}$ is a nondecreasing sequence in ω such that $\zeta_n \longrightarrow \zeta^* \in \omega$ as $n \longrightarrow \infty$, then there exists $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ such that $\left(\zeta_{n(k)}, \zeta^*\right) \in E(G)$ for each $k \in \mathbb{N}$.*

If Υ is continuous, then S and T have a common fixed point $\zeta^* \in \omega$.

Proof. Define

$$\alpha_K(\zeta, \eta) = \begin{cases} K^2, & (\zeta, \eta) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The proof follows from the proof of Theorem 5. □

Corollary 5. *Let (ω, P_b, K) be a complete partial b-metric space. Let $S, T : \omega \longrightarrow \omega$ be two self-mappings such that:*
(i) (S, T) is a generalized (Υ, Λ)-contraction pair of mappings, i.e., there exist a comparison function Υ and a function $\Lambda \in \Phi$ such that for $\zeta, \eta \in \omega$,

$$\Lambda\left(\alpha_K(\zeta, \eta) P_b\left(S\left(\zeta\right), T\left(\eta\right)\right)\right) \leq \Upsilon\left(\Lambda\left(U_{P_b}\left(\zeta, \eta\right)\right)\right).$$

(ii) S and T are P_b-continuous.

If Υ is continuous, there exists a common fixed point, e.g. $\zeta^* \in \omega$.

Proof. It follows as the same lines in proof of Theorem 5. □

4. Applications

4.1. Application to Nonlinear Matrix Equations

Denote by $J(n)$ the set of all $n \times n$ Hermitian matrices, $Q(n)$ by the set of all $n \times n$ Hermitian positive definite matrices and $S(n)$ by the set of all $n \times n$ positive semi-definite matrices. $A > 0$ (respectively, $A \geq 0$) means $A \in Q(n)$ (respectively, $A \in S(n)$). The spectral norm is denoted by $\|.\|$, i.e.,

$$\|E\| = \sqrt{\mu^+\left(E^* E\right)},$$

where $\mu^+\left(E^* E\right)$ is the greatest eigenvalue of the matrix $E^* E$. The Ky Fan norm is given as

$$\|E\|_1 = \sum_{i=1}^{n} S_i(E),$$

where $\{S_1(E), S_2(E), \cdots, S_n(E)\}$ is the set of the singular values of E. Moreover,

$$\|B\|_1 = tr\left((B^*B)^{\frac{1}{2}}\right),$$

The set $(J(n), P_b)$ is a complete partial b-metric space, where

$$P_b(A,B) = \|B-A\|_1^2 + L = (tr(B-A))^2 + L,\ L > 0.$$

Take the system of nonlinear matrix equations:

$$\begin{cases} X = \pi + \sum\limits_{i=1}^{m} E_i^*\gamma(X)E_i \\ X = \pi + \sum\limits_{i=1}^{m} E_i^*\delta(X)E_i, \end{cases} \tag{24}$$

where π is a positive definite matrix, E_1,..., E_m are $n \times n$ matrices and γ, δ are mappings from $J(n)$ to $J(n)$ which maps $Q(n)$ into $Q(n)$.

Theorem 11. *Let $\pi \in Q(n)$ and $\gamma, \delta : J(n) \longrightarrow J(n)$ be a mapping which maps $Q(n)$ into $Q(n)$. Suppose that there exists $M > 0$ such that $\sum\limits_{i=1}^{m} E_iE_i^* < MI_n$. Assume that either $\sum\limits_{i=1}^{m} E_i^*\gamma(\pi)E_i > 0$, or $\sum\limits_{i=1}^{m} E_i^*\delta(\pi)E_i > 0$ such that for all X, Y,*

$$\|\gamma(X) - \delta(Y)\|_1^2 \leq \frac{1}{M^2}\frac{U_{P_b}(X,Y)}{U_{P_b}(X,Y)+1} - \sigma,\ \sigma > 0,$$

where

$$U_{P_b}(X,Y) = \max\left\{P_b(X,Y), P_b(X,\Gamma X), P_b(Y,\Phi Y), \frac{1}{2K}\left[P_b(X,\Phi Y) + P_b(Y,\Gamma X)\right]\right\}.$$

Then, the matrix in Equation (24) has a solution in $Q(n)$.

Proof. Define $\Gamma, \Phi : J(n) \longrightarrow J(n)$, $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ and $\Upsilon : (0,\infty) \longrightarrow (0,\infty)$ by

$$\Gamma(X) = \pi + \sum_{i=1}^{m} E_i^*\gamma(X)E_i, \text{ and } \Phi(X) = \pi + \sum_{i=1}^{m} E_i^*\delta(X)E_i,$$

$$\Lambda(t) = t, t > 0 \text{ and } \Upsilon(t) = \frac{t}{t+1}, t > 0, \text{ respectively.}$$

Then, a common fixed point of Γ and Φ is a solution of Equation (24). Let $X, Y \in J(n)$ with $X \neq Y$. Then, for $P_b(X,Y) > 0$, we have

$$\begin{aligned}
\|\Phi(Y)-\Gamma(X)\|_1 &= tr\left(\Phi(Y)-\Gamma(X)\right)= \\
&= \sum_{i=1}^{m} tr\left(E_iE_i^*\left(\Phi(Y)-\Gamma(X)\right)\right) \\
&= tr\left(\left(\sum_{i=1}^{m} E_iE_i^*\right)\left(\Phi(Y)-\Gamma(X)\right)\right) \\
&\leq \left\|\sum_{i=1}^{m} E_iE_i^*\right\| \|\Phi(Y)-\Gamma(X)\|_1 \\
&\leq \frac{\left\|\sum_{i=1}^{m} E_iE_i^*\right\|}{M}\sqrt{\frac{U_{P_b}(X,Y)}{U_{P_b}(X,Y)+1}-\sigma} \\
&< \sqrt{\frac{U_{P_b}(X,Y)}{U_{P_b}(X,Y)+1}-\sigma},
\end{aligned}$$

and so

$$\|\Phi(Y)-\Gamma(X)\|_1^2 < \frac{U_{P_b}(X,Y)}{U_{P_b}(X,Y)+1}-\sigma.$$

This implies,

$$d(\Gamma(X),\Phi(Y)) < \frac{U_{P_b}(X,Y)}{U_{P_b}(X,Y)+1},$$

which implies

$$\Lambda\left(d(\Gamma(X),\Phi(Y))\right) \leq \frac{\Lambda\left(U_{P_b}(X,Y)\right)}{\Lambda\left(U_{P_b}(X,Y)\right)+1}.$$

Consequently,

$$\Lambda\left(d(\Gamma(X),\Phi(Y))\right) \leq \Upsilon\left(\Lambda\left(U_{P_b}(X,Y)\right)\right).$$

Therefore, all conditions of Corollary 5 immediately hold. Thus, Γ and Φ have a common fixed point and hence the system in Equation (24) of matrix equations has a solution in $Q(n)$. □

Example 4. *Consider the system of nonlinear matrix equations:*

$$\begin{cases} X = \pi + \sum_{i=1}^{2} E_i^*\gamma(X)E_i \\ X = \pi + \sum_{i=1}^{2} E_i^*\delta(X)E_i \end{cases},$$

where π, E_1 and E_2 are given by,

$$\pi = \begin{pmatrix} 0.1 & 0.01 & 0.01 \\ 0.01 & 0.1 & 0.01 \\ 0.01 & 0.01 & 0.1 \end{pmatrix},\ E_1 = \begin{pmatrix} 0.4 & 0.01 & 0.01 \\ 0.01 & 0.4 & 0.01 \\ 0.01 & 0.01 & 0.4 \end{pmatrix}$$

$$\text{and } E_2 = \begin{pmatrix} 0.6 & 0.01 & 0.01 \\ 0.01 & 0.6 & 0.01 \\ 0.01 & 0.01 & 0.6 \end{pmatrix}.$$

Define γ and $\delta : J(3) \longrightarrow J(3)$ by

$$\gamma(X) = \frac{X}{2} \text{ and } \delta(X) = \frac{X}{3}.$$

Define Γ *and* $\Phi : J(3) \longrightarrow J(3)$, *by* $\Gamma(X) = \pi + \sum_{i=1}^{2} E_i^* \gamma(X) E_i$ *and* $\Phi(X) = \pi + \sum_{i=1}^{2} E_i^* \delta(X) E_i$. *Then, conditions of Theorem 11 are satisfied for* $M = \frac{3}{5}$ *and* $\sigma = \frac{1}{2}$.

4.2. Application to Functional Equations

Here, applying our obtained results, we solve a functional equation arising in dynamic programming.

Consider U and V two Banach spaces, $W \subseteq U$, $D \subseteq V$ and

$$\begin{aligned} \xi &: W \times D \longrightarrow W \\ g, u &: W \times D \longrightarrow \mathbb{R} \\ \Gamma, \Psi &: W \times D \times \mathbb{R} \longrightarrow \mathbb{R} \end{aligned}$$

For more details on dynamic programming, we refer to [36–39]. Suppose that W and D represent the state and decision spaces, respectively. The problem of related dynamic programming is reduced to solve the functional equations

$$p(\zeta) = \sup_{\eta \in D} \{ g(\zeta, \eta) + \Gamma(\zeta, \eta, p(\xi(\zeta, \eta))) \}, \text{ for } \zeta \in W \tag{25}$$

$$q(\zeta) = \sup_{\eta \in D} \{ u(\zeta, \eta) + \Psi(\zeta, \eta, q(\xi(\zeta, \eta))) \}, \text{ for } \zeta \in W. \tag{26}$$

We ensure the existence and uniqueness of a common and bounded solution of Equations (25) and (26). Denote by $B(W)$ the set of all bounded real valued functions on W. Consider,

$$P_b(h, k) = \left\| (h-k)^2 \right\|_\infty + L = \sup_{\zeta \in W} |h\zeta - k\zeta|^2 + L, \; L > 0, \tag{27}$$

for all $h, k \in B(W)$. Assume that:

$(B1)$: Γ, Ψ, g and u are bounded and continuous.

$(B2)$: For $\zeta \in W$, $h \in B(W)$ and $b > 0$, take $E, A : B(W) \longrightarrow B(W)$ as

$$Eh(\zeta) = \sup_{\eta \in D} \{ g(\zeta, \eta) + \Gamma(\zeta, \eta, h(\xi(\zeta, \eta))) \}, \tag{28}$$

$$Ah(\zeta) = \sup_{\eta \in D} \{ u(\zeta, \eta) + \Psi(\zeta, \eta, h(\xi(\zeta, \eta))) \}. \tag{29}$$

Moreover, for every $(\zeta, \eta) \in W \times D$, $h, k \in B(W)$, $t \in W$ and $\sigma > 0$ implies

$$|\Gamma(\zeta, \eta, h(t)) - \Psi(\zeta, \eta, k(t))|^2 \leq \frac{U_{P_b}(h(t), k(t))}{2} - \sigma \tag{30}$$

where

$$\begin{aligned} U_{P_b}((h(t), k(t)) &= \max\{ P_b(h(t), k(t)), P_b(h(t), Eh(t)), P_b(k(t), Ak(t)), \\ &\quad \frac{P_b(h(t), Ak(t)) + P_b(k(t), Eh(t))}{2K} \}. \end{aligned}$$

Theorem 12. *Assume that Conditions* $(B1)$ *and* $(B2)$ *hold. Then, Equations (25) and (26) have a common and bounded solution in* $B(W)$.

Proof. Note that $(B(W), P_b)$ is a complete partial bMS with constant $K = 4$. By $(B1)$, E, A are self-mappings of $B(W)$. Given $\lambda > 0$ and $h_1, h_2 \in B(W)$. Choose $\zeta \in W$ and $\eta_1, \eta_2 \in D$ such that

$$Eh_1 < g(\zeta,\eta_1) + \Gamma(\zeta,\eta_1,h_1(\xi(\zeta,\eta_1)) + \lambda \tag{31}$$
$$Ah_2 < g(\zeta,\eta_2) + \Psi(\zeta,\eta_2,h_2(\xi(\zeta,\eta_2)) + \lambda. \tag{32}$$

Further from Equations (31) and (32), we have

$$Eh_1 \geq g(\zeta,\eta_2) + \Gamma(\zeta,\eta_2,h_1(\xi(\zeta,\eta_2)) \tag{33}$$
$$Ah_2 \geq g(\zeta,\eta_1) + \Psi(\zeta,\eta_1,h_2(\xi(\zeta,\eta_1)). \tag{34}$$

Then, Equations (31) and (34) together with Equation (30) imply

$$\begin{aligned} Eh_1(\zeta) - Ah_2(\zeta) &< \Gamma(\zeta,\eta_1,h_1(\xi(\zeta,\eta_1))) - \Psi(\zeta,\eta_1,h_2(\xi(\zeta,\eta_1))) + \lambda \\ &\leq |\Gamma(\zeta,\eta_1,h_1(\xi(\zeta,\eta_1))) - \Psi(\zeta,\eta_1,h_2(\xi(\zeta,\eta_1)))| + \lambda \\ &\leq \sqrt{\frac{U_{P_b}(h_1(\zeta),h_2(\zeta))}{2} - \sigma} + \lambda. \end{aligned} \tag{35}$$

Then, Equations (32) and (33) together with Equations (30) imply

$$\begin{aligned} Ah_2(\zeta) - Eh_1(\zeta) &\leq \Gamma(\zeta,\eta_2,h_1(\xi(\zeta,\eta_2)) - \Psi(\zeta,\eta_2,h_2(\xi(\zeta,\eta_2)) + \lambda \\ &\leq |\Gamma(\zeta,\eta_2,h_1(\xi(\zeta,\eta_2)) - \Psi(\zeta,\eta_2,h_2(\xi(\zeta,\eta_2))| + \lambda \\ &\leq \sqrt{\frac{U_{P_b}(h_1(\zeta),h_2(\zeta))}{2} - \sigma} + \lambda, \end{aligned} \tag{36}$$

where

$$\begin{aligned} U_{P_b}((h_1(\zeta),h_2(\zeta)) = \max\{&P_b(h_1(\zeta),h_2(\zeta)), P_b(h_1(\zeta),Eh_1(\zeta)), P_b(h_2(\zeta),Ah_2(\zeta)), \\ &\frac{P_b(h_1(\zeta),Ah_2(\zeta)) + P_b(h_2(\zeta),Eh_1(\zeta))}{2K}\}. \end{aligned}$$

From Equations (35) and (36) and since $\lambda > 0$ is arbitrary, we obtain

$$|Eh_1(\zeta) - Ah_2(\zeta)| \leq \sqrt{\frac{U_{P_b}(h_1(\zeta),h_2(\zeta))}{2} - \sigma}.$$

Thus,

$$|Eh_1(\zeta) - Ah_2(\zeta)|^2 + \sigma \leq \frac{U_{P_b}(h_1(\zeta),h_2(\zeta))}{2}. \tag{37}$$

The inequality in Equation (37) implies

$$P_b(Eh_1(\zeta),Ah_2(\zeta)) \leq \frac{U_{P_b}(h_1(\zeta),h_2(\zeta))}{2}. \tag{38}$$

Taking $\Lambda(t) = t$ and $\Upsilon(t) = \frac{t}{2}$ for $t > 0$, we get

$$\Lambda(P_b(Eh_1(\zeta),Ah_2(\zeta))) \leq \Upsilon\left(\Lambda\left(U_{P_b}(h_1(\zeta),h_2(\zeta))\right)\right). \tag{39}$$

Therefore, all conditions of Corollary 5 immediately hold. Thus, there exists a common fixed point of E and A, e.g. $h^* \in B(W)$, that is, $h^*(\zeta)$ is a common solution of Equations (25) and (26). □

Example 5. *Let $U = V = \mathbb{R}$, $W = D = [0,\infty)$. Define $\xi : W \times D \longrightarrow W$, $g, u : W \times D \longrightarrow \mathbb{R}$, and $\Gamma, \Psi : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$ by,*

$$\xi(\zeta,\eta) = \begin{cases} \frac{\zeta \sin(1-\zeta-\eta^2)}{2+\zeta+\eta\zeta}, & \zeta^2+\eta^2 < 1, \\ \frac{1}{1+\zeta^2+\eta\eta^2}, & \zeta^2+\eta^2 \geq 1, \end{cases}$$
$$g(\zeta,\eta) = \frac{\zeta^2}{1+\zeta\eta},\ u(\zeta,\eta) = \frac{-\zeta^3}{1+\zeta+\eta},\ \text{and}$$
$$\Gamma(\zeta,\eta,t) = \frac{t}{1+|\sin(\zeta+\eta)|},$$
$$\Psi(\zeta,\eta,t) = \frac{|t|}{1+|t|+\zeta\eta},$$

where $(\zeta,\eta) \in W \times D$, $(\zeta,\eta,t) \in W \times D \times \mathbb{R}$ and $h, k \in B(W)$ with $h(t) = k(t) = t$. Define $E, A : B([0,\infty)) \to B([0,\infty))$, by

$$\begin{aligned} Eh(\zeta) &= \sup_{\eta\in[0,\infty)}\{g(\zeta,\eta) + \Gamma(\zeta,\eta,h(\xi(\zeta,\eta)))\}, \\ Ak(\zeta) &= \sup_{\eta\in[0,\infty)}\{u(\zeta,\eta) + \Psi(\zeta,\eta,k(\xi(\zeta,\eta)))\}. \end{aligned}$$

Then, Assumptions $(B1)$ and $(B2)$ of Theorem 12 are fulfilled, with $\sigma = \frac{1}{2}$, $\zeta, \eta \geq 50$ and $t = 5$. It follows from Theorem 12 that Equations (25) and (26) have a common and bounded solution in $B(W)$.

5. Conclusions

In this paper, we have provided common fixed theorems for generalized $(\alpha_K^*, \Upsilon, \Lambda)$-contraction multivalued pair of mappings in α_K-complete partial b-metric spaces. Our results are extensions of recent fixed point theorems of Wardowski [21], Piri and Kumam [17], Jleli et al. [9,10] and Liu et al. [13] and some other results. Moreover, we applied our main results to solve systems of functional equations and nonlinear matrix equations. It would be interesting to apply our given concepts and results for generalized metric spaces.

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