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# Breakable Semihypergroups

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Received: 14 December 2018; Accepted: 12 January 2019; Published: 16 January 2019

**Abstract:** In this paper, we introduce and characterize the breakable semihypergroups, a natural generalization of breakable semigroups, defined by a simple property: every nonempty subset of them is a subsemihypergroup. Then, we present and discuss on an extended version of Rédei's theorem for semi-symmetric breakable semihypergroups, proposing a different proof that improves also the theorem in the classical case of breakable semigroups.

**Keywords:** breakable semigroup; semihypergroup; hyperideal; semi-symmetry

## 1. Introduction

Breakable semigroups, introduced by Rédei [1] in 1967, have the property that every nonempty subset of them is a subsemigroup. It was proved that they are semigroups with empty Frattini-substructure [1]. For a structure $S$ (i.e., a group, a semigroup, a module, a ring or a field), the set of those elements which may be omitted from each generating system (containing them) of $S$ is a substructure of the same kind of $S$, called the Frattini-substructure of $S$. However, as mentioned in the book [1], there are some exceptions. The first one is when the Frattini-substructure is the empty set and this is the case of breakable semigroups, unit groups, zero modules or zero rings. The second one concerns the skew fields having the Frattini-substructure zero [1]. Based on the definition, it is easy to see that a semigroup $S$ is breakable if and only if $xy \in \{x, y\}$ for any $x, y \in S$, i.e., the product of any two elements of the given semigroup is always one of the considered elements. Another characterization of these semigroups is given by Tamura and Shafer [2], using the associated power semigroup, i.e., a semigroup $S$ is breakable if and only if its power semigroup $\mathcal{P}^*(S)$ is idempotent. An idempotent semigroup is a semigroup $S$ that satisfies the identity $a^2 = a$ for any $a \in S$. A complete description of breakable semigroups was given by Rédei [1], writing them as a special decomposition of left-zero and right-zero semigroups (see Theorem 1).

The power set, i.e., the family of all subsets of the initial set, has many roles in algebra, one of them being in hyperstructures theory, where the power set $\mathcal{P}(S)$ is the codomain of any hyperoperation on a nonempty set $S$, i.e., a mapping $S \times S \longrightarrow \mathcal{P}(S)$. If the support set $S$ is endowed with a binary associative operation, i.e., $(S, \cdot)$ is a semigroup, then this operation can be extended also to the set of nonempty subsets of $S$, denoted by $\mathcal{P}^*(S)$, in the most natural way: $A \star B = \{a \cdot b \mid a \in A, b \in B\}$. Thereby, $(\mathcal{P}^*(S), \star)$ becomes a semigroup, called the power semigroup of $S$. Similarly, if $(S, \circ)$ is a semihypergroup, then we can define on the power set a binary operation

$$A \star B = \bigcup_{a \in A, b \in B} a \circ b, \quad \text{for all} \quad A, B \in \mathcal{P}^*(S)$$

which is again associative (see Theorem 5). Going more in deep now, if we have a group $(G, \cdot)$ and we extend the operation to the set $\mathcal{P}^*(G)$ as before, then a new operation is defined on

$\mathcal{P}^*(G)$: $A \circ B = \{a \cdot b \mid a \in A, b \in B\}$. A nonempty subset $\mathcal{G}$ of $\mathcal{P}^*(G)$ is called an $HX$-group [3] on $G$, if $(\mathcal{G}, \circ)$ is a group. Similarly, on the group $(G, \cdot)$, one may define a hyperoperation by $a\hat{\circ}b = \{a \cdot b \mid a \in A, b \in B\}$, where $A, B \in \mathcal{P}^*(G)$, called by Corsini [4] the Chinese hyperoperation. An overview on the links between $HX$-groups and hypergroups has recently proposed by Cristea et al. [5].

Having in mind these connections between semigroups and semihypergroups and the importance of the power set and the decomposition of a set in the classical algebra, in this paper we would like to direct the reader's attention to a new concept, that one of breakable semihypergroup. The rest of the paper is structured as follows. In Section 2 we recall the breakable semigroups and the fundamental semigroups associated with semihypergroups. The main part of the paper is covered by Section 3, where we define the breakable semihypergroups and we present their characterizations using the power set and a generalization of Rédei's theorem for semi-symmetric semihypergroups, that permits to decompose them in a certain way. This decomposition is similar with that one proposed by Rédei's for semigroups, but slightly modified, to cover all the types of algebraic semihypergroups, by consequence all the types of algebraic semigroups. We have noticed that for some semigroups the Rédei's theorem does not work, while our proposed decomposition solves the problem. Besides we show that the set of all hyperideals of a breakable semi-symmetric semihypergroup is a chain. The semi-symmetry property plays here a fundamental role. This property holds for the classical structures, while in the hyperstructures has a significant meaning: the cardinalities of the hyperproducts of two elements $x \circ y$ and $y \circ x$ are the same for each pair of elements $(x, y)$ in the considered hyperstructure $(H, \circ)$. Clearly this is evident for commutative hyperstructures. At the end of the paper, some conclusive ideas and new lines of research are included.

## 2. Preliminaries

Since we like to have the keywords of this note clearly specified and laid out, in this section we recall some definitions and properties of semigroups and semihypergroups. For more details on both arguments the reader is refereed to [1,2,6] for the classical algebraic structures and [7–10] for the algebraic hyperstructures.

A semigroup $(S, \cdot)$ is called a *left zero semigroup*, by short an l-semigroup, if each element of it is a left zero element, i.e., for any $x \in S$, we have $x \cdot y = x$ for all $y \in S$. Similarly, a *right zero semigroup*, or an *r-semigroup*, is a semigroup in which each element is a zero right element, i.e., for any $x \in S$, we have $x \cdot y = y$ for all $y \in S$.

In 1967, Rédei [1] gave the definition of *breakable semigroups*, as a subclass of the semigroups having an empty Frattini-substructure.

**Definition 1.** *A semigroup $S$ is breakable if every non-empty subset of $S$ is a subsemigroup.*

It is easy to see that a semigroup $(S, \cdot)$ is breakable if and only if $x \cdot y \in \{x, y\}$ for any $x, y \in S$.

A complete description of the structure of a breakable semigroup is given by Theorem 50 in [1].

**Theorem 1.** *A semigroup $S$ is breakable if and only if, it can be partitioned into classes and the set of classes can be ordered in such a way that every class constitutes an l-semigroup or an r-semigroup, and for any two elements $x \in C$ and $y \in C'$ of two different classes $C, C'$, with $C < C'$, we have $x \cdot y = y \cdot x = y$.*

Moreover, if $(S, \cdot)$ is a semigroup, then it is obvious that the set $\mathcal{P}^*(S)$ of all non-empty subsets of $S$ can be endowed with a semigroup structure, too, called the *power semigroup*, where the binary operation is defined as follows: for $A, B \in \mathcal{P}^*(S)$, $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$. Then a breakable semigroup can be characterized also using properties of its power semigroup, as shown by Tamura and Shafer [2].

**Theorem 2.** *A semigroup $S$ is breakable if and only if its power semigroup is idempotent, i.e., $X = X^2$ for all $X \in \mathcal{P}^*(S)$.*

On the other hand, the set $\mathcal{P}^*(S)$ is the codomain of any hyperoperation defined on the support set $S$, i.e., a mapping $S \times S \longrightarrow \mathcal{P}^*(S)$. Now, if we start with a semihypergroup $(S, \circ)$, till now only a classical operation was defined on $S$, and not a hyperoperation, so the power set is again a semigroup, as we will show later on in Theorem 5.

The other natural and crucial connection between hyperstructures and classical structures is represented by the *strongly regular relations*. More exactly, on any semihypergroup $(S, \circ)$ one can define the relation $\beta$ and its transitive closure $\beta^*$, and define a suitable operation on the quotient $S/\beta^*$ in order to endow it with a semigroup structure, called the *fundamental semigroup* related to $S$. Here below we recall the construction, introduced by Koskas [11] and studied mainly by Freni [12], who proved that $\beta = \beta^*$ on hypergroups. For all natural numbers $n > 1$, define the relation $\beta_n$ on a semihypergroup $(S, \circ)$, as follows: $a\beta_n b$ if and only if there exist $x_1, \ldots, x_n \in S$ such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$. Take $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in S\}$ is the diagonal relation on $S$. Denote by $\beta^*$ the transitive closure of $\beta$. The relation $\beta^*$ is a strongly regular relation. On the quotient $S/\beta^*$ define a binary operation as follows: $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$ for all $c \in \beta^*(a) \circ \beta^*(b)$. Moreover, the relation $\beta^*$ is the smallest equivalence relation on a semihypergroup $S$, such that the quotient $S/\beta^*$ is a semigroup. The quotient $S/\beta^*$ is called the *fundamental semigroup*.

## 3. Breakable Semihypergroups

In this section, based on the notion of breakable semigroup introduced by Rédei [1], we define and characterize breakable semihypergroups. We present a generalization of Rédei's theorem for semi-symmetric semihypergroups.

In a classical structure (semigroup, monoid, group, ring, etc.) the composition of two elements is always another element of the support set. This property is not conserved in a hyperstructure, but it is extended in such a way that the result of the composition of two elements—called hypercomposition—is a subset of the support set. This means that, for two elements $x, y \in S$, the cardinalities of the compositions $x \cdot y$ and $y \cdot x$ in a classical algebraic structure are always equal (being both 1), while in a hyperstructure they could be greater than 1 and also different one from another. For this reason we introduce the next concept.

**Definition 2.** *A semihypergroup $(S, \circ)$ is called semi-symmetric if $|x \circ y| = |y \circ x|$ for every $x, y \in S$.*

It is clear that any commutative semihypergroup is also semi-symmetric.

**Definition 3.** *A semihypergroup $S$ is called breakable if every non-empty subset of $S$ is a subsemihypergroup.*

Obviously, every breakable semigroup can be considered as a breakable semihypergroup, by consequence l-semigroups and r-semigroups are examples of breakable semihypergroups.

A hyperoperation "$\circ$" on a nonempty set $S$, satisfying the property $x, y \in x \circ y$ for all elements $x, y \in S$, is called *extensive* (by J. Chvalina and his group of researchers [13–15]) or *closed* (by Ch. Massouros [16]). The most simple hyperoperation of this type was defined by the first time by Konguetsof [17] around 70's as $x \circ y = \{x, y\}$ for all $x, y \in S$. More than 20 years later, this hyperoperation was re-considered by G.G. Massouros et al. [18,19] in the framework of automata theory, proving the following result.

**Theorem 3.** *Let $H$ be a non-empty set [19]. For every $x, y \in H$ define $x \star_B y = \{x, y\}$. Then $(H, \star_B)$ is a join hypergroup.*

G.G. Massouros called this hyperstructure a *B-hypergroup*, after the binary result that the hyperoperation gives.

**Example 1.** *Consider* $S = (\{1, 2, 3\}, \circ)$ *defined by the following Cayley table*

| $\circ$ | **1** | **2** | **3** |
|---|---|---|---|
| 1 | 1 | 1 | $\{1,3\}$ |
| 2 | $\{1,2\}$ | 2 | $\{2,3\}$ |
| 3 | $\{1,3\}$ | 3 | 3 |

*Then S is a breakable semihypergroup.*

**Example 2.** *Consider* $S = (\{1, 2, 3, 4, 5\}, \circ)$ *defined by the following Cayley table*

| $\circ$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 2 | 2 | $\{2,3\}$ | 2 | $\{2,5\}$ |
| 3 | 3 | $\{2,3\}$ | 3 | 3 | $\{3,5\}$ |
| 4 | 4 | 2 | 3 | 4 | 5 |
| 5 | 5 | $\{2,5\}$ | $\{3,5\}$ | 5 | 5 |

*Then S is a breakable semihypergroup.*

Notice that in both examples the hyperoperation is extensive. Moreover, both are semihypergroups, but not hypergroups, since the reproduction axiom does not hold. The next theorem gives a characterization of breakable hypergroups.

**Theorem 4.** *A hypergroup* $(H, \circ)$ *is breakable if and only if it is a B-hypergroup.*

**Proof.** First, suppose that $(H, \circ)$ is a breakable hypergroup. For any two distinct elements $x$ and $y$ of $H$, by left reproducibility, there exists $z \in H$ such that $y \in x \circ z$. Since $H$ is breakable, it follows that $\{x, z\}$ is a subsemihypergroup, so $x \circ z \subseteq \{x, z\}$. It follows that $y \in \{x, z\}$ and thus $y = z$. Therefore $y \in x \circ y$. Similarly, using the right reproducibility, one proves that $x \in x \circ y$. So we obtain $x \circ y = \{x, y\}$, i.e., $(H, \circ)$ is a $B$-hypergroup.

Conversely, the other implication is evident. □

Similarly to the classic case, one can characterize the breakable semihypergroups using the associated power semigroup.

**Theorem 5.** *Let* $(S, \circ)$ *be a semihypergroup. Then the following assertions hold:*

$(I)$ $(\mathcal{P}^*(S), \star)$ *is a semigroup, where the binary operation* $\star$ *is defined by:*

$$A \star B = \bigcup_{a \in A, b \in B} a \circ b, \quad \textit{for all} \quad A, B \in \mathcal{P}^*(S).$$

$(II)$ $(S, \circ)$ *is breakable if and only if* $(\mathcal{P}^*(S), \star)$ *is idempotent.*

**Proof.** (I) The binary operation $\star$ is associative since, for every non empty subsets $A, B, C$ of $S$ we have

$$\begin{aligned} A \star (B \star C) &= A \star \Big( \bigcup_{b \in B, c \in C} b \circ c \Big) \\ &= \bigcup_{a \in A, b \in B, c \in C} a \circ (b \circ c) \\ &= \bigcup_{a \in A, b \in B, c \in C} (a \circ b) \circ c \\ &= \Big( \bigcup_{a \in A, b \in B} a \circ b \Big) \star C \\ &= (A \star B) \star C. \end{aligned}$$

(II) Let $(S, \circ)$ be breakable and $A \subseteq S$. Then $A$ is a subsemihypergroup of $S$, that is $A \star A \subseteq A$. On the other hand, for every $a \in A$ we have $a = a \circ a \subseteq A \star A$. Thus $A \star A = A$, so $(\mathcal{P}^*(S), \star)$ is idempotent. Conversely, suppose that $(\mathcal{P}^*(S), \star)$ is idempotent. Then, for every non empty subset $A$ of $S$, we have $A \star A = A$, so $A$ is a subsemihypergroup, meaning that $S$ is breakable. □

**Proposition 1.** *The fundamental semigroup of a breakable semihypergroup is breakable, too.*

**Proof.** Let $S$ be a breakable semihypergroup and $(S/\beta^*, \odot)$ the associated fundamental semigroup. We know that, for $x, y \in S$, $\beta^*(x) \odot \beta^*(y) = \beta^*(z)$, whenever $z \in x \circ y \subseteq \{x, y\}$, because $(S, \circ)$ is breakable. So $\beta^*(x) \odot \beta^*(y) \in \{\beta^*(x), \beta^*(y)\}$, meaning that $(S/\beta^*, \odot)$ is breakable, too. □

Now it is the time to go back to Rédei's theorem and try to find a generalization in the broader context of semihypergroups. Notice here the significance of the notion of semi-symmetric semihypergroup.

**Theorem 6.** *A semi-symmetric semihypergroup $(S, \circ)$ is breakable if and only if it can be partitioned into classes, i.e., $S = \bigcup_{\gamma \in \Gamma} S_\gamma$, where $\Gamma$ is a chain and all $S_\gamma$ are pairwise disjoint l-semigroups, r-semigroups or B-hypergroups. Moreover, for every $x \in S_\alpha$ and $y \in S_\beta$, with $\alpha < \beta$, we have $x \circ y = y \circ x = y$.*

**Proof.** "$\Longrightarrow$" Suppose that $(S, \circ)$ is a breakable semi-symmetric semihypergroup. Then, for any $x, y \in S$, the sets $\{x\}$ and $\{x, y\}$ are semi-symmetric semihypergroups, so

$$x^2 = x \tag{1}$$

and

$$x \circ y = x \ \text{ or } \ x \circ y = y \ \text{ or } \ x \circ y = \{x, y\}. \tag{2}$$

We will prove the theorem in several steps.

Step 1. First we define on $S$ three relations as follows:

$$x \sim_l y \Longleftrightarrow x \circ y = y, y \circ x = x. \tag{3}$$

$$x \sim_r y \Longleftrightarrow x \circ y = x, y \circ x = y. \tag{4}$$

$$x \sim_h y \Longleftrightarrow x \circ y = y \circ x = \{x, y\}. \tag{5}$$

In [1], it was proved that $\sim_l$ and $\sim_r$ are equivalences. We show now that also $\sim_h$ is an equivalence. The reflexivity holds because of (1) and the simmetry is evident. For proving the transitivity, take $x, y, z \in S$ such that $x \sim_h y$ and $y \sim_h z$, thus

$$x \circ y = y \circ x = \{x, y\} \quad \text{and} \quad y \circ z = z \circ y = \{y, z\}.$$

Hence it follows that

$$\begin{aligned} &\{x, y\} \cup x \circ z = x \circ y \cup x \circ z = x \circ \{y, z\} = x \circ (y \circ z) = (x \circ y) \circ z = \\ &= x \circ z \cup y \circ z = x \circ z \cup \{y, z\}. \end{aligned}$$

Thus $\{x, z\} \subseteq x \circ z \subseteq \{x, z\}$ (because $S$ is breakable), implying that $x \circ z = \{x, z\}$, i.e., $x \sim_h z$. Therefore, $\sim_h$ is an equivalence relation on $S$.

Define the corresponding partitions of $S$ related to $\sim_l, \sim_r$ and $\sim_h$ by $\mathcal{C}_l, \mathcal{C}_r$ and $\mathcal{C}_h$, respectively. Based on relations (3)–(5), we can notice that each class in $\mathcal{C}_l$, $\mathcal{C}_r$ and $\mathcal{C}_h$ is a maximal l-semigroup, r-semigroup and $B$-hypergroup, respectively. Indeed, for example, let $H$ be a $B$-subhypergroup such that $\hat{x}_h \subseteq H \subseteq S$, where $\hat{x}_h \in \mathcal{C}_h$. Then, for every $y \in H$, we have $x \circ y = y \circ x = \{x, y\}$, meaning that $y \in \hat{x}_h$, so $\hat{x}_h = H$. Thus the class $\hat{x}_h$ represented by $x$ is maximal.

Step 2. We show that if any two classes of $\mathcal{C}_l, \mathcal{C}_r$ or $\mathcal{C}_h$ have a common element, then one of them contains only one element. Let us assume, in contrast, that there exist two classes $\hat{x}_l \in \mathcal{C}_l$ and $\hat{z}_h \in \mathcal{C}_h$, both with more than one element, such that $\hat{x}_l \cap \hat{z}_h \neq \emptyset$. Thus there exists $y \in \hat{x}_l \cap \hat{z}_h$. It means that $\{x, y\}$ is an l-semigroup and $\{y, z\}$ is a $B$-hypergroup. Then

$$(x \circ z) \circ y = x \circ (z \circ y) = x \circ \{z, y\} = x \circ z \cup x \circ y = x \circ z \cup \{y\} \tag{6}$$

and

$$y \circ (x \circ z) = (y \circ x) \circ z = x \circ z. \tag{7}$$

On the other hand, because of (2), we have $x \circ z = x$ or $z \in x \circ z$. If $x \circ z = x$, using (6), we get $y = x \circ y = \{x, y\}$, which is impossible, because $x \neq y$. If $z \in x \circ z$, then by (7), we have $\{y, z\} = y \circ z \subseteq y \circ (x \circ z) = x \circ z$, so $y \in x \circ z$, which is again a contradiction, because of (2). Similarly, the other cases can be verified.

Step 3. Based on the assertion proved in the previous step, we may define on $S$ a new partition: we take the classes, of cardinality at least 2, of $\mathcal{C}_l, \mathcal{C}_r$ and $\mathcal{C}_h$, and then the singleton classes of all the other elements of $S$ (we read here that all the other elements are put each one in a different class). We denote the corresponding equivalence relation by $\sim$ and the class of $x$ with respect to $\sim$ by $\overline{x}$.

Take $x$ and $y$ from two different classes, i.e., $x \nsim y$. Since $S$ is a breakable semihypergroup, it follows that $\{x, y\}$ is a subsemihypergroup, so relation (2) is verified. If $x \circ y = x$, then since $S$ is semi-symmetric, it follows that $y \circ x = x$ or $y \circ x = y$. If $y \circ x = y$, it means that $x \sim_r y$ and thus $x \sim y$, which is false. So $x \circ y = y \circ x = x$. Similarly, if $x \circ y = y$ it follows that $y \circ x = y$. Thereby, for $x \nsim y$, we get

$$x \circ y = y \circ x = x \quad \text{or} \quad x \circ y = y \circ x = y. \tag{8}$$

Step 4. We show that for any different elements $x_1, x_2, y$ of $S$ such that $x_1 \sim x_2 \nsim y$, we have

$$\text{either} \quad x_i \circ y = y \circ x_i = y \quad \text{or} \quad x_i \circ y = y \circ x_i = x_i \tag{9}$$

for $i = 1, 2$. Since $x_1 \sim x_2$, the set $\{x_1, x_2\}$ is an l-semigroup, an r-semigroup or a $B$-hypergroup. Let us assume that $\{x_1, x_2\}$ is an l-semigroup, i.e., we have $x_1 \circ x_2 = x_2$ and $x_2 \circ x_1 = x_1$. Besides, from (8), we have $x_i \circ y = y \circ x_i$, for $i = 1, 2$.

Now, by contrast, if we suppose that the assertion is false, then, because of (2) with a suitably order, we have

$$x_1 \circ y = y \circ x_1 = x_1, \;\; x_2 \circ y = y \circ x_2 = y.$$

Hence $x_1 = x_1 \circ y = x_1 \circ (y \circ x_2) = (x_1 \circ y) \circ x_2 = x_1 \circ x_2 = x_2$, which is a contradiction, so relation (9) is now proved. Similarly, relation (9) holds whenever $\{x_1, x_2\}$ is an r-semigroup or a $B$-hypergroup.

Step 5. On the set of all classes $\overline{x}$ define an ordering relation $<$ as follows:

$$\overline{x} < \overline{y} \Longleftrightarrow x \circ y = y \circ x = y. \tag{10}$$

First we prove that the relation is well-defined, i.e., it does not depend on the representatives $x$ and $y$. Take $x \sim x'$ and $y \sim y'$. By using (9) for $x_1 = x, x_2 = x'$ and $y$, then for $x_1 = y, x_2 = y'$ and $y = x$, respectively, we get $x' \circ y = y \circ x' = y$ and $x \circ y' = y' \circ x = y'$.

The reflexivity and the symmetry are evident. It remains to prove the transitivity. Assume that $\overline{x} < \overline{y}$ and $\overline{y} < \overline{z}$. By definition of $<$, these two relations mean that $x \circ y = y \circ x = y$ and $y \circ z = z \circ y = z$. It follows that $x \circ z = x \circ (y \circ z) = (x \circ y) \circ z = y \circ z = z$ and similarly, $z \circ x = z$, meaning that $\overline{x} < \overline{z}$.

Besides, from (8), for $x \sim y$, it follows that either $\overline{x} < \overline{y}$ or $\overline{y} < \overline{x}$ always holds, so the order $<$ is total.

"$\Longleftarrow$" The converse implication is obvious.

Now the proof is completed. □

**Remark 1.** *The structure of the proof of Theorem 6 is similar to that one proposed by Rédei [1] for the decomposition of breakable semigroups, but it was obviously extended to hyperstructure environment. Moreover, in the original proof, Rédei considered in Step 3 a different partition of the initial semigroup $S$, i.e., he considered the classes of cardinality at least 2 of $\mathcal{C}_l$ and $\mathcal{C}_r$, and then the class of all the other elements of $S$. But doing in this way, not all the breakable semigroups are decomposed as is requested by Theorem 1, as we can notice here below.*

*Consider on the set $S = \{1, 2\}$ the operation $x \cdot y = \max\{x, y\}$. It is clear that $S$ is a breakable semigroup and using the above mentioned partition, we have to consider 1 and 2 in the same class (the last one, "of all the other elements," let's say), since 1 and 2 are not equivalent with respect to both relations $\sim_l$ and $\sim_r$. So the partition will be $\{\{1, 2\}\}$, which is not an l-semigroup or an r-semigroup, obtaining thus a contradiction. On the other way, if we consider the partition of $S$ as in Theorem 6 in Step 3, i.e., we take the classes, of cardinality at least 2, of $\mathcal{C}_l, \mathcal{C}_r$ and $\mathcal{C}_h$, and then the singleton classes of all the other elements of $S$, we get another partition of $S$ as $S = S_\alpha \cup S_\beta$, where $S_\alpha = \{1\}$ and $S_\beta = \{2\}$, both being l-semigroups (or r-semigroups), so Rédei's theorem is verified also in this particular case.*

In the following examples we will show the decomposition of some breakable semihypergroups obtained using Theorem 6.

**Example 3.** *Let $\Gamma = \{\alpha, \beta\}$, $\alpha < \beta$, $S_\alpha = \{1, 2\}$ be a l-semigroup and $S_\beta = \{3, 4\}$ be a B-hypergroup. Then $(\{1, 2, 3, 4\}, \circ)$ is a breakable semihypergroup with the following Cayley table:*

| $\circ$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 4 |
| 2 | 2 | 2 | 3 | 4 |
| 3 | 3 | 3 | 3 | $\{3, 4\}$ |
| 4 | 4 | 4 | $\{3, 4\}$ | 4 |

**Example 4.** *Let $\Gamma = \{\alpha, \beta, \gamma, \delta\}$, $\alpha < \beta < \gamma < \delta$, $S_\alpha = \{1,2,3\}$ and $S_\gamma = \{6,7\}$ be B-hypergroups, $S_\beta = \{4,5\}$ be an l-semigroup and $S_\delta = \{8,9\}$ be an r-semigroup. Then $(\{1,2,\ldots,9\}, \circ)$ is a breakable semihypergroup with the following Cayley table:*

| $\circ$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $\{1,2\}$ | $\{1,3\}$ | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | $\{1,2\}$ | 2 | $\{2,3\}$ | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | $\{1,3\}$ | $\{2,3\}$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 4 | 4 | 4 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 5 | 5 | 5 | 4 | 5 | 6 | 7 | 8 | 9 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | $\{6,7\}$ | 8 | 9 |
| 7 | 7 | 7 | 7 | 7 | 7 | $\{6,7\}$ | 7 | 8 | 9 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

**Example 5.** *Consider the following binary hyperoperation on the set of integers:*

$$\forall m,n \in \mathbb{Z}, m \circ n = \begin{cases} n & \text{if } m,n < 0 \\ 0 & \text{if } m = n = 0 \\ \{m,n\} & \text{if } m,n > 0 \\ \max\{m,n\} & \text{otherwise.} \end{cases}$$

*Then $(\mathbb{Z}, \circ)$ is a breakable semihypergroup, since it is sufficient to take $\Gamma = \{\alpha, \beta, \gamma\}$, with $\alpha < \beta < \gamma$, $S_\alpha = \mathbb{Z}^-$ as an l-semigroup, $S_\beta = \{0\}$ as an r-semigroup and $S_\gamma = \mathbb{N}$ as a B-hypergroup.*

The notion of ideal of a semigroup was extended to the hyperstructures for the first time by Hasankhani [20], defining the concept of *left (right) ideal* in a hypergroupoid, that was after changed into *hyperideal*, in order to keep the meaning of the hyperoperation.

**Definition 4.** *Let $(H, \circ)$ be a hypergroupoid. A non empty set $A$ of $H$ is called a left hyperideal if, for $x \in A$, it follows that $y \circ x \subseteq A$ for any $y \in H$. Similarly, $A$ is a right hyperideal if, for $x \in A$, it follows that $x \circ y \subseteq A$ for any $y \in H$. Moreover $A$ is called a hyperideal of $H$ if it is both a left and a right hyperideal.*

**Theorem 7.** *Let $S$ be a breakable semi-symmetric semihypergroup. Then the set of all hyperideals of $S$ together with the inclusion is a chain.*

**Proof.** Let $(S, \circ)$ be a breakable semi-symmetric semihypergroup. Then by Theorem 6, there exists an equivalence relation $\sim$ on $S$ such that the set of classes $\mathcal{C}$ with respect to it can be ordered in such a way that, for every distinct classes $\overline{x}$ and $\overline{y}$, we have

$$\overline{x} < \overline{y} \Longleftrightarrow x \circ y = y \circ x = y. \tag{11}$$

Please note that the definition of $\mathcal{C}$ is equivalent with

$$x \sim y \Longrightarrow x \circ y \cup y \circ x = \{x, y\}, \quad x, y \in S. \tag{12}$$

We claim that, if $I$ is a hyperideal of $S$, then

$$I = \bigcup_{x \in I} \overline{x}.$$

Indeed, for every $x, x' \in S$, where $x' \sim x \in I$, we have $\{x, x'\} = x \circ x' \cup x' \circ x \subseteq I$, so the claim is proved.

Furthermore, (11) implies that, for every hyperideal $I$ of $S$, we have

$$x \in I, y \in S, \overline{x} < \overline{y} \Longrightarrow y \in I, \tag{13}$$

hence $\overline{y} \subseteq I$.

Now, let $I$ and $J$ be distinct hyperideals of $S$. We will prove that either $I \subseteq J$ or $J \subseteq I$. To do this, first we will show that either $J \setminus I \neq \emptyset$ or $I \setminus J \neq \emptyset$, i.e., just one of the assertions holds. In contrast, let us suppose that $J \setminus I \neq \emptyset$ and $I \setminus J \neq \emptyset$, therefore there exist $a_0 \in I \setminus J$ and $b_0 \in J \setminus I$. From (13) it follows that

$$\overline{a_0} < \overline{b} \text{ for any } b \in J \text{ and } \overline{b_0} < \overline{a} \text{ for any } a \in I. \tag{14}$$

Indeed, if $\overline{b} < \overline{a_0}$, with $b \in J$, then $a_0 \in J$ (since $J$ is a hyperideal), which is false. So $\overline{a_0} < \overline{b}$, for any $b \in J$. Similarly the other relation holds. This implies that $a_0 < b_0$ and $b_0 < a_0$, hence $\{a_0, b_0\} \subseteq \overline{a_0} = \overline{b_0} \subseteq I \cap J$, which is impossible.

Thereby, for two distinct hyperideals $I$ and $J$, we have either $J \setminus I \neq \emptyset$ or $I \setminus J \neq \emptyset$. Without loss of generality, let $I \setminus J \neq \emptyset$ and take $b \in J$. Then, by (14), we have $\overline{a_0} < \overline{b}$, with $a_0 \in I$; this implies that $b \in I$, thus $J \subset I$. Similarly, if $J \setminus I \neq \emptyset$, then $I \subseteq J$. We can conclude thus, that the set of the hyperideals of $S$ is a chain with respect to the inclusion. □

**Corollary 1.** *The set of all ideals of a breakable semigroup is a chain.*

## 4. Conclusions

In this paper, we have started the study of breakable semihypergroups, based on the classical concept of breakable semigroups. In a breakable semihypergroup, each nonempty subset is a subsemihypergroup. If we search for the same property in hypergroups (so semihypergroups satisfying also the reproduction axiom), we obtain that there is only one class of breakable hypergroups and this is that of B-hypergroups. Moreover, we have proved that a breakable semi-symmetric semihypergroup can be decomposed in classes that are ordered in such a way that each class is an l-semigroup, an r-semigroup or a B-hypergroup. At the end, we have proved that the set of all hyperideals of a semi-symmetric breakable semihypergroup is a chain.

The properties of the breakable semihypergroups, in particular the proposed decomposition, suggest several new lines of research. A first one could be a generalization of the classical notion of Frattini-substructure, so the study of the Frattini-subhyperstructure. Another perspective could be related to the role of the power set, so the set of subsets of the support set. It is well known that the operation on a semigroup can be extended to the family of nonempty subsets of the semigroup, endowing it with a semigroup structure, called the associated power semigroup. Now, if the support set is a semihypergroup, then we can similarly extend the hyperoperation to an operation on the power set, which remains associative. In our future work we intend to define a hyperoperation on the power set and investigate its properties, aiming to define the power semihypergroup.

**Author Contributions:** Conceptualization, D.H.; Investigation, D.H.; Methodology, I.C.; Supervision, I.C.; Writing original draft, D.H.; Writing review and editing, I.C.

**Funding:** The second author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0285).

**Conflicts of Interest:** The authors declare no conflict of interest.

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