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Local Convergence of a Family of Weighted-Newton Methods

Ramandeep Behl [1,*], Ioannis K. Argyros [2], J.A. Tenreiro Machado [3] and Ali Saleh Alshomrani [1]

[1] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; aszalshomrani@kau.edu.sa
[2] Department of Mathematics Sciences, Cameron University, Lawton, OK 73505, USA; iargyros@cameron.edu
[3] Institute of Engineering, Polytechnic of Porto Department of Electrical Engineering, 4200-072 Porto, Portugal; jtm@isep.ipp.pt
* Correspondence: ramanbehl87@yahoo.in

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Abstract: This article considers the fourth-order family of weighted-Newton methods. It provides the range of initial guesses that ensure the convergence. The analysis is given for Banach space-valued mappings, and the hypotheses involve the derivative of order one. The convergence radius, error estimations, and results on uniqueness also depend on this derivative. The scope of application of the method is extended, since no derivatives of higher order are required as in previous works. Finally, we demonstrate the applicability of the proposed method in real-life problems and discuss a case where previous studies cannot be adopted.

Keywords: Banach space; weighted-Newton method; local convergence; Fréchet-derivative; ball radius of convergence

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1. Introduction

In this work, $\mathbb{B}_1$ and $\mathbb{B}_2$ denote Banach spaces, $\mathbb{A} \subseteq \mathbb{B}_1$ stands for a convex and open set, and $\varphi : \mathbb{A} \to \mathbb{B}_2$ is a differentiable mapping in the Fréchet sense. Several scientific problems can be converted to the expression. This paper addresses the issue of obtaining an approximate solution s_* of:

$$\varphi(x) = 0, \tag{1}$$

by using mathematical modeling [1–4]. Finding a zero s_* is a laborious task in general, since analytical or closed-form solutions are not available in most cases.

We analyze the local convergence of the two-step method, given as follows:

$$\begin{aligned} y_j &= x_j - \delta\varphi'(x_j)^{-1}\varphi(x_j), \\ x_{n+1} &= x_j - A_j^{-1}\big(c_1\varphi(x_j) + c_2\varphi(y_j)\big), \end{aligned} \tag{2}$$

where $x_0 \in \mathbb{A}$ is a starting point, $A_j = \alpha\varphi'(x_j) + \beta\varphi'\left(\frac{x_j+y_j}{2}\right) + \gamma\varphi'(y_j)$, and $\alpha,\ \beta,\ \gamma,\ \delta,\ c_1,\ c_2 \in \mathbb{S}$, where $\mathbb{S} = \mathbb{R}$ or $\mathbb{S} = \mathbb{C}$. The values of the parameters $\alpha,\ \gamma,\ \beta$, and c_1 are given as follows:

$$\alpha = -\frac{1}{3}c_2(3\delta^2 - 7\delta + 2),\ \ \beta = -\frac{4}{3}c_2(2\delta - 1),$$
$$\gamma = \frac{1}{3}c_2(\delta - 2) \text{ and } c_1 = -c_2(\delta^2 - \delta + 1),\ \text{for } \delta \neq 0,\ c_2 \neq 0.$$

Comparisons with other methods, proposed by Cordero et al. [5], Darvishi et al. [6], and Sharma [7], defined respectively as:

$$\begin{aligned} w_j &= x_j - \varphi'(x_j)^{-1}\varphi(x_j), \\ x_{n+1} &= w_j - B_j^{-1}\varphi(w_j), \end{aligned} \tag{3}$$

$$\begin{aligned} w_j &= x_j - \varphi'(x_j)^{-1}\varphi(x_j), \\ z_j &= x_j - \varphi'(x_j)^{-1}\big(\varphi(x_j) + \varphi(w_j)\big), \\ x_{n+1} &= x_j - C_j^{-1}\varphi(x_j), \end{aligned} \tag{4}$$

$$\begin{aligned} y_j &= x_j - \frac{2}{3}\varphi'(x_j)^{-1}\varphi(x_j), \\ x_{n+1} &= x_j - \frac{1}{2}D_j^{-1}\varphi'(x_j)^{-1}\varphi(x_j), \end{aligned} \tag{5}$$

where:

$$\begin{aligned} B_j &= 2\varphi'(x_j)^{-1} - \varphi'(x_j)^{-1}\varphi'(w_j)\varphi'(x_j)^{-1}, \\ C_j &= \frac{1}{6}\varphi'(x_j) + \frac{2}{3}\varphi'\left(\frac{x_j + w_j}{2}\right) + \frac{1}{6}\varphi'(z_j), \\ D_j &= -I + \frac{9}{4}\varphi'(y_j)^{-1}\varphi'(x_j) + \frac{3}{4}\varphi'(x_j)^{-1}\varphi'(y_j), \end{aligned}$$

were also reported in [8]. The local convergence of Method (2) was shown in [8] for $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^m$ and $\mathbb{S} = \mathbb{R}$, by using Taylor series and hypotheses reaching up to the fourth Fréchet-derivative. However, the hypothesis on the fourth derivative limits the applicability of Methods (2)–(5), particularly because only the derivative of order one is required. Let us start with a simple problem. Set $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$ and $\mathbb{A} = [-\frac{5}{2}, \frac{3}{2}]$. We suggest a function $\varphi : \mathbb{A} \to \mathbb{R}$ as:

$$\varphi(x) = \begin{cases} 0, & x = 0 \\ x^3 ln x^2 + x^5 - x^4, & x \neq 0 \end{cases},$$

which further yield:

$$\begin{aligned} \varphi'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ \varphi''(x) &= 12x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ \varphi'''(x) &= 12 \ln x^2 + 60x^2 - 12x + 22, \end{aligned}$$

where the solution is $s_* = 1$. Obviously, the function $\varphi'''(x)$ is unbounded in the domain $\mathbb{A}$. Therefore, the results in [5–9] and Method (2) cannot be applicable to such problems or its special cases that require the hypotheses on the third- or higher order derivatives of φ. Without a doubt, some of the iterative method in Brent [10] and Petkovíc et al. [4] are derivative free and are used to locate zeros of functions. However, there have been many developments since then. Faster iterative methods have been developed whose convergence order is determined using Taylor series or with the technique introduce in our paper. The location of the initial points is a "shot in the dark" in these references; no uniqueness results or estimates on $\|x_n - x_*\|$ are available. Methods on abstract spaces derived from the ones on the real line are also not addressed.

These works do not give a radius of convergence, estimations on $\|x_j - s_*\|$, or knowledge about the location of s_*. The novelty of this study is that it provides this information, but requiring only the derivative of order one for method (2). This expands the scope of utilization of (2) and similar methods. It is vital to note that the local convergence results are very fruitful, since they give insight into the difficult operational task of choosing the starting points/guesses.

Otherwise, with the earlier approaches: (i) use the Taylor series and high-order derivative; (ii) have no clue about the choice of the starting point x_0; (iii) have no estimate in advance about the number of iterations needed to obtain a predetermined accuracy; and (iv) have no knowledge of the uniqueness of the solution.

The work is laid out as follows: we give the convergence of the iterative scheme (2) with the main Theorem 1 is given in Section 2. Six numerical problems are discussed in Section 3. The final conclusions are summarized in Section 4.

2. Convergence Study

This section starts by analyzing the convergence of Scheme (2). We assume that $L > 0$, $L_0 > 0$, $M \geq 1$ and $\gamma,\ \alpha,\ \beta,\ \delta,\ c_1,\ c_2 \in \mathbb{S}$. We consider some maps/functions and constant numbers. Therefore, we assume the following functions g_1, p, and h_p on the open interval $[0, \frac{1}{L_0})$ by:

$$\begin{aligned} g_1(t) &= \frac{1}{2(1-L_0 t)}(Lt + 2M|1-\delta|),\\ p(t) &= \frac{L_0}{|\alpha+\beta+\gamma|}\left(|\alpha| + \frac{|\beta|}{2}\left(\frac{|\beta|}{2} + |\gamma|\right) g_1(t)\right) t,\ \text{for}\ \alpha+\beta+\gamma \neq 0,\\ h_p(t) &= p(t) - 1, \end{aligned}$$

and the values of r_1 and r_A are given as follows:

$$r_1 = \frac{2(M|1-\delta|-1)}{},\ \ r_A = \frac{2}{L+2L_0}.$$

Consider that:

$$M|1-\delta| < 1. \tag{6}$$

It is clear from the function g_1, parameters r_1 and r_A, and Equation (6), that $0 < r_1 \leq r_A < \frac{1}{L_0}$, $g_1(r_1) = 1$, and $0 \leq g_1(t) < 1$, for each $t \in [0,\ r_1)$ and $h_p(0) = -1$ and $h_p(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. On the basis of the classical intermediate value theorem, the function h_p has at least one zero in the open interval $\left(0,\ \frac{1}{L_0}\right)$. Let us call r_p as the smallest zero. We suggest some other functions g_2 and h_2 on the interval $[0,\ r_p)$ by means of the expressions:

$$\begin{aligned} g_2(t) =&\frac{1}{2(1-L_0 t)}\Bigg[Lt + \frac{2M^2\big(|\alpha-1|+|\beta|+|\gamma|\big)\big(|1-c_1|+|c_2|g_1(t)\big)}{|\alpha+\beta+\gamma|(1-L_0 t)(1-p(t))}\\ &+\frac{2M\big(|1-c_1|+|c_2|g_1(t)\big)}{1-L_0 t}\Bigg] \end{aligned}$$

and:

$$h_2(t) = g_2(t) - 1.$$

Suppose that:

$$M\big(|1-c_1| + c_2 M|1-\delta|\big)\left(1 + \frac{M\big(|\alpha-1|+|\beta|+|\gamma|\big)}{|\alpha+\beta+\gamma|}\right) < 1. \tag{7}$$

Then, we have by Equation (7) that $h_2(0) < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$ by the definition of r_p. We recall r_2 as the least zero of h_2 on $(0,\ r_p)$.
Define:

$$r = \min\{r_1,\ r_2\}. \tag{8}$$

Then, notice that for all $t \in [0,\ r)$:

$$0 < r < r_A, \tag{9}$$

$$0 \leq g_1(t) < 1, \tag{10}$$

$$0 \le p(t) < 1, \tag{11}$$

$$0 \le g_2(t) < 1. \tag{12}$$

Assume that $Q(x, \delta) = \left\{ y \in \mathbb{B}_1 : \|x - y\| < \delta \right\}$. We can now proceed with the local convergence study of (2) adopting the preceding notations.

Theorem 1. *Let us assume that $\varphi : \mathbb{A} \subset \mathbb{B}_1 \to \mathbb{B}_2$ is a differentiable operator. In addition, we consider that there exist $s_* \in \mathbb{A}$, $L > 0$, $L_0 > 0$, $M \ge 1$ and the parameters $\alpha,\ \beta,\ \gamma,\ c_1,\ c_2 \in \mathbb{S}$, with $\alpha + \beta + \gamma \ne 0$, are such that:*

$$\varphi(s_*) = 0, \quad \varphi'(s_*)^{-1} \in L(\mathbb{B}_2,\ \mathbb{B}_1), \tag{13}$$

$$\|\varphi'(s_*)^{-1}(\varphi'(s_*) - \varphi'(x)\| \le L_0\|s_* - x\|, \ \ \forall\ x \in \mathbb{A}. \tag{14}$$

Set $x,\ y \in \mathbb{A}_0 = \mathbb{A} \cap Q\left(s_*,\ \frac{1}{L_0}\right)$ so that:

$$\|\varphi'(s_*)^{-1}(\varphi'(y) - \varphi'(x))\| \le L\|y - x\|, \ \ \forall\ y,\ x \in \mathbb{A}_0 \tag{15}$$

$$\|\varphi'(s_*)^{-1}\varphi'(x)\| \le M, \quad \forall\ x \in \mathbb{A}_0, \tag{16}$$

satisfies Equations (6) and (7), the condition:

$$\bar{Q}(s_*, r) \subset \mathbb{A}, \tag{17}$$

holds, and the convergence radius r is provided by (8). The obtained sequence of iterations $\{x_j\}$ generated for $x_0 \in Q(s_*,\ r) - \{x^*\}$ by (2) is well defined. In addition, the sequence also converges to the required root s_*, remains in $Q(s_*,\ r)$ for every $n = 0, 1, 2, \ldots$, and:

$$\|y_j - s_*\| \le g_1(\|x_j - s_*\|)\|x_j - s_*\| \le \|x_j - s_*\| < r, \tag{18}$$

$$\|x_{n+1} - s_*\| \le g_2(\|x_j - s_*\|)\|x_j - s_*\| < \|x_j - s_*\|, \tag{19}$$

where the g functions were described previously. Moreover, the limit point s_* of the obtained sequence $\{x_j\}$ is the only root of $\varphi(x) = 0$ in $\mathbb{A}_1 := \bar{Q}(s_*,\ T) \cap \mathbb{A}$, and T is defined as $T \in [r, \frac{2}{L_0})$.

Proof. We prove the estimates (18)–(19), by mathematical induction. Adopting the hypothesis $x_0 \in Q(s_*, r) - \{x^*\}$ and Equations (6) and (14), it results:

$$\|\varphi'(s_*)^{-1}(\varphi'(x_0) - \varphi'(s_*))\| \le L_0\|x_0 - s_*\| < L_0 r < 1. \tag{20}$$

Using Equation (20) and the results on operators by [1–3] that $\varphi'(x_0) \ne 0$, we get:

$$\|\varphi'(x_0)^{-1}\varphi'(s_*)\| \le \frac{1}{1 - L_0\|x_0 - s_*\|}. \tag{21}$$

Therefore, it is clear that y_0 exists. Then, by using Equations (8), (10), (15), (16), and (21), we obtain:

$$\begin{aligned} \|y_0 - s_*\| &= \|(x_0 - s_* - \varphi'(x_0)^{-1}\varphi(x_0)) + (1-\delta)\varphi'(x_0)^{-1}\varphi(x_0)\| \\ &\le \|\varphi'(x_0)^{-1}\varphi'(s_*)\|\|\textstyle\int_0^1 \varphi'(x^*)^{-1}[\varphi'(s_* + \theta(x_0 - s_*)) - \varphi'(x_0)](x_0 - s_*)d\theta\| \\ &\quad + \|\varphi'(x_0)^{-1}\varphi'(s_*)\|\|\textstyle\int_0^1 \varphi'(x^*)^{-1}\varphi'(s_* + \theta(x_0 - s_*))(x_0 - s_*)d\theta\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - s_*\|)} + \frac{M|1-\delta|\|x_0 - s_*\|}{1 - L_0\|x_0 - s_*\|} \\ &= g_1(\|x_0 - s_*\|)\|x_0 - s_*\| < \|x_0 - s_*\| < r, \end{aligned} \tag{22}$$

illustrating that $y_0 \in Q(s_*, r)$ and Equation (18) is true for $j = 0$.

Now, we demonstrate that the linear operator A_0 is invertible. By Equations (8), (10), (14), and (22), we obtain:

$$\begin{aligned} &\|((\alpha+\beta+\gamma)\varphi'(s_*))^{-1}(A_0 - (\alpha+\beta+\gamma)\varphi'(s_*))\| \\ &\leq \frac{L_0}{|\alpha+\beta+\gamma|}\left[|\alpha|\|x_0 - s_*\| + \frac{|\beta|}{2}(\|x_0 - s_*\| + \|y_0 - s_*\|) + |\gamma|\|y_0 - s_*\|\right] \\ &\leq \frac{L_0}{|\alpha+\beta+\gamma|}\left[|\alpha| + \frac{|\beta|}{2}\left(\frac{|\beta|}{2} + |\gamma|\right) g_1(\|x_0 - s_*\|)\|x_0 - s_*\|\right] \\ &= p(\|x_0 - s_*\|) < p(r) < 1. \end{aligned} \tag{23}$$

Hence, $A_0^{-1} \in L(\mathbb{B}_2,\ \mathbb{B}_1)$,

$$\|A_0^{-1}\varphi'(s_*)\| \leq \frac{1}{|\alpha+\beta+\gamma|(1 - p(\|x_0 - s_*\|))}, \tag{24}$$

and x_1 exists. Therefore, we need the identity:

$$\begin{aligned} x_1 - s_* =& x_0 - s_* - \varphi'(x_0)^{-1}\varphi(x_0) - \varphi'(x_0)^{-1}((1-c_1)\varphi(x_0) + c_2\varphi(y_0)) \\ &+ \varphi'(x_0)^{-1}(A_0 - \varphi'(x_0))A_0^{-1}(c_1\varphi(x_0) + c_2\varphi(y_0)). \end{aligned} \tag{25}$$

Further, we have:

$$\begin{aligned} \|x_1 - s_*\| \leq& \|x_0 - s_* - \varphi'(x_0)^{-1}\varphi(x_0)\| + \|\varphi'(x_0)^{-1}((1-c_1)\varphi(x_0) + c_2\varphi(y_0))\| \\ &+ \|\varphi'(x_0)^{-1}\varphi'(s_*)\|\|\varphi'(s_*)^{-1}(A_0 - \varphi'(x_0))\|\|A_0^{-1}\varphi'(s_*)\|\|\varphi'(s_*)^{-1}(c_1\varphi(x_0) + c_2\varphi(y_0))\| \\ &\leq \frac{L\|x_0 - s_*\|^2}{2(1 - L_0\|x_0 - s_*\|)} + \frac{M(|1-c_1|\|x_0 - s_*\| + |c_2|\|y_0 - s_*\|)}{1 - L_0\|x_0 - s_*\|} \\ &+ \frac{M^2(|\alpha - 1| + |\beta| + |\gamma|)(|1-c_1| + |c_2|g_1(\|x_0 - s_*\|))\|x_0 - s_*\|}{|\alpha+\beta+\gamma|(1 - L_0\|x_0 - s_*\|)(1 - p(\|x_0 - s_*\|))} \\ &\leq g_2(\|x_0 - s_*\|)\|x_0 - s_*\| < \|x_0 - s_*\| < r, \end{aligned} \tag{26}$$

which demonstrates that $x_1 \in Q(s_*\ r)$ and (19) is true for $j = 0$, where we used (15) and (21) for the derivation of the first fraction in the second inequality. By means of Equations (21) and (16), we have:

$$\begin{aligned} \|\varphi(s_*)^{-1}\varphi(x_0)\| &= \|\varphi'(s_*)^{-1}(\varphi(x_0) - \varphi(s_*))\| \\ &= \left\|\int_0^1 \varphi'(s_*)^{-1}\varphi'(s_* + \theta(x_0 - s_*))d\theta\right\| \leq M\|x_0 - s_*\|. \end{aligned}$$

In the similar fashion, we obtain $\|\varphi'(s_*)^{-1}\varphi(y_0)\| \leq M\|y_0 - s_*\| \leq Mg_1(\|x_0 - s_*\|)\|x_0 - s_*\|$ (by (22)) and the definition of $\mathbb{A}$ to arrive at the second section. We reach (18) and (19), just by changing x_0, z_0, y_0, and x_1 by x_j, z_j, y_j, and x_{j+1}, respectively. Adopting the estimates $\|x_{j+1} - s_*\| \leq q\|x_j - s_*\| < r$, where $q = g_2(\|x_0 - s_*\|) \in [0, 1)$, we conclude that $x_{j+1} \in Q(s_*,\ r)$ and $\lim_{j\to\infty} x_j = s_*$. To illustrate the unique solution, we assume that $y_* \in \mathbb{A}_1$, satisfying $\varphi(y_*) = 0$ and $U = \int_0^1 \varphi'(y_* + \theta(s_* - y_*))d\theta$. From Equation (14), we have:

$$\begin{aligned} &\|\varphi'(s_*)^{-1}(U - \varphi'(s_*))\| \leq \|\int_0^1 L_0|y_* + \theta(s_* - y_*) - s_*\|d\theta \\ &\leq \int_0^1 (1-t)\|y_* - s_*\|d\theta \leq \frac{L_0}{2}T < 1. \end{aligned} \tag{27}$$

It follows from Equation (27) that U is invertible. Therefore, the identity $0 = \varphi(y_*) - \varphi(s_*) = U(y_* - s_*)$ leads to $y_* = s_*$. □

3. Numerical Experiments

Herein, we illustrate the previous theoretical results by means of six examples. The first two are standard test problems. The third is a counter problem where we show that the previous results are not applicable. The remaining three examples are real-life problems considered in several disciplines of science.

Example 1. *We assume that $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3,\ \mathbb{A} = \bar{Q}(0,\ 1)$. Then, the function φ is defined on $\mathbb{A}$ for $u = (x_1,\ x_2,\ x_3)^T$ as follows:*

$$\varphi(u) = \left(e_1^x - 1,\ x_2 - \frac{1}{2}(1-e)x_2^2,\ x_3\right)^T. \tag{28}$$

We yield the following Fréchet-derivative:

$$\varphi'(u) = \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & (e-1)x_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is important to note that we have $s_* = (0,\ 0,\ 0)^T,\ L_0 = e - 1 < L = e^{\frac{1}{L_0}}, \delta = 1, M = 2, c_1 = 1,$ and $\varphi'(s_*) = \varphi'(s_*)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By considering the parameter values that were defined in Theorem 1, we get the different radii of convergence that are depicted in Tables 1 and 2.

Table 1. Radii of convergence for Example 1, where $L_0 < L$.

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1,\ r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.382692	0.0501111	0.0501111
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.382692	0.334008	0.334008
3	1	1	1	0	0.382692	0.382692	0.382692
4	1	1	1	$\frac{1}{100}$	0.382692	0.342325	0.342325
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.382692	0.325413	0.325413

Table 2. Radii of convergence for Example 1, where $L_0 = L = e$ by [3,11].

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1,\ r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.245253	0.0326582	0.0326582
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.245253	0.213826	0.213826
3	1	1	1	0	0.245253	0.245253	0.245253
4	1	1	1	$\frac{1}{100}$	0.245253	0.219107	0.219107
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.245253	0.208097	0.208097

Example 2. *Let us consider that $\mathbb{B}_1 = \mathbb{B}_2 = C[0,\ 1],\ \mathbb{A} = \bar{Q}(0,\ 1)$ and introduce the space of continuous maps in $[0,\ 1]$ having the max norm. We consider the following function φ on $\mathbb{A}$:*

$$\varphi(\phi)(x) = \varphi(x) - 5\int_0^1 x\tau\phi(\tau)^3 d\tau, \tag{29}$$

which further yields:

$$\varphi'(\phi(\mu))(x) = \mu(x) - 15\int_0^1 x\tau\phi(\tau)^2\mu(\tau)d\tau, \text{ for each } \mu \in \mathbb{A}.$$

We have $s_* = 0,\ L = 15,\ L_0 = 7.5, M = 2, \delta = 1,$ *and* $c_1 = 1$. *We will get different radii of convergence on the basis of distinct parametric values as mentioned in Tables 3 and 4.*

Table 3. Radii of convergence for Example 2, where $L_0 < L$.

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1, r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.0666667	0.00680987	0.00680987
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.0666667	0.0594212	0.0594212
3	1	1	1	0	0.0666667	0.0666667	0.0666667
4	1	1	1	$\frac{1}{100}$	0.0666667	0.0609335	0.0609335
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.0666667	0.0588017	0.0588017

Table 4. Radii of convergence for Example 2, where $L_0 = L = 15$ by [3,11].

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1, r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.0444444	0.00591828	0.00591828
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.0444444	0.0387492	0.0387492
3	1	1	1	0	0.0444444	0.0444444	0.0444444
4	1	1	1	$\frac{1}{100}$	0.0444444	0.0397064	0.0397064
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.0444444	0.0377112	0.0377112

Example 3. *Let us return to the problem from the Introduction. We have* $s_* = 1,\ L = L_0 = 96.662907,$ $M = 2,\ \delta = 1,$ *and* $c_1 = 1$. *By substituting different values of the parameters, we have distinct radii of convergence listed in Table 5.*

Table 5. Radii of convergence for Example 3.

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1, r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.00689682	0.000918389	0.000918389
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.00689682	0.00601304	0.00601304
3	1	1	1	0	0.00689682	0.00689682	0.00689682
4	1	1	1	$\frac{1}{100}$	0.00689682	0.00616157	0.00616157
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.00689682	0.0133132	0.0133132

Example 4. *The chemical reaction [12] illustrated in this case shows how* W_1 *and* W_2 *are utilized at rates* $q_* - Q_*$ *and* Q_**, respectively, for a tank reactor (known as CSTR), given by:*

$$W_2 + W_1 \to W_3$$
$$W_3 + W_1 \to W_4$$
$$W_4 + W_1 \to W_5$$
$$W_5 + W_1 \to W_6$$

Douglas [13] analyzed the CSTR problem for designing simple feedback control systems. The following mathematical formulation was adopted:

$$K_C \frac{2.98(x+2.25)}{(x+1.45)(x+2.85)^2(x+4.35)} = -1,$$

where the parameter K_C has a physical meaning and is described in [12,13]. For the particular value of choice $K_C = 0$, we obtain the corresponding equation:

$$\varphi(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875. \tag{30}$$

The function φ has four zeros $s_* = (-1.45, -2.85, -2.85, -4.35)$. Nonetheless, the desired zero is $s_* = -4.35$ for Equation (30). Let us also consider $\mathbb{A} = [-4.5, -4]$.

Then, we obtain:

$$L_0 = 1.2547945,\ L = 29.610958,\ M = 2,\ \delta = 1,\ c_1 = 1.$$

Now, with the help of different values of the parameters, we get different radii of convergence displayed in Table 6.

Table 6. Radii of convergence for Example 4.

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1, r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.0622654	0.00406287	0.00406287
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.0622654	0.0582932	0.0582932
3	1	1	1	0	0.0622654	0.0622654	0.0622654
4	1	1	1	$\frac{1}{100}$	0.0622654	0.0592173	0.0592173
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.0622654	0.0585624	0.0585624

Example 5. *Here, we assume one of the well-known Hammerstein integral equations (see pp. 19–20, [14]) defined by:*

$$x(s) = 1 + \frac{1}{5}\int_0^1 F(s,t)x(t)^3 dt,\ \ x \in C[0,1],\ s,t \in [0,1], \tag{31}$$

where the kernel F is:

$$F(s,t) = \begin{cases} s(1-t), s \le t, \\ (1-s)t, t \le s. \end{cases}$$

We obtain (31) by using the Gauss–Legendre quadrature formula with $\int_0^1 \phi(t)dt \simeq \sum_{k=1}^{8} w_k \phi(t_k)$, where t_k and w_k are the abscissas and weights, respectively. Denoting the approximations of $x(t_i)$ with x_i $(i = 1, 2, 3, ..., 8)$, then it yields the following 8×8 system of nonlinear equations:

$$5x_i - 5 - \sum_{k=1}^{8} a_{ik} x_k^3 = 0, \ i = 1, 2, 3..., 8,$$

$$a_{ik} = \begin{cases} w_k t_k (1 - t_i), & k \le i, \\ w_k t_i (1 - t_k), & i < k. \end{cases}$$

The values of t_k and w_k can be easily obtained from the Gauss–Legendre quadrature formula when $k = 8$. The required approximate root is:

$$s_* = (1.002096\ldots,\ 1.009900\ldots,\ 1.019727\ldots,\ 1.026436\ldots,\ 1.026436\ldots,\ 1.019727\ldots,\ 1.009900\ldots,\ 1.002096\ldots)^T.$$

Then, we have:

$$L_0 = L = \frac{3}{40},\ M = 2,\ \delta = 1,\ c_1 = 1$$

and $\mathbb{A} = Q(s_*, 0.11)$. By using the different values of the considered disposable parameters, we have different radii of convergence displayed in Table 7.

Table 7. Radii of convergence for Example 5.

Cases	**Different Values of Parameters That Are Defined in Theorem 1**						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1, r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	8.88889	1.18366	1.18366
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	8.88889	7.74984	7.74984
3	1	1	1	0	8.88889	8.88889	8.88889
4	1	1	1	$\frac{1}{100}$	8.88889	7.94127	7.94127
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	8.88889	7.54223	7.54223

Example 6. *One can find the boundary value problem in [14], given as:*

$$y'' = \frac{1}{2}y^3 + 3y' - \frac{3}{2-x} + \frac{1}{2},\ \ y(0) = 0,\ y(1) = 1. \tag{32}$$

We suppose the following partition of $[0,\ 1]$:

$$x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_j, \text{ where } \ x_{i+1} = x_i + h,\ h = \frac{1}{j}.$$

In addition, we assume that $y_0 = y(x_0) = 0,\ y_1 = y(x_1),\ \ldots,\ y_{j-1} = y(x_{j-1})$ and $y_j = y(x_j) = 1$. Now, we can discretize this problem (32) relying on the first- and second-order derivatives, which is given by:

$$y'_k = \frac{y_{k+1} - y_{k-1}}{2h},\ y''_k = \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2},\ k = 1,\ 2,\ \ldots,\ j-1.$$

Hence, we find the following general $(j-1) \times (j-1)$ nonlinear system:

$$y_{k+1} - 2y_k + y_{k-1} - \frac{h^2}{2}y_k^3 - \frac{3}{2 - x_k}h^2 - \frac{1}{h^2} = 0,\ k = 1, 2, \ldots, j-1.$$

We choose the particular value of $j = 7$ that provides us a 6×6 nonlinear systems. The roots of this nonlinear system are $s_* = \big(0.07654393\ldots,\ 0.1658739\ldots,\ 0.2715210\ldots,\ 0.3984540\ldots,\ 0.5538864\ldots,\ 0.7486878\ldots\big)^T$, and the results are mentioned in Table 8.

Then, we get that:

$$L_0 = 73,\ L = 75,\ M = 2,\ \delta = 1,\ c_1 = 1,$$

and $\mathbb{A} = Q(s_*, 0.15)$.
With the help of different values of the parameters, we have the different radii of convergence listed in Table 8.

Table 8. Radii of convergence for Example 6.

Cases	Different Values of Parameters That Are Defined in Theorem 1						
	α	β	γ	c_2	r_1	r_2	$r = \min\{r_1, r_2\}$
1	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	0.00904977	0.00119169	0.00119169
2	$-\frac{2}{3}$	$\frac{4}{3}$	-100	$\frac{1}{100}$	0.00904977	0.00789567	0.00789567
3	1	1	1	0	0.00904977	0.00904977	0.00904977
4	1	1	1	$\frac{1}{100}$	0.00904977	0.00809175	0.00809175
5	10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$	0.00904977	0.00809175	0.00809175

Remark 1. *It is important to note that in some cases, the radii r_i are larger than the radius of $Q(s_*, r)$. A similar behavior for Method (2) was noticed in Table 7. Therefore, we have to choose all $r_i = 0.11$ because Expression (17) must be also satisfied.*

4. Concluding Remarks

The local convergence of the fourth-order scheme (2) was shown in earlier works [5,6,8,15] using Taylor series expansion. In this way, the hypotheses reach to four-derivative of the function φ in the particular case when $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^m$ and $S = \mathbb{R}$. These hypotheses limit the applicability of methods such (2). We analyze the local convergence using only the first derivative for Banach space mapping. The convergence order can be found using the computational order of convergence (COC)or the approximate computational order of convergence $(ACOC)$ (Appendix A), avoiding the computation of higher order derivatives. We found also computable radii and error bounds not given before using Lipschitz constants, expanding, therefore, the applicability of the technique. Six numerical problems were proposed for illustrating the feasibility of the new approach. Our technique can be used to study other iterative methods containing inverses of mapping such as (3)–(5) (see also [1–9,11–45]) and to expand their applicability along the same lines.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism
COC	Computational order of convergence
(COC)	Approximate computational order of convergence

Appendix A

Remark

(a) The procedure of studying local convergence was already given in [1,2] for similar methods. Function $M(t) = M = 2$ or $M(t) = 1 + L_0 t$, since $0 \leq t < \frac{1}{L_0}$ can be replaced by (16). The convergence radius r cannot be bigger than the radius r_A for the Newton method given in this paper. These results are used to solve autonomous differential equations. The differential equation plays an important role in the study of network science, computer systems, social networking systems, and biochemical systems [46].

In fact, we refer the reader to [46], where a different technique is used involving discrete samples from the existence of solution spaces. The existence of intervals with common solutions, as well as disjoint intervals and the multiplicity of intervals with common solutions is also shown. However, this work does not deal with spaces that are continuous and multidimensional.

(b) It is important to note that the scheme (2) does not change if we adopt the hypotheses of Theorem 1 rather than the stronger ones required in [5–9]. In practice, for the error bounds, we adopt the following formulas [22] for the computational order of convergence (COC), when the required root is available, or the approximate computational order of convergence $(ACOC)$, when the required root is not available in advance, which can be written as:

$$\xi = \frac{\ln \frac{\|x_{k+2} - s_*\|}{\|x_{k+1} - s_*\|}}{\ln \frac{\|x_{k+1} - s_*\|}{\|x_k - s_*\|}}, \quad k = 0, 1, 2, 3 \ldots,$$

$$\xi^* = \frac{\ln \frac{\|x_{k+2} - x_{k+1}\|}{\|x_{k+1} - x_k\|}}{\ln \frac{\|x_{k+1} - x_k\|}{\|x_k - x_{k-1}\|}}, \quad k = 1, 2, 3, \ldots,$$

respectively. By means of the above formulas, we can obtain the convergence order without using estimates on the high-order Fréchet derivative.

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