Singularities for One-Parameter Developable Surfaces of Curves

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Abstract: Developable surfaces, which are important objects of study, have attracted a lot of attention from many mathematicians. In this paper, we study the geometric properties of one-parameter developable surfaces associated with regular curves. According to singularity theory, the generic singularities of these developable surfaces are classified—they are swallowtails and cuspidal edges. In addition, we give some examples of developable surfaces which have symmetric singularity models.

Keywords: one-parameter developable surfaces; singularity; one-parameter support functions

1. Introduction

The study of developable surfaces has many practical applications. There is much literature about developable surfaces, (see, e.g., [1–5]). Many cartographic projections involve projecting the Earth onto a developable surface and then "unrolling" the surface into a region on the plane. Since developable surfaces may be constructed by bending flat sheets, they are also important in manufacturing objects from cardboard, plywood, and sheet metal. In mathematics, developable surfaces are typically defined as surfaces which can be developed into planes without distorting the surface metric. There is some literature about developable surfaces of space curves from the viewpoint of singularity theory [1,2]. The *tangent developable surface* of a space curve is a ruled surface, which is formed by the space curve's tangent lines. In algebraic geometry, tangent developable surfaces play an important role in the duality theory [6]. In [1], the author investigated the relationship between the singularities of tangent developable surfaces and some types of space curves. He also gave a classification of tangent developable surfaces by using the local topological property. On the other hand, S. Izumiya et al. introduced the rectifying developable surfaces of space curves in [2], where they showed that a regular curve is a geodesic of its rectifying developable surface and revealed the relationship between singularities of the rectifying developable surface and geometric invariants. The geometric invariants can also characterize the contact between a space curve and a helix. In this sense, the study of the singularities of developable surfaces is an interesting subject.

In the present paper, we investigate one-parameter developable surfaces, which are related to the space curves, as a fundamental case for the research of the highest dimensional manifolds in Euclidean 3-space. We investigated the singularities of hypersurfaces in semi-Euclidean space [7–10]. However, at least to the best of our knowledge, there exists little literature concerning the singularities of one-parameter developable surfaces related to regular space curves in Euclidean space. Therefore, we study this problem in the present paper. In the frame of space curves, we define one-parameter developable surfaces. When the parameter is fixed, the sections of one-parameter developable surfaces are developable surfaces. Moreover, the tangent developable surfaces and the rectifying developable

surfaces are sections of one-parameter developable surfaces. We also define the one-parameter support functions on regular space curves, which can be used to study the geometric properties of one-parameter developable surfaces. In fact, one-parameter developable surfaces are the discriminant sets of these functions. The main result, Theorem 2, shows that the singularities of developable surfaces are A_k-singularities ($k = 2, 3$) of these functions.

The organization of this paper is as follows: We review the concepts of ruled surfaces in Euclidean space in Section 2. In Section 3, the one-parameter developable surfaces of a space curve are defined, and we obtain two geometric invariants of the curve. We also get singularities of one-parameter developable surfaces (Theorem 1), and Theorem 2 gives the classification of these singularities in this section. The preparations for the proof of Theorem 2 are in Sections 4 and 5. In the last section, we give some examples to illustrate the main results in this paper.

2. Basic Notation

Let $\mathbb{R}^3$ be 3-dimensional Euclidean space and $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$. We denote their standard inner product by $x \cdot y$, and the norm of x is denoted by $\|x\|$. Let $\gamma : I \to \mathbb{R}^3$ be a curve and the tangent vector respect to t is $\dot{\gamma}(t) = d\gamma/dt(t)$. The *arc-length* is $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| dt$ and $\|\gamma'(s)\| = \|d\gamma/ds\| = 1$. We define three unit vectors $t(s) = \gamma'(s)$, $n(s) = \gamma''(s)/\|\gamma''(s)\|$, and $b(s) = t(s) \times n(s)$. Then, the Frenet-Serret formula is as follows:

$$\begin{cases} t'(s) = \kappa(s)n(s) \\ n'(s) = -\kappa(s)t(s) + \tau(s)b(s) \\ b'(s) = -\tau(s)n(s), \end{cases}$$

where $\kappa(s)$ is the curvature function and $\tau(s)$ is the torsion function.

We now introduce developable surfaces and ruled surfaces. Suppose that $\gamma : I \to \mathbb{R}^3$ be a curve and $\xi : I \to \mathbb{R}^3 \setminus \{0\}$ be a C^∞-mapping. A surface $F_{(\gamma,\xi)} : I \times \mathbb{R} \to \mathbb{R}^3$ is defined by

$$F_{(\gamma,\xi)}(s, u) = \gamma(s) + u\xi(s),$$

then $F_{(\gamma,\xi)}$ is a *ruled surface*, and γ and ξ are called the *base curve* and *director curve*, respectively. For a fixed $s \in I$, $\gamma(s) + u\xi(s)$ is the *ruling*. A *developable surface* is a ruled surface with vanishing Gaussian curvature. It's well known that $F_{(\gamma,\xi)}$ is a developable surface if and only if $\det(\gamma'(s), \xi(s), \xi'(s)) = 0$.

$F_{(\gamma,\xi)}$ is called a *cylinder* if the director curve ξ has a fixed direction. We denote that $\widetilde{\xi}(s) = \xi(s)/\|\xi(s)\|$, then $F_{(\gamma,\xi)}$ is a cylinder if and only if $\widetilde{\xi}'(s) \equiv 0$. If a ruled surface $F_{(\gamma,\xi)}$ is not a cylinder, a *striction curve* of $F_{(\gamma,\xi)}$ is defined by

$$c(s) = \gamma(s) - \frac{\gamma'(s) \cdot \widetilde{\xi}'(s)}{\widetilde{\xi}'(s) \cdot \widetilde{\xi}'(s)} \widetilde{\xi}(s).$$

It is known that the singularities of ruled surface $F_{(\gamma,\xi)}$ (not a cylinder) are located on its striction curve [11]. We say $F_{(\gamma,\xi)}$ is a *cone* if and only if the striction curve c is constant.

3. One-Parameter Developable Surfaces

We consider the one-parameter developable surfaces of space curves in this section. Let γ be a space curve. We consider a spherical vector $L : [0, \frac{\pi}{2}] \times I \to S^2$, which is defined by

$$L(s, \theta) = \frac{\tau(s)t(s) - \kappa(s)\sin\theta\cos\theta n(s) + \kappa(s)\cos^2\theta b(s)}{\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta}},$$

where $\tau^2(s) + \kappa^2(s)\cos^2\theta \neq 0$. We assume, throughout the whole paper, that $\tau^2(s) + \kappa^2(s)\cos^2\theta \neq 0$ for any $(s, \theta) \in I \times [0, \frac{\pi}{2}]$. We write $L(s, \theta) = L_\theta(s)$, and define a map $D_\theta : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$D_\theta(s, u) = \gamma(s) + uL_\theta(s) = \gamma(s) + u\frac{\tau(s)t(s) - \kappa(s)\sin\theta\cos\theta n(s) + \kappa(s)\cos^2\theta b(s)}{\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta}}.$$

We call D_θ a *one-parameter developable surface* of γ. We can easily check that $D_{\frac{\pi}{2}}(s, u)$ is the tangent developable surface of γ and $D_0(s, u)$ is the rectifying developable surface of γ.

For any $\theta_0 \in [0, \frac{\pi}{2}]$, we have

$$L'_{\theta_0}(s) = \frac{(\kappa\tau^2\sin\theta_0 + \kappa^3\sin\theta_0\cos^2\theta_0 + \kappa\tau'\cos\theta_0 - \kappa'\tau\cos\theta_0)(\kappa\cos\theta_0 t + \tau\sin\theta_0 n - \tau\cos\theta_0 b)}{(\tau^2 + \kappa^2\cos^2\theta_0)^{\frac{3}{2}}}.$$

So, we have

$$\det\big(\gamma'(s), L_{\theta_0}(s), L'_{\theta_0}(s)\big)$$

$$= \big(\kappa\tau^2\sin\theta_0 + \kappa^3\sin\theta_0\cos^2\theta_0 + \kappa\tau'\cos\theta_0 - \kappa'\tau\cos\theta_0\big)\det\Big(t, \frac{\tau t - \kappa\sin\theta_0\cos\theta_0 n + \kappa\cos^2\theta_0 b}{\sqrt{\tau^2 + \kappa^2\cos^2\theta_0}},$$

$$\frac{\kappa\cos\theta_0 t + \tau\sin\theta_0 n - \tau\cos\theta_0 b}{(\tau^2 + \kappa^2\cos^2\theta_0)^{\frac{3}{2}}}\Big)$$

$$= 0$$

for all $s \in I$. This means D_{θ_0} is a developable surface. For this reason, we call D_θ the one-parameter developable surfaces of γ. Moreover, we introduce two invariants as follows:

$$\delta(s) = \frac{\kappa(s)\sin\theta_0(\tau^2(s) + \kappa^2(s)\cos^2\theta_0) + \cos\theta_0(\kappa(s)\tau'(s) - \kappa'(s)\tau(s))}{\tau^2(s) + \kappa^2(s)\cos^2\theta_0},$$

$$\sigma(s) = \frac{\tau(s)}{\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta_0}} - \frac{d}{ds}\left(\frac{\kappa(s)\cos\theta_0}{\delta(s)\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta_0}}\right), \quad (\text{when } \delta(s) \neq 0).$$

Since

$$L'_{\theta_0}(s) = \frac{(\kappa\tau^2\sin\theta_0 + \kappa^3\sin\theta_0\cos^2\theta_0 + \kappa\tau'\cos\theta_0 - \kappa'\tau\cos\theta_0)(\kappa\cos\theta_0 t + \tau\sin\theta_0 n - \tau\cos\theta_0 b)}{(\tau^2 + \kappa^2\cos^2\theta_0)^{\frac{3}{2}}}(s)$$

$$= \delta(s)\frac{\kappa\cos\theta_0 t + \tau\sin\theta_0 n - \tau\cos\theta_0 b}{\sqrt{\tau^2 + \kappa^2\cos^2\theta_0}}(s),$$

so that $\delta(s) = 0$ if and only if $L'_{\theta_0}(s) = \mathbf{0}$. We can also calculate that

$$\frac{\partial D_{\theta_0}}{\partial s}(s, u) \times \frac{\partial D_{\theta_0}}{\partial u}(s, u) = -\left(\frac{\kappa(s)\cos\theta_0}{\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta_0}} + u\delta(s)\right)(\sin\theta_0 b(s) + \cos\theta_0 n(s)).$$

Thus, that $(s_0, u_0) \in I \times \mathbb{R}$ is a singular point of D_{θ_0} is equivalent to

$$\frac{\kappa(s_0)\cos\theta_0}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}} + u_0\frac{\kappa(s_0)\sin\theta_0(\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0) + \cos\theta_0(\kappa(s_0)\tau'(s_0) - \kappa'(s_0)\tau(s_0))}{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0} = 0.$$

If $\theta_0 \neq \frac{\pi}{2}$ and (s_0, u_0) is a singular point of D_{θ_0}, then we have $u_0 \neq 0$; that is, D_{θ_0} has no singular points on the base curve $\gamma(s)$. We have the following result for $\delta(s)$ and $\sigma(s)$:

Theorem 1. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve. Then the following holds:*
(A) For any $\theta_0 \in [0, \frac{\pi}{2}]$, the following statements are equivalent:
  (1) D_{θ_0} is a cylinder,
  (2) $\delta(s) = 0$ for all $s \in I$.
(B) If $\delta(s) \neq 0$ for all $s \in I$, then the following statements are equivalent:

(3) $\boldsymbol{D}_{\theta_0}$ is a conical surface,

(4) $\sigma(s) = 0$ for all $s \in I$.

(C) The singularities of one-parameter developable surfaces $\boldsymbol{D}_\theta$ are $\boldsymbol{\gamma}(s)$, $\boldsymbol{D}_{\frac{\pi}{2}}$ and

$$\{\boldsymbol{D}_\theta(s,u)|\kappa(s)\cos\theta\sqrt{\tau^2(s)+\kappa^2(s)\cos^2\theta}+u[\kappa(s)\sin\theta(\tau^2(s)+\kappa^2(s)\cos^2\theta)+\cos\theta(\kappa(s)\tau'(s)-\kappa'(s)\tau(s))]=0\}.$$

Proof. (A) By definition, the developable surface is a cylinder if and only if the director vector is a constant vector and $\boldsymbol{L}_{\theta_0}(s)$ is the director vector of $\boldsymbol{D}_{\theta_0}$. Since

$$\boldsymbol{L}'_{\theta_0}(s) = \delta(s)\frac{\kappa\cos\theta_0\boldsymbol{t}+\tau\sin\theta_0\boldsymbol{n}-\tau\cos\theta_0\boldsymbol{b}}{\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}(s),$$

then $\boldsymbol{D}_{\theta_0}$ is a cylinder if and only if $\delta(s) = 0$ for all $s \in I$.

(B) We consider the striction curve $\boldsymbol{c}(s)$ which is defined by

$$\boldsymbol{c}(s) = \boldsymbol{\gamma}(s) - \frac{\boldsymbol{\gamma}(s)\cdot\boldsymbol{L}'_{\theta_0}(s)}{\boldsymbol{L}'_{\theta_0}(s)\cdot\boldsymbol{L}'_{\theta_0}(s)}\boldsymbol{L}_{\theta_0}(s) = \boldsymbol{\gamma}(s) - \frac{\kappa(s)\cos\theta_0}{\delta(s)\sqrt{\tau^2(s)+\kappa^2(s)\cos^2\theta_0}}\boldsymbol{L}_{\theta_0}(s).$$

Then (B)-(3) is equivalent to $\boldsymbol{c}'(s) = 0$, for all $s \in I$. We can calculate that

$$\begin{aligned}
\boldsymbol{c}' &= \boldsymbol{\gamma}' - \frac{d}{ds}\left(\frac{\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)\boldsymbol{L}_{\theta_0} - \frac{\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\boldsymbol{L}'_{\theta_0} \\
&= \boldsymbol{t} - \frac{\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\frac{\delta(\kappa\cos\theta_0\boldsymbol{t}+\tau\sin\theta_0\boldsymbol{n}-\tau\cos\theta_0\boldsymbol{b})}{\sqrt{\tau^2+\kappa^2\cos^2\theta_0}} - \frac{d}{ds}\left(\frac{\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)\boldsymbol{L}_{\theta_0} \\
&= \frac{\tau(\tau\boldsymbol{t}-\kappa\cos\theta_0\sin\theta_0\boldsymbol{n}+\kappa\cos^2\theta_0\boldsymbol{b})}{\tau^2+\kappa^2\cos^2\theta_0} - \frac{d}{ds}\left(\frac{\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)\boldsymbol{L}_{\theta_0} \\
&= \sigma(s)\boldsymbol{L}_{\theta_0}.
\end{aligned}$$

It follows that (B)-(3) and (B)-(4) are equivalent.

(C) By straightforward calculation, we have

$$\frac{\partial\boldsymbol{D}_\theta}{\partial s} = \boldsymbol{t} + u\frac{[\kappa\sin\theta(\tau^2+\kappa^2\cos^2\theta)+\cos\theta(\kappa\tau'-\kappa'\tau)](\tau\sin\theta\boldsymbol{n}+\kappa\cos\theta\boldsymbol{t}-\tau\cos\theta\boldsymbol{b})}{(\tau^2+\kappa^2\cos^2\theta)^{\frac{3}{2}}},$$

$$\frac{\partial\boldsymbol{D}_\theta}{\partial u} = \frac{\tau\boldsymbol{t}-\kappa\sin\theta\cos\theta\boldsymbol{n}+\kappa\cos^2\theta\boldsymbol{b}}{\sqrt{\tau^2+\kappa^2\cos^2\theta}},$$

$$\frac{\partial\boldsymbol{D}_\theta}{\partial\theta} = u\frac{\kappa^2\tau\cos\theta\sin\theta\boldsymbol{t}+(\kappa\tau^2\sin^2\theta-\kappa^3\cos^4\theta-\kappa\tau^2\cos^2\theta)\boldsymbol{n}-(\kappa^3\sin\theta\cos^3\theta+2\kappa\tau^2\cos\theta\sin\theta)\boldsymbol{b}}{(\tau^2+\kappa^2\cos^2\theta)^{\frac{3}{2}}}.$$

We can obtain the singularities of $\boldsymbol{D}_\theta$ if the above three vectors are linearly dependent, which is equivalent to

$$\frac{u\kappa\cos\theta[u(\kappa\sin\theta(\tau^2+\kappa^2\cos^2\theta)+\cos\theta(\kappa\tau'-\kappa'\tau))+\kappa\cos\theta\sqrt{\tau^2+\kappa^2\cos^2\theta}]}{(\tau^2+\kappa^2\cos^2\theta)^{\frac{3}{2}}} = 0.$$

This means that $u = 0$ or $\cos\theta = 0$ or

$$\kappa(s)\cos\theta\sqrt{\tau^2(s)+\kappa^2(s)\cos^2\theta}+u[\sin\theta\kappa(s)(\tau^2(s)+\kappa^2(s)\cos^2\theta)-\cos\theta(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))]=0.$$

Therefore, (C) holds. $\square$

We give relationships between the singularities of one-parameter developable surfaces of unit speed curves and the above two invariants, as follows:

Theorem 2. *Let $\gamma : I \to \mathbb{R}^3$ be a space curve. Then, we have the following:*
(1) (s_0, u_0) is a regular point of $\mathbf{D}_{\theta_0}$ if and only if

$$u_0 \delta(s_0) + \frac{\kappa(s_0)\cos\theta_0}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}} \neq 0.$$

(2) Suppose (s_0, u_0) is a singular point of $\mathbf{D}_{\theta_0}$, then $\mathbf{D}_{\theta_0}$ is locally diffeomorphic to the cuspidal edge at (s_0, u_0) if
  (i) $\delta(s_0) \neq 0$, $\sigma(s_0) \neq 0$ and

$$u_0 = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}},$$

or
  (ii) $\delta(s_0) = \cos\theta_0 = 0$ and

$$u_0 \neq \frac{\kappa(s_0)\tau(s_0)}{\kappa(s_0)\tau'(s_0) + \kappa'(s_0)\tau(s_0)}.$$

(3) Suppose (s_0, u_0) is a singular point of $\mathbf{D}_{\theta_0}$, then $\mathbf{D}_{\theta_0}$ is locally diffeomorphic to the swallowtail at (s_0, u_0) if $\delta(s_0) \neq 0$, $\sigma(s_0) = 0$, $\sigma'(s_0) \neq 0$ and

$$u_0 = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}}.$$

Here $SW = \{(x_1, x_2, x_3)|x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail, $C = \{(x_1, x_2, x_3)|x_1^2 = x_2^3\}$ is the cusp and $C \times \mathbb{R}$ is the cuspidal edge (see Figures 1–3).
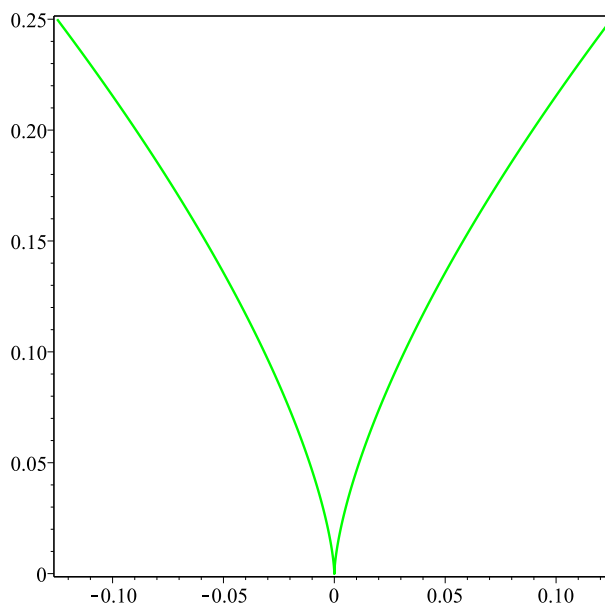

Figure 1. Cusp.

Figure 2. Cuspidal edge.

Figure 3. Swallowtail.

4. One-Parameter Support Functions

For a space curve $\gamma : I \to \mathbb{R}^3$, we introduce a function

$$G : I \times [0, \frac{\pi}{2}] \times \mathbb{R}^3 \to \mathbb{R}$$

by $G(s, \theta, x) = \left(\cos \theta n(s) + \sin \theta b(s) \right) \cdot \left(x - \gamma(s) \right)$. G is called the *one-parameter support function* of γ, with respect to the unit normal vector $\cos \theta n(s) + \sin \theta b(s)$. We denote $g_{\theta_0, x_0}(s) = G(s, \theta_0, x_0)$ for any $(\theta_0, x_0) \in [0, \frac{\pi}{2}] \times \mathbb{R}^3$. Then, we have the following proposition:

Proposition 3. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve and $g_{\theta_0, x_0}(s) = \left(x_0 - \gamma(s) \right) \cdot \left(\cos \theta_0 n(s) + \sin \theta_0 b(s) \right)$ the one-parameter support function. Then, the following statements hold:*
(1) $g_{\theta_0, x_0}(s_0) = 0$ if and only if there exist $v, u \in \mathbb{R}$ such that

$$x_0 - \gamma(s_0) = u t(s_0) + v \left(\sin \theta_0 n(s_0) - \cos \theta_0 b(s_0) \right).$$

(2) $g_{\theta_0, x_0}(s_0) = g'_{\theta_0, x_0}(s_0) = 0$ if and only if there exists $u \in \mathbb{R}$, such that

$$x_0 - \gamma(s_0) = u \frac{\tau(s_0) t(s_0) - \kappa(s_0) \sin \theta_0 \cos \theta_0 n(s_0) + \kappa(s_0) \cos^2 \theta_0 b(s_0)}{\sqrt{\tau^2(s_0) + \kappa^2(s_0) \cos^2 \theta_0}}.$$

(A) Suppose $\delta(s_0) \neq 0$. Then, we have the following:
(3) $g_{\theta_0, x_0}(s_0) = g'_{\theta_0, x_0}(s_0) = g''_{\theta_0, x_0}(s_0) = 0$ if and only if

$$x_0 - \gamma(s_0) = \frac{-\kappa(s_0) \cos \theta_0}{\delta(s_0) \sqrt{\tau^2(s_0) + \kappa^2(s_0) \cos^2 \theta_0}} \frac{t(s_0) \tau(s_0) - \kappa(s_0) \sin \theta_0 \cos \theta_0 n(s_0) + \kappa(s_0) \cos^2 \theta_0 b(s_0)}{\sqrt{\kappa^2(s_0) \cos^2 \theta_0 + \tau^2(s_0)}}.$$

(4) $g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = g''_{\theta_0,x_0}(s_0) = g^{(3)}_{\theta_0,x_0}(s_0) = 0$ *if and only if* $\sigma(s_0) = 0$ *and*

$$x_0 - \gamma(s_0) = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}} \frac{\tau(s_0)t(s_0) - \kappa(s_0)\sin\theta_0\cos\theta_0 n(s_0) + \kappa(s_0)\cos^2\theta_0 b(s_0)}{\sqrt{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)}}.$$

(5) $g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = g''_{\theta_0,x_0}(s_0) = g^{(3)}_{\theta_0,x_0}(s_0) = g^{(4)}_{\theta_0,x_0}(s_0) = 0$ *if and only if* $\sigma(s_0) = \sigma'(s_0) = 0$ *and*

$$x_0 - \gamma(s_0) = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)}} \frac{\tau(s_0)t(s_0) - \kappa(s_0)\sin\theta_0\cos\theta_0 n(s_0) + \kappa(s_0)\cos^2\theta_0 b(s_0)}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}}.$$

(B) Suppose $\delta(s_0) = 0$. *Then, the following statements hold:*

(6) $g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = g''_{\theta_0,x_0}(s_0) = 0$ *if and only if* $\cos\theta_0 = 0$ $(\theta_0 = \frac{\pi}{2})$ *and there exists* $u \in \mathbb{R}$, *such that*

$$x_0 - \gamma(s_0) = u t(s_0).$$

(7) $g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = g''_{\theta_0,x_0}(s_0) = g^{(3)}_{\theta_0,x_0}(s_0) = 0$ *if and only if* $\cos\theta_0 = 0$ $(\theta_0 = \frac{\pi}{2})$ *and*

$$x_0 - \gamma(s_0) = \frac{\kappa(s_0)\tau(s_0)}{\kappa(s_0)\tau'(s_0) + \kappa'(s_0)\tau(s_0)} t(s_0).$$

Proof. Since $g_{\theta_0,x_0}(s) = (x_0 - \gamma(s)) \cdot (\cos\theta_0 n(s) + \sin\theta_0 b(s))$, we have the following:

(i) $g'_{\theta_0,x_0} = (x_0 - \gamma) \cdot (-\kappa\cos\theta_0 t + \tau\cos\theta_0 b - \tau\sin\theta_0 n)$,

(ii) $g''_{\theta_0,x_0} = \kappa\cos\theta_0 + (x_0 - \gamma) \cdot [(\kappa\tau\sin\theta_0 - \kappa'\cos\theta_0)t - (\tau^2\cos\theta_0 + \kappa^2\cos\theta_0 + \tau'\sin\theta_0)n$
$\quad - (\tau^2\sin\theta_0 - \tau'\cos\theta_0)b]$,

(iii) $g'''_{\theta_0,x_0} = 2\kappa'\cos\theta_0 - \kappa\tau\sin\theta_0 + (x_0 - \gamma) \cdot [(\kappa'\tau\sin\theta_0 - \kappa''\cos\theta_0 + \kappa\tau'\sin\theta_0 + \kappa\tau^2\cos\theta_0 + \kappa^3\cos\theta_0)t$
$\quad + (\kappa^3\tau\sin\theta_0 - 3\tau\tau'\cos\theta_0 - 3\kappa\kappa'\cos\theta_0 - \tau''\sin\theta_0 + \tau^3\sin\theta_0)n$
$\quad + (\tau''\cos\theta_0 - \kappa^2\tau\cos\theta_0 - 3\tau\tau'\sin\theta_0 - \tau^3\cos\theta_0)b]$,

(iv) $g^{(4)}_{\theta_0,x_0} = 3\kappa''\cos\theta_0 - 2\kappa'\tau\sin\theta_0 - \kappa^3\cos\theta_0 - 2\kappa\tau'\sin\theta_0 - \kappa\tau^2\cos\theta_0$
$\quad + (x_0 - \gamma) \cdot [(\kappa''\tau\sin\theta_0 - \kappa'''\cos\theta_0 + 2\kappa'\tau'\sin\theta_0 + \kappa'\tau^2\cos\theta_0 + 2\kappa\tau''\sin\theta_0 + 6\kappa^2\kappa'\cos\theta_0$
$\quad - \kappa^4\tau\sin\theta_0 - \kappa\tau^3\sin\theta_0 + 5\kappa\tau\tau'\cos\theta_0)t$
$\quad + (\kappa\kappa'\tau\sin\theta_0 - 4\kappa\kappa''\cos\theta_0 + \kappa^2\tau'\sin\theta_0 + \kappa^4\cos\theta_0 + 2\kappa^2\tau^2\cos\theta_0 + 3\kappa^2\kappa'\tau\sin\theta_0$
$\quad + \kappa^3\tau'\sin\theta_0 - 4\tau\tau''\cos\theta_0 - 3(\kappa')^2\cos\theta_0 - 3(\tau')^2\cos\theta_0 + \tau^4\cos\theta_0 + 6\tau^2\tau'\sin\theta_0)n$
$\quad + (\tau'''\cos\theta_0 - 3(\tau')^2\sin\theta_0 - 4\tau\tau''\sin\theta_0 - 5\kappa\kappa'\tau\cos\theta_0 - \kappa^2\tau'\cos\theta_0$
$\quad + \tau^4\sin\theta_0 - 6\tau^2\tau'\cos\theta_0\kappa^3\tau^2\sin\theta_0)b]$.

By definition, $g_{\theta_0,x_0}(s_0) = 0$ if and only if $x_0 - \gamma(s_0) = u t(s_0) + a n(s_0) + b b(s_0)$ and $a\cos\theta_0 + b\sin\theta_0 = 0$, where $u, a, b \in \mathbb{R}$. We write $a = v\sin\theta_0$ and $b = v\cos\theta_0$, where v is a real number. Then, we have

$$x_0 - \gamma(s_0) = u t(s_0) + v(\sin\theta_0 n(s_0) - \cos\theta_0 b(s_0)).$$

Therefore, (1) holds.

By (i), $g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = 0$ if and only if

$$x_0 - \gamma(s_0) = u t(s_0) + v(\sin\theta_0 n(s_0) - \cos\theta_0 b(s_0))$$

and $u\kappa \cos\theta_0 + v\tau = 0$. Since $\kappa \neq 0$ and $\kappa^2 \cos\theta_0^2 + \tau^2 \neq 0$, then there exists $u \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u \frac{\tau(s_0)\boldsymbol{t}(s_0) - \kappa(s_0)\sin\theta_0 \cos\theta_0 \boldsymbol{n}(s_0) + \cos^2\theta_0 \kappa(s_0)\boldsymbol{b}(s_0)}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}}.$$

Therefore, (2) holds.

By (ii), $g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = g''_{\theta_0,x_0}(s_0) = 0$ if and only if there exists $u \in \mathbb{R}$, such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u \frac{\tau(s_0)\boldsymbol{t}(s_0) - \kappa(s_0)\sin\theta_0 \cos\theta_0 \boldsymbol{n}(s_0) + \kappa(s_0)\cos^2\theta_0 \boldsymbol{b}(s_0)}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}}$$

and

$$\kappa(s_0)\cos\theta_0 + u \frac{\kappa(s_0)\sin\theta_0(\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0) - \cos\theta_0(\kappa'(s_0)\tau(s_0) - \kappa(s_0)\tau'(s_0))}{\sqrt{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)}} = 0.$$

It follows

$$\kappa(s_0)\cos\theta_0 + u\delta(s_0)\sqrt{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)} = 0.$$

Thus,

$$\delta(s_0) = \frac{\kappa(s_0)\sin\theta_0(\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)) - \cos\theta_0(\kappa'(s_0)\tau(s_0) - \kappa(s)\tau'(s_0))}{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)} \neq 0$$

and

$$u = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}},$$

or $\delta(s_0) = 0$ and $\cos\theta_0 = 0$. This completes the proof of (A)-(3) and (B)-(6).

Suppose $\delta(s_0) \neq 0$. By (iii), we have

$$g_{\theta_0,x_0}(s_0) = g'_{\theta_0,x_0}(s_0) = g''_{\theta_0,x_0}(s_0) = g'''_{\theta_0,x_0}(s_0) = 0$$

if and only if

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)}} \frac{\boldsymbol{t}(s_0)\tau(s_0) - \kappa(s_0)\sin\theta_0 \cos\theta_0 \boldsymbol{n}(s_0) + \kappa(s_0)\cos^2\theta_0 \boldsymbol{b}(s_0)}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}}$$

and

$$\left[2\kappa'\cos\theta_0 - \kappa\tau\sin\theta_0 - \frac{\kappa\cos\theta_0}{\delta(\tau^2 + \kappa^2\cos^2\theta_0)}(\kappa'\tau^2\sin\theta_0 + 2\kappa\tau\tau'\sin\theta_0 + 3\kappa^2\kappa'\sin\theta_0\cos^2\theta_0 - \tau\kappa''\cos\theta_0 \right.$$
$$\left. + \kappa\tau''\cos\theta_0)\right](s_0) = 0.$$

We rewrite $\sigma(s)$ as following:

$$\sigma(s) = -\sqrt{\tau^2 + \kappa^2\cos^2\theta_0}\left[2\kappa'\cos\theta_0 - \kappa\tau\sin\theta_0 - \frac{\kappa\cos\theta_0}{\delta(\tau^2 + \kappa^2\cos^2\theta_0)}(\kappa'\tau^2\sin\theta_0\right.$$
$$\left. + 3\kappa^2\kappa'\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\tau'\sin\theta_0 - \tau\kappa''\cos\theta_0 + \kappa\tau''\cos\theta_0)\right](s).$$

Therefore, we have (A)-(4). By similar arguments as above, we have (A)-(5).

Suppose $\delta(s_0) = 0$. By (iii), $g_{\theta_0, x_0}(s_0) = g'_{\theta_0, x_0}(s_0) = g''_{\theta_0, x_0}(s_0) = g'''_{\theta_0, x_0}(s_0) = 0$ if and only if $\cos \theta_0 = 0$ and

$$-\kappa(s_0)\tau(s_0) + u\left(\kappa(s_0)\tau'(s_0) + \kappa'(s_0)\tau(s_0)\right) = 0,$$

where $u \in \mathbb{R}$. Since $\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0 \neq 0$ and $\cos\theta_0 = 0$, we have $u = \frac{\kappa(s_0)\tau(s_0)}{\kappa'(s_0)\tau(s_0) + \kappa(s_0)\tau'(s_0)}$. Therefore, we obtain (B)-(7). □

5. Unfoldings of One-Parameter Support Functions

In this section, by using the unfolding theory of functions, we give a classification for singularities of the one-parameter developable surface of γ.

Suppose that $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ, and write $f(s) = F_{x_0}(s, x_0)$. F is called an *r-parameter unfolding* of f. We say that f has an A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$, for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. If $f^{(p)}(s_0) = 0$, for all $1 \leq p \leq k$, we also say that f has an $A_{\geq k}$-singularity at s_0. Suppose f has an A_k-singularity $(k \geq 1)$ at s_0 and F be an r-parameter unfolding of f; then, we write the $(k-1)$-jet of $\partial F / \partial x_i$ at s_0 as

$$j^{(k-1)}\frac{\partial F}{\partial x_i}(s, x_0)(s_0) = \sum_{j=1}^{k+1} \alpha_{ji}(s - s_0)^j, (i = 1, \ldots, r).$$

We call F an *R-versal unfolding* of f if the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \leq r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$. The *discriminant set* of F is defined to be

$$D_F = \{x \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0\}.$$

A well-known classification [12–15] follows.

Theorem 4. *Let $f(s)$ have A_k-singularity at s_0 and $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$. If F is an R-versal unfolding of f, then we have the following statements:*
(1) If $k = 2$, then D_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.
(2) If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.

By Proposition 3, we get the discriminant set of the one-parameter support function $G(s, \theta, x)$, as follows:

$$D_G = \left\{ \gamma(s) + u \frac{t(s)\tau(s) - \kappa(s)\sin\theta\cos\theta n(s) + \kappa(s)\cos^2\theta b(s)}{\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta}} \,\middle|\, s, u \in \mathbb{R}, \theta \in [0, \frac{\pi}{2}] \right\}.$$

We have the following proposition:

Proposition 5. *Let $\gamma : I \to \mathbb{R}^3$ be a space curve. If g_{θ_0, x_0} has the A_k-singularity $(k = 2, 3)$ at (s_0) and $\delta(s_0) \neq 0$ for $k = 3$, then $G(s, \theta_0, x_0)$ is an R-versal unfolding of g_{θ_0, x_0}.*

Proof. Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$, $x = (x_1, x_2, x_3)$ and $\cos\theta_0 n(s) + \sin\theta_0 b(s) = (l_1(s), l_2(s), l_3(s))$. Then, we have

$$G(s, \theta_0, x) = l_1(s)(x_1 - \gamma_1(s)) + l_2(s)(x_2 - \gamma_2(s)) + l_3(s)(x_3 - \gamma_3(s)),$$

and

$$\frac{\partial G}{\partial x_i}(s, \boldsymbol{x}) = l_i(s), \quad (i = 1, 2, 3).$$

Therefore, the 2-jet is as follows:

$$j^2 \frac{\partial G}{\partial x_i}(s_0, \boldsymbol{x}_0) = l_i(s_0) + l_i'(s_0)(s - s_0) + \frac{1}{2} l_i''(s_0)(s - s_0)^2.$$

We consider the following 3×3 matrix:

$$A = \begin{pmatrix} l_1(s_0) & l_2(s_0) & l_3(s_0) \\ l_1'(s_0) & l_2'(s_0) & l_3'(s_0) \\ l_1''(s_0) & l_2''(s_0) & l_3''(s_0) \end{pmatrix} = \begin{pmatrix} \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{b}(s_0) \\ \cos\theta_0 \boldsymbol{n}'(s_0) + \sin\theta_0 \boldsymbol{b}'(s_0) \\ \cos\theta_0 \boldsymbol{n}''(s_0) + \sin\theta_0 \boldsymbol{b}''(s_0) \end{pmatrix}.$$

By the Frenet-Serret formula, we have

$$\cos\theta_0 \boldsymbol{n}'(s_0) + \sin\theta_0 \boldsymbol{b}'(s_0) = -\kappa(s_0)\cos\theta_0 \boldsymbol{t}(s_0) + \tau(s_0)\cos\theta_0 \boldsymbol{b}(s_0) - \tau(s_0)\sin\theta_0 \boldsymbol{n}(s_0),$$

$$\cos\theta_0 \boldsymbol{n}''(s_0) + \sin\theta_0 \boldsymbol{b}''(s_0) = \left(\tau(s_0)\kappa(s_0)\sin\theta_0 - \kappa'(s_0)\cos\theta_0\right)\boldsymbol{t}(s_0) - \left[(\kappa^2(s_0) + \tau^2(s_0))\cos\theta_0\right.$$
$$\left. + \tau'(s_0)\sin\theta_0\right]\boldsymbol{n}(s_0) + \left(\tau'(s_0)\cos\theta_0 - \tau^2(s_0)\sin\theta_0\right)\boldsymbol{b}(s_0).$$

Since the orthonormal frame $\{\boldsymbol{t}(s_0), \boldsymbol{b}(s_0), \boldsymbol{n}(s_0)\}$ is a basis of $\mathbb{R}^3$, then the rank of

$$A = \begin{pmatrix} \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{b}(s_0) \\ \cos\theta_0 \boldsymbol{n}'(s_0) + \sin\theta_0 \boldsymbol{b}'(s_0) \\ \cos\theta_0 \boldsymbol{n}''(s_0) + \sin\theta_0 \boldsymbol{b}''(s_0) \end{pmatrix}$$

is equal to the rank of

$$\begin{pmatrix} 0 & \cos\theta_0 & \sin\theta_0 \\ -\kappa(s_0)\cos\theta_0 & -\sin\theta_0\tau(s_0) & \tau(s_0)\cos\theta_0 \\ \tau(s_0)\kappa(s_0)\sin\theta_0 - \cos\theta_0\kappa'(s_0) & -(\kappa^2(s_0) + \tau^2(s_0))\cos\theta_0 - \tau'(s_0)\sin\theta_0 & \tau'(s_0)\cos\theta_0 - \tau^2(s_0)\sin\theta_0 \end{pmatrix}.$$

This means $\text{rank} A = 3$, if and only if

$$\kappa(s_0)\sin\theta_0(\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0) - \cos\theta_0(\kappa'(s_0)\tau(s_0) - \kappa(s_0)\tau'(s_0)) \neq 0.$$

The above inequality is equivalent to $\delta(s_0) \neq 0$. Moreover, the rank of

$$\begin{pmatrix} \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{b}(s_0) \\ \cos\theta_0 \boldsymbol{n}'(s_0) + \sin\theta_0 \boldsymbol{b}'(s_0) \end{pmatrix} = \begin{pmatrix} \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{b}(s_0) \\ -\kappa(s_0)\cos\theta_0 \boldsymbol{t}(s_0) - \tau(s_0)\sin\theta_0 \boldsymbol{n}(s_0) + \tau(s_0)\cos\theta_0 \boldsymbol{b}(s_0) \end{pmatrix}$$

is always two, under the condition $\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0) \neq 0$.

Therefore, G is an R-versal unfolding of g_{θ_0, x_0} if g_{θ_0, x_0} has A_k-singularity ($k = 2, 3$) at s_0. $\square$

Proof of Theorem 2. By direct calculation, we have

$$\frac{\partial \boldsymbol{D}_{\theta_0}}{\partial s}(s, u) \times \frac{\partial \boldsymbol{D}_{\theta_0}}{\partial u}(s, u) = -\left(\frac{\kappa(s)\cos\theta_0}{\sqrt{\tau^2(s) + \kappa^2(s)\cos^2\theta_0}} + u\delta(s)\right)(\sin\theta_0 \boldsymbol{b}(s) + \cos\theta_0 \boldsymbol{n}(s)).$$

Then, that (s_0, u_0) is a regular point of $\boldsymbol{D}_{\theta_0}$ is equivalent to

$$\frac{\kappa(s_0)\cos\theta_0}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}} + u_0\delta(s_0) \neq 0.$$

Thus, statement (1) holds.

By Proposition 3-(2), D_G is the image of the one-parameter developable surfaces of γ.

Suppose $\delta(s_0) \neq 0$. By Proposition 3-(A)-(3), (4), and (5), g_{θ_0,x_0} has an A_2-type singularity (respectively, an A_3-type singularity) at $s = s_0$ if and only if

$$u_0 = -\frac{\kappa(s_0)\cos\theta_0}{\delta(s_0)\sqrt{\kappa^2(s_0)\cos^2\theta_0 + \tau^2(s_0)}} \quad (*)$$

and $\sigma(s_0) \neq 0$ (respectively, $(*)$, $\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$). By Theorem 4 and Proposition 5, we have (2)-(i) and (3).

Suppose $\delta(s_0) = 0$. By Proposition 3-(B)-(6) and (7), g_{θ_0,x_0} has an A_2-type singularity if and only if $\cos\theta_0 = 0$ and

$$u_0 \neq \frac{\kappa(s_0)\tau(s_0)}{\kappa(s_0)\tau'(s_0) + \kappa'(s_0)\tau(s_0)}.$$

Following from Theorem 4 and Proposition 5, we obtain (2)-(ii). This completes the proof. □

6. Examples

In this section, we construct the one-parameter developable surfaces associated with a space curve and two sections of the one-parameter developable surfaces. The two sections are the tangent developable surface and the rectifying developable surface of the curve. They are also the wavefronts of the curve.

Example 1. *Let* $\gamma(s) = (-\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}s)$, *where s is the arc-length parameter. Then*

$$t(s) = \left(-\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\right), \quad n(s) = (\sin s, \cos s, 0), \quad b(s) = \left(-\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\right).$$

We can calculate that $\kappa(s) = \frac{\sqrt{2}}{2}$ and $\tau(s) = -\frac{\sqrt{2}}{2}$. Therefore, the one-parameter developable surfaces of γ is as follows:

$$D_\theta(s,u) = \gamma(s) + u\frac{-t(s) - \sin\theta\cos\theta n(s) + \cos^2\theta b(s)}{\sqrt{1 + \cos^2\theta}}.$$

The tangent developable surface of γ is as follows:

$$D_{\frac{\pi}{2}}(s,u) = \left(-\frac{\sqrt{2}}{2}\sin s - \frac{\sqrt{2}}{2}u\cos s, \frac{\sqrt{2}}{2}u\sin s - \frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}u\right).$$

In this case, $\delta(s) \equiv \frac{\sqrt{2}}{2}$ and $\sigma(s) \equiv -1$. By Theorem 2 (2)-(i), we have the cuspidal edge singularities at $u = 0$ (Figure 4). The rectifying developable surface of γ is as follows:

$$D_0(s,u) = \left(-\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}s - \sqrt{2}u\right).$$

In this case, $\delta(s) \equiv 0$. By Theorem 1, the rectifying developable surface of γ is a cylinder (Figure 5).

Example 2. *Let* $\gamma(s) = (\frac{1}{2}s^2, \frac{1}{4}s^4, \int_0^s \sqrt{1 - t^2 - t^6}dt)$, *where $s \in (-0.68, 0.68)$. Then, we have*

$$t(s) = (s, s^3, \sqrt{1 - s^2 - s^6}),$$

$$n(s) = \left(\frac{\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{3s^2\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{-s(1 + 3s^4)}{\sqrt{1 + 9s^4 - 4s^6}}\right),$$

$$b(s) = \left(\frac{s^2(-3 + 2s^2)}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{1 + 2s^6}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{2s^3\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}}\right).$$

We can calculate that $\kappa(s) = \frac{\sqrt{1-s^2-s^6}}{\sqrt{1+9s^4-4s^6}}$ and $\tau(s) = \frac{2s(4s^8-12s^6-4s+3)}{(1+9s^4-4s^6)\sqrt{1-s^2-s^6}}$. *Therefore, the one-parameter developable surface of γ is as follows:*

$$\boldsymbol{D}_\theta(s,u) = \boldsymbol{\gamma}(s) + u\frac{2s(4s^8-12s^6-4s+3)\boldsymbol{t}(s) - (1-s^2-s^6)\sqrt{1+9s^4-4s^6}\cos\theta(\sin\theta\boldsymbol{n}(s)-\cos\theta\boldsymbol{b}(s))}{\sqrt{4s^2(4s^8-12s^6-4s+3)^2+(1-s^2-s^6)(1+9s^4-4s^6)\cos^2\theta}}.$$
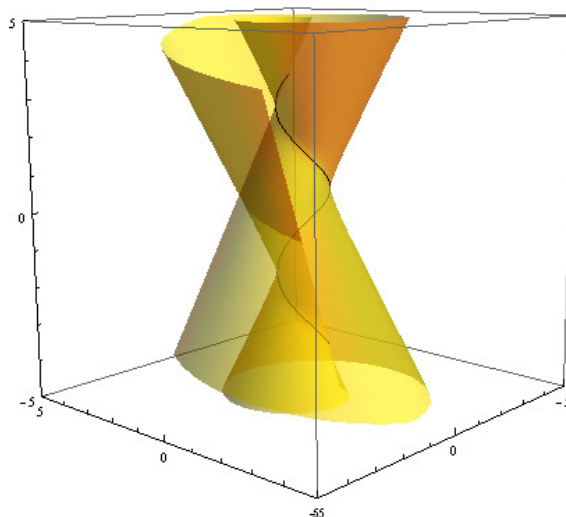
Figure 4. γ and $\boldsymbol{D}_{\frac{\pi}{2}}(s,u)$ of Example 1.
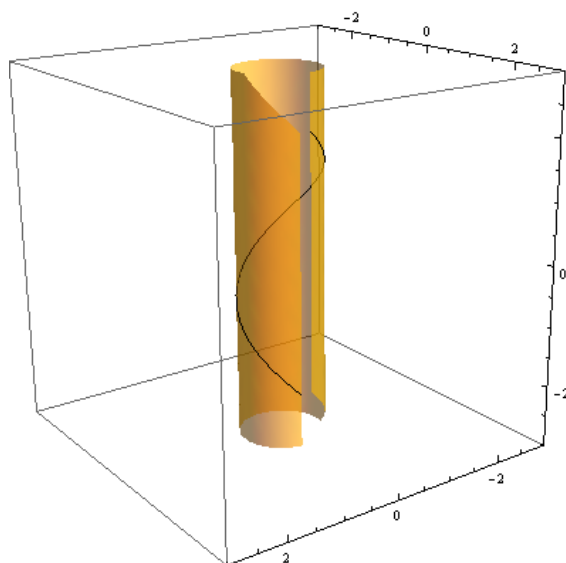
Figure 5. γ and $\boldsymbol{D}_0(s,u)$ of Example 1.

The tangent developable surface of γ is as follows:

$$\boldsymbol{D}_{\frac{\pi}{2}}(s,u) = \left(\frac{1}{2}s^2+us, \frac{1}{4}s^4+us^3, \int_0^s \sqrt{1-t^2-t^6}dt + u\sqrt{1-s^2-s^6}\right).$$

In this case, $\delta(s) = \frac{\sqrt{1+9s^4-4s^6}}{\sqrt{1-s^2-s^6}} \neq 0$ when $s \in (-0.68, 0.68)$. Since $\tau^2(s) + \kappa^2(s)\cos^2\theta \neq 0$, then $\sigma(s) \equiv 1$. By Theorem 2 (2)-(i), we have the cuspidal edge singularities are at $u = 0$ if $s \neq 0$ (Figure 6). The rectifying developable surface of γ is as follows:

$$D_0(s,u) = \left(\frac{1}{2}s^2 + \frac{u2s^2(4s^8 - 12s^6 - 4s + 3) + us^2(1 - s^2 - s^6)(-3 + 2s^2)}{\sqrt{4s^2(4s^8 - 12s^6 - 4s + 3)^2 + (1 - s^2 - s^6)(1 + 9s^4 - 4s^6)}},\right.$$

$$\frac{1}{4}s^4 + \frac{u2s^4(4s^8 - 12s^6 - 4s + 3) + u(1 - s^2 - s^6)(1 + 2s^6)}{\sqrt{4s^2(4s^8 - 12s^6 - 4s + 3)^2 + (1 - s^2 - s^6)(1 + 9s^4 - 4s^6)}},$$

$$\left.\int_0^s \sqrt{1 - t^2 - t^6}\,dt + \frac{u2s(4s^8 - 12s^6 - 4s + 3)\sqrt{1 - s^2 - s^6} + u2s^3(1 - s^2 - s^6)\sqrt{1 - s^2 - s^6}}{\sqrt{4s^2(4s^8 - 12s^6 - 4s + 3)^2 + (1 - s^2 - s^6)(1 + 9s^4 - 4s^6)}}\right).$$

In this case, we have

$$\delta(0) = \left.\frac{\tau'(s)}{\kappa(s)}\right|_{s=0} = \left.\frac{2(3 - 12s^2 - 73s^4 + 126s^6 - 84s^8 - 130s^{10} + 252s^{12} - 120s^{14} + 32s^{16} + 24s^{18})}{(1 - s^2 - s^6)(1 + 9s^4 - 4s^6)^{\frac{5}{2}}}\right|_{s=0} = 6$$

and

$$\sigma(0) = \left[\frac{s(1 + 9s^4 - 4s^6)^{\frac{3}{2}}}{(3 - 12s^2 - 73s^4 + 126s^6 - 84s^8 - 130s^{10} + 252s^{12} - 120s^{14} + 32s^{16} + 24s^{18})^2}\left(9 + 281s^2\right.\right.$$

$$- 1144s^4 - 405s^6 + 6397s^8 - 9474s^{10} + 10,962s^{12} - 10,698s^{14} + 6396s^{16} + 1854s^{18} - 8180s^{20}$$

$$\left.\left.+ 6616s^{22} - 1728s^{24} + 536s^{26} + 144s^{28}\right)\right]\Bigg|_{s=0} = 0.$$

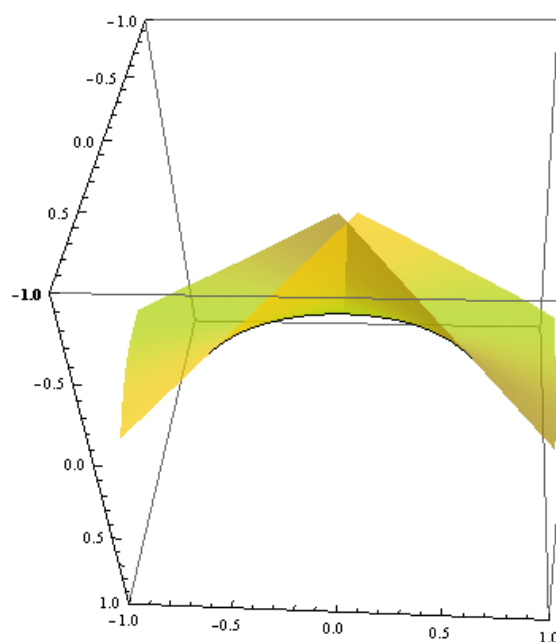*We can also calculate $\sigma'(0) = 1$. By Theorem 2 (3), we have the swallowtail singularities at $\left(0, -\frac{1}{6}\right)$ (Figure 7).*

Figure 6. γ and $D_{\frac{\pi}{2}}(s,u)$ of Example 2.

Figure 7. γ and $D_0(s, u)$ of Example 2.

Conflicts of Interest: The authors declare no conflict of interest.

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