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Periodic Orbits of Third Kind in the Zonal $J_2 + J_3$ Problem

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Abstract: In this work, the periodic orbits' phase portrait of the zonal $J_2 + J_3$ problem is studied. In particular, we center our attention on the periodic orbits of the third kind in the Poincaré sense using the averaging theory of dynamical systems. We find three families of polar periodic orbits and four families of inclined periodic orbits for which we are able to state their explicit expressions.

Keywords: periodic orbits; perturbed Kepler problem; frozen orbits; zonal problem; averaging theory

MSC: 70F07; 70F15

1. Introduction and Theoretical Background

In the *Méthodes Nouvelles de la Mécanique Céleste* [1], the seminal work of the modern qualitative theory of dynamical systems, Henri Poincaré considered the investigation of periodic solutions as a principal topic of interest and a key point for classifying the solutions of a system. In his investigations on the Restricted Three Body Problem (RTBP), he classified the periodic orbits into three kinds. The first kind comprises those that are generated by the planar circular orbits of the unperturbed Kepler problem. The second kind comprise those generated by the planar elliptical orbits of the Kepler problem, and the third kind comprises those generated by the spatial circular orbits of the Keplerian system with no null inclination. See for instance [2] for a recent study on the extension of the orbits of the first and second kind from the RTBP to the perturbed RTBP.

In astrodynamics arises the necessity to determine the periodic orbits of a great amount for such dynamical systems, which are used for modeling space missions of different types. The importance of the determination of these orbits in astrodynamics scientific missions is great, for instance, the periodic polar orbits whose inclination is $\frac{\pi}{2}$ are used for monitoring practically all of the surface of the planet of the satellite object of the mission. A great number of satellites use such orbits, for example the DMSP series of the USA and the next generation, named NPOEES (National Polar-orbiting Operational Environmental Satellite System); see [3,4] for more details.

A classical approach in many investigations about these topics is the determination of a special type of orbit called a frozen orbit; see [5]. A frozen orbit is characterized by the absence of long-term changes in orbital eccentricity and the argument of perigee. This type of orbit maintains almost constant altitude over any particular point. The use of second-order averaging in perturbed Keplerian problems is the main tool for developing this study. Frozen orbits correspond to equilibria of a double averaged

system and are almost periodic solutions of the full (non-averaged) problem with the eccentricity and argument of the perigee of such orbits remaining close to a stationary value. Frozen orbits in the zonal $J_2 + J_3$ problem are investigated in [6,7] and exposed in [8]. For more details about the zonal problem, see the classical work of [9]. Other recent works where the J_2 problem and the zonal RTBP were studied with a non-averaging approach are [10–12] respectively.

The main aim of the present work is to study sufficient conditions for the existence of periodic orbits of the third kind in the $J_2 + J_3$ zonal problem. For this purpose, we are inspired by the methodology developed in [13], and explain in detail its most relevant aspects in Appendix A. Finally, we state the existence of three families of polar periodic orbits and the existence of four families of inclined periodic orbits. Moreover, we prove the existence of a bifurcation on the periodic polar orbits; more specifically, we identify the energies that produce a change in the number of polar orbits passing from three families to four.

2. Main Results

2.1. Hamiltonian Description of the Model: The Zonal $J_2 + J_3$ Problem

We consider \mathbb{R}^3 with the euclidean norm $\|\cdot\|$. In the phase space $T^*(\mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^3)$, we denote by $(\mathbf{Q}, \mathbf{P}) = (Q_1, Q_2, Q_3, P_1, P_2, P_3)$ the canonical coordinates of position and moment, and $\omega = d\mathbf{Q} \wedge d\mathbf{P}$ the canonical symplectic form. The Kepler Hamiltonian corresponding to a body subjected to central forces is:

$$\mathcal{H}_0 = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} \quad (1)$$

In the following, we shall use the Delaunay variables for studying the periodic orbits of the Hamiltonian system associated with the Hamiltonian (1), which is called the zonal $J_2 + J_3$ problem; see [13–18] for more details on the Delaunay variables and this astrophysical problem. Thus, if $\{l, g, k, L, G, K\}$ are the action angle coordinates of Delaunay, where l is the mean anomaly, g is the argument of the periapsis of the unperturbed elliptical orbit measured in the invariant plane, k is the longitude of the ascending node, L is the square root of the major semi-axis of the unperturbed elliptical orbit, G is the modulus of the total angular momentum, and K is the third component of the angular momentum, then the Hamiltonian (1) has the form:

$$\mathcal{H}_0 = -\frac{1}{2L^2} \quad (2)$$

Regarding the term of the potential corresponding to the non-sphericity of the major body (see [14]), we have:

$$\mathcal{H}_1 = \frac{1}{r} \left(\sum_{n=2}^3 \left(\frac{R}{r} \right)^n J_n p_n(\sin \phi) \right)$$

where J_2 and J_3 are the harmonic coefficients, p_2 and p_3 are the Legendre polynomials of the second and third degree, respectively, ϕ is the latitude of the orbit with respect to the equator, and R is the equatorial radius of the major body.

In terms of orbital elements, the terms of the Legendre polynomials are (see [19]):

$$\begin{aligned} p_2(\sin \phi) &= \frac{1}{2}(3s^2 \sin^2(f + g) - 1) \\ p_3(\sin \phi) &= \frac{5}{2}s^3 \sin^3(f + g) - \frac{3}{2}s \sin(f + g) \end{aligned}$$

where f is the true anomaly and $s = \sin(i)$, where i is the inclination, so that, using the expressions of the Delaunay coordinates:

$$\begin{aligned} L &= \sqrt{a} \\ G &= L\sqrt{1-e^2} \\ K &= G \cos i \end{aligned} \quad (3)$$

and the relations:

$$\begin{aligned} r &= a(1 - e \cos E), \\ l &= E - e \sin E \\ \sin f &= \frac{a\sqrt{1-e^2} \sin E}{r}, \quad \cos f = \frac{a(\cos E - e)}{r} \end{aligned}$$

where a is the semi-major axis, e is the eccentricity of the unperturbed elliptic orbit, and E is the eccentricity anomaly, \mathcal{H}_1 can be written as a function of E, g, L, G , and K (for the explicit expression of \mathcal{H}_1 , see Appendix B).

2.2. Averaged Potential $\langle \mathcal{H}_1 \rangle$

Due to \mathcal{H}_0 only depending on the action variable L , the potential \mathcal{H}_1 has to be averaged by the angle variable l , and since the dependence of l in \mathcal{H}_1 is through the eccentricity anomaly E by the Kepler equation, we must make the corresponding change of variable in the integral:

$$\begin{aligned} \langle \mathcal{H}_1 \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1(l, g, L, G, K) dl \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1(E - e \sin E, g, L, G, K) (1 - e \cos E) dE, \end{aligned}$$

and operating,

$$\langle \mathcal{H}_1 \rangle = -\frac{R^2}{G^7 L^3} \left(2G^2(G^2 - 3K^2)J_2 + 3(G^2 - 5K^2) \sin(g) \varphi(K, G) \phi(G, L) R J_3 \right)$$

where $\varphi(K, G) = \sqrt{1 - \frac{K^2}{G^2}}$ and $\phi(G, L) = \sqrt{1 - \frac{G^2}{L^2}}$.

2.3. Equilibrium Points of the Differential System

Let $h^* < 0$ be an energy level $\mathcal{H} = h^*$, $\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*)) = (-2h^*)^{3/2}$.

The differential system (A2) with respect to the mean anomaly l is (see [13]):

$$\begin{aligned} \frac{dG}{dl} &= \varepsilon \frac{\{G, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*))} = \frac{-\varepsilon}{2\sqrt{2}(-h^*)^{3/2}} \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial g} = -\varepsilon f_1(g, G, K), \\ \frac{dg}{dl} &= \varepsilon \frac{\{g, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*))} = \frac{\varepsilon}{2\sqrt{2}(-h^*)^{3/2}} \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial G} = \varepsilon f_2(g, G, K), \\ \frac{dK}{dl} &= \varepsilon \frac{\{K, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*))} = \frac{\varepsilon}{2\sqrt{2}(-h^*)^{3/2}} \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial K} = \varepsilon f_3(g, G, K), \\ \frac{dK}{dl} &= \varepsilon \frac{\{K, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*))} = \frac{\varepsilon}{2\sqrt{2}(-h^*)^{3/2}} \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial k} = 0 \Rightarrow K = k^*. \end{aligned}$$

Note that the differential equation $dK/dt = 0$, because we are working in the invariant set, and $K = k^*$.

We remark that the functions f_i have been written in the variables (g, G, K) because we are working on the energy level $\mathcal{H} = h^*$, and consequently, $L = \frac{1}{\sqrt{-2h^*}} + O(\varepsilon)$.

Since from the last equation, we obtain the constant value of $K = k^*$, the equilibrium points of this system are obtained solving the system $f_i = 0$ in the variables g and G . For this reason, in general, the system will not have a solution for arbitrary values of k^* and h^* , and consequently, it is more convenient to work with the whole system:

$$f_i(L, G, K, g) = 0, \quad i = 1, 2, 3 \quad (4)$$

which in general, will allow us to compute three variables depending on the fourth one.

Finally, we observe that the system:

$$\frac{\partial \langle \mathcal{H}_1 \rangle}{\partial g} = 0, \quad \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial G} = 0, \quad \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial K} = 0$$

is equivalent to the system:

$$3\sqrt{G^2 - K^2}(G^2 - 5K^2)\sqrt{L^2 - G^2}\cos(g) = 0 \quad (5)$$

$$\begin{aligned} -2G^3(G^2 - 5K^2)\sqrt{G^2 - K^2}\sqrt{L^2 - G^2}J_2 + J_3(4G^6 - 40K^4L^2 \\ - 5G^4(7K^2 + L^2) + G^2(35K^4 + 41K^2L^2))R\sin(g) = 0 \end{aligned} \quad (6)$$

$$K[-4G^3\sqrt{G^2 - K^2}LJ_2 + (-11G^2 + 15K^2)\sqrt{L^2 - G^2}\sin(g)RJ_3] = 0 \quad (7)$$

Since $G^2 \neq K^2$ and $G^2 \neq L^2$ (otherwise, $\langle \mathcal{H}_1 \rangle$ is not differentiable), (5) implies that either $G^2 - 5K^2 = 0$ or $\cos(g) = 0$.

Considering that $\frac{G}{K} = \cos i$, the first equation leads to critical inclination $i = 63.4^\circ$ [9,20].

Let us see that this critical inclination does not lead to any solution of the two remaining equations: Substituting into (6), we have that:

$$-40K^4J_3(5K^2 - L^2)R\sin(g) = 0$$

and as $L^2 \neq G^2$ and $5K^2 - L^2 \neq 0$, (6) is verified when $\sin(g) = 0$. If we replace now $\sin(g) = 0$ in (7), it would have to be:

$$-4G^3K\sqrt{G^2 - K^2}LJ_2 = -4G^3K\sqrt{G^2 - \frac{1}{5}G^2}LJ_2 = -\frac{8}{\sqrt{5}}G^4KLJ_2 = 0$$

which is not possible.

Therefore, $g = \pm \frac{\pi}{2}$ in order to satisfy (5). Substituting into (6) and (7), we obtain that:

$$\begin{aligned} -2G^3(G^2 - 5K^2)\sqrt{G^2 - K^2}\sqrt{L^2 - G^2}LJ_2 + RJ_3(4G^6 - 40K^4L^2 \\ - 5G^4(7K^2 + L^2) + G^2(35K^4 + 41K^2L^2)) = 0 \end{aligned} \quad (8)$$

$$K[-4G^3\sqrt{G^2 - K^2}LJ_2 + RJ_3(-11G^2 + 15K^2)\sqrt{L^2 - G^2}] = 0 \quad (9)$$

An obvious solution of (9) is $K = 0$, that is the inclination is $i = \pm \frac{\pi}{2}$ (polar orbits). Let us remember that in the averaged system, $K = k^*$, that is we are studying a particular value of the constant k^* . Replacing $K = 0$ in (8), we obtain:

$$-G^4(2J_2LG^2\sqrt{L^2 - G^2} + J_3R(-4G^2 + 5L^2)) = 0$$

Since $G \neq 0$, it must be:

$$2J_2LG^2\sqrt{L^2 - G^2} = -J_3R(-4G^2 + 5L^2)$$

and squaring, we obtain:

$$4J_2^2 L^2 G^4 (L^2 - G^2) = J_3^2 R^2 (16G^4 + 25L^4 - 40G^2 L^2)$$

Substituting $L = \sqrt{a}$ and $G = L\sqrt{1 - e^2}$, we obtain:

$$4a^2 e^2 (1 - e^2)^2 J_2^2 = (1 + 4e^2)^2 J_3^2 R^2$$

and, therefore, ($a > 0$):

$$a = \frac{(1 + 4e^2) J_3 R}{2e(1 - e^2) J_2} = \frac{1 + 4e^2}{2e(1 - e^2) \tau}$$

where $\tau = \frac{J_2}{J_3 R}$.

If the energy level $h^* < 0$, then $L = \sqrt{a} = \frac{1}{\sqrt{-2h^*}}$, that is $a = \frac{1}{-2h^*}$, and we have:

$$\tau 2a = \frac{\tau}{-h^*} = \frac{1 + 4e^2}{e(1 - e^2)}$$

Let us consider now the case $g = \frac{\pi}{2}$, $K \neq 0$.

The Equation (9) can be written as:

$$-2G^3 \sqrt{G^2 - K^2} L \tau = -\frac{(-11G^2 + 15K^2) \sqrt{L^2 - G^2}}{2} \quad (10)$$

with $\tau = \frac{J_2}{J_3 R}$, and substituting in (8),

$$\begin{aligned} & -(-11G^2 + 15K^2) \frac{\sqrt{L^2 - G^2}}{2} \sqrt{L^2 - G^2} (G^2 - 5K^2) \\ & + (4G^6 - 40K^4 L^2 - 5G^4 (7K^2 + L^2) + G^2 (35K^4 + 41K^2 L^2)) = 0 \end{aligned}$$

and simplifying,

$$(G^4 + 12G^2 K^2 - 5K^4) L^2 - (3G^4 + 5K^4) G^2 = 0$$

hence,

$$L^2 = \frac{G^2 (3G^4 + 5K^4)}{G^4 + 12G^2 K^2 - 5K^4} \quad (11)$$

The denominator, in the coordinates (a, e, i) , is:

$$a(1 - e^2)^2 (1 + 12 \cos^2(i) - 5 \cos^4(i)) > 0$$

if we substitute (11) in the square of Equation (10), we obtain:

$$4G^6 (G^2 - K^2) \frac{G^2 (3G^4 + 5K^4)}{G^4 + 12G^2 K^2 - 5K^4} \tau = \frac{(11G^2 - 15K^2)^2}{4} \frac{G^2 (3G^4 + 5K^4)}{G^4 + 12G^2 K^2 - 5K^4} - G^2$$

and simplifying, we have:

$$\frac{G(G^2 - K^2)}{G^4 + 12G^2 K^2 - 5K^4} (8G^6 (3G^4 + 5K^4) \tau - (11G^2 - 15K^2)^2 (G^2 - 5K^2)) = 0$$

and since $G \neq 0$, $G^2 \neq K^2$, and $G^4 + 12G^2K^2 - 5K^4 \neq 0$, it is equivalent to:

$$1125K^6 + (-1875 + 40G^4\tau)G^2K^4 + 935G^4K^2 - 121G^6 = 0 \quad (12)$$

which is a third-degree equation in K^2 with at least one real solution $K^2 > 0$. Bearing in mind that, for (3), $K = G \cos(i)$, substituting in the previous equation and simplifying, we obtain the equation:

$$1125 \cos^6(i) + 935 \cos^2(i) - 121 + (40G^4\tau - 1875) \cos^4(i) = 0$$

If we solve this equation in terms of G^2 , we obtain the positive solution:

$$G^2 = \frac{\sqrt{-1125 \cos^2(i) + 121 \sec^4(i) - 935 \sec^2(i) + 1875}}{2\sqrt{10}\sqrt{\tau}}$$

Obviously, $\psi(i) = -1125 \cos^2(i) + 121 \sec^4(i) - 935 \sec^2(i) + 1875$ has a discontinuity in $\frac{\pi}{2}$, is symmetric with respect to the line $i = \frac{\pi}{2}$, and vanishes at $\arccos\left(\pm\sqrt{\frac{11}{15}}\right)$ and $\arccos\left(\pm\sqrt{\frac{1}{5}}\right)$, such that $\psi(i)$ is real if and only if $i \in (\arccos\left(\sqrt{\frac{1}{5}}\right), \frac{\pi}{2})$ or $i \in (\frac{\pi}{2}, \arccos\left(-\sqrt{\frac{1}{5}}\right))$.

The statement of our first main result is:

Theorem 1 (Equilibrium points). *For the system (4), there are two families of equilibrium points:*

(a) $K = 0$ (polar orbits). This family is parameterized by:

$$\begin{aligned} L &= \sqrt{\frac{1 + 4e^2}{2e(1 - e^2)}}, \\ G &= \sqrt{\frac{1 + 4e^2}{2e}}, \\ g &= \pm \frac{\pi}{2} \end{aligned}$$

(b) $K \neq 0$. This family is parameterized by:

$$\begin{aligned} G &= \left(\frac{\sqrt{1875 - 1125 \cos^2(i) - 935 \sec^2(i) + 121 \sec^4(i)}}{2\sqrt{10}\tau} \right)^{1/2}, \\ L &= G \left(\frac{3 + 5 \cos^4(i)}{1 + 12 \cos^2(i) + 5 \cos^4(i)} \right)^{1/2}, \\ K &= G \cos(i), \\ g &= \pm \frac{\pi}{2}, \end{aligned}$$

where the inclination $i \in (\arccos\left(\sqrt{\frac{1}{5}}\right), \frac{\pi}{2}) \cup (\frac{\pi}{2}, \arccos\left(-\sqrt{\frac{1}{5}}\right))$ (let us observe that $i = \arccos\left(\sqrt{\frac{1}{5}}\right)$ is, again, the critical inclination).

Proof. The proof is a consequence of all calculations and reasonings previously stated in Section 2.3. \square

The equilibrium points of the previous theorem give rise to periodic orbits of the perturbed problem, except in a finite number of values of the parameter e , which characterize them. As a second main result, we are going to calculate the periodic orbits in Theorem 2.

Lemma 1. For every $h^* > 0$ such that $\frac{\tau}{-h^*} > \sqrt{\frac{83}{8} + \frac{13\sqrt{65}}{8}}$, there are two values of $e \in (0, 1) / \left\{ \sqrt{\frac{-7+\sqrt{65}}{8}} \right\}$ for which $\frac{\tau}{-h^*} = \frac{1+4e^2}{e(1-e^2)} = f(e)$.

Proof. It is obvious that:

$$\lim_{e \rightarrow 0^+} \frac{1+4e^2}{e(1-e^2)} = +\infty = \lim_{e \rightarrow 1^-} \frac{1+4e^2}{e(1-e^2)}$$

and, if we derive:

$$\frac{d}{de} \left(\frac{1+4e^2}{e(1-e^2)} \right) = \frac{-1+7e^2+4e^4}{e^2(1-e^2)^2}$$

Let us see that it only cancels for $e = \sqrt{\frac{-7+\sqrt{65}}{8}}$.

Since $f\left(\sqrt{\frac{-7+\sqrt{65}}{8}}\right) = \sqrt{\frac{83}{8} + \frac{13\sqrt{65}}{8}}$ for any value of $\frac{\tau}{-h^*}$, the line $y = \frac{\tau}{-h^*}$ intersects with the graph of $y = f(e)$ in two points, and due to the coefficient of degree zero being negative, it has a positive real root, which is unique, by Descartes' rule.

□

Theorem 2 (Periodic orbits). For each energy level $h^* < 0$ such that:

$$\frac{\tau}{-h^*} > \sqrt{\frac{83+13\sqrt{65}}{8}}$$

there are four 2π -periodic solutions $\gamma_\varepsilon^{\pm,i}(l) = (g(l, \varepsilon), k(l, \varepsilon), L(l, \varepsilon), G(l, \varepsilon), K(l, \varepsilon))$ such that:

$$\gamma_\varepsilon^{\pm,i}(0) \rightarrow \left(\pm \frac{\pi}{2}, k_0, \frac{1}{\sqrt{-2h^*}}, \sqrt{\frac{1-e_i^2}{-2h^*}}, 0 \right)$$

$i = 1, 2$, where $e_1 < e_2$, with the two values of $e \in (0, 1)$ for which:

$$\frac{1+4e_i^2}{e_i(1-e_i^2)} = \frac{\tau}{-h^*}$$

except for energy values h_1^*, h_2^* :

$$\begin{aligned} \frac{\tau}{-h_1^*} &= \frac{1}{10} \sqrt{\frac{3}{2}(953+157\sqrt{41})} \\ \frac{\tau}{-h_2^*} &= 11\sqrt{\frac{3}{2}} \end{aligned}$$

for which there are only three periodic solutions corresponding to the equilibrium points:

$$\begin{aligned} &\left(\frac{\pi}{2}, k_0, \frac{1}{\sqrt{-2h_1^*}}, \sqrt{\frac{1-e_2^2}{-2h_1^*}}, 0 \right) \\ &\left(\frac{-\pi}{2}, k_0, \frac{1}{\sqrt{-2h_1^*}}, \sqrt{\frac{1-e_1^2}{-2h_1^*}}, 0 \right) \end{aligned}$$

and:

$$\left(\frac{\pi}{2}, k_0, \frac{1}{\sqrt{-2h_2^*}}, \sqrt{\frac{1-e_i^2}{-2h_2^*}}, 0 \right)$$

$$\left(\frac{-\pi}{2}, k_0, \frac{1}{\sqrt{-2h_2^*}}, \sqrt{\frac{1-e_1^2}{-2h_2^*}}, 0 \right)$$

Proof. We have to see at what equilibrium points computed in the previous section, (g_0, G_0, k^*) , the Jacobian is not null, that is,

$$\left| \frac{\partial(f_1, f_2, f_3)}{\partial(g, G, K)} \right|_{(g_0, G_0, k^*)} \neq 0$$

For the family (a) of the previous theorem, the equilibrium points are given by $g = \pm \frac{\pi}{2}$, and the equation:

$$\left| \frac{\partial(f_1, f_2, f_3)}{\partial(g, G, k)} \right|_{(\pm \frac{\pi}{2}, G_0, K)} = 0$$

it is an algebraic curve in the (G, K) plane; therefore, the equilibrium points on which the Jacobian is zero are the intersection points of this curve with the algebraic curves:

$$f_2(\pm \frac{\pi}{2}, G, K) = 0$$

$$f_3(\pm \frac{\pi}{2}, G, K) = 0$$

and since the intersection of algebraic curves is a finite number of points, we can always find equilibrium points in which the Jacobian does not vanish.

If we write Equation (11) in coordinates (a, e, i) , we obtain:

$$a = \frac{a(1 - e^2)(3 + 5 \cos^4(i))}{1 + 12 \cos^2(i) - 5 \cos^4(i)}$$

where:

$$1 - e^2 = \frac{1 + 12 \cos^2(i) - 5 \cos^4(i)}{3 + 5 \cos^4(i)} \tag{13}$$

and since $e \in (0, 1)$, $1 - e^2 \in (0, 1)$, which only happens when $i \in (\arccos(\frac{1}{\sqrt{5}}), \pi - \arccos(\frac{1}{\sqrt{5}}))$, $i \in (-\pi + \arccos(\frac{1}{\sqrt{5}}), -\arccos(\frac{1}{\sqrt{5}}))$.

Now, if we write (6) in coordinates (a, e, i) , we have:

$$-121a(1 - e^2) + 935a(1 - e^2) \cos^2(i) - 5(375 - 8a^2(1 - e^2)^2\tau) \cos^4(i) + 1125a(1 - e^2) \cos^6(i) = 0$$

and hence, we can obtain the value of $a > 0$:

$$a = \frac{1}{80 \cos^4(i)(1 - e^2)\tau} (-\alpha + \sqrt{\alpha^2 + 300000\tau \cos^8(i)})$$

with:

$$\alpha = 121 + 935 \cos^2(i) + 1125 \cos^6(i)$$

To specify more, let us take the case $g = \pm \frac{\pi}{2}$ and $K = 0$.

As we have seen, the equilibrium points are given, in coordinates (a, e) , by:

$$a = \frac{(1 + 4e^2)J_3R}{2e(1 - e^2)J_2} \tag{14}$$

and the Jacobian is:

$$\left| \frac{\partial(f_1, f_2, f_3)}{\partial(g, G, k)} \right|_{(\pm \frac{\pi}{2}, G, 0)} = \frac{1}{32768G^{19}L^3(G^2 - L^2)} \left(27J_3R^7(8G^4\sqrt{L^2 - G^2}J_2L - 8G^2\sqrt{L^2 - G^2}J_2L \mp (20G^4J_3R - 51G^2J_3LR + 30J_3RL^4) (\pm 4G^2J_2L + 11\sqrt{L^2 - G^2}J_3R) \right)$$

Writing this Jacobian in coordinates (a, e) and replacing the value of a given by (14), we have that this Jacobian vanishes for the real and positive values:

$$\begin{cases} e = \frac{1}{2}\sqrt{\frac{1}{6}(-5 + \sqrt{41})} \approx 0.2418 & \text{for } g = \frac{\pi}{2} \\ e = \sqrt{\frac{2}{3}} \approx 0.8165, \quad e = \frac{1}{8}\sqrt{(-7 + \sqrt{65})} \approx 0.3644 & \text{for } g = -\frac{\pi}{2} \end{cases}$$

□

3. Conclusions

In this paper, the periodic orbits of the third kind have been studied in the zonal $J_2 + J_3$ problem using averaging theory. In this way, we start writing explicitly the Kepler Hamiltonian in the $J_2 + J_3$ -zone as a function of the coordinates of Delaunay and the averaged Hamiltonian, which allows us to obtain all the stationary points of the averaged problem both for polar orbits ($K = 0$), parameterized by the eccentricity e , and for non-polar orbits ($K \neq 0$), parameterized by the inclination (Theorem 1). Both families of equilibrium points give rise to periodic orbits, whose explicit expression is given in Theorem 2.

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Appendix A

In this Appendix, we present the methodology that has been developed in [13], which we have used for proving the main results of this paper. If we consider the Hamiltonians family:

$$\mathcal{H}(I_1, \dots, I_n, \theta_1, \dots, \theta_n) = \mathcal{H}_0(I_1) + \varepsilon \mathcal{H}_1(I_1, \dots, I_n, \theta_1, \dots, \theta_n), \tag{A1}$$

where $(I_i, \theta_i), i = 1, \dots, n$, are pairs of action angle variables and ε is a small parameter. The following result is the key for computing periodic orbits of the Hamiltonian system that is associated with the Hamiltonian (A1).

Theorem A1. We define:

$$\langle \mathcal{H}_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1(I_1, \dots, I_n, \theta_1, \dots, \theta_n) d\theta_1,$$

and we consider the differential system on the energy level $\mathcal{H}^{-1}(h^*)$, with $h^* \in \mathbb{R}$ and $\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*)) \neq 0$:

$$\begin{aligned} \frac{dI_i}{d\theta_1} &= \varepsilon \frac{\{I_i, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*))} = \varepsilon f_{i-1}(I_2, \dots, I_n, \theta_2, \dots, \theta_n) \quad i = 2, \dots, n, \\ \frac{d\theta_i}{d\theta_1} &= \varepsilon \frac{\{\theta_i, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}'_0(\mathcal{H}_0^{-1}(h^*))} = \varepsilon f_{i+n-2}(I_2, \dots, I_n, \theta_2, \dots, \theta_n) \quad i = 2, \dots, n, \end{aligned} \quad (\text{A2})$$

where $\{\cdot, \cdot\}$ denote the usual Poisson bracket of the Hamiltonian system given by (A1).

This differential system is a Hamiltonian system with Hamiltonian $\varepsilon \langle \mathcal{H}_1 \rangle$. If $\varepsilon \neq 0$ is sufficiently small, then for every equilibrium point $p = (I_2^0, \dots, I_n^0, \theta_2^0, \dots, \theta_n^0)$ of System (A2) for which (f_1, \dots, f_{2n-2}) are a local coordinates system in a neighborhood of p , there exists a 2π -periodic solution $\gamma_\varepsilon(\theta_1) = (I_1(\theta_1, \varepsilon), I_2(\theta_1, \varepsilon), (I_n(\theta_1, \varepsilon), \theta_2(\theta_1, \varepsilon), \dots, \theta_n(\theta_1, \varepsilon)))$ of the Hamiltonian system associated with the Hamiltonian (A1) taking as the independent variable the angle θ_1 such that $\gamma_\varepsilon(0) \rightarrow (\mathcal{H}_0^{-1}(h^*), I_2^0, \dots, I_n^0, \theta_2^0, \dots, \theta_n^0)$ when $\varepsilon \rightarrow 0$.

From this theorem, it is immediately deduced that, once the energy level h^* is determined to satisfy the condition of the statement, the problem of finding the periodic orbits is reduced to computing, first, the averaged Hamiltonian $\langle \mathcal{H}_1 \rangle$, the Poisson brackets $f_{i-1} = \{I_i, \langle \mathcal{H}_1 \rangle\}, \dots, f_{i+n-2} = \{\theta_i, \langle \mathcal{H}_1 \rangle\}$, and the stationary points of the system (A2):

$$\mathcal{S} = \left\{ p \in \mathbb{R}^{2n-2} : f_i(p) = 0 \quad i = 1, \dots, 2n-2 \right\}$$

and after that, it is necessary to verify in which of this points the functions (f_1, \dots, f_{2n-2}) form a system of local coordinates.

Therefore, the development of the paper will be the attainment of these stages for the Hamiltonian \mathcal{H}_1 corresponding to the perturbation, due to the non-sphericity of the central body.

The proof of the existence of periodic and quasi-periodic solutions of a system of differential equations in general is a difficult problem. The use of averaging theory constitutes a tool that provides sufficient conditions for the existence of periodic solutions of systems of perturbed differential equations.

Appendix B. Explicit Expression of the Hamiltonian \mathcal{H}_1 Written as a Function of the Variables $E, g, L, G,$ and K

$$\begin{aligned} \mathcal{H}_1 = & J_2 \left[\frac{R^2}{8G^2L^8 (-1 + \phi(G, L) \cos(E))^5} \left(2G^2L^2 - 6K^2L^2 - 2(G^2 - 3K^2)(G^2 - L^2) \cos^2(E) \right. \right. \\ & - 3(G^2 - K^2)(-3G^2 + 3L^2 + (G^2 + L^2) \cos(2E)) \cos(2g) \\ & + 4\phi(G, L) \cos(E)L^2(-G^2 + 3K^2 + 3(G^2 - K^2) \cos(2g)) \\ & \left. \left. + 24GL\phi(G, L) \cos(g) \sin(g) \sin(E)(-G^2 + K^2) + 6GL \sin(2E) \sin(2g)(G^2 - K^2) \right) \right] \\ & - J_3 \left[\frac{R^3 \phi(K, G)}{16G^2L^{11} (-1 + \phi(G, L) \cos(E))^7} \left(3G(G^2 - 5K^2)(G^2 - 3L^2 + 4\phi(G, L)L^2 \cos(E) \right. \right. \\ & + (G^2 - L^2) \cos(2E)) \cos(g) \sin(E) + 5G(G^2 - K^2)(-7G^2 + 9L^2 - 12L^2\phi(G, L) \cos(E) \\ & + (G^2 + 3L^2) \cos(2E)) \cos(3g) \sin(E) + 6\phi(G, L)L^3 \sin(g)(G^2 - 5K^2) \\ & + 6L \cos(E) \sin(g)(2G^4 - 10G^2K^2 - 3G^2L + 15K^2L^2) \\ & + 6L\phi(G, L) \cos^2(E) \sin(g)(-G^4 + 5G^2K^2 + 3G^2L^2 - 15K^2L^2) \\ & + 6L \cos^3(E) \sin(g)(G^4 - 5G^2K^2 - G^2L^2 + 5K^2L^2) \\ & + 5L\phi(G, L) \sin(3g)(5G^4 - 5G^2K^2 - 3G^2L^2 + 3K^2L^2) \\ & + 5L \cos(E) \sin(3g)(-9G^4 + 9G^2K^2 + 7G^2L^2 - 7K^2L^2) \\ & + 20L^3\phi(G, L) \cos^2(E) \sin(3g)(-G^2 + K^2) \\ & + 5L\phi(G, L) \cos(2E) \sin(3g)(-3G^4 + 3G^2K^2 - G^2L^2 + K^2L^2) \\ & \left. \left. + 5L \cos(E) \cos(2E) \sin(3g)(3G^4 - 3G^2K^2 + G^2L^2 - K^2L^2) \right) \right]. \end{aligned}$$

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