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The Existence of Symmetric Positive Solutions of Fourth-Order Elastic Beam Equations

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Abstract: In this study, we consider the eigenvalue problems of fourth-order elastic beam equations. By using Avery and Peterson's fixed point theory, we prove the existence of symmetric positive solutions for four-point boundary value problem (BVP). After this, we show that there is at least one positive solution by applying the fixed point theorem of Guo-Krasnosel'skii.

Keywords: four-point boundary value problem; fixed point theorem; symmetric positive solution; cone

1. Introduction

Consider the problem of the fourth-order four-point boundary value as given below. We will examine the existence of symmetric positive solutions for the following problem:

$$u''''(z) + \lambda y(z)(f\big(z, u(z), u'(z)\big) = 0, z \in [0,1] \tag{1}$$

$$u(0) = 0, u(1) = 0$$

$$\mu\, u''(\omega_1) + \alpha u'''(\omega_1)=0,\ \gamma\, u''(\omega_2) + \beta u'''(\omega_2)=0. \tag{2}$$

which defines the corruptions of an elastic beam with two stable endpoints, where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous; y: $(0, 1) \to [0,\infty)$ is symmetric on (0, 1) and possibly singular at $z = 0$ and $z = 1$; λ is called an eigenvalue and nontrivial solution as that λ is called an eigenfunction, with $\lambda > 0$; μ, α, γ and β are nonnegative constants; $0 \le \omega_1 \le \omega_2 \le 1$ and $f(.,u)$ is symmetric on [0, 1] for all u $\in [0,\infty)$, $f(z,u) = f(1-z,u)$, for each $(z,u) \in [0,1] \times [0,\infty)$.

$$u(z) = u(1-z), z \in [0,1] \text{ and } u(z) > 0, z \in (0,1).$$

Fourth-order ordinary differential equations have important applications in engineering and physical sciences as they form the models related to bending or deformation of elastic beams. These problems, especially used in material mechanics, define the deformation of an elastic beam with two fixed endpoints. Building beams in buildings and bridge construction requires serious calculations to ensure the safety of the structure. As part of these calculations, it is important to evaluate the maximum deviations in the beams. The main objective is to provide a solution to the problem that the beam can safely support the intended load. Calculations often require complex and difficult operations. At this stage, different techniques in numerical integration and applied mathematics are applied.

Equation (1) is called the beam equation and is examined under different boundary conditions. Two or more point boundary value problems for these equations have attracted a significant amount of attention. Two-point boundary value problems have been studied extensively. Multi-point boundary value problems have also started to be examined in the literature. Many authors have investigated the beam equation under various boundary conditions and with different approaches.

These studies include Tymoshenko's study on elasticity [1], Soedel's study on the deformation of the structure on monograms [2] and Dulácska study on the effects of [3] soil settlement.

Adequate conditions for the presence and absence of positive solutions for three-point boundary value problems were established by Graef et al. [4]. Siddiqi and Ghazala [5] determined the solution of the system of fourth-order boundary value problems using the non-cubic non-polynomial spline method. Ghazala and Hamood [6] used the fourth-degree solution to solve the reproducing kernel method (RKM) discrete boundary value problem. In the periodic beam equation examined with nonlinear boundary conditions, they used the Rayleigh–Ritz approach method. Avery and Peterson fixed point theorem, Iteration method and Leray–Schauder theorem have been used to solve these problems (See Gupta [7], Feng and Webb [8], Ma [9]).

In reference [9], Ma considered the fourth-order boundary problem with the two-point boundary conditions as follows:

$$y^{(4)}(z) - f(z, y(z), y''(z)) = 0, 0 \le z \le 1,$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

Zhong and Chen [10] investigated the fourth-order nonlinear differential equation as follows:

$$y^{(4)}(z) - f(z, y(z), y''(z)) = 0, 0 \le z \le 1,$$

with the ensuing four-point boundary value condition:

$$y(0) = y(1) = 0, c_1 y''(\omega_1) - c_2 y'''(\omega_1) = 0, c_3 y''(\omega_2) + c_4 y'''(\omega_2) = 0,$$

where $f \in C\big([0,1] \times [0,\infty) \times (-\infty, 0], [0,\infty)\big)$; c_1, c_2, c_3, c_4 are positive constants; and $\omega_1 < \omega_2$, $(\omega_1, \omega_2) \in (0,1)$.

In addition, Chen et al. [11] studied the following problem:

$$y^{(4)}(z) = f(z, y),\ 0 < z < 1$$

$$y(0) = y(1) = 0,\ c_1 y(\omega_1) - c_2 y(\omega_1) = 0, c_3 y(\omega_2) + c_4 y(\omega_2) = 0,$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$; μ is a positive constant, with (μ < 1); and $k^{\mu} f(z, y) \ge f(z, \mu y)$, for any $k \in (0,1)$. In fact, Agarwal [12], Cabada [13] as well as De Coster and Sanchez [14] applied the upper and lower solution method for the fourth-order equation under Lidstone boundary conditions or other conditions:

$$y^{(4)}(z) = \lambda f(z, y) + \mu g(z, y), 0 \le z \le 1,$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

In the case where f is nonlinear, it obtains an infinite number of solutions under symmetrical conditions for $\lambda = 1$ and $\mu = 0$.

In reference [15], they proved that the boundary value problem has at least one positive solution in the superlinear case, i.e., max $f_0 = 0$ and min $f_\infty = \infty$, or in the sublinear case, i.e., min $f_0 = \infty$ and max $f_\infty = 0$. These values can be as low as min f_0 = max f_0 = max f_∞ = min $f_\infty \in \{0, \infty\}$. The different studies examining this include those by Liu [16], Sun [17], Han [18], Wei and Pang [19], Zhong et al. [10] and Yao [20]. On the other hand, Adomian decomposition method was used to solve linear and nonlinear ordinary differential equations by Biazar and Shafiof [21] and Mestrovic [22]. This method provides the solution in a fast convergent series with computable terms. However, in order to solve boundary value problems using Adomian decomposition method (ADM), some unknown parameters must be determined and therefore, nonlinear algebraic equations must be solved. Geng and Cui [23] developed a method for solving nonlinear quadratic two-point BVP with the combination of ADM and RKM. Further detail can be found in studies of fourth-order boundary value problems [23–31].

In this article, our problem relates to the classical bending theory of flexible elastic beams on a nonlinear basis. Here, non-linear $f(x)$ represents the force exerted on the elastic beam. We can use the expanding method to solve fourth-order four-point BVP, which was developed by Geng and Cui [23]. In our study, we have considered the following two problems: the first one can have a value as low as max f_0 = max f_∞ = 0 or min f_∞ = min f_0 = ∞. It can be as low as min f_0 = max f_0= max f_∞ = min f_∞ ∈ {0,∞}. In the conventional bending theory, the distortion of a flexible beam at both endpoints is given in Figure 1. Based on the theory, the in-plane displacement status in the x direction of the two parts can be determined. F is used to represent the shear force along the beam. The shear force is used to calculate the shear stress on the cross-section of the beam. The maximum shear stress occurs at the neutral axis of the beam. This is the case of the superlinear problem.

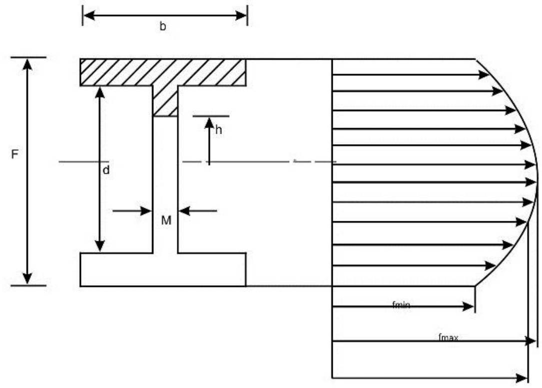

Figure 1. Based on the classical beam theory, the in-plane displacement of the two parts in the x direction is shown. The maximum deviation in beams is f_{max} where $f_{max}=\frac{3V}{2A}$ (A = bh is the cross-sectional area, V=dM/dx).

The aim of this present study is to establish sufficient conditions for fourth-order nonlinear (1) and (2) problems with four-point boundary value conditions of min f_0 = max f_0= max f_∞ = min f_∞ ∈ {0,∞}, or max f_0, max f_∞, min f_0, min f_∞ ∈ {0, ∞} and to prove that the boundary value problem has at least one positive symmetric solution in the superlinear state. In contrast to the other studies, the Krasnosel'skii [15] method was used and the results were shown in the last section and an example was given.

2. Preliminaries and Lemmas

Definition 1 ([19]). *E is a real Banach space that takes a shut convex set $P \subset E$, which is non-empty. This is defined as a contour of E in the following two conditional cases:*
(1) $x \in P, \lambda > 0$ *implies* $\lambda x \in P$;
(2) $x \in P, -x \in P$ *implies* $x = 0$.

Definition 2 ([19]). *If an operator is continuous, it is constantly constrained and the maps are set in restricted clusters.*

Definition 3 ([19]). *Let P be a cone on E in Banach space. The function $\mathfrak{s}$ is said to be concave on P, if $\mathfrak{s}: P \to [0,\infty)$ is continuous and:*

$$\mathfrak{s}(rx + (1-r)y) \geq r\mathfrak{s}(x) + (1-r)\mathfrak{s}(y)$$

for all $x, y \in P$ *and* $r \in [0,1]$.

Definition 4 ([17]). *If* $u(z) = u(1 - z), z \in [0,1]$, *the function* u *is symmetric.*

Definition 5 [17]). *If u [0, 1] is also positive and symmetric and corresponds to the solution of BVP (1) and (2), u is called the symmetric function.*

Theorem 1. *Let E be a real Banach space, Ω be a bounded explicit subset of $E, 0 \in \Omega$ and $T: \bar{\Omega} \to E$ be a completely continuous operator. Thus, there exists $x \in \partial\Omega$ such that $Tx = \lambda x$ where $\lambda > 1$, or there exists a fixed point $x^* \in \bar{\Omega}$.*

Let χ and ϕ denote convex functions on P, η nonnegative concave functions and Φ nonnegative continuous functions on P. The following convex clusters are available, with $a, b, c,$ and d positive real numbers:

$$P(\chi, d) = \{x \in P \mid \chi(x) < d\},$$

$$P(\chi, \eta, b, d) = \{x \in P \mid b \leq \eta(x), \chi(x) \leq d\},$$

$$P(\chi, \varphi, \eta, b, c, d) = \{x \in P \mid b \leq \alpha(x), \varphi(x) \leq c, \chi(x) \leq d\},$$

$$R(\chi, \Phi, a, d) = \{x \in P \mid a \leq \Phi(x), \chi(x) \leq d\}.$$

According to Avery and Peterson [32], we use the well-known fixed point theorem as shown below in (1) and (2) to investigate positive solutions to the problem.

Theorem 2. *Let E be a real Banach space and $P \subset E$ be a cone in E. Let χ and ϕ denote convex functions on P, η nonnegative concave functions and Φ nonnegative continuous functions on P convincing $\Phi(\lambda x) \leq \lambda\Phi(x)$ for $\lambda \in [0, 1]$. Thus, for some positive numbers B and d, we have the following:*

$$\eta(x) \leq \Phi(x) \text{ and } \|x\| \leq B\chi(x),$$

for all $x \in \overline{P(\chi, d)}$. Presume: $\overline{P(\chi, d)} \to \overline{P(\chi, d)}$ is an entirely continuous operator and there are positive numbers a, b and c with $a < b$ so that
$(C1)$ $x \in P(\chi, \varphi, \eta, b, c, d)|\eta(x) > b \neq \emptyset$ and $\eta(Tx) > b$ for $x \in P(\chi, \varphi, \eta, b, c, d)$;

(C2) $\eta(Tx) > b$ for $x \in P(\chi, \alpha, b, d)$ with $\varphi(Tx) > c$;

(C3) $0 \notin R(\chi, \Phi, a, d)$ and $\Phi(Tx) < a$ for $x \in R(\chi, \Phi, a, d)$ with $\phi(x) = a$.

where T is at least three fixed points $x_1, x_2, x_3 \in P(\chi, d)$, such that

$$\chi(x_i) \leq d \text{ for } i = 1, 2, 3;\ b < \eta(x_1);$$

$$a < \Phi(x_2) \text{ with } \eta(x_2) < b;\ \Phi(x_3) < a.$$

The after lemma is described below.

Lemma 1. *For $u \in E$, $\|u\|_\infty \leq \|u'\|_\infty$, where $\|u\|_\infty = sup_{t\in[0,1]} |u(z)|$.*
Thus, E is a Banach space when it is endowed with the norm $\|u\| = \|u'\|_\infty$.

Lemma 2. *Let $x \in C[0, 1] = \{x \in C[0, 1], x(z) \geq 0, z \in [0, 1]\}$ and $\delta = \mu\beta + \alpha\gamma + \mu\gamma(\omega_2 - \omega_1) > 0$, $0 \leq \omega_1 < \omega_2 \leq 1$. Thus, the BVP is:*

$$v''(z) = x(z), \qquad 0 < z < 1, \tag{3}$$

$$\mu v(\omega_1) - \alpha v'(\omega_1) = 0,\ \gamma v(\omega_2) + \beta v'(\omega_2) = 0 \tag{4}$$

for which, there is only one solution:

$$v(z) = \lambda\left[\int_0^{\omega_1}(s - z)x(s)ds + \frac{1}{\delta}\int_{\omega_1}^{\omega_2}(\mu(\omega_1 - z) - \alpha)(\beta + \gamma(\omega_2 - z))\, g(s)ds + \int_0^z(z - s)x(s)ds\right] \tag{5}$$

where $\delta = \mu\beta + \alpha\gamma + \mu\gamma(\omega_2 - \omega_1)$.

Proof. (3) is known as

$$v(z) = M + Nz + \int_0^t (z - s)x(s)ds. \quad (6)$$

where M can be N constants. Using boundary conditions, we describe (4) as:

$$M = \int_0^{\omega_1} sx(s)ds + \frac{\mu\omega_1 - \alpha}{\delta}\int_{\omega_1}^{\omega_2}\big(\beta + \gamma(\omega_2 - s)\big)x(s)ds \quad (7)$$

and

$$N = -\int_0^{\omega_1} x(s)ds + \frac{\mu}{\delta}\int_{\omega_1}^{\omega_2}\big(\beta + \gamma(\omega_2 - s)\big)x(s)ds \quad (8)$$

Substituting (7) and (8) into (6), we acquire (5).

Let $G(z,s)$ be the Green function of the following differential equation:

$$u''(z) = x(z), \quad 0 < z < 1,$$

$$u''(z) = x(z), \quad 0 < z < 1,$$

$$u(0) = u(1) = 0$$

Thus, $G(z,s) \geq 0$ for $0 \leq z,s \leq 1$ and:

$$G(z,s) = \begin{cases} s(1 - z), & 0 \leq s \leq z \leq 1, \\ z(1 - s), & 0 \leq z \leq s \leq 1. \end{cases}$$

Here, we know that:

$$0 \leq G(z,s) \leq G(s,s) = s(1-s), \quad 0 \leq z,s \leq 1 \quad (9)$$

and

$$G(z,s) \geq pG(s,s) = ps(1-s), \quad z \in [p, 1 - p], \quad s \in [0,1],$$

where $0 < p < \min\{\omega_1, 1 - \omega_2\} < \frac{1}{2}$. Thus, $u(z) \geq p\|u\|_0$, $z \in [p, 1 - p]$.

Let $P = \{u \in C^2[0,1];\ u(0) = u(1) = 0\}$, which is endowed with the indenting $u \leq v$ if $u(z) \leq v(z)$ for all $z \in [0, 1]$, and the norm:

$$\|u\| = \max\{\|u\|_0, \|u'\|_0\}$$

where $\|u\|_0 = max_{z\in[0,1]} |u(z)|$. □

Lemma 3. *If* $u \in P$, $\|u\|_0 \leq \frac{3}{4}\|u'\|_0$.

Define $T: E \to P$ *by:*

$$(Tu)(z) = \lambda\int_0^1 G(z,s)(Lu)(s)ds, \quad (10)$$

where

$$(Lu)(s): \lambda\left[\int_{\omega_1}^{s}(\xi - s)y(\xi)f\big(\xi, u(\xi), u'(\xi)\big)d\xi + \frac{1}{\delta}\int_{\omega_1}^{\omega_2}(\alpha - \mu(\omega_1 - s)))(\gamma(\omega_2 - \xi) + \beta)\,y(\xi)f\big(\xi, u(\xi), u'(\xi)\big)d\xi\right], \quad (11)$$

and $G(z,s)$ *as in (2.7),* $(Tu)(z) = Tu(1 - z)$, $0 \leq z \leq \frac{1}{2}$, $Tu \in P$.

3. The Existence of Positive Solutions

In this section, we will investigate positive solutions to a cone related to our problem described in (1) and (2) by giving sufficient conditions for λ and u. After this, we will continue to examine the existence of these solutions.

Let $E = C^2[0,1]$ be a Banache space of whole continuous functions as a norm:

$$\|u\|_0 = \max_{z\in[0,1]} |u(z)|, \quad E = \{u \in C^2[0,1]: u(0) = u(1) = 0\}.$$

$P = \{u \in E: u(z) \geq 0, u$ symmetric, concave and nonnegative valued on $[0,1]\}$.

Let the nonnegative, increasing, continuous functionals χ, Φ, φ and η be $\chi(u) = \max_{0\leq z\leq 1}|u'(z)|$:

$$\Phi(u) = \varphi(u) = \max_{0\leq z\leq 1}|u(z)|, \quad \eta(u) = \min_{p\leq z\leq 1-p}|u(z)|.$$

Lemma 4. *Let $u(z)$ be symmetric on (0, 1), with μ, α, γ and β being nonnegative constants. Thus, the only solution $u(z)$ of the BVP described in (1) and (2) is symmetric on (0, 1).*

Proof. From (5), we have:

$$u(z) = \int_{\omega_1}^{\omega_2} G(z,s)f\big(s,u(s),u'(s)\big)ds.$$

Therefore, we know:

$$u(1-z) = \int_{\omega_1}^{\omega_2} G(1-z,s)f\big(s,u(s),u'(s)\big)ds.$$

We have $u(z) = u(1-z)$ and thus, $f\big(1-z,u(1-z)\big) = f\big(z,u(z)\big)$.

$$u(1-z) = \int_{\omega_1}^{\omega_2} G(1-z,s)f(s,u(s),u'(s))ds,$$

$$= \int_{\omega_1}^{\omega_2} G(1-z,1-s)f(1-s,u(1-s),u'(1-s))d(1-s),$$

$$= \int_{\omega_1}^{\omega_2} G(z,s)f\big(s,u(s),u'(s)\big)ds$$

$$= u(z).$$

The BVP described in (1) is the only symmetric solution:

$$u(z)= \int_{\omega_1}^{\omega_2} G(z,s)\int_{\omega_1}^{\omega_2} L(s,\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi ds$$

where

$$y(z) = u''(z) = \int_{\omega_1}^{\omega_2} L(z,s)f\big(s,u(s),u'(s)\big)ds.$$

$T: P \to P$ is continuous and similarly, we obtain $(Tu)(1-z) = (Tu)(z)$.

We accept the following assumptions:

(B1) $y: [0,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous;

(B2) $h \in C[0,1], h(z) \leq 0, \forall z \in [0,\omega_1]$, $h(z) \geq 0$, $\forall z \in [\omega_1,\omega_2], h(z) \leq 0, \forall z \in [\omega_2,1]$, and $h(z)$ are not equal to zero in any subrange of [0, 1]. □

Lemma 5. *Assume that there is (B1) and (B2). If $\alpha \geq \mu\omega_1$ and $\beta \geq \gamma(1-\omega_2)$, $T: P \to P$ are completely continuous.*

Proof. For each $z \in [0,1]$, we contemplate three situations:

Case (i): $z \in [0, \omega_1]$. For any $u \in P$, we have from (11), (B1), (B2) and $\alpha \geq \mu\omega_1$ that

$$(Lu)(z){:}\,\lambda\left[\int_z^{\omega_1}(z-\xi)y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi + \frac{1}{\delta}\int_{\omega_1}^{\omega_2}(\alpha-\mu(\omega_1-z)))(\gamma(\omega_2-\xi)+\beta)\,y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi\right] \quad (12)$$

Case (ii): $z \in [\omega_1,\omega_2]$. For each $u \in P$, (11) is available and we have from (B1), (B2) and $\alpha \geq \mu\omega_1$ that

$$(Lu)(z) = \lambda\left(\int_{\omega_1}^{z}((\gamma(\omega_2-\xi)+\beta)\right)y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi \quad (13)$$

$$+\frac{1}{\delta}\int_{z}^{\omega_2}(\alpha-\mu(\omega_1-z)))(\gamma(\omega_2-\xi)+\beta)\ y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi)$$

$$=\lambda(\frac{1}{\delta}\int_{\omega_1}^{z}(\alpha+\mu(\xi-\omega_1)))(\gamma(\omega_2-z)+\beta)\,y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi$$

$$+\frac{1}{\delta}\int_{z}^{t}(\alpha+\mu(z-\omega_1)))(\gamma(\omega_2-\xi)+\beta)\ y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi)\ \geq 0.$$

Case (iii): $z\in[\omega_2,1]$. For each, $u\in P$, (2.9) is available and we have from (B1), (B2) and $\beta\geq(1-\omega_2)\gamma$ that

$$(Lu)(z)=\lambda(\int_{\omega_1}^{\omega_2}(\xi-z)y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi+\int_{\omega_2}^{z}(\xi-z)\,y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi+\frac{1}{\delta}\int_{\omega_1}^{\omega_2}(\alpha-\mu(\omega_1-z)))(\gamma(\omega_2-\xi)+\beta)\ y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi)=\lambda(\frac{1}{\delta}\int_{\omega_1}^{\omega_2}\big(\alpha+\mu(\xi-\omega_1)\big)(-\gamma(z-\omega_2)+\beta)\,y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi+\int_{\omega_2}^{z}(\xi-z)\ y(\xi)f\big(\xi,u(\xi),u'(\xi)\big)d\xi)\geq 0. \quad (14)$$

Thus, from (13) and (14), we get:

$$(Lu)(z)\geq 0,\qquad z\in[0,1]. \quad (15)$$

Therefore, we acquire:

$$(T\,u)(z)\geq 0,\qquad z\in[0,1]. \quad (16)$$

Evidently, this becomes $(T\,u)(0)=(T\,u)(1)=0$ and $(T\,u)'(z)=-(Lu)(z)\leq 0, z\in[0,1]$. Therefore, $T:P\to P$. Furthermore, the Arzera–Ascoli theorem suggests that T is completely continuous.

Note. By $\delta=\mu\beta+\alpha\gamma+\mu\gamma(\omega_2-\omega_1)>0$, $\alpha\geq\mu\omega_1$ and $\beta\geq\gamma(1-\omega_2)$, we have $\alpha>0$ and $\beta>0$.

For whole $u\in P$, this becomes:

$$p(\varphi)\leq\vartheta(u)\leq\varphi(u)=\Phi(u),\ \|u\|=max\{\varphi(u),\Phi(u)\}\ \vartheta(u) \quad (17)$$

It is assumed that $0<a<b\leq 3/4\,pd$ are constant values to reach our main result:

(B3) $(z,u,v)\leq\frac{d}{M\lambda}$, for $(z,u,v)\in[0,1]\times[0,\frac{3}{4}d]\times[-d,0]$,

(B4) $(z,u,v)\leq\frac{b}{m\lambda}$, for $(z,u,v)\in[p,1-p]\times\left[b,\frac{b}{p}\right]\times[-d,0]$,

(B5) $f(z,u,v)\leq\frac{a}{N\lambda}$, for $(z,u,v)\in[0,1]\times[0,a]\times[-d,0]$, where $\lambda>0$ and

$$M=\int_0^{\omega_1}-\xi y(\xi)d\xi+(\omega_2-\omega_1+\frac{\alpha\beta}{\delta})\int_{\omega_1}^{\omega_2}y(\xi)d\xi,$$

$$N=(\frac{{\omega_1}^2}{2}-\frac{{\omega_1}^2}{6})\int_0^{\omega_1}-\xi y(\xi)d\xi+\frac{1}{6}\Big(\omega_2-\omega_1+\frac{\alpha\beta}{\delta}\Big)\int_{\omega_1}^{\omega_2}y(\xi)d\xi,$$

$$m=\min\{m_1,m_2\},$$

$$m_1=\frac{\alpha}{\delta}\int_{\omega_1}^{\omega_2-p}G(p,s)ds\int_s^{\omega_2-p}(\gamma(\omega_2-\xi)+\beta)y(\xi)d\xi,$$

$$m_2=\frac{\alpha}{\delta}\int_{\omega_1}^{\omega_2-p}G(\omega_2-p,s)ds\int_s^{\omega_2-p}(\gamma(\omega_2-\xi)+\beta)y(\xi)d\xi$$

Theorem 3. *Let $\alpha\geq\mu\omega_1$ and assume that (B1)–(B5) exist. Thus, BVP (1) and (2) becomes the minimum positive solutions.*

$$\max_{0\leq z\leq 1}|u_i'(z)|\leq d,\ \text{for}\ i=1,2,\ldots$$

$$\min_{p\leq z\leq 1-p}|u_1(z)|>b,a<\max_{0\leq z\leq 1}|u_2(z)|,\ \text{with}\ \min_{p\leq z\leq 1-p}|u_2(z)|<b.$$

Proof. $T: P \to P$, with the Arzela–Ascoli theorem, we show that T is continuous. Let us now explain that all the conditions of the Theorem 2 are fulfilled. If $u \in P(\vartheta, d)$, $\vartheta(u) = \max_{0\le z\le 1}|u_i'(z)| \le d$. We have $\max_{0\le z\le 1}|u(z)| \le \frac{3}{4}\, d$, According to the assumption of (B3), $f(z,u(z),u'(z)) \le \frac{d}{M}$. On the other hand, from (13) and (14), we have:

$$\max_{0\le z\le \omega_1}(Lu)(z) \le \lambda(\int_0^{\omega_1} -\xi y(\xi) f(\xi,u(\xi),u'(\xi))d\xi + \frac{1}{\delta}\int_{\omega_1}^{\omega_2} \alpha(\gamma(\omega_2-\xi)+\beta)\, y(\xi) f(\xi,u(\xi),u'(\xi))d\xi) \le \lambda(\int_0^{\omega_1} -\xi y(\xi) f(\xi,u(\xi),u'(\xi))d\xi + \frac{1}{\delta}\int_{\omega_1}^{\omega_2} \alpha(\gamma(\omega_2-\xi)+\beta)\, y(\xi) f(\xi,u(\xi),u'(\xi))d\xi \le \lambda(\int_0^{\omega_1} -\xi y(\xi) f(\xi,u(\xi),u'(\xi))d\xi + \frac{1}{\delta}(\gamma(\omega_2-\omega_1)\beta\int_{\omega_1}^{\omega_2} y(\xi) f(\xi,u(\xi),u'(\xi))d\xi) \tag{18}$$

and

$$\max_{0\le z\le \omega_1}(Lu)(z) \le \frac{\lambda}{\delta}\int_{\omega_1}^{z}(\alpha+\mu(z-\omega_1))(\gamma(\omega_2-\xi)+\beta) y(\xi) f(\xi,u(\xi),u'(\xi))d\xi + \frac{\lambda}{\delta}\int_{z}^{\omega_2}(\alpha+\mu(z-\omega_1))(\gamma(\omega_2-\xi)+\beta)\, y(\xi) f(\xi,u(\xi),u'(\xi))d\xi = \frac{\lambda}{\delta}(\alpha+\mu(z-\omega_1))(\gamma(\omega_2-\xi)+\beta) y(\xi) f(\xi,u(\xi),u'(\xi))d \le \frac{\lambda}{\delta}\left(\alpha+\mu(\omega_2-\omega_1)\right)(\gamma(\omega_2-\omega_1)+\beta)\int_{\omega_1}^{\omega_2} y(\xi) f(\xi,u(\xi),u'(\xi))d\xi \tag{19}$$

If (18) and (19) are combined, we obtain the following:

$$(Tu) = \max_{z\in[0,1]}|(Tu)'(z)| = \max_{z\in[0,1]}|(Lu)(z)|$$
$$= max\left\{\max_{0\le z\le \omega_1}|(Lu)(z)|, \max_{\omega_1\le z\le \omega_2}|(Lu)(z)|\right\}$$
$$\le \lambda\int_0^{\omega_1} -\xi y(\xi) f(\xi,u(\xi),u'(\xi))d\xi + \frac{\lambda}{\delta}\left(\alpha+\mu(\omega_2-\omega_1)\right)(\gamma(\omega_2-\omega_1)\beta)\int_{\omega_1}^{\omega_2} y(\xi) f(\xi,u(\xi),u'(\xi))d\xi = \lambda\int_0^{\omega_1} -\xi y(\xi) f(\xi,u(\xi),u'(\xi))d\xi + \lambda(\omega_2-\omega_1+\frac{\alpha\beta}{\delta})\int_{\omega_1}^{\omega_2} y(\xi) f(\xi,u(\xi),u'(\xi))d\xi \le \frac{\lambda d}{M}(\int_0^{\omega_1} -\xi y(\xi)d\xi + \lambda(\omega_2-\omega_1+\frac{\alpha\beta}{\delta})\int_{\omega_1}^{\omega_2} y(\xi)d\xi = \frac{d}{M}M = d \tag{20}$$

Hence, $T: \overline{P(\chi,d)} \to \overline{P(\chi,d)}$.

Thus, according to Theorem 2, we choose $(z) = \frac{b}{p}$, $0 \le z \le 1$. It is simple to see that $u(z) = \frac{b}{p} \in P(\chi,\varphi,\eta,b,\frac{b}{p},d)$ and $\alpha(u) = \alpha(\frac{b}{p}) > b$, and so $\{u \in P(\chi,\varphi,\eta,b,\frac{b}{p},d) \mid \alpha(u) > b\} \ne \emptyset$. Therefore, if $u \in P(\chi,\varphi,\eta,b,\frac{b}{p},d)$, $b \le u(z) \le \frac{b}{p}$, $-d \le u'(z) \le 0$ for $p \le z \le (1-p)$. From supposition (B4), we have $f(z,u(z),u'(z)) > \frac{b}{p}$ for $p \le z \le 1-p$ and by the terms on α and the cone P, we have to separate the two states by pursuing:

State (i): $\eta(Tu) = (Tu)(p)$. From (12), we have:

$$\eta(Tu) = (Tu)(p) = \lambda\int_0^1 G(p,s)\,(Lu)(s)ds \ge \lambda\int_{\omega_1}^{\omega_2} G(p,s)(Lu)(s)ds,$$
$$\ge \frac{\lambda}{\delta}\int_{\omega_1}^{\omega_2} G(p,s)\int_s^{\omega_2}\left(\alpha+\mu(\omega_2-\omega_1)\right)(\gamma(\omega_2-\omega_1)+\beta) y(\xi) f(\xi,u(\xi),u'(\xi))d\xi$$
$$\ge \frac{\lambda\alpha}{\delta}\int_p^{1-p} G(p,s)\int_s^{1-p}(\gamma(\omega_2-\omega_1)+\beta) y(\xi) f(\xi,u(\xi),u'(\xi))d\xi \tag{21}$$
$$\ge \lambda\frac{\alpha}{\delta}\frac{b}{m}\int_{\omega_1}^{1-p} G(p,s)\int_s^{1-p}(\gamma(\omega_2-\omega_1)+\beta) y(\xi) f(\xi,u(\xi),u'(\xi))d\xi$$
$$= \frac{b}{m}m_1 \ge b.$$

State (ii): $\eta(Tu) = (Tu)(1-p)$, which is the same as:

$$\eta(Tu) = (Tu)(1-p) \ge \lambda\int_0^1 G(1-p,s)\,(Lu)(s)$$
$$\ge \lambda\int_{\omega_1}^{\omega_2} G(1-p,s)(Lu)(s)ds, \tag{22}$$

$$\geq \frac{\lambda}{\delta} \int_{\omega_1}^{\omega_2} G(1-p,s) \int_s^{\omega_2} \big(\alpha + \mu(\omega_2-\omega_1)\big)(\gamma(\omega_2-\omega_1)+\beta) y(\xi) f\big(\xi,u(\xi),u'(\xi)\big) d\xi$$

$$\geq \lambda \frac{\alpha}{\delta} \int_p^{1-p} G(1-p,s) \int_s^{1-p} (\gamma(\omega_2-\omega_1)+\beta) y(\xi) f\big(\xi,u(\xi),u'(\xi)\big) d\xi$$

$$\geq \lambda \frac{\alpha}{\delta} \frac{b}{m} \int_{\omega_1}^{1-p} G(1-p,s) \int_s^{1-p} (\gamma(\omega_2-\omega_1)+\beta) y(\xi) f\big(\xi,u(\xi),u'(\xi)\big) d\xi$$

$$= \frac{b}{m} m_2 \geq b.$$

$$\eta(Tu) > b, \forall u \in P((\chi,\ \eta, b, \frac{b}{p}, d).$$

This proves that the requirement (C1) of Theorem 2 is fulfilled. Furthermore, from (20) and $b \leq \frac{3}{4}\, pd$, we have:

$$\eta(Tu) \ \geq\ p\varphi(Tu) \ >\ p\ \frac{b}{p} \ =\ b,$$

for all $u \ \in\ P(\chi,\eta,b,d)$ with $\varphi(Tu) \ >\ \frac{b}{p}$. Thus, the requirement (C1) of Theorem 2 is fulfilled. After all, we showed that Theorem 2 fulfils (C3). It is clear that Φ (0) = 0 < a, 0 ∉ R (χ, Φ, a, d). Presume that $u \in R(\chi,\Phi,a,d)$ with $\Phi(u) = a$. Moreover, according to the assumptions (B5), (20) and (21), we obtain the following:

$$\Phi(Tu) = \max_{z\in[0,1]} |(Tu)(z)| = \lambda \max_{z\in[0,1]} \int_0^1 G(z,s)\,(Lu)(s) ds$$

$$\leq \lambda \int_0^1 G(s,s)\,(Lu)(s) ds$$

$$\leq \lambda \int_0^{\omega_1} G(s,s)\, ds \max_{s\in[0,\omega_1]} |(Lu)(s)| + \lambda \int_{\omega_1}^{\omega_2} G(s,s) ds \max_{s\in[\omega_1,\omega_2]} |(Lu)(s)|$$

$$\leq\ \lambda \int_0^{\omega_1} G(s,s)\, ds (\int_s^{\omega_1} (s-\xi\,) y\,(\xi) f\big(\xi,u(\xi),u'(\xi)\big) d\xi \ +\ \lambda \frac{\alpha}{\delta} (\gamma(\omega_2-\xi) +$$

$$\beta) \int_{\omega_1}^{\omega_2} y(\xi) f\big(\xi,u(\xi),u'(\xi)\big) d\xi\ ) \ +\ \frac{\lambda}{\delta}\big(\alpha+\mu(\omega_2-\omega_1)\big)(\gamma(\omega_2-\xi) +$$

$$\beta) \int_{\omega_1}^{\omega_2} G(s,s) ds\ \int_{\omega_1}^{\omega_2} y(\xi) f\big(\xi,u(\xi),u'(\xi)\big) d\xi\ )$$

$$<\ \ \frac{\lambda a}{N} \int_0^{\omega_1} G(s,s)\, ds \int_0^{\omega_1} -\xi y(\xi) d\xi + \ \ (\ \ \omega_2-\omega_1 \ \ + \ \ \frac{\alpha\beta}{\delta}) \frac{\lambda a}{N} \int_0^{\omega_1} G(s,s)\, ds \int_{\omega_1}^{\omega_2} y(\xi) d\xi\ ) +$$

$$\lambda(\omega_2-\ \omega_1 + \frac{\alpha\beta}{\delta}) \frac{a}{N} \int_{\omega_1}^{\omega_2} G(s,s)\, ds \int_{\omega_1}^{\omega_2} y(\xi) d\xi\ )$$

$$= \lambda \int_0^{\omega_1} G(s,s)\, ds \int_0^{\omega_1} -\xi y(\xi) d\xi + (\omega_2-\omega_1 + \frac{\alpha\beta}{\delta}) \frac{\lambda a}{N} \int_0^{\omega_1} G(s,s)\, ds \int_{\omega_1}^{\omega_2} y(\xi) d\xi\ )$$

$$= \lambda \frac{a}{N} \left\{ \left(\frac{\omega_1^2}{2} - \frac{\omega_1^3}{2} \right) \right\} \int_0^{\omega_1} -\xi y(\xi) d\xi + \frac{1}{6} (\omega_2 - \omega_1 + \frac{\alpha\beta}{\delta}) \int_{\omega_1}^{\omega_2} y(\xi) d\xi\ ) = a.$$

That is, the condition (C3) of Theorem 2 is ensured. As a result, it is indicated that there is at least one positive solution to the problem (1) and (2) so that:

$$\max_{0\leq z\leq 1} |u_i{}'(z)| \leq d, \text{ for } i = 1,2,\ldots \min_{p\leq z\leq 1-p} |u_1(z)| > b,$$

$$a\ <\ \max_{0\leq z\leq 1} |u_2(z)|, \text{ with } \min_{p\leq z\leq 1-p} |u_2(z)| < b,$$

and the proof is complete.

We are now using the fixed point theorem of Guo and Krasnosel'skii to prove the main results. This theorem is first described below.

Theorem 4 ([17]). *E is a Banach space and P, E is a cone. Let Ω_1 and Ω_2 be the obvious subset of E with 0 $\in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. T: $P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$, it is an entirely continuous operator. Furthermore, assume that the following conditions are met:*

(A) $\|Tu\| \leq \|u\|$, $\forall u \in P \cap \partial \Omega_1$ *and* $\|Tu\| \geq \|u\|$, $\forall u \in P \cap \partial \Omega_2$ *or*
(B) $\|Tu\| \geq \|u\|$, $\forall u \in P \cap \partial \Omega_1$ *and* $\|Tu\| \leq \|u\|$, $\forall u \in P \cap \partial \Omega_2$ *holds. Thus,* T *has a fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

4. Main Results

In this part, we argue against the presence of a positive solution of BVP (1) and (2). For suitability, we set the following:

$$\max f_0 = \lim_{-v \to 0} \max_{z \in [0,1]} sup_{u \in [0,+\infty]} \frac{f(z,u,v)}{-v},$$

$$\min f_0 = \lim_{-v \to 0} \min_{z \in [0,1]} inf_{u \in [0,+\infty]} \frac{f(z,u,v)}{-v}$$

$$\max f_\infty = \lim_{-v \to +\infty} \max_{z \in [0,1]} sup_{u \in [0,+\infty]} \frac{f(z,u,v)}{-v}$$

$$\min f_\infty = \lim_{-v \to +\infty} \min_{z \in [0,1]} inf_{u \in [0,+\infty]} \frac{f(z,u,v)}{-v}$$

Throughout this section, we assume that $\mu, \alpha, \gamma, \beta$ are nonnegative constants, $0 \leq \omega_1 < \omega_2 \leq 1$ and $\delta = \mu\beta + \alpha\gamma + \mu\gamma(\omega_2 - \omega_1) > 0$, $-\mu\omega_1 + \alpha \geq 0$, $\gamma(\omega_2 - 1) + \beta \geq 0$.

Lemma 6. *Suppose* $-\mu\omega_1 + \alpha \geq 0$, $\gamma(\omega_2 - 1) + \beta \geq 0$. *This means that* $T(P) \subset P$ *, where* $\tau_1 = \omega_1 + \frac{1}{4}(\omega_2 - \omega_1)$, $\tau_2 = \omega_2 - \frac{1}{4}(\omega_2 - \omega_1)$. *P is a cone in E and If* $u \in P, min(-u'(z))_{z \in \{\tau_1, \tau_2\}} \geq \lambda \frac{1}{4}\|u\|$.

Theorem 5. *Assume that* $f \in C([0,1] \times [0,\infty) \times (-\infty,0],[0,\infty))$ *and* f *is sublinear, i.e., min* $f_\infty = \infty$ *and max* $f_0 = 0$. *Furthermore, there is at least one positive solution of BVP (1) and (2).*

Proof. Since $\max f_0 = \infty$, for any ϵ satisfying $\frac{1}{4}\epsilon \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) d\xi \leq 1$, there exists $R_1 > 0$ so that:

$$(z,u,v) \leq \epsilon(-v), \text{ for } z \in [0,1], \ u \in [0,+\infty], \ 0 \leq -v \leq R_1 \quad (23)$$

Set $\Omega_{R_1} = \{u \in P: \|u\| < R_1\}$.
If so, for $u \in \partial\Omega_{R_1}$, we can get the folowing from Lemma 4, Lemma 6 and using (23):

$$-(Tu)'(z) = \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) f(\xi, u(\xi), u'(\xi))\, d\xi$$

$$\geq \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) f(\xi, u(\xi), u'(\xi))\, d\xi \geq \epsilon \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) (-u'(\xi))\, d\xi$$

$$\geq \frac{1}{4}\epsilon\|u\| \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) d\xi$$

$$\geq \|u\|.$$

This is also true:

$$\|Tu\| \geq \|u\|, \qquad \text{for } u \in \partial\Omega_{R_1} \quad (24)$$

From this point onwards, $\max f_\infty = 0$ for each ϵ^* filling $\epsilon^* \int_{\tau_1}^{\tau_2} G(\xi,\xi)(-u'(\xi))\, d\xi \leq 1$,
Set $\Omega_{R_2} = \{u \in P: \|u\| < R_2\}$ for $u \in \partial\Omega_{R_2}$.

$$f(z,u,v) \leq \epsilon^*(-v), \text{for } z \in [0,1], u \in [0,+\infty], -v \geq R \quad (25)$$

$$-(Tu)'(z) = \lambda \int_{\tau_1}^{\tau_2} G(\xi,\xi) f(\xi, u(\xi), u'(\xi))\, d\xi$$

$$\leq \lambda\epsilon^* \int_{\tau_1}^{\tau_2} G(\xi,\xi)(-u'(\xi))\, d\xi$$

$$\leq \|u'\|_\infty = \|u\|,$$

and thus:

$$\|Tu\| \geq \|u\|, \text{for } u \in \ \partial\Omega_{R_2}. \tag{26}$$

Hence from (24), (26) and Theorem 4, the operator T can obtain a fixed point u in $\bar{\Omega}_{R_2}\backslash\Omega_{R_1}$ where u is a positive solution of (1) and (2) and satisfies $R_1 \leq \|u\| \leq R_2$. □

Theorem 6. *Suppose that* $f \in C([0,1] \times [0,\infty) \times (-\infty,0],[0,\infty))$ *and* f *is sublinear, i.e., min* $f_0 =$ ∞ *and max* $f_\infty = 0$*. Furthermore, there is at least one positive solution of BVP (1) and (2).*

Proof. Since min $f_0 = \infty$, for any ϵ satisfying $\frac{1}{4}\ \epsilon * \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) d\xi \ \leq 1$, there exists $R_1 > 0$ such that:

$$f(z,u,v) \leq \epsilon * (-v), \text{for } \ z \in [0,1], \ u \ \in \ [0,+\infty], \ 0 \leq -v \leq \ R_1. \tag{27}$$

Set $\Omega_{R_1} = \{u \in P \colon \|u\| \ < \ R_1\}$. For u $\in \ \partial\Omega_{R_1}$, we can obtain the following from (23) and (27):

$$-(Tu)'(z) \ = \ \lambda \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) f(\xi, u(\xi), u'(\xi))\, d\xi$$

$$\geq \lambda \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) f(\xi, u(\xi), u'(\xi))\, d\xi$$

$$\geq \lambda\epsilon * \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) (-u'(\xi))\, d\xi$$

$$\geq \lambda\ \frac{1}{4}\ \epsilon^* \|u\| \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) d\xi$$

$$\geq \|u\|,$$

And thus:

$$\|Tu\| \geq \|u\|, \text{for u} \in \partial\Omega_{R_1}. \tag{28}$$

If max $f_\infty = \ 0$, for any ϵ^* convincing $\epsilon^* \int_{\tau_1}^{\tau_2} G(\xi,\xi)\big(-u'(\xi)\big)\, d\xi \leq 1$ there exists $R^* > R_1$ such that:

$$f(z,u,v) \leq \epsilon * (-v), \ \text{for } z \in \ [0,1], \quad u \in [0,+\infty], \ -v \geq R^*. \tag{29}$$

Since max $f_\infty \ = \ 0$, two cases can be examined.

Case (i): Presume that $f\ (z,u,v)$ is infinite. Thus, for any ϵ^* satisfying $\frac{1}{4}\epsilon * \int_{\tau_1}^{\tau_2} G\left(\frac{1}{2}(\tau_1 - \tau_2), \xi\right) d\xi \ \leq 1$, there exists $R^* > \ R_1$ such that:

$$f(z,u,v) \leq \epsilon^*\ (-v), \ \text{ for } \ z \in [0,1], \quad u \ \in \ [0,+\infty], \ -v \geq \ R^*. \tag{30}$$

Let $f^*\ (q)\colon [0,\infty) \ \to \ [0,\infty)$ define the function by:

$$f^*\ (q) \ = \max\{\ f\ (z,u,v)\colon z \ \in \ [0,1], \ \ 0 \ \leq u \ \leq q, \ 0 \ \leq -v \ \leq q\},$$

where $\lim\limits_{q\to+\infty} \frac{f^*\ (q)}{q} = 0$ and:

$$f^*\ (q) \leq \epsilon^* q \text{ for } \ q > R^*. \tag{31}$$

If $R_2 > \ R^*$, (30) and (31) can be written as

$$f\ (z,u,v) \ \leq f^*(R_2) \leq \ \epsilon^* R_2 \text{ for } \ z \ \in \ [0,1], 0 \ \leq u \ \leq \ R_2, 0 \ \leq -v \ \leq R_2. \tag{32}$$

If $u \in P$ with $\|u\| \ = \ R_2$, from Lemma 1, we know that:

$$\|u\|_\infty \ \leq \ R_2. \tag{33}$$

Thus, from (32), (33) and Lemma 6, we get the following for $u \ \in \ P$ and $\|u\| \ = \ R_2$:

$$-(Tu)''(z) \ = \ \int_{\tau_1}^{\tau_2} G(\xi,\xi) f(\xi, u(\xi), u'(\xi))\, d\xi$$

$$\leq \ \epsilon^* R_2 \int_{\tau_1}^{\tau_2} G(\xi,\xi)(u''(\xi))\, d\xi \leq \ R_2 = \|u\|.$$

Case (ii): Suppose $f(z,u,v)$ is bounded, i.e., There is a Q number such that $f(z,u,v) \leq Q$ $(Q > 0)$. Get $R_2 > \max\{Q \int_{\tau_1}^{\tau_2} G(\xi,\xi)\, d\xi,\ R_1\}$. For $u \in P$ and $\|u\| = R_2$, we obtain the following:

$$-(Tu)''(z) = \int_{\tau_1}^{\tau_2} G(\xi,\xi) f(\xi, u(\xi), u'(\xi))\, d\xi$$

$$\leq Q \int_{\tau_1}^{\tau_2} G(\xi,\xi)\, d\xi \leq R_2 = \|u\|.$$

Therefore, in both cases, we can set $\Omega_{R_2} = \{u \in P: \|u\| < R_2\}$ so that

$$\|Tu\| \leq \|u\|, \qquad \text{for } u \in \partial\Omega_{R_2}. \tag{34}$$

Hence, from (28), (34) and Theorem 5, T seems to be a fixed point, $u\ is\ \bar{\Omega}_{R_2} \backslash \Omega_{R_1}$, u is a positive solution of (1) and (2) and satisfies $R_1 \leq \|u\| \leq R_2$.

Example 1. *Consider the following boundary value problem system:*

$$u''''(z) + \lambda z \left[\left(\frac{\sqrt{z}\sqrt{1-z}}{2}\right) (u')^2 + In(u(z)+1) \right] = 0,\ z \in (0,1)$$

$$u(0) = 0, \qquad u(1) = 0$$

$$\frac{1}{2} u''\left(\frac{1}{4}\right) - u'''\left(\frac{1}{4}\right) = 0,\ \frac{2}{3} u''\left(\frac{1}{3}\right) + 2u'''\left(\frac{1}{3}\right) = 0,$$

where λ *is a clearly a positive parameter:*

$$f(z,u,v) = \left(\frac{\sqrt{z}\sqrt{1-z}}{2}\right) (v)^2 + In(u(z)+1),\ y(z) = z,\ \omega_1 = \frac{1}{4},\ \omega_2 = \frac{1}{3} \text{ and } \mu = \frac{1}{2}, \alpha = 1,$$
$$\gamma = \frac{2}{3}, \beta = 2$$

There is a nonnegative constant satisfying $\mu\beta + \alpha\gamma + \mu\gamma(\omega_2 - \omega_1) > 0$, $0 \leq \omega_1 < \omega_2 \leq 1$.

$f(z,u)$ is symmetric on [0, 1] for all $u \in [0,\infty)$.

$$\int_{\frac{13}{48}}^{\frac{15}{48}} G\left(\frac{1}{2}\left(\frac{13}{48} - \frac{15}{48}\right), s\right) ds \leq 1,\ \min f_\infty = \infty \text{ and } \max f_0 = 0.$$

Namely, this is:

$$\max f_0 = \lim_{-v \to 0} \frac{f(z,u,v)}{-v} = \lim_{-v \to 0} \frac{\left(\frac{\sqrt{z}\sqrt{1-z}}{2}\right)(v)^2 + In(u(z)+1)}{-v} = 0,$$

$$\min f_\infty = \lim_{-v \to +\infty} \frac{f(z,u,v)}{-v} = \lim_{-v \to +\infty} \frac{\left(\frac{\sqrt{z}\sqrt{1-z}}{2}\right)(v)^2 + In(u(z)+1)}{-v} = \infty.$$

Thus, according to Theorem 5, BVP has at least one positive solution.

5. Conclusions

The four-point fourth-order boundary value problem is used to calculate the deflection at any point in the beam. The maximum deviation is determined by the symmetry of the beam. If it is unclear where the maximum deviation occurs, it is possible to determine this with the above-mentioned equations. The equation used in this study determines the location and time of the change in the beam slope. The problem examined is used to solve the deflection in the beam.

In this paper, we first evaluated the properties of the symmetrical solutions for the fourth-order boundary value problems [4–11,15–19]. We have examined the requirements. After this, we used the fixed point index as a different method compared to the fourth-order boundary value problems [4,9,10]. In Section 3, we investigated the existence of symmetric positive solutions by giving sufficient conditions for λ and u in our problem. We have proved with the Lemma 4 and Theorem 3 that under certain conditions, the problem has a minimum symmetric positive solution with Avery and Peterson's fixed point theorem. Unlike other studies, in the last section, we showed that in a superlinear case of a non-linear problem with four-point boundary value conditions, there is at least one positive symmetric solution obtained by the method of Krasnosel'skii [15]. We have given an example that proves Theorem 5.

In previous studies, two-point and three-point boundary value problems are generally emphasized. This study is different because it is a four-point boundary value problem and the symmetric solution is the solution. The given boundary conditions indicate that the non-linear beam rests against two ends with an elastic response. Moreover, the solution of (1) and (2) gives the balance of the beam that will prevent it from bending when exposed to a force. This corresponds to the bending moment in the tip in physics. The symmetrical solution means that the beams are exposed to slow oscillations in the elastic bearings. As part of these calculations, the maximum deviations in the beams will be evaluated and the beam will be supported in a safe manner. Therefore, these results will shed light for engineers in some typical beam deformation problems and are a useful contribution to the current literature.

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