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Positive Solutions of a Fractional Thermostat Model with a Parameter

Xinan Hao * and Luyao Zhang

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China; shuxuequfu@126.com
* Correspondence: haoxinan2004@163.com

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Abstract: We study the existence, multiplicity, and uniqueness results of positive solutions for a fractional thermostat model. Our approach depends on the fixed point index theory, iterative method, and nonsymmetry property of the Green function. The properties of positive solutions depending on a parameter are also discussed.

Keywords: positive solution; fractional thermostat model; fixed point index; dependence on a parameter

1. Introduction

In this paper, we investigate a fractional nonlocal boundary value problem (BVP)

$$\begin{cases} {}^cD^{\alpha}_{0+}x(t) + \lambda g(t)f(x(t)) = 0, \quad t \in (0,1), \\ x'(0) = 0, \ \beta {}^cD^{\alpha-1}_{0+}x(1) + x(\eta) = 0, \end{cases} \tag{1}$$

where $1 < \alpha \le 2$, $\beta > 0$, $0 \le \eta \le 1$, $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$, ${}^cD^{\alpha}_{0+}$ is the Gerasimov–Caputo fractional derivative of order α, $\lambda > 0$ is a parameter, $f \in C([0,+\infty),[0,+\infty))$, $g \in C((0,1),[0,+\infty))$, and $0 < \int_0^1 g(t)dt < +\infty$.

One motivation is that the thermostat model

$$\begin{cases} x''(t) + g(t)f(t,x(t)) = 0, \quad t \in (0,1), \\ x'(0) = 0, \ \beta x'(1) + x(\eta) = 0, \end{cases} \tag{2}$$

which is a special case with $\alpha = 2$ and $\lambda = 1$, has been discussed by Infante and Webb [1,2]. They established multiplicity results of BVP (2). These types of problems have been investigated by various scholars, see References [3–17].

Recently, the thermostat model was extended to the fractional case

$$\begin{cases} {}^cD^{\alpha}_{0+}x(t) + f(t,x(t)) = 0, \quad t \in (0,1), \ \alpha \in (1,2], \\ x'(0) = 0, \ \beta {}^cD^{\alpha-1}_{0+}x(1) + x(\eta) = 0, \end{cases} \tag{3}$$

where $\beta > 0$, $0 \le \eta \le 1$, $f \in C([0,1] \times [0,+\infty),[0,+\infty))$. Nieto and Pimentel [18] proved the existence of positive solutions based on the Krasnosel'skii fixed point theorem. Cabada and Infante [19] discussed the multiplicity results of positive solutions for BVP (3).

In Reference [20], Shen, Zhou, and Yang studied a fractional thermostat model

$$\begin{cases} {}^cD^{\alpha}_{0+}x(t) + \lambda f(t,x(t)) = 0, \quad t \in (0,1), \ 1 < \alpha \le 2, \\ x'(0) = 0, \ \beta {}^cD^{\alpha-1}_{0+}x(1) + x(\eta) = 0, \end{cases}$$

where $\beta > 0$, $0 \leq \eta \leq 1$, $\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}$, $\lambda > 0$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. The authors obtained intervals of parameter λ that correspond to at least one and no positive solutions. Similar fractional thermostat problems have been studied in References [21–24].

In this paper, we deal with positive solutions for the fractional thermostat model (1). The existence, multiplicity, and uniqueness results are established by the fixed point index theory and iterative method. The properties of positive solutions depending on a parameter are also discussed. Some of the ideas in this paper are from References [25,26]. Let us remark that the definition of the Gerasimov–Caputo derivative was first introduced and applied by Gerasimov in 1947 and then by Caputo in 1967, see for example, the overview by Novozhenova in Reference [27]. For details on the theory and applications of the fractional derivatives and integrals and fractional differential equations, see References [28–31].

2. Preliminaries

Lemma 1 ([20])**.** *Given $u(t) \in C(0,1) \cap L^1(0,1)$, the solution of the problem*

$$\begin{cases} {}^cD_{0+}^{\alpha}x(t) + u(t) = 0, \quad t \in (0,1), \\ x'(0) = 0, \ \beta\, {}^cD_{0+}^{\alpha-1}x(1) + x(\eta) = 0 \end{cases}$$

is

$$x(t) = \int_0^1 G(t,s)u(s)ds, \quad t \in [0,1],$$

where

$$G(t,s) = \begin{cases} \beta - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq \eta, \ s \leq t, \\ \beta + \dfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq \eta, \ s \geq t, \\ \beta - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \eta \leq s \leq 1, \ s \leq t, \\ \beta, & \eta \leq s \leq 1, \ s \geq t, \end{cases}$$

and $G(t,s)$ satisfies:

(i) $G(t,s) : [0,1] \times [0,1] \to (0,+\infty)$ *is continuous;*
(ii) $\frac{\partial}{\partial t}G(t,s) \leq 0, \ t,s \in [0,1]$;
(iii) $\gamma\overline{G} = \underline{G} \leq G(1,s) \leq G(t,s) \leq G(0,s) \leq \overline{G}, \quad t,s \in [0,1]$,

where

$$\gamma = \frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha) + \eta^{\alpha-1}}, \quad \underline{G} = \frac{\beta\Gamma(\alpha) - (1-\eta)^{\alpha-1}}{\Gamma(\alpha)}, \quad \overline{G} = \frac{\beta\Gamma(\alpha) + \eta^{\alpha-1}}{\Gamma(\alpha)}.$$

Denote $E = C[0,1]$ and $\|x\| = \sup_{t\in[0,1]} |x(t)|$. We define the cone

$$P = \{x \in E : x(t) \geq 0, \ \inf_{t\in[0,1]} x(t) \geq \gamma\|x\|\}.$$

For any $0 < r < +\infty$, let $P_r = \{x \in P : \|x\| < r\}$. We define $T : (0,+\infty) \times E \to E$ as

$$T(\lambda,x)(t) = \lambda \int_0^1 G(t,s)g(s)f(x(s))ds, \quad t \in [0,1].$$

It is obvious from Lemma 1 that if $x \in P$ is a fixed point of operator T, then x is a positive solution of Problem (1). By regularity arguments, we can show that T is completely continuous and $T(P) \subset P$.

Define the linear operator $L : E \to E$ by

$$Lx(t) = \int_0^1 G(t,s)g(s)x(s)ds, \quad t \in [0,1].$$

By the Krein–Rutman theorem, we see that the spectral radius $r(L)$ of the operator L is positive, and L has positive eigenfunction φ_1 corresponding to its first eigenvalue $\mu_1 = (r(L))^{-1}$.

Lemma 2 ([32]). *Let P be a cone in Banach space E. Suppose that $T : P \to P$ is a completely continuous operator. (i) If $Tu \neq \mu u$ for any $u \in \partial P_r$ and $\mu \geq 1$, then $i(T, P_r, P) = 1$. (ii) If $Tu \neq u$ and $\|Tu\| \geq \|u\|$ for any $u \in \partial P_r$, then $i(T, P_r, P) = 0$.*

Denote

$$f_0 = \lim_{s\to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{s\to\infty} \frac{f(s)}{s}, \quad A = \int_0^1 G(0,s)g(s)ds, \quad l = \min_{s\in(0,\infty)} \frac{f(s)}{s}.$$

We assume that:

(H_1) *f is nondecreasing on $[0, +\infty)$;*
(H_2) *there exists a function $\phi : (0,1] \to [0,1]$ continuous nondecreasing, such that $f(\kappa x) \geq \phi(\kappa) f(x)$ for $0 < \kappa < 1,\ x > 0$, and $F(\kappa) := \frac{\kappa}{\phi(\kappa)}$ is strictly increasing on $(0,1]$ and $F(1) = 1$.*

Lemma 3. *Suppose that (H_1) holds, $f_0 = \infty$ and $l > 0$. If $0 < \lambda_1 < \lambda_2 < \frac{1}{lA}$, then there exist $x_1, x_2 \in P \setminus \{\theta\},\ x_1 \leq x_2$, such that $T(\lambda_1, x_1)(t) = x_1(t)$ and $T(\lambda_2, x_2)(t) = x_2(t)$.*

Proof. Assume $s_0 \in (0,\infty)$ such that $f(s_0) = ls_0$. Since $0 < \lambda_1 < \lambda_2 < \frac{1}{lA}$, we have $l < \frac{1}{\lambda_2 A} < \frac{1}{\lambda_1 A}$. We define

$$x_0(t) = \frac{s_0}{A} \int_0^1 G(t,s)g(s)ds, \quad t \in [0,1],$$

then

$$\|x_0\| = x_0(0) = s_0, \quad x_0(t) \geq \frac{s_0}{A} \int_0^1 \gamma G(0,s)g(s)ds = \gamma\|x_0\|, \quad t \in [0,1].$$

Therefore, $x_0 \in P$ and $\|x_0\| = s_0$. Direct computations yield

$$\begin{aligned} T(\lambda_1, x_0)(t) =& \lambda_1 \int_0^1 G(t,s)g(s)f(x_0(s))ds \leq \lambda_1 \int_0^1 G(t,s)g(s)f(\|x_0\|)ds \\ =& \lambda_1 l s_0 \int_0^1 G(t,s)g(s)ds < \frac{s_0}{A} \int_0^1 G(t,s)g(s)ds = x_0(t), \quad t \in [0,1]. \end{aligned}$$

Define

$$x_1^1(t) = T(\lambda_1, x_0)(t),\ x_1^j(t) = T(\lambda_1, x_1^{j-1})(t) = T^j(\lambda_1, x_0)(t), \quad j = 2,3,\cdots,\ t \in [0,1].$$

Direct calculations show that $x_0 > x_1^1 > x_1^2 > \cdots > x_1^j > x_1^{j+1} > \cdots \geq \theta$. Hence, sequence $\{x_1^j\}_{j=1}^\infty$ is decreasing and bounded from below, $\lim_{j\to\infty} x_1^j(t)$ exists and convergence is uniform for $t \in [0,1]$. Assume that $\lim_{j\to\infty} x_1^j = x_1$, we claim that $x_1(t) > 0$. Otherwise, since $x_1 \in P$, $x_1(t) = 0$, i.e., $\lim_{j\to\infty} x_1^j(t) = 0,\ t \in [0,1]$, and hence from $x_1^j \in P$, we deduce $\|x_1^j\| \to 0$. Since $f_0 = \infty$, for any $H > \frac{1}{\lambda_1 \gamma A}$, there is integral $Z > 0$ such that for $j > Z$, we have $f(x_1^j(t)) > Hx_1^j(t)$, and hence

$$
\begin{aligned}
x_1^{j+1}(0) =& \lambda_1 \int_0^1 G(0,s)g(s)f(x_1^j(s))ds \\
>& \lambda_1 H\gamma \int_0^1 G(0,s)g(s)\|x_1^j\|ds \\
\geq& x_1^j(0)\lambda_1 H\gamma A > x_1^j(0).
\end{aligned}
$$

The contradiction shows that $x_1 \in P \setminus \{\theta\}$ and $x_1 = T(\lambda_1, x_1)$.

Similarly, from $x_2^1(t) = T(\lambda_2, x_0)(t)$ and $x_2^j(t) = T(\lambda_2, x_2^{j-1})(t),\ j = 2, 3, \cdots$, we deduce

$$x_0 > x_2^1 > x_2^2 > \cdots > x_2^j > x_2^{j+1} > \cdots \geq \theta,$$

$\lim_{j\to\infty} x_2^j = x_2 \in P \setminus \{\theta\}$, and $x_2 = T(\lambda_2, x_2)$. It follows from $x_1^1 = T(\lambda_1, x_0) < T(\lambda_2, x_0) = x_2^1$ and the monotonicity of f that $x_1^j \leq x_2^j,\ j = 2, 3, \cdots$. Therefore, $x_1 \leq x_2$. □

Lemma 4. *If $f_\infty = \infty$, then for any $\mu > 0$, the set $S_\mu = \{x \in P : T(\lambda, x) = x,\ \lambda \in [\mu, \infty)\}$ is bounded.*

Proof. Otherwise, there exists $x_n \in S_\mu$ corresponding to $\lambda_n \in [\mu, \infty)$ such that

$$T(\lambda_n, x_n) = x_n, \quad \lim_{n\to\infty} \|x_n\| = \infty.$$

Because $f_\infty = \infty$, there is $X > 0$ such that $f(s) > Hs$ for $s > X$, where $H > \frac{1}{\mu\gamma A}$. Since $\lim_{n\to\infty} \|x_n\| = \infty$, there exists $N_0 > 0$ such that $\|x_n\| > \frac{X}{\gamma}$ for $n > N_0$, and $x_n(t) \geq \gamma\|x_n\| > X,\ t \in [0,1]$. Then, for any $n > N_0$, we obtain

$$\|x_n\| > \lambda_n \int_0^1 G(0,s)g(s)Hx_n(s)ds > \mu H\gamma\|x_n\|A > \|x_n\|,$$

which is absurd, and hence S_μ is bounded. □

Lemma 5. *Assume that (H_1) holds, and that $f_0 = f_\infty = \infty$. Then, T admits a fixed point for $\lambda = \frac{1}{lA}$.*

Proof. Choosing a sequence $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots < \frac{1}{lA}$ such that $\lim_{n\to\infty} \lambda_n = \frac{1}{lA}$. By Lemma 3, there exists a nondecreasing sequence $\{x_n\}_{n=1}^\infty \subset P \setminus \{\theta\}$ such that $x_n = T(\lambda_n, x_n)$. By Lemma 4, we know that $\{x_n\}_{n=1}^\infty$ is uniformly bounded and equicontinuous. By using the Arzela–Ascoli theorem, we can prove that there exists $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that $x_{n_k} \to \widetilde{x} \in E$ uniformly on $[0,1]$. Therefore, x_{n_k} satisfies

$$x_{n_k}(t) = T(\lambda_{n_k}, x_{n_k})(t) = \lambda_{n_k} \int_0^1 G(t,s)g(s)f(x_{n_k}(s))ds, \quad t \in [0,1].$$

Passing to the limit as $k \to \infty$, we obtain

$$\widetilde{x}(t) = \frac{1}{lA} \int_0^1 G(t,s)g(s)f(\widetilde{x}(s))ds, \quad t \in [0,1].$$

Hence, $\widetilde{x} = T\left(\frac{1}{lA}, \widetilde{x}\right)$. □

Lemma 6. *Assume that (H_1) holds, and that $f(0) > 0$. Then, for any $x \in P$, there exist $U_x \geq V > 0$ such that*

$$VK_\lambda \leq T(\lambda, x)(t) \leq U_x K_\lambda, \quad t \in [0,1],$$

where

$$K_\lambda = \lambda \int_0^1 g(t)dt.$$

Proof. By (H_1), for any $x \in P$ and $t \in [0,1]$, we have

$$T(\lambda, x)(t) \geq \underline{G} f(0)\lambda \int_0^1 g(t)dt := VK_\lambda,$$

and

$$T(\lambda, x)(t) \leq \overline{G} f(\|x\|)\lambda \int_0^1 g(t)dt := U_x K_\lambda.$$

□

3. Main Results

Theorem 1. *Assume that $f_\infty = \infty$ and $0 < f_0 < \infty$. Then, for any $0 < \lambda < \frac{\mu_1}{f_0}$, BVP (1) admits a positive solution.*

Proof. Since $0 < \lambda < \frac{\mu_1}{f_0}$, there exist $\varepsilon > 0$ small enough and $r > 0$ such that $\lambda(f_0 + \varepsilon) < \mu_1$, and $\frac{f(s)}{s} < f_0 + \varepsilon$ for $s \in (0, r]$. We claim that

$$T(\lambda, x) \neq \mu x, \quad x \in \partial P_r, \ \mu \geq 1.$$

Otherwise, there exist $x_0 \in \partial P_r$ and $\mu_0 \geq 1$ such that $T(\lambda, x_0) = \mu_0 x_0$. Since $0 < \gamma r \leq x_0(t) \leq \|x_0\| = r$, we have

$$\mu_0 x_0(t) \leq \lambda(f_0 + \varepsilon) \int_0^1 G(t,s)g(s)x_0(s)ds = \lambda(f_0 + \varepsilon)Lx_0(t),$$

then $Lx_0(t) \geq \frac{\mu_0}{\lambda(f_0+\varepsilon)} x_0(t)$. Thus, $r(L) \geq \frac{\mu_0}{\lambda(f_0+\varepsilon)} \geq \frac{1}{\lambda(f_0+\varepsilon)}$. It follows that $\mu_1 \leq \lambda(f_0 + \varepsilon)$, which is a contradiction. Then, $i(T, P_r, P) = 1$.

Next, we prove that $i(T, P_R, P) = 0$ for some $R > r$. In fact, $f_\infty = \infty$ implies that $f(s) > Ms$ for some large $R_1 > 0$ and $s \geq R_1$, where $M > (\lambda\gamma A)^{-1}$. Let $R > \max\{r, \frac{R_1}{\gamma}\}$. For $x \in \partial P_R$, we have $x(t) \geq \gamma\|x\| = \gamma R > R_1, \ t \in [0,1]$, then

$$\|T(\lambda, x)\| \geq \lambda M \int_0^1 G(0,s)g(s)x(s)ds \geq \lambda M\gamma\|x\|A > \|x\|.$$

Hence, $i(T, P_R, P) = 0$, and $i(T, P_R \setminus \overline{P}_r, P) = -1$. Therefore, T admits a fixed point $x^* \in P_R \setminus \overline{P}_r$. □

Theorem 2. *Assume that (H_1) holds, and that $f_0 = f_\infty = \infty$. Then, BVP (1) has at least one and two positive solutions for $\lambda = \frac{1}{lA}$ and $\lambda \in (0, \frac{1}{lA})$, respectively.*

Proof. By Lemma 5, BVP (1) admits a positive solution for $\lambda = \frac{1}{lA}$. For $\lambda \in (0, \frac{1}{lA})$, by Lemmas 3 and 5, there exist $\tilde{x}, \ x_\lambda \in P \setminus \{\theta\}, \ x_\lambda \leq \tilde{x}$ such that

$$T\left(\frac{1}{lA}, \tilde{x}\right)(t) = \tilde{x}(t), \quad T(\lambda, x_\lambda)(t) = x_\lambda(t), \quad t \in [0,1].$$

If $x_\lambda = \tilde{x}$, we have

$$T(\lambda, x_\lambda) = x_\lambda = \tilde{x} = T\left(\frac{1}{lA}, \tilde{x}\right) = T\left(\frac{1}{lA}, x_\lambda\right).$$

This contradiction shows that $x_\lambda < \tilde{x}$.

Define $\Omega_1 = \{x \in E : -r < x(t) < \tilde{x}(t),\ t \in [0,1]\}$, where $r > 0$ is the same as in the first part of Theorem 1. For any $x \in P \cap \partial\Omega_1$, we obtain $\|x\| = \|\tilde{x}\|$, and

$$\|T(\lambda, x)\| < \frac{1}{lA} \int_0^1 G(0,s)g(s)f(\tilde{x}(s))ds = \tilde{x}(0) = \|\tilde{x}\|.$$

Therefore,

$$\|T(\lambda, x)\| < \|x\|, \quad x \in P \cap \partial\Omega_1.$$

As in the proof in Theorem 1, there is $R > 0$ large enough such that

$$\|T(\lambda, x)\| > \|x\|, \quad x \in P \cap \partial\Omega_2,$$

where $\Omega_2 = \{x \in E : \|x\| < R\}$. By compression expansion fixed point theorem, we see that T has a fixed point $\overline{x}_\lambda \in P \cap (\Omega_2 \setminus \overline{\Omega}_1)$. Since $x_\lambda \in \Omega_1,\ x_\lambda \neq \overline{x}_\lambda$, problem (1) has a second positive solution. □

Theorem 3. *Assume that (H_1) and (H_2) hold, and that $f(0) > 0$. Then, for any $\lambda \in (0, \infty)$, BVP (1) admits a unique positive solution $\dot{x}_\lambda(t)$, and $\dot{x}_\lambda(t)$ satisfies:*

(i) $\dot{x}_\lambda(t)$ *is nondecreasing with respect to* λ*;*
(ii) $\lim_{\lambda \to 0^+} \|\dot{x}_\lambda\| = 0,\ \lim_{\lambda \to \infty} \|\dot{x}_\lambda\| = \infty$*;*
(iii) $\|\dot{x}_\lambda - \dot{x}_{\lambda_0}\| \to 0$ *as* $\lambda \to \lambda_0$*.*

Proof. Since T is nondecreasing, for $u \in P$, we have

$$T(\lambda, \kappa x)(t) \geq \phi(\kappa)\lambda \int_0^1 G(t,s)g(s)f(x(s))ds = \phi(\kappa)T(\lambda, x)(t), \quad t \in [0,1]. \tag{4}$$

Define $\widehat{x}(t) = K_\lambda$, where K_λ is given by Lemma 6, then $\widehat{x} \in P$ and $VK_\lambda \leq T(\lambda, \widehat{x})(t) \leq U_x K_\lambda$. Denote

$$\overline{V} = \sup\{\mu : \mu K_\lambda \leq T(\lambda, \widehat{x})(t)\}, \quad \overline{U} = \inf\{\mu : \mu K_\lambda \geq T(\lambda, \widehat{x})(t)\},$$

then $\overline{V} \geq V$ and $\overline{U} \leq U_x$. Select $\widetilde{V}$ and $\widetilde{U}$ so that

$$0 < \widetilde{V} < \min\{1, F^{-1}(\overline{V})\}, \quad 0 < \frac{1}{\widetilde{U}} < \min\left\{1, F^{-1}\left(\frac{1}{\overline{U}}\right)\right\}.$$

We define

$$x_1(t) = \widetilde{V}K_\lambda,\ x_{k+1}(t) = T(\lambda, x_k)(t), \quad t \in [0,1],\ k = 1, 2, \cdots,$$

$$y_1(t) = \widetilde{U}K_\lambda,\ y_{k+1}(t) = T(\lambda, y_k)(t), \quad t \in [0,1],\ k = 1, 2, \cdots.$$

Combining the properties of T and (4), we get

$$\widetilde{V}K_\lambda = x_1(t) \leq x_2(t) \leq \cdots \leq x_k(t) \leq \cdots \leq y_k(t) \leq \cdots \leq y_2(t) \leq y_1(t) = \widetilde{U}K_\lambda. \tag{5}$$

Let $d = \frac{\widetilde{V}}{\widetilde{U}}$, obviously $0 < d < 1$. We claim that

$$x_k(t) \geq \phi_{k-1}(d)y_k(t), \quad t \in [0,1],\ k = 1, 2, \cdots, \tag{6}$$

where $\phi_0(d) = d,\ \phi_k(d) = \phi(\phi_{k-1}(d)),\ k = 1, 2, \cdots$. In fact, $x_1(t) = dy_1(t) = \phi_0(d)y_1(t),\ t \in [0,1]$. Suppose $x_n(t) \geq \phi_{n-1}(d)y_n(t)$ for $t \in [0,1]$, then

$$x_{n+1}(t) \geq T(\lambda, \phi_{n-1}(d)y_n)(t) \geq \phi(\phi_{n-1}(d))T(\lambda, y_n)(t) = \phi_n(d)y_{n+1}(t).$$

Hence, it follows by induction that (6) is true. According to (5) and (6), one has

$$0 \leq x_{n+m}(t) - x_n(t) \leq y_n(t) - x_n(t) \leq (1 - \phi_{n-1}(d))y_1(t) = (1 - \phi_{n-1}(d))\widetilde{U}K_\lambda,$$

where $m \geq 0$ is an integer. Thus,

$$\|x_{n+m} - x_n\| \leq \|y_n - x_n\| \leq (1 - \phi_{n-1}(d))\widetilde{U}K_\lambda. \tag{7}$$

We claim that $\lim_{n\to\infty} \phi_n(d) = 1$. From (H_2) and $0 < d < 1$, we see that $\phi(d) \in (d, 1)$ and $d = \phi_0(d) < \phi_1(d) < \cdots < \phi_n(d) < \cdots < 1$. Sequence $\{\phi_n(d)\}_{n=1}^{\infty}$ is increasing and bounded, there is $p \in [d, 1]$ such that $\lim_{n\to\infty} \phi_n(d) = p$. By the continuity of ϕ and $\phi_n(d) = \phi(\phi_{n-1}(d))$, we conclude that $p = \phi(p)$, i.e., $F(p) = 1$. It follows that $p = 1$. Inequality (7) implies that there exists $\overline{x} \in P$ such that $\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} y_n(t) = \overline{x}(t)$ for $t \in [0, 1]$. Clearly, $\overline{x}(t)$ is a positive solution of problem (1).

Suppose that $\bar{x}_1(t)$ and $\bar{x}_2(t)$ are positive solutions of problem (1), then $T(\lambda, \bar{x}_1)(t) = \bar{x}_1(t)$ and $T(\lambda, \bar{x}_2)(t) = \bar{x}_2(t),\ t \in [0, 1]$. Define $\tilde{\delta} = \sup\{\delta : \bar{x}_1(t) \geq \delta\bar{x}_2(t)\}$, then $\bar{x}_1(t) \geq \tilde{\delta}\bar{x}_2(t)$. We claim that $\tilde{\delta} \geq 1$. Otherwise, $\tilde{\delta} < 1$. Assumption (H_2) implies that $f(\tilde{\delta}\bar{x}_2(t)) > \varphi(\tilde{\delta})f(\bar{x}_2(t)),\ t \in [0, 1]$. Since f is nondecreasing,

$$\bar{x}_1(t) = T(\lambda, \bar{x}_1)(t) \geq T(\lambda, \tilde{\delta}\bar{x}_2)(t) > \phi(\tilde{\delta})T(\lambda, \bar{x}_2)(t) > \tilde{\delta}\bar{x}_2(t), \quad t \in [0, 1],$$

a contradiction. Then, $\bar{x}_1(t) \geq \bar{x}_2(t)$ for $t \in [0, 1]$. Similarly, $\bar{x}_2(t) \geq \bar{x}_1(t)$. Therefore, $\bar{x}_1(t) = \bar{x}_2(t),\ t \in [0, 1]$. This proves the uniqueness result.

Next, we show that $(i) - (iii)$ hold. Let

$$(Hx)(t) = \int_0^1 G(t, s)g(s)f(x(s))ds, \quad t \in [0, 1],$$

then $T(\lambda, x) = \lambda Hx$. Since $P^o = \{x \in P : x(t) > 0,\ t \in [0, 1]\}$ is nonempty, the operator $H : P^o \to P^o$ is increasing, and $H(\kappa x) \geq \phi(\kappa)Hx$ for $0 < \kappa < 1$. Let $\omega = \frac{1}{\lambda}$. We now write $Hx_\omega = \omega x_\omega$ instead of $\lambda Hx_\lambda = x_\lambda$. Assume $0 < \omega_1 < \omega_2$, then $x_{\omega_1} \geq x_{\omega_2}$. Indeed, denote $\overline{\omega} = \sup\{t > 0 : x_{\omega_1} \geq tx_{\omega_2}\}$, then $\overline{\omega} \geq 1$. Otherwise $0 < \overline{\omega} < 1$. Direct computations yield $\omega_1 x_{\omega_1} = Hx_{\omega_1} \geq H(\overline{\omega}x_{\omega_2}) \geq \phi(\overline{\omega})Hx_{\omega_2} = \phi(\overline{\omega})\omega_2 x_{\omega_2}$, then $x_{\omega_1} \geq \frac{\omega_2}{\omega_1}\phi(\overline{\omega})x_{\omega_2} > \overline{\omega}x_{\omega_2}$. This is a contradiction to the definition of $\overline{\omega}$. Thus, $\overline{\omega} \geq 1,\ x_{\omega_1} \geq x_{\omega_2}$, and further

$$x_{\omega_1} = \frac{1}{\omega_1}Hx_{\omega_1} \geq \frac{1}{\omega_1}Hx_{\omega_2} = \frac{\omega_2}{\omega_1}x_{\omega_2} \gg x_{\omega_2}, \quad 0 < \omega_1 < \omega_2. \tag{8}$$

Then, $x_\omega(t)$ is strong decreasing in ω, that is, $x_\lambda(t)$ is strong increasing in λ. Let $\omega_2 = \omega$ and fix ω_1 in (8), for $\omega > \omega_1$, we have $x_{\omega_1} \geq \frac{\omega}{\omega_1}x_\omega$, and

$$\|x_\omega\| \leq \frac{N\omega_1}{\omega}\|x_{\omega_1}\|,$$

where $N > 0$ is a normal constant of cone P. Because $\omega = \frac{1}{\lambda}$, then $\lim_{\lambda\to 0^+} \|x_\lambda\| = 0$. Let $\omega_1 = \omega$ and fix ω_2 in (8), we obtain $\lim_{\lambda\to+\infty} \|x_\lambda\| = +\infty$.

Finally, for given ω_0, by (8), we have

$$x_\omega \ll x_{\omega_0}, \quad \omega > \omega_0. \tag{9}$$

Let $t_\omega = \sup\{t > 0 : x_\omega \geq tx_{\omega_0},\ \omega > \omega_0\}$, then $0 < t_\omega < 1$ and $x_\omega \geq t_\omega x_{\omega_0}$. Direct computations yield $\omega x_\omega = Hx_\omega \geq H(t_\omega x_{\omega_0}) \geq \phi(t_\omega)Hx_{\omega_0} = \phi(t_\omega)\omega_0 x_{\omega_0}$. By the definition of t_ω, we have $\frac{\omega_0}{\omega}\phi(t_\omega) \leq t_\omega$, and

$$t_\omega \geq F^{-1}\left(\frac{\omega_0}{\omega}\right), \quad \forall \omega > \omega_0. \tag{10}$$

Combining (9) with (10), one has that

$$\|x_{\omega_0} - x_\omega\| \leq N\left[1 - F^{-1}\left(\frac{\omega_0}{\omega}\right)\right]\|x_{\omega_0}\| \to 0, \quad \omega \to \omega_0 + 0.$$

Similarly, $\|x_{\omega_0} - x_\omega\| \to 0$, $\omega \to \omega_0 - 0$. Hence, $\|x_{\omega_0} - x_\omega\| \to 0$ as $\omega \to \omega_0$. □

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